

# Sensorics exercises

Prof. Dr.-Ing. O. Nelles  
Measurement and Control Engineering - Mechatronics  
University of Siegen

12. Juni 2019

## 1 Introduction to Measurement Techniques

Mathematical Introduction

### Solution 1.1 Calculating with complex numbers

- a)  $\frac{1+2i}{3i+2} = \frac{(1+2i)(2-3i)}{13} = \frac{2+4i-3i-6i^2}{13} = \frac{8}{13} + \frac{1}{13}i$
- b)  $\frac{(1+2i)(2-3i)}{(1+2i)(2-2i)} = \frac{(2-3i)(2+2i)}{8} = \frac{4+4i-6i-6i^2}{8} = \frac{5}{4} - \frac{1}{4}i$
- c)  $\|2 + 3i\| = \sqrt{(4 + 9)} = \sqrt{13}$
- d)  $\|\frac{2+3i}{2i+2}\| = \frac{\sqrt{4+9}}{\sqrt{4+4}} = \sqrt{\frac{13}{8}}$
- e)  $\|(2 + i)^3\| = \|(2 + i)\|^3 = (\sqrt{5})^3 = 5^{\frac{3}{2}}$
- f)  $e^{i\pi} = -1$
- g)  $e^{i2\pi+2} = e^2$

### Solution 1.2 Magnitude and Phase

- a) Magnitude:  
 $\|1 + i\| = \sqrt{2}$

Phase:

Mit  $\text{atan}(\frac{1}{1}) = +45^\circ$  ergibt sich:

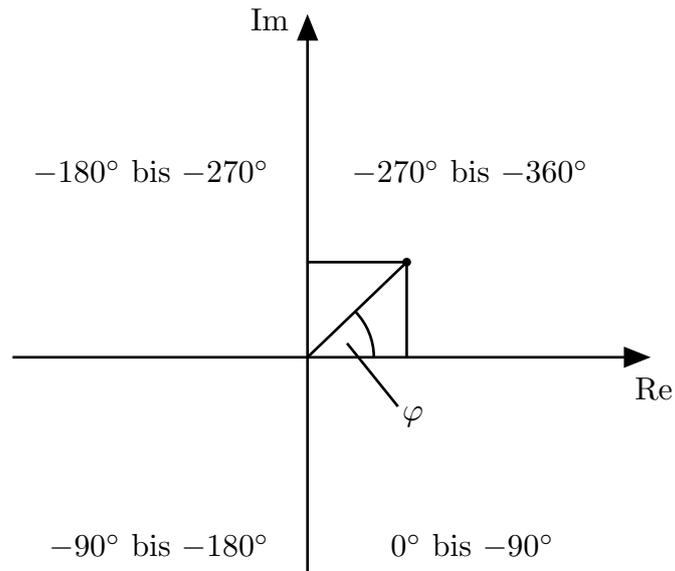
$$\arg(1 + i) = \varphi = -360^\circ + 45^\circ = -315^\circ$$

- b) Magnitude:  
 $\|1 - i\| = \sqrt{2}$

Phase:

Mit  $\text{atan}(\frac{1}{-1}) = -45^\circ$  ergibt sich:

$$\arg(1 - i) = \varphi = 0^\circ - 45^\circ = -45^\circ$$



c) Magnitude:

$$\left\| \frac{1}{1+2i\omega} \right\| = \frac{1}{\sqrt{1^2+4\omega^2}}$$

Phase:

$$\varphi = \arg\left(\frac{1}{1+2i\omega}\right) = \arg(1) - \arg(1+2i\omega)$$

$$\arg(1) = \operatorname{atan}(0) = 0^\circ$$

$$-\arg(1+2i\omega) = -\operatorname{atan}\left(\frac{2\omega}{1}\right) = 0^\circ$$

$$\text{F\AA}_{\frac{1}{4}} \omega \rightarrow 0: \varphi \rightarrow 0^\circ$$

$$\text{F\AA}_{\frac{1}{4}} \omega \rightarrow \infty: \varphi \rightarrow 90^\circ$$

d) Magnitude:

$$\|2e^{i\pi}\| = \|2\| \|e^{i\pi}\| = 2$$

Phase:

$$\arg(2e^{i\pi}) = \arg(2) + \arg(e^{i\pi}) = 0^\circ + 180^\circ = 180^\circ$$

### Solution 1.3 Complex Plane

a)  $\|z\| \leq 1$

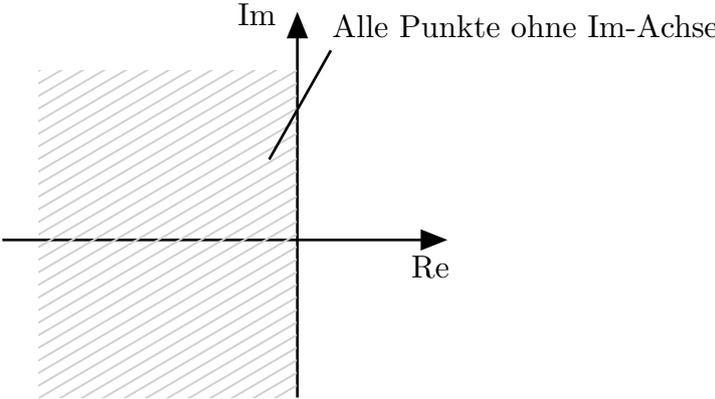
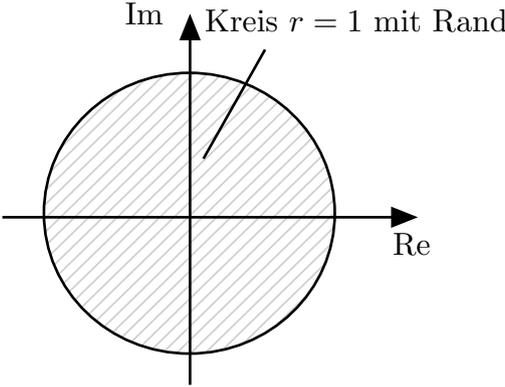
$$\sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2} \leq 1$$

$$x = \operatorname{Re}(z)$$

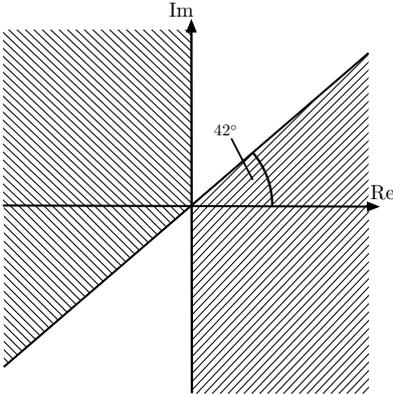
$$y = \operatorname{Im}(z)$$

$$\sqrt{x^2 + y^2} \leq 1$$

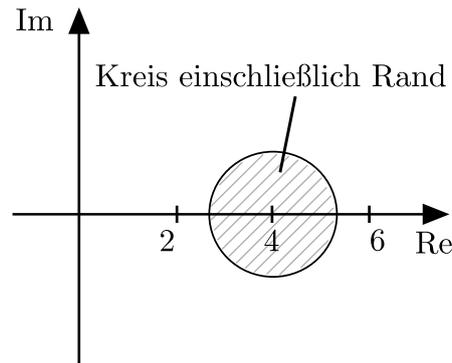
b)  $\operatorname{Re}(z) < 0$



c)  $\frac{\text{Im}(z)}{\text{Re}(z)} < 0.9$



d)  $\|z - 4\|^2 \leq 2$



### Solution 1.4 Continuous Time Systems

For the following transfer functions make a statement, about poles/zeros (i), about stability (ii), if the system is oscillatory (iii), whether the system is of minimal phase (iv), whether the system is proper or strictly proper (v), whether the system is linear (vi).

a)

$$G_1 = \frac{s + 1}{s^2 + 2s + 1}$$

b)

$$G_2 = \frac{s + 2}{s^2 + 0.2s + 1}$$

c)

$$G_3 = \frac{s^2}{s}$$

d)

$$G_4 = \frac{(s - 1)(s - 2)}{s + 3}$$

e)

$$G_5 = \frac{s + 1}{s - 1}$$

## 2 Measurement of Electrical Quantities

### Solution 2.1 Measuring voltage and current

a) Calculate  $I_0$ !

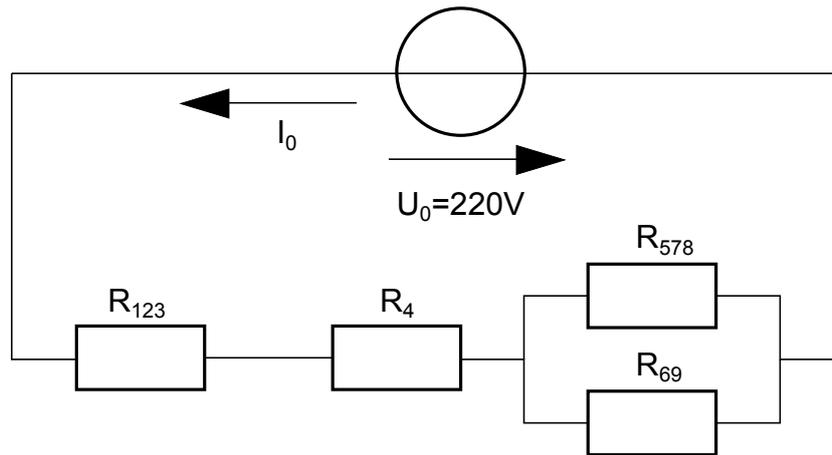


Figure 1: Linear electrical circuit.

$$I_0 = \frac{U_0}{R_{all}}$$

$$R_{all} = R_{123} + R_4 + \frac{1}{\frac{1}{R_{578}} + \frac{1}{R_{69}}}$$

$$R_{123} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}} = \frac{R_1 R_2 R_3}{R_1 R_2 + R_2 R_3 + R_1 R_3} = \frac{20 \cdot 40 \cdot 40 \cdot \Omega^3}{(20 \cdot 40 + 40^2 + 20 \cdot 40) \cdot \Omega^2} = 10\Omega$$

$$R_{578} = R_5 + \frac{1}{\frac{1}{R_7} + \frac{1}{R_8}} = 20\Omega + \frac{90 \cdot 45 \cdot \Omega^2}{(90 + 45) \cdot \Omega} = 50\Omega$$

$$R_{69} = R_6 + R_9 = 50\Omega$$

$$\Rightarrow R_{all} = 10\Omega + 20\Omega + \frac{50^2 \Omega^2}{100\Omega} = 55\Omega$$

$$\Rightarrow I_0 = \frac{220V}{55\Omega} = 4A$$

- b) Imagine to measure  $I_0$  by a moving coil mechanism with an internal resistance of  $R_m = 1\Omega$ . What value will be displayed? What relative error in current measurement occurs?

$$I_M = \frac{U_0}{R_{all} + R_M}$$

$$I_M = \frac{220V}{(55 + 1)\Omega} = 3.929A$$

$$E_{rI} = \frac{I_0 - I_M}{I_0}$$

$$E_{rI} = \frac{(4 - 3.929)A}{4} = 0.0179$$

- c) Calculate  $U_4$ !  
See figure 1.

$$U_4 = I_0 \cdot R_4 = 4A \cdot 20\Omega = 80V$$

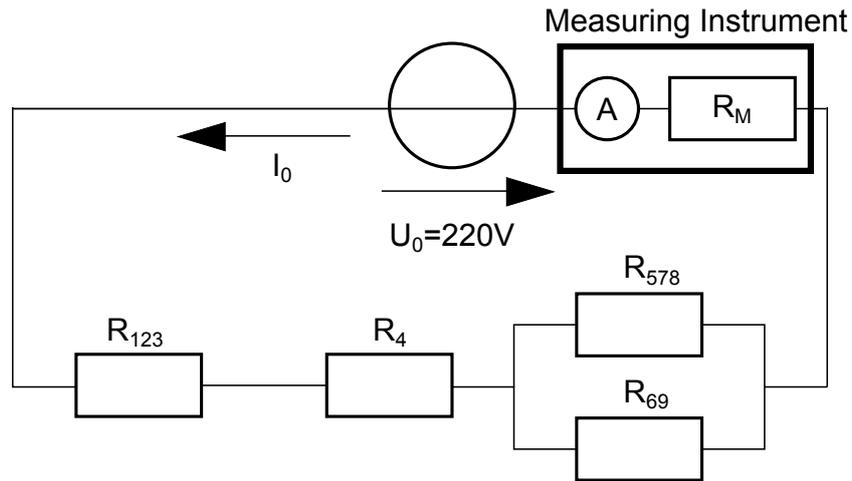


Figure 2: Electrical circuit with current measuring instrument.

- d) Imagine to measure  $U_4$  by a moving coil mechanism. Determine the internal resistance  $R_m$  for which the relative error in voltage measurement is less than 1%!

$$E_{rV} = \frac{U_4 - U_{M4}}{U_4}$$

$$U_4 = I_0 \cdot R_4$$

$$U_{M4} = \frac{R_4 R_M}{R_4 + R_M} \cdot I_0$$

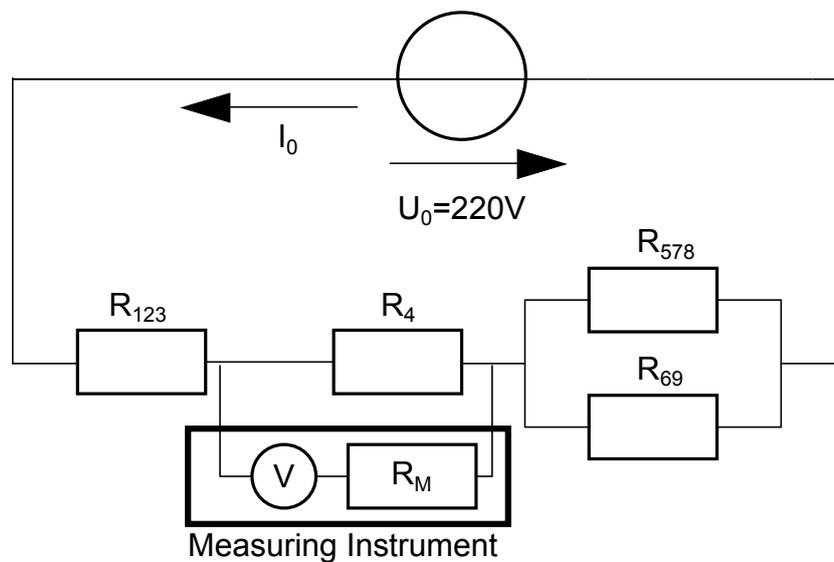


Figure 3: Electrical circuit with current measuring instrument.

$$\begin{aligned} \Rightarrow E_{rV} &\geq \frac{I_0 R_4 - \frac{R_4 R_M}{R_4 + R_M} I_0}{R_4 I_0} \\ E_{rV} &\geq 1 - \frac{R_4 R_M}{R_4 (R_4 + R_M)} \\ E_{rV} (R_4 + R_M) &\geq R_4 + R_M - R_M \\ R_M &\geq \frac{R_4 (1 - E_{rV})}{E_{rV}} \\ R_M &\geq \frac{20 \Omega (1 - 0.01)}{0.01} = 1980 \Omega \end{aligned}$$

Question to the audience: Which way to measure current would you prefer? The direct or indirect method, where indirect means to measure the voltage drop and calculate the corresponding current.

Answer: The indirect method, because the realization of high resistances is much easier and cheaper than the realization of very small and accurate resistances.

## Solution 2.2 Operational Amplifier Transfer Functions

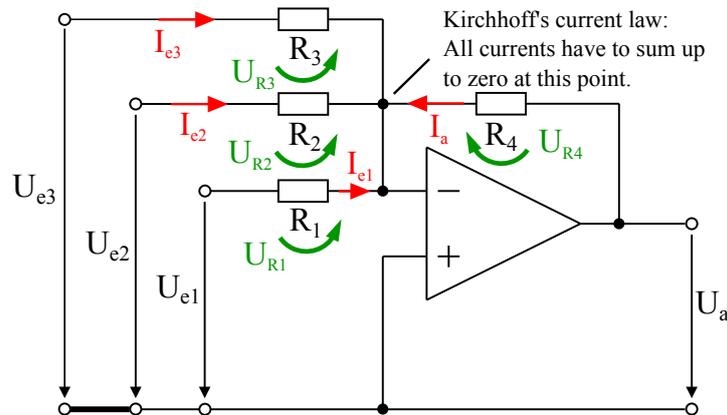


Figure 4: Voltage drop and current designations for the OpAMP circuit (a).

a) OpAmpCircuit 1

$$I_{e1} + I_{e2} + I_{e3} + I_a = 0 \quad (1)$$

$\Rightarrow$  Calculate currents  $I_{e1} - I_{e3}$  and  $I_a$ !

Mathematical expressions of currents should only depend on  $U_{e1}$ ,  $U_{e2}$ ,  $U_{e3}$  and  $U_a$

$\hookrightarrow$  Goal is to calculate the the equation describing the relationship between the inputs and the output:  $U_a = f(U_{e1}, U_{e2}, \dots U_{en})$

1)

$(U_{e1} - \text{Mesh})$  : Except of  $R_1$  there is no voltage drop!

**Kirchhoff's voltage law**  $\Rightarrow U_{e1} = U_{R1}$

$$\Rightarrow I_{e1} = \frac{U_{R1}}{R_1} \quad (2)$$

2)

$$(U_{e2} - \text{Mesh}) : I_{e2} = \frac{U_{R2}}{R_2} \text{ (analogous to mesh 1)} \quad (3)$$

3)

$$(U_{e3} - \text{Mesh}) : I_{e3} = \frac{U_{R3}}{R_3} \text{ (analogous to mesh 1)} \quad (4)$$

4)

$$(U_a - \text{Mesh}) : I_a = \frac{U_a}{R_4} \text{ (analogous to mesh 1)} \quad (5)$$

(2), (3), (4), (5) into (1) leads to:

$$\frac{U_{e1}}{R_1} + \frac{U_{e2}}{R_2} + \frac{U_{e3}}{R_3} + \frac{U_a}{R_4} = 0 \quad | - \frac{U_{e1}}{R_1}; - \frac{U_{e2}}{R_2}; \dots$$

$$\frac{U_a}{R_4} = - \frac{U_{e1}}{R_1} - \frac{U_{e2}}{R_2} - \frac{U_{e3}}{R_3} \quad | \cdot R_4$$

$$U_a = - \frac{R_4}{R_1} U_{e1} - \frac{R_4}{R_2} U_{e2} - \frac{R_4}{R_3} U_{e3}$$

b) OpAmp circuit 2

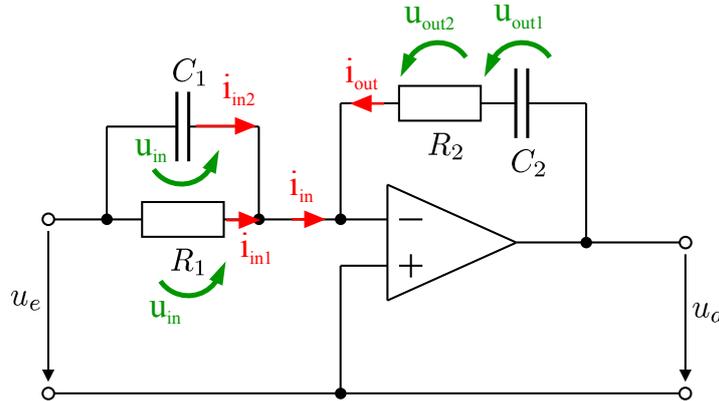


Figure 5: Voltage drop and current designations for the OpAMP circuit (b).

In general the electrical quantities of a capacitor can be calculated as follows:

$$u(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$$

$$i(t) = C \dot{u}(t) .$$

Connection of in- and output side:

$$i_{in} + i_{out} = 0 \tag{1}$$

Same goal as before:  $u_a = f(u_e)$ ; here all values depend on time!

Equations for input-side:

$$\begin{aligned} i_{in} &= i_{in1} + i_{in2} && \text{From now on } i_{in} \hat{=} i_{in}(t) \\ i_{in1} &= \frac{u_{in}}{R_1} && \text{From now on } u_{in} \hat{=} u_{in}(t) \\ i_{in2} &= C_1 \cdot \dot{u}_{in} && \text{From now on } u_{out} \hat{=} u_{out}(t) \\ \Rightarrow i_{in} &= \frac{u_{in}}{R_1} + C_1 \dot{u}_{in} && \text{From now on } i_{out} \hat{=} i_{out}(t) \end{aligned} \tag{2}$$

Equations for output-side:

$$\begin{aligned} u_{out} &= u_{out1} + u_{out2} \\ u_{out1} &= \frac{1}{C_2} \int_0^t i_{out}(\tau) d\tau ; u_{out2} = R_2 i_{out} \\ \Rightarrow u_{out} &= \frac{1}{C_2} \int_0^t i_{out}(\tau) d\tau + R_2 i_{out} \\ i_{out}(t) &= \frac{u_{out}}{R_2} - \frac{1}{C_2 R_2} \int_0^t i_{out} d\tau \end{aligned} \tag{3}$$

Equation 2 and 3 put in equation 1 equals to:

$$\frac{u_{in}}{R_1} + C_1 \dot{u}_{in} + \frac{u_{out}}{R_2} - \frac{1}{C_2 R_2} \int_0^t i_{out} d\tau = 0 \quad (4)$$

Solving Eqn. 4 for  $u_{out}$ :

$$u_{out} = \frac{R_2}{C_2 R_2} \int_0^t i_{out} d\tau - \frac{R_2}{R_1} u_{in} - C_1 R_2 \dot{u}_{in} \quad (5)$$

From Eqn. 1 we know, that  $i_{out} = -i_{in}$ . Together with Eqn. 2 we get:

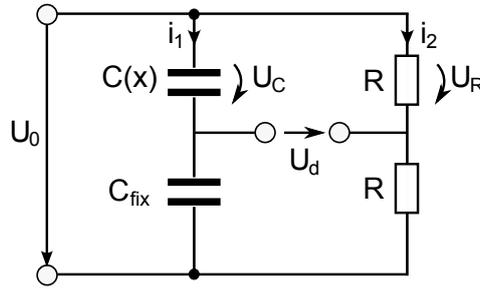
$$i_{out} = -\frac{1}{R_1} u_{in} - C_1 \dot{u}_{in} \quad (6)$$

Equation 6 in 5:

$$\begin{aligned} u_{out} &= \frac{R_2}{C_2 R_2} \int_0^t \left(-\frac{1}{R_1} u_{in} - C_1 \dot{u}_{in}\right) d\tau - \frac{R_2}{R_1} u_{in} - C_1 R_2 \dot{u}_{in} \\ u_{out} &= -\frac{1}{C_2 R_1} \int_0^t u_{in} d\tau - \frac{C_1}{C_2} u_{in} - \frac{R_2}{R_1} u_{in} - C_1 R_2 \dot{u}_{in} \end{aligned} \quad (7)$$

Finally we get:

$$u_{out}(t) = - \underbrace{\left(\frac{C_1}{C_2} + \frac{R_2}{R_1}\right)}_P u_{in}(t) - \underbrace{\frac{1}{R_1 C_2} \int_0^t u_{in}(\tau) d\tau}_I - \underbrace{R_2 C_1 \dot{u}_{in}}_D \quad (8)$$

**Solution 2.3 Bridge Circuit**Figure 6: Bridge circuit with variable capacitance  $c(x)$ .

Goal:  $U_d = f(C(x), C_{fix}, R, U_0)$

Complex impedance of a capacitor with capacitance  $C$ :

$$Z_C = \frac{-j}{\omega C}.$$

We are looking at the upper mesh and sum up all voltage drops:

$$U_d = U_R - U_C \quad (1)$$

The voltage drop over the capacitor and the resistance can be calculated as follows:

$$\begin{aligned} U_R &= \frac{R}{2R} U_0 \\ &= \frac{1}{2} U_0 \text{ and} \end{aligned} \quad (2)$$

$$\begin{aligned} U_C &= \frac{Z_{C(x)}}{Z_{C(x)} + Z_{C_{fix}}} U_0 \\ &= \frac{\frac{-j}{C(x)\omega}}{\frac{-j}{C(x)\omega} + \frac{-j}{C_{fix}\omega}} U_0 \\ &= \frac{C_{fix}}{C_{fix} + C(x)} U_0 \end{aligned} \quad (3)$$

(2) and (3) in (1):

$$U_R = \frac{1}{2} U_0 - \frac{C_{fix}}{C_{fix} + C(x)} U_0 \quad (4)$$

**Solution 2.4 Measurement of AC Quantities**

a) If the meter is used to measure the voltage, what value would be displayed?  
 1 Volt (The instrument shows only the mean value of an alternating voltage.)  
 → See script page 32

b) If the meter is used to measure the current, what value would be displayed?  
 1 A (The instrument shows only the mean value of an alternating current.)

c) In general a sinusoidal signal looks like the following equation:

$$y(t) = A \cdot \sin(\omega t + \varphi) + b$$

Equation for the voltage:

$$u(t) = 2 \sin\left(2\pi \frac{1}{T} \cdot t + \frac{2\pi}{8}\right) + 1$$

Equation for the current:

$$i(t) = 3 \sin\left(2\pi \frac{1}{T} \cdot t + 0\right) + 1$$

d)

$$X_{RMS} = \sqrt{\frac{1}{T} \int_0^T x^2(t) dt}$$

$$\Rightarrow U_{RMS} = \sqrt{\frac{1}{T} \int_0^T u^2(t) dt} \quad \text{and} \quad I_{RMS} = \sqrt{\frac{1}{T} \int_0^T i^2(t) dt}$$

$$i^2(t) = \left(3 \sin\left(2\pi \frac{1}{T} t\right) + 1\right)^2$$

$$\Leftrightarrow 9 \sin^2\left(2\pi \frac{1}{T} t\right) + 6 \sin\left(2\pi \frac{1}{T} t\right) + 1$$

$$I_{RMS} = \sqrt{\frac{1}{T} \int_0^T \left[9 \sin^2\left(2\pi \frac{1}{T} t\right) + 6 \sin\left(2\pi \frac{1}{T} t\right) + 1\right] dt}$$

$$\Leftrightarrow \sqrt{\frac{1}{T} \left[ \underbrace{\int_0^T 9 \sin^2\left(2\pi \frac{1}{T} t\right) dt}_{I_1} + \underbrace{\int_0^T 6 \sin\left(2\pi \frac{1}{T} t\right) dt}_{I_2} + \underbrace{\int_0^T 1 dt}_{I_3} \right]}$$

$$I_1 = \int_0^T 9 \sin^2\left(2\pi \frac{1}{T} t\right) dt$$

with the power reduction formula:

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2} \Rightarrow \sin^2\left(2\pi \frac{1}{T} t\right) = \frac{1 - \cos\left(4\pi \frac{1}{T} t\right)}{2}$$

$$\begin{aligned} &\Leftrightarrow \frac{9}{2} \int_0^T 1 - \cos\left(4\pi \frac{1}{T}t\right) dt \\ &\Leftrightarrow \frac{9}{2} \left[ t \Big|_0^T - \frac{T}{4\pi} \sin\left(4\pi \frac{1}{T}t\right) \Big|_0^T \right] \\ &\Leftrightarrow \frac{9}{2}T = I_1 \end{aligned}$$

$I_2 = 0$  (Integration of sine over a full cycle)

$$I_3 = T$$

$$\begin{aligned} I_{RMS} &= \sqrt{\frac{1}{T} \left[ \frac{9}{2}T + T \right]} \\ &= \sqrt{\frac{11}{2}} \end{aligned}$$

$U_{RMS}$  analogous!

$$\begin{aligned} \Rightarrow I_{RMS} &= \sqrt{\frac{1}{T} \left[ \frac{4}{2}T + T \right]} \\ &= \sqrt{3} \end{aligned}$$

### 3 Measurement of Non-Electrical Quantities

#### Solution 3.1 Fill Level Measuring

a) Draw an expressive sketch.

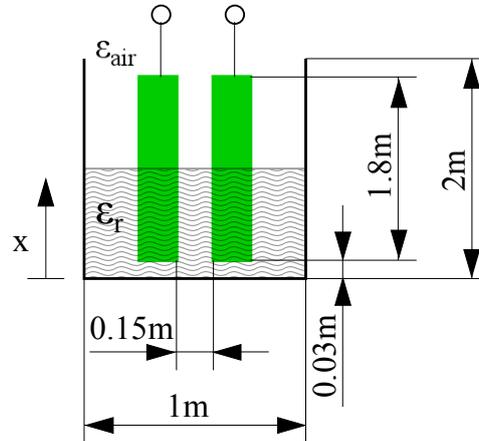


Figure 7: Sketch: Fill level of a tank

$$\epsilon_0 = 8.85 \cdot 10^{-12} \frac{As}{Vm}$$

$$\epsilon_{air} = 1$$

$$\epsilon_r = 3$$

b) Derive a function of the capacity  $C$  depending on the fill level  $x$ .

In general:

$$C = \frac{\epsilon_0 \epsilon_r A}{d},$$

with the plate separation  $d$  and the one-sided area  $A$ . The overall capacity follows the rule of 2 capacitors in parallel:

$$C = C_1 + C_2$$

Case by case analysis ( $w_p$ : plate width,  $x$ : fill level,  $l$ : plate length):

1)  $x \leq 0.03m$  :

$$C = \frac{\epsilon_0 \epsilon_{air} w_p \cdot l}{d}$$

2)  $0.03 < x < 1.83m$  :

$$C = \frac{\epsilon_0 \epsilon_r w_p}{d} (x - 0.03) + \frac{\epsilon_0 \epsilon_{air} w_p}{d} (1.83 - x)$$

$$\Leftrightarrow \frac{\epsilon_0 w_p}{d} [\epsilon_r x - \epsilon_r 0.03 + 1.83 \epsilon_{air} - x \epsilon_{air}]$$

$$\Leftrightarrow \frac{\epsilon_0 w_p}{d} (\epsilon_r - \epsilon_{air}) x + \frac{\epsilon_0 w_p}{d} (1.83 \epsilon_{air} - 0.03 \epsilon_r)$$

$$3) \quad x \geq 1.83\text{m}$$

$$C = \frac{\epsilon_0 \epsilon_r w_p \cdot l}{d}$$

$$C(x) = \begin{cases} \frac{\epsilon_0 \epsilon_{air} w_p \cdot l}{d} & \text{for } x \leq 0.03\text{m} \\ \frac{\epsilon_0 w_p}{d} (\epsilon_r - \epsilon_{air}) x + \frac{\epsilon_0 w_p}{d} (1.83 \epsilon_{air} - 0.03 \epsilon_r) & \text{for } 0.03\text{m} < x < 1.83\text{m} \\ \frac{\epsilon_0 \epsilon_r w_p \cdot l}{d} & \text{for } x \geq 1.83\text{m} \end{cases}$$

c) Sketch the curve of the capacity  $C$  depending on the fill level  $x$ . Up to  $x = 0.03\text{m}$ ,

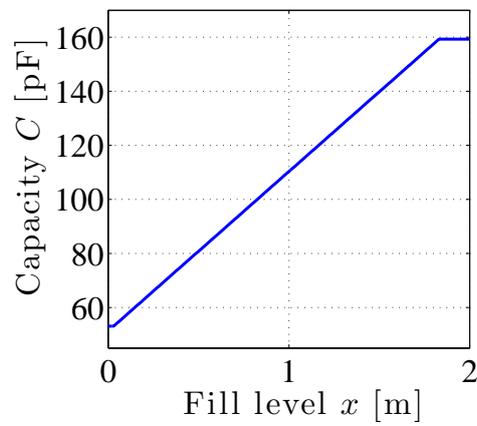


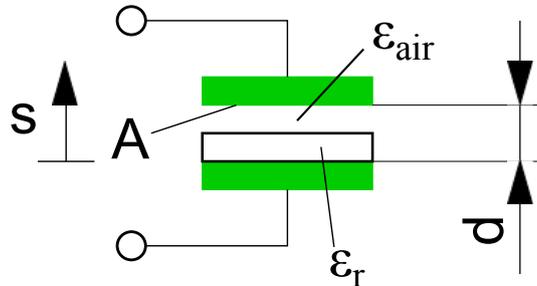
Figure 8: Sketch of  $C(x)$

the value is  $C = 53.1\text{pF}$ . From  $x = 1.83\text{m}$  the value stays at  $C = 159.3\text{pF}$ .

**Solution 3.2 Thickness Measurement**

a) Derive formula

Sketch:

Figure 9: Sketch of  $C(x)$ 

The air and the plastic film can be treated as a series connection of 2 capacitors:

$$\frac{1}{C_{tot}} = \sum_{i=1}^N \frac{1}{C_i} \stackrel{N=2}{\Rightarrow} \frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$C_1 = \frac{\epsilon_r \epsilon_0 A}{s} \text{ (plastic film)} ; C_2 = \frac{\epsilon_{air} \epsilon_0 A}{d-s} \text{ (Air)}$$

$$\frac{1}{C} = \frac{s}{\epsilon_r \epsilon_0 A} + \frac{d-s}{\epsilon_{air} \epsilon_0 A} \quad | \cdot T$$

$$\Leftrightarrow \frac{1}{C} = \frac{s\epsilon_{air} + (d-s)\epsilon_r}{\epsilon_r \epsilon_{air} \epsilon_0 A} \quad | \cdot T$$

$$\Leftrightarrow \frac{\epsilon_r \epsilon_{air} \epsilon_0 A}{C} = (\epsilon_{air} - \epsilon_r)s + d\epsilon_r \quad | \cdot T$$

$$\Leftrightarrow s = \frac{\epsilon_r \epsilon_{air} \epsilon_0 A}{C(\epsilon_{air} - \epsilon_r)} - \frac{d\epsilon_r}{\epsilon_{air} - \epsilon_r}$$

b) Determine reasonable lower and upper bounds for the capacity  $C$ :

Looking at the general equation

$$C = \frac{\epsilon_r \epsilon_0 A}{d}, \quad (5)$$

one can see, that the capacity increases with an increasing permittivity. Here the permittivity of air is lower than the permittivity of the plastic film. So we can conclude, that the minimum capacity is obtained, if no plastic film is present and the maximum capacity is obtained if no air is left between the two capacitor plates.

$$C_{min} = 33.2 \text{ nF and}$$

$$C_{max} = 100 \text{ nF}$$

C [nF]	33,2	40	60	80	100
s [ $\mu\text{m}$ ]	0	204	536	702	802

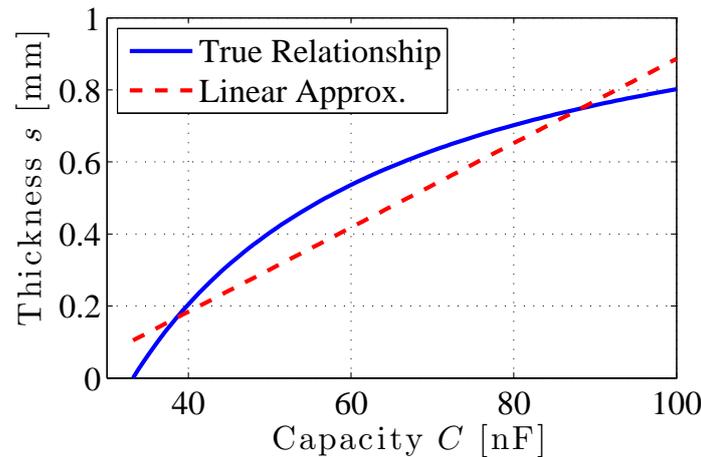


Figure 10: Sketch of  $C(x)$

c) Error function

$$\begin{aligned} E(c) &= s(c) - g(c) \\ &= \frac{\epsilon_r \epsilon_{air} \epsilon_0 A}{C(\epsilon_{air} - \epsilon_r)} - \frac{d\epsilon_r}{\epsilon_{air} - \epsilon_r} - 11700C + 3 \cdot 10^{-4} \\ &= \frac{K_1}{C K_2} - K_3 - 11700C + 3 \cdot 10^{-4} \end{aligned}$$

$$\text{with } K_1 = \epsilon_r \epsilon_{air} \epsilon_0 A = 7,9650 \cdot 10^{-11}$$

$$K_2 = -2 = \epsilon_{air} - \epsilon_r$$

$$K_3 = \frac{d\epsilon_r}{\epsilon_{air} - \epsilon_r} = -0,0012$$

The global maximum of the absolute error is either on one edge or in between  $C \in [33,2 ; 100]$  nF

$$\text{Left edge: } E(C=33,2\text{nF}) = -8,7988 \cdot 10^{-5} \text{m} = -87,988 \mu\text{m}$$

$$\text{Right edge: } E(C=100\text{nF}) = -6,8250 \cdot 10^{-5} = -68,250 \mu\text{m}$$

To find the (local) maximum (maximum because the real function lays above the regression line, see figure) of the error function we have to derive  $E(C)$  with respect to  $C$ :

$$E'(C) = \frac{dE(C)}{dC} = \frac{-K_1}{C^2 K_2} - 11700$$

To find local extrema we have to:

$$\begin{aligned}
 E'(C) &\stackrel{!}{=} 0 \\
 0 &= \frac{-K_1}{C^2 K_2} - 11700 && \left| + 11700 \right. \\
 \Leftrightarrow 11700 &= \frac{-K_1}{C^2 K_2} && \left| \cdot C_2; : 11700 \right. \\
 \Leftrightarrow C^2 &= \frac{-K_1}{K_2 11700} && \left| \sqrt{\quad} \right. \\
 \Leftrightarrow C_{max} &= \sqrt{\frac{-K_1}{K_2 11700}} \Rightarrow 5,8342 \cdot 10^{-8} = 58,342 nF \\
 E(C_{max}) &= 1,3479 \cdot 10^{-4} = 134,79 \mu m \leftarrow \text{maximum error!} \\
 \Rightarrow E_r &= \frac{g(C) - s(C)}{s(C)} = \boxed{-0,2605}
 \end{aligned}$$

**Solution 3.3 Torque Measurement with Strain Gauges**

a) Sketch

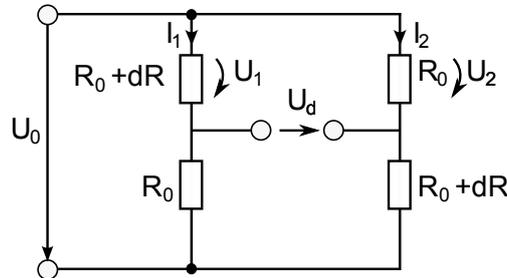


Figure 11: Electrical circuit with current measuring instrument.

b) Transfer function  $U_d$ 

$$I_1 = \frac{U_0}{2R_0 + \Delta R} \quad (1)$$

$$I_2 = \frac{U_0}{2R_0 + \Delta R} \quad (2)$$

$$U_1 = (R_0 + \Delta R)I_1;$$

$$U_2 = R_0 I_2$$

$$U_d = U_2 - U_1$$

$$= R_0 \cdot I_2 - (R_0 + \Delta R)I_1 \quad | \text{ with (1) and (2)}$$

$$= \frac{R_0 \cdot U_0}{2R_0 + \Delta R} - \frac{R_0 + \Delta R}{2R_0 + \Delta R} U_0$$

$$= \frac{R_0 - R_0 - \Delta R}{2R_0 + \Delta R} U_0$$

$$= \frac{-\Delta R}{2R_0 + \Delta R} U_0$$

→ Question: Can anybody explain why we got the minus sign in the numerator?

⇒ Assumed direction of  $U_d$  is false! Voltage drop over  $(R + \Delta R)$  is bigger !

c)  $M_A(U_d)$ :

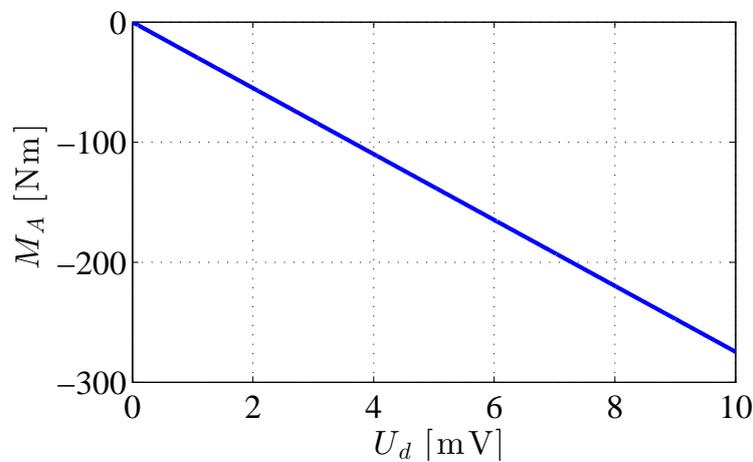
$$U_d = \frac{-\Delta R}{2R_0 + \Delta R} U_0 \quad (3)$$

$$\Delta R = \frac{R_0 k M_A}{\pi r^3 G} \quad (4)$$

4 in 3:

$$\begin{aligned}
 U_d &= \frac{-R_0 k M_A}{\pi r^3 G (2R_0 + \frac{R_0 k M_A}{\pi r^3 G})} U_0 && \left| \begin{array}{l} T \\ \text{Denominator} \end{array} \right. \\
 U_d &= \frac{-R_0 k M_A U_0}{2R_0 \pi r^3 G + R_0 k M_A} \\
 \Leftrightarrow 2R_0 \pi r^3 G U_d + R_0 k M_A U_d &= -R_0 k M_A U_0 && \left| \begin{array}{l} +R_0 k M_A U_0; -2\pi r^3 G R_0 U_d \end{array} \right. \\
 \Leftrightarrow R_0 k M_A U_d + R_0 k M_A U_0 &= -2R_0 \pi r^3 G U_d && \left| \begin{array}{l} T \end{array} \right. \\
 \Leftrightarrow M_A (U_d + U_0) R_0 k &= -2R_0 \pi r^3 G U_d && \left| \begin{array}{l} : (U_d + 1) R_0 \end{array} \right. \\
 \Leftrightarrow M_A &= \frac{-2R_0 \pi r^3 G}{(U_0 + U_d) R_0 k} U_d
 \end{aligned}$$

$U_d$ [mV]	0	2	4	6	8	10
$M_A$ [Nm]	0	-55	-110	-165	-220	-275

Figure 12: Sketch: Torque  $M_A(U_d)$ 

→ Within the chosen range of  $U_d$  is nearly linear!

## 4 Digital Measurement Techniques

### Solution 4.1

MATLAB!

### Solution 4.2

MATLAB!

### Solution 4.3 Subsampling of a Time Signal

a)

$$y(t) = \sin(2\pi \cdot 10Hz \cdot t) \quad f_0 = 9Hz$$

Shannon's sampling theorem:

$$\begin{aligned} f_0 &> 2f_{max} \\ 9Hz &> 2 \cdot 10Hz \quad \text{!} \end{aligned}$$

Because of the sampling we can transform the time  $t$ :

$$\begin{aligned} t &= \frac{1}{f_0}k = T_0k \quad \text{with } k = 0, 1, 2, \dots \\ \Rightarrow y(k) &= \sin\left(2\pi \frac{10Hz}{f_0}k\right) \\ &= \sin\left(2\pi \frac{10}{9}k\right) \\ &= \sin\left(2\pi\left(1 + \frac{1}{9}\right)k\right) \\ &= \sin\left(2\pi k + \frac{2\pi}{9}k\right) \end{aligned}$$

Because the sine is  $2\pi$ -periodic, this is the same as:

$$\begin{aligned} y(k) &= \sin\left(\frac{2\pi}{9}k\right) \\ &= \sin\left(2\pi \cdot \underbrace{1Hz}_{\text{measured frequency}} \cdot \underbrace{\frac{1}{9Hz}k}_t\right) \end{aligned}$$

2nd way to calculate the measured frequency is missing just right now... See handwritten notes!

We know that:

$$\omega_m = \omega_s \pm l\omega_0$$

$$\text{with } l = \dots, -2, -1, 0, +1, \dots$$

$$\Leftrightarrow f_m = f_s \pm lf_0$$

$$\text{Here: } \left| \frac{f_0}{2} \right| = 4,5Hz$$

$$f_m = 10Hz \pm l \cdot 9Hz$$

Find  $l$  such that

$$-\frac{f_0}{2} < f_m < \frac{f_0}{2}$$

$$\Rightarrow l = -1 \Rightarrow f_m = 1Hz$$

## 5 Measurement Errors and Statistics

### Solution 5.1 Correlation Analysis

a) Determine the coordinates of all data points in figure 1 and list them in a table.

Tabelle 1: Coordinates of all data points.

$k$	1	2	3	4	5
$x$	-1	-0.5	0	0.5	1
$y$	1.1	0.2	0	0.2	1.1

b) Calculate the correlation coefficient for all data points of figure 1.

Equation to calculate the correlation coefficient:

$$\rho_{xy} = \frac{\frac{1}{N-1} \sum_{k=1}^N (x(k) - \bar{x})(y(k) - \bar{y})}{\sigma_x \sigma_y} . \quad (5)$$

Because we do not know the mean values of  $x$ ,  $y$  and their standard deviations, these values are approximated as follows:

$$\bar{z} = \frac{1}{N} \sum_{k=1}^N z(k) , \quad (6)$$

$$\hat{\sigma}_z = \sqrt{\frac{1}{N-1} \sum_{k=1}^N (z(k) - \bar{z})^2} . \quad (7)$$

The approximated values of the mean and the standard deviation of  $x$  are:

$$\bar{x} = \frac{1}{5} [(-1) + (-0.5) + 0 + 0.5 + 1] = 0 . \quad (8)$$

$$\hat{\sigma}_x = \sqrt{\frac{1}{5-1} [(-1)^2 + (-0.5)^2 + 0^2 + 0.5^2 + 1^2]} = \sqrt{\frac{2.5}{4}} \approx 0.79 . \quad (9)$$

The mean value as well as the standard deviation for  $y$  are missing:

$$\bar{y} = \frac{1}{5} [1.1 + 0.2 + 0 + 0.2 + 1.1] = \frac{2.6}{5} = 0.52 . \quad (10)$$

$$\hat{\sigma}_y = \sqrt{\frac{1}{5-1} [(1.1 - 0.52)^2 + (0.2 - 0.52)^2 + (0 - 0.52)^2 + \dots]} \quad (11)$$

$$\dots + (0.2 - 0.52)^2 + (1.1 - 0.52)^2] \approx \sqrt{\frac{1.15}{4}} \approx 0.54 . \quad (12)$$

The numerator  $\rho_{xy2} \cdot \sigma_x \sigma_y$  is calculated in the following:

$$\rho_{xy} \cdot \sigma_x \sigma_y = \frac{1}{5-1} [-1 \cdot (1.1 - 0.52) + -0.5 \cdot (0.2 - 0.52) + \dots] \quad (13)$$

$$0 \cdot (0 - 0.52) + 0.5 \cdot (0.2 - 0.52) + 1 \cdot (1.1 - 0.52)] \quad (14)$$

$$= \frac{0}{4} = 0 . \quad (15)$$

The correlation coefficient is exactly zero, which means, that there is no (linear) correlation between  $x$  and  $y$ . Even though the correlation is very low, we see that in fact there is a dependency between the two variables.

- c) Now calculate two correlation coefficients for figure 1. For the first one use all data points, where the  $x$  values are smaller or equal to zero. For the second one use all data points, where the  $x$  values are bigger or equal to zero.

Correlation coefficient for all data points where  $x \leq 0$ :

New values arise for the mean and standard deviation of  $x$  and  $y$  as well as for the numerator  $\rho_{xyL} \cdot \sigma_{xL} \sigma_{yL}$ :

$$\bar{x}_L = \frac{1}{3} [(-1) + (-0.5) + 0] = -0.5 . \quad (16)$$

$$\hat{\sigma}_{xL} = \sqrt{\frac{1}{3-1} [(-1 - (-0.5))^2 + (-0.5 - (-0.5))^2 + \dots]} \quad (17)$$

$$\dots + (0 - (-0.5))^2] = 0.5 . \quad (18)$$

$$\bar{y}_L = \frac{1}{3} [0 + 0.2 + 1.1] \approx 0.43 . \quad (19)$$

$$\hat{\sigma}_{yL} = \sqrt{\frac{1}{3-1} [(0 - 0.43)^2 + (0.2 - 0.43)^2 + (1.1 - 0.43)^2]} \approx 0.59 . \quad (20)$$

$$\rho_{xyL} \cdot \sigma_{xL} \sigma_{yL} = \frac{1}{3-1} [(-1 - (-0.5)) \cdot (1.1 - 0.43) + \dots] \quad (21)$$

$$\dots + (-0.5 - (-0.5)) \cdot (0.2 - 0.43) + \dots \quad (22)$$

$$\dots + (0 - (-0.5)) \cdot (0 - 0.43)] \approx -0.28 . \quad (23)$$

With these values the correlation coefficient for the left hand side is given by:

$$\rho_{xyL} \approx \frac{-0.28}{0.5 \cdot 0.59} \approx -0.95 . \quad (24)$$

The correlation coefficient for all data points where  $x \geq 0$ :

Nearly all values from the left hand side maintain except for the mean of  $x$  and the numerator  $\rho_{xyR} \cdot \sigma_{xR} \sigma_{yR}$ :

$$\bar{x}_R = \frac{1}{3} [0 + 0.5 + 1] = 0.5 . \quad (25)$$

$$\rho_{xyR} \cdot \sigma_{xR} \sigma_{yR} = \frac{1}{3-1} [(1-0.5) \cdot (1.1-0.43) + \dots] \quad (26)$$

$$\dots + (0.5-0.5) \cdot (0.2-0.43) + \dots \quad (27)$$

$$\dots + (0-0.5) \cdot (0-0.43)] \approx 0.28 . \quad (28)$$

The absolute value of the correlation coefficient for the right hand side maintains:

$$\rho_{xyR} \approx \frac{0.28}{0.5 \cdot 0.59} \approx 0.95 . \quad (29)$$

For both cases we get an absolute correlation coefficient near the possible maximum (which is 1). This indicates a high correlation between  $x$  and  $y$  if we do not incorporate all measured samples. Conclusion: High correlation values only reveal a linear correlation between variables. More complex dependencies can not be revealed.

### Solution 5.2 Autocorrelation Function

- a) Figure out which positive time shifts  $\tau$  are likely to have the highest autocorrelation function value, except for  $\tau = 0$ .

In general the following two equations are used to calculate the auto-correlation functions for positive time-shifts  $\tau$ :

$$r_{xyBias}(\tau) = \frac{1}{N} \sum_{k=1}^{N-|\tau|} x(k)x(k+\tau), \quad (30)$$

$$r_{xy}(\tau) = \frac{1}{N-|\tau|} \sum_{k=1}^{N-|\tau|} x(k)x(k+\tau). \quad (31)$$

By looking at figures 13 and 14 we realize, that there should be a maximum for the biased as well as for the unbiased auto-correlation function at  $\tau_1 = 1$ ,  $\tau_2 = 4$ ,  $\tau_3 = 8$  and  $\tau_4 = 12$ . We should keep in mind that the values left to calculate the

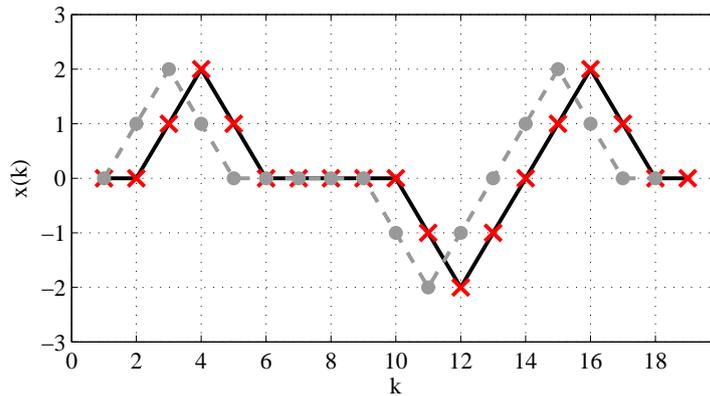


Figure 13: Function together with time-shifted function  $\tau = 1$ .

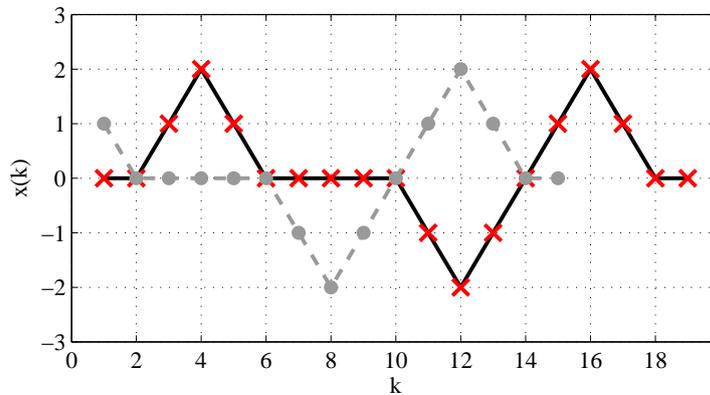


Figure 14: Function together with time-shifted function  $\tau = 4$ .

auto-correlation function decreases with an increasing time-shift  $\tau$ .

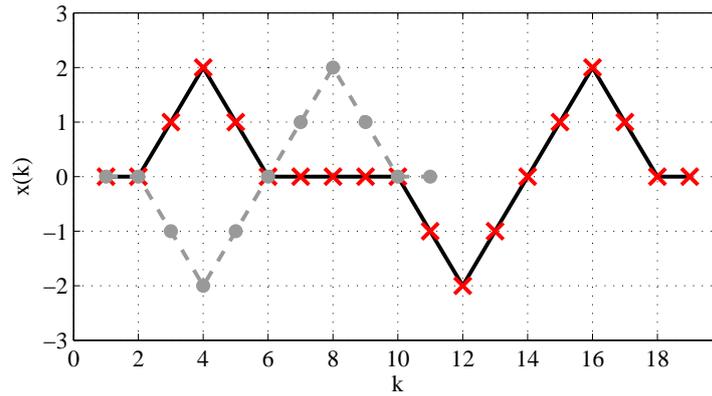


Figure 15: Function together with time-shifted function  $\tau = 8$ .

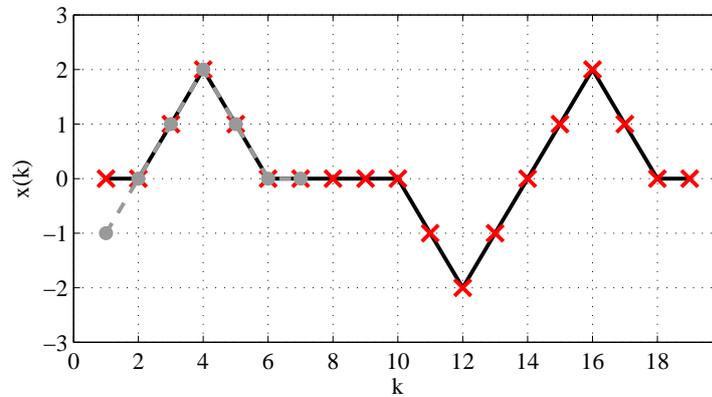


Figure 16: Function together with time-shifted function  $\tau = 12$ .

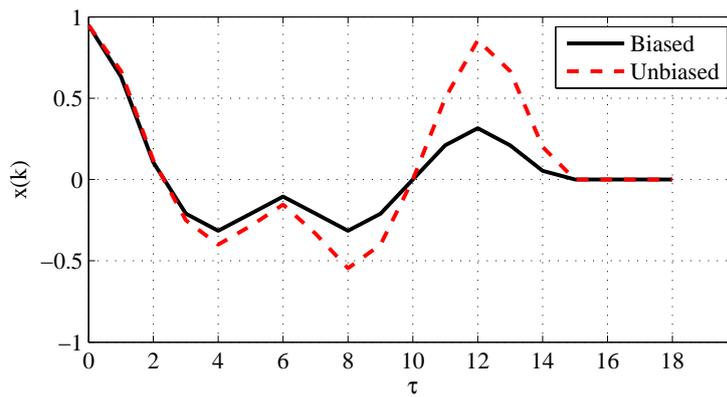


Figure 17: Biased and unbiased auto-correlation functions.

- b) Calculate the autocorrelation function values for the time shifts found in a) with the unbiased equation.

For the time-shift  $\tau = 1$  six summands are not zero ( $N - |\tau| = 18$ ):

$$r_{xx}(\tau = 1) = \frac{1}{18}[x(3) \cdot x(3 + 1) + x(4) \cdot x(4 + 1) + x(11) \cdot x(11 + 1) + \dots] \quad (32)$$

$$\dots + x(12) \cdot x(12 + 1) + x(15) \cdot x(15 + 1) + x(16) \cdot x(16 + 1)] \quad (33)$$

$$= \frac{1}{18}[6 \cdot 2] \quad (34)$$

$$= \frac{12}{18} \approx 0.67 \quad (35)$$

For the time-shift  $\tau = 4$  only three summands are not zero ( $N - |\tau| = 15$ ):

$$r_{xx}(\tau = 4) = \frac{1}{15}[x(11) \cdot x(11 + 4) + x(12) \cdot x(12 + 4) + x(13) \cdot x(13 + 4)] \quad (36)$$

$$= \frac{1}{15}[1 \cdot (-1) + 2 \cdot (-2) + 1 \cdot (-1)] \quad (37)$$

$$= \frac{-6}{15} \approx -0.40 \quad (38)$$

For the time-shift  $\tau = 8$  the value for  $N - |\tau| = 11$  changes:

$$r_{xx}(\tau = 8) = \frac{-6}{11} \approx -0.55 \quad (39)$$

For the time-shift  $\tau = 12$  the value for  $N - |\tau| = 7$  changes as well as the sign of the resulting numerator:

$$r_{xx}(\tau = 12) = \frac{1}{7}[x(3) \cdot x(3 + 12) + x(4) \cdot x(4 + 12) + x(5) \cdot x(5 + 12)] \quad (40)$$

$$= \frac{1}{7}[1 \cdot 1 + 2 \cdot 2 + 1 \cdot 1] \quad (41)$$

$$= \frac{6}{7} \approx 0.86 \quad (42)$$

- c) Calculate the autocorrelation function values for the time shifts found in a) with the biased equation.

For this step we can use the results from b). All summands stay the same, but the denominator  $N - |\tau|$  becomes  $N$  and therefore stays the same for all three time-shifts.  $\tau_1 = 1$ :

$$r_{xxBiased}(\tau = 1) = \frac{12}{19} \approx 0.63 \quad (43)$$

For the time-shifts  $\tau = 4$  and  $\tau = 8$  the auto-correlation value is ( $N = 19$ ):

$$r_{xxBiased}(\tau = 4) = r_{xxBiased}(\tau = 8) = \frac{-6}{19} \approx -0.32 \quad (44)$$

For the time-shift  $\tau = 12$  only three summands are not zero ( $N = 19$ ):

$$r_{xxBiased}(\tau = 12) = \frac{6}{19} \approx 0.32 \quad (45)$$

In case of the biased auto-correlation estimation the division through  $N$  instead of  $N - |\tau|$  leads to a systematic error. The absolute value of this estimation is always smaller than the unbiased one. This takes into account, that the uncertainty of the estimation increases due to an increase in the variance error as will be explained in the following. The number of values that are available to estimate the auto-correlation function decreases with an increasing time-shift  $\tau$ . Therefore the calculation is based on fewer data points, which leads to a lower averaging effect and which makes the result more sensitive to noisy data.

**Examples for Auto-Correlation Functions**

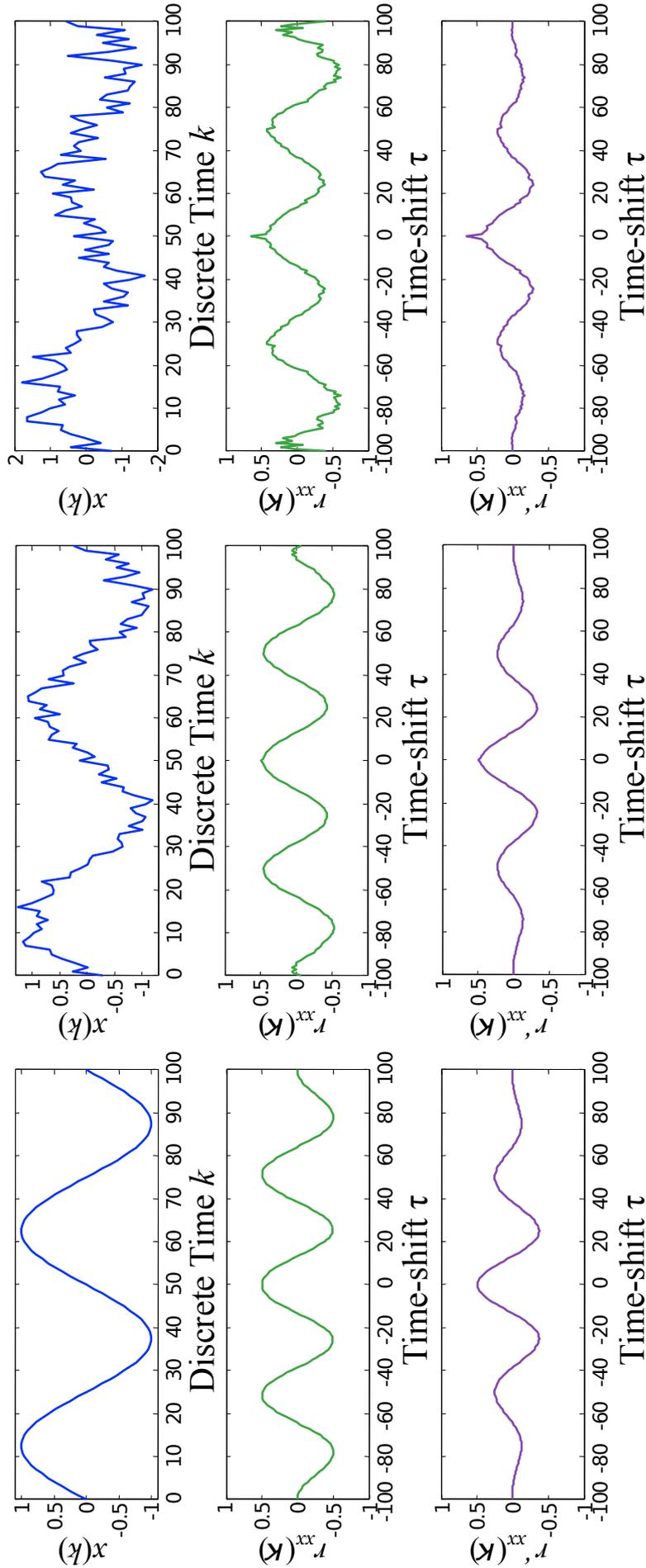


Figure 18: Biased versus unbiased Auto-Correlation Functions.

**Solution 5.3 Correlation Coefficients**

Figure 2 (a):  $\rho_{xy} = 0.9$ .

Figure 2 (b):  $\rho_{xy} = -1$ .

Figure 2 (c):  $\rho_{xy} = -0.4$ .

Figure 2 (d):  $\rho_{xy} = 0$ .

**Solution 5.4 Resistive Position Sensor**

a) Sketch

$$R(s) = R_0 \frac{s}{l}; \quad (46)$$

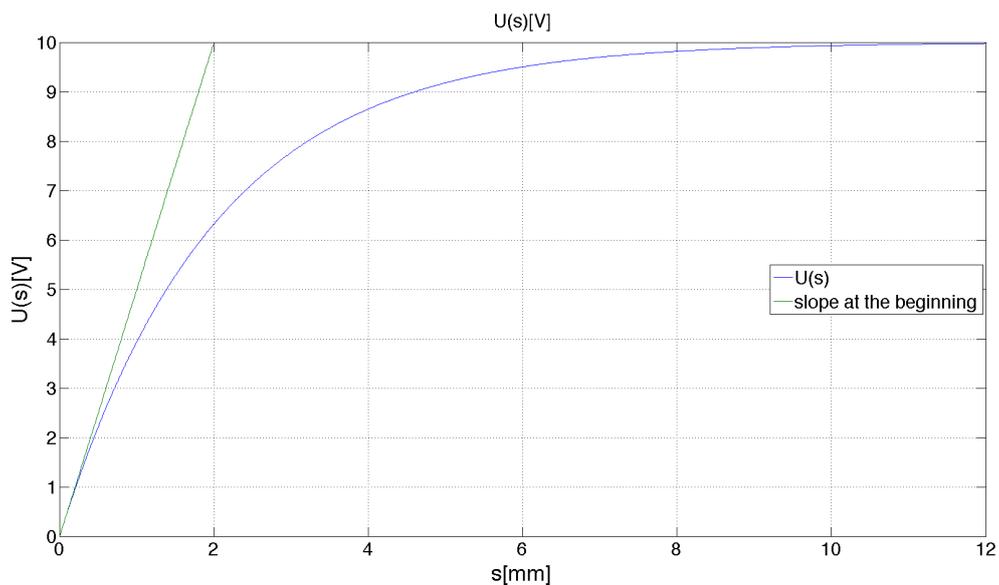
$$U(s) = U_0 \left( 1 - e^{-5 \frac{R(s)}{R_0}} \right) \quad (47)$$

(1) in (2):

$$U(s) = U_0 \left( 1 - e^{-5 \frac{R_0 s}{R_0 l}} \right) \quad (48)$$

To sketch the transfer function  $U(s)$  qualitatively, we need to know the slope of the curve at the beginning ( $s=0$ )

$$\begin{aligned} \frac{d(U(s))}{ds} &= \frac{dU_0}{ds} - \frac{d(U_0 e^{-5 \frac{s}{l}})}{ds} \\ &= 0 - \left( -\frac{5}{l} U_0 e^{-5 \frac{s}{l}} \right) \\ &= \frac{5}{l} U_0 e^{-5 \frac{s}{l}} \\ \frac{dU(s=0)}{ds} &= \frac{5}{l} U_0 \\ &= \frac{5}{10 \text{ mm}} U_0 = \frac{1}{2} \frac{U_0}{\text{mm}} = 5 \frac{\text{V}}{\text{mm}} \end{aligned}$$

Figure 19: Sketch: Function and slope of  $U(s)$

b) Gaussian error propagation:  $\Delta U_s \approx \frac{d(U_s)}{ds} \cdot \Delta s$

$$\Delta y = \frac{\partial f}{\partial x_1} \Delta x_1 + \frac{\partial f}{\partial x_2} \Delta x_2 + \dots + \frac{\partial f}{\partial x_n} \Delta x_n \quad \text{Generalized equation}$$

$$\frac{d(U(s))}{ds} = \frac{5}{l} U_0 e^{-5\frac{s}{l}}$$

$$\Rightarrow \Delta U_s \approx \frac{5}{l} U_0 e^{-5\frac{s_0}{l}} \underbrace{(s - s_0)}_{\Delta s}$$

**Solution 5.5 Errorpropagation at Measuring Errors**

$s(t, T) = \frac{1}{2}t\sqrt{\kappa_L R_L T} \rightarrow 2$  dependent variables!

a) Specify the Gaussian error propagation  $\Delta s$  for the ultrasonic position sensor!

$$\Delta s = \sqrt{\left(\frac{\partial s}{\partial t} \Delta t\right)^2 + \left(\frac{\partial s}{\partial T} \Delta T\right)^2}$$

$$\frac{\partial s}{\partial t} = \frac{1}{2}\sqrt{\kappa_L R_L T} \quad ; \quad \frac{\partial s}{\partial T} = \frac{1}{2}t \cdot \frac{1}{2\sqrt{\kappa_L R_L T}} \cdot R_L \kappa_L$$

$$\begin{aligned} \Rightarrow \Delta s &= \sqrt{\frac{1}{4}\kappa_L R_L T \cdot \Delta t^2 + \frac{1}{16} \frac{t^2 R_L^2 \kappa_L^2 \Delta T^2}{\kappa_L R_L T}} \\ &= \sqrt{\frac{1}{4}\kappa_L R_L T \cdot \Delta t^2 + \frac{1}{16} \frac{t^2 R_L \kappa_L}{T} \Delta T^2} \end{aligned}$$

b) Compare the values of the expressions within the square root of the Gaussian error propagation in case of  $\Delta t = 0.1t$  and  $\Delta T = 0.1T$  for any operating point  $s_i(t_i, T_i)$ .

To compare the two terms underneath the square root, we calculate the ratio:

$$\begin{aligned} F &= \frac{\left(\frac{\partial s}{\partial t} \Delta t\right)^2}{\left(\frac{\partial s}{\partial T} \Delta T\right)^2} \\ &= \frac{\frac{1}{4}\kappa_L R_L T \cdot \Delta t^2}{\frac{1}{16} \frac{t^2 R_L \kappa_L}{T} \Delta T^2} \\ &= \frac{4T^2 \Delta t^2}{t^2 \Delta T^2} \\ &= \frac{4\Delta t^2 T^2}{t^2 \Delta T^2} \quad \Delta t = 0.1t \Leftrightarrow \frac{\Delta t}{t} = 0.1 \quad ; \quad \Delta T = 0.1T \Leftrightarrow \frac{T}{\Delta T} = 10 \\ &= 4 \cdot (0.1)^2 \cdot (10)^2 \\ &= 4 \cdot 10^{-2} \cdot 10^2 \\ &= \underline{4} \end{aligned}$$

The same relative error (0.1) causes a 4 times higher value for the first summand than for the second one. An accurate measuring of the run time is very important for this type of sensor!

### Solution 5.6 Temperature Measurement

a) Recommendation: Number of intervals  $n_i = \sqrt{N}$

N: Number of measurements Here:  $25 = N \rightarrow n_i = 5$

Here:  $25 = N \Rightarrow n_i = 5$

Lower boundary: minimum of measurements  $T_{min} = 49.49^\circ C$  Upper boundary: maximum of measurements  $T_{max} = 50.23^\circ C$ .

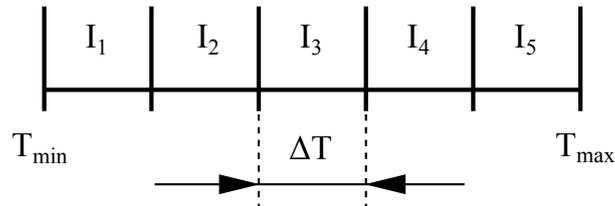


Figure 20: Intervals for the histogram.

All intervals have the same width  $\Delta T = \frac{T_{max} - T_{min}}{n_i}$   
 $\Rightarrow \Delta T = \frac{50.23 - 49.49}{5} = \frac{0.74}{5} = 0.148$

I<sub>1</sub> : 49.49 – 49.638

Number of values within interval: 2

I<sub>2</sub> : 49.638 – 49.786

Number of values within interval: 4

I<sub>3</sub> : 49.786 – 49.934

Number of values within interval: 4

I<sub>4</sub> : 49.934 – 50.082

Number of values within interval: 11

I<sub>5</sub> : 50.082 – 50.23

Number of values within interval: 4

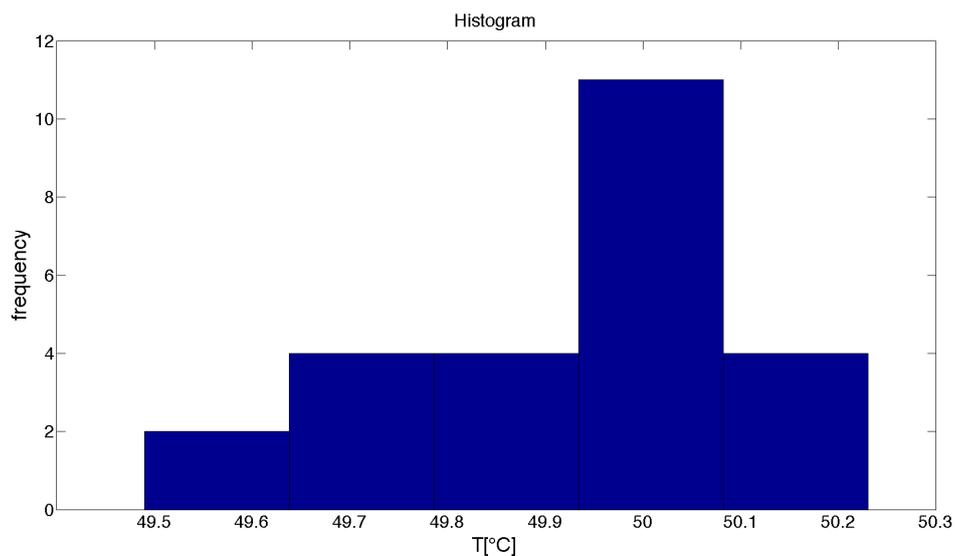


Figure 21: Histogram of the measured temperatures.

- b) Calculate the sample mean and the sample variance!

Convention:  $T(i)$  is the  $i$ -th measured temperature.

Sample mean:

$$\begin{aligned}\bar{T} &= \frac{1}{N} \sum_{i=1}^N T(i) \\ &= 49.918^\circ\text{C}\end{aligned}$$

Sample variance:

$$\begin{aligned}s_T^2 &= \frac{1}{N-1} \sum_{i=1}^N (T(i) - \bar{T})^2 \\ &= 0.0329(\text{°C})^2\end{aligned}$$

- c) Calculate in what range the true temperature will be with a probability of 99.7%!

Since the variance  $s_T^2$  is estimated from the measurements in table 1, we should use the student's t-distribution. The factor  $c$  for the t-distribution can be obtained from page 162 in the script. Therefore we look at the column for a confidence level of  $1 - \alpha = 99.7\%$ . Furthermore the line for  $N = 20$  measurements is closest to our case, where the number of measurements to determine the standard deviation was  $N = 25$ .

$$\begin{aligned}c &= 3.42 \\ \bar{T} - c \cdot \frac{s_T}{\sqrt{N}} &< T < \bar{T} + c \cdot \frac{s_T}{\sqrt{N}}\end{aligned}$$

For this new measurement, where we measured a temperature of  $75^\circ\text{C}$ , only one measurement exists. So  $N = 1$  and  $\bar{T} = 75^\circ\text{C}$ . Important: To determine the factor  $c$  for a demanded confidence level,  $N$  is the number of measurements, that were used to estimate the standard deviation  $s_T$ .

Now we can calculate the range:

$$\begin{aligned}75 - 3,42 \cdot \frac{0.1814}{\sqrt{1}} &< T < 75 + 3.42 \cdot \frac{0.1814}{\sqrt{1}} \\ 74.38^\circ\text{C} &< T < 75.62^\circ\text{C}\end{aligned}$$

- d) What can be done to improve the accuracy of the temperature measurement without changing the measurement instrument?

We can carry out more than just one temperature measurement to decrease the factor  $\frac{s_T}{\sqrt{N}}$ .

## 6 Static and Dynamic Behavior of Sensors

### Solution 6.1 Linearization and Inversion

a) Inversion of the equation :  $R = R_0 e^{b \cdot \left(\frac{1}{T} - \frac{1}{T_0}\right)}$

characteristic has to be strictly monotonic!

Given equation:  $R = f(T)$

Inverted equation:  $T = f(R)$

$$\begin{aligned}
 R &= R_0 e^{b \cdot \left(\frac{1}{T} - \frac{1}{T_0}\right)} && | \ln \\
 \Leftrightarrow \ln R &= \ln R_0 + b \cdot \left(\frac{1}{T} - \frac{1}{T_0}\right) && | - \ln R_0; : b \\
 \Leftrightarrow \frac{\ln R - \ln R_0}{b} &= \frac{1}{T} - \frac{1}{T_0} && | + \frac{1}{T_0} \\
 \Leftrightarrow \frac{\ln R - \ln R_0}{b} + \frac{1}{T_0} &= \frac{1}{T} && | ()^{-1} \\
 \Leftrightarrow T &= \left(\frac{1}{b} \ln \frac{R}{R_0} + \frac{1}{T_0}\right)^{-1} \\
 \Leftrightarrow T &= \left(\frac{T_0 \ln \frac{R}{R_0} + b}{T_0 b}\right)^{-1} \\
 \Leftrightarrow T &= \frac{T_0 b}{T_0 \ln \frac{R}{R_0} + b}
 \end{aligned}$$

- only for strictly monotonic relationships
- + valid in the whole range
- Sensitivity can change
- often true relationship not known (no inversion possible)

b) Linearize the equation around  $T_{OP}$ !

At first some definitions:

$$y = f(\vec{x}) \text{ with } \vec{x} = [x_1 \ x_2 \ x_3 \ \dots \ x_n]$$

$$\text{Operating Point: } \vec{x}_0 = [x_{1,0} \ x_{2,0} \ x_{3,0} \ \dots \ x_{n,0}]$$

General equation for linear approximation of a function that depends on  $n$  variables:

$$\tilde{y} = f(\vec{x}_0) + \left. \frac{\partial f}{\partial x_1} \right|_{\vec{x}_0} (x_1 - x_{1,0}) + \left. \frac{\partial f}{\partial x_2} \right|_{\vec{x}_0} (x_2 - x_{2,0}) + \left. \frac{\partial f}{\partial x_n} \right|_{\vec{x}_0} (x_n - x_{n,0})$$

Here we want to linearize

$$R(T) = R_0 e^{b \cdot \left(\frac{1}{T} - \frac{1}{T_0}\right)} \Rightarrow \tilde{R} = R(T_{OP}) + \left. \frac{\partial R}{\partial T} \right|_{T_{OP}} [T - T_{OP}]$$

Therefore we have to calculate the partial differential as well as the resistance at the operating point  $T_{OP}$ .

$$\frac{\partial R}{\partial T} = R_0 e^{b\left(\frac{1}{T} - \frac{1}{T_0}\right)} \cdot \left(-\frac{1}{T^2} \cdot b\right)$$

Finally this leads to the following linearized equation:

$$\begin{aligned} \tilde{R} &= \underbrace{R_0 e^{b\left(\frac{1}{T_{OP}} - \frac{1}{T_0}\right)}}_{R(T_{OP})} + \underbrace{(-1) \frac{R_0 b}{T_{OP}^2} e^{b\left(\frac{1}{T_{OP}} - \frac{1}{T_0}\right)}}_{\left.\frac{\partial R}{\partial T}\right|_{T_{OP}}} [T - T_{OP}] \\ &= \left(1 - \frac{b}{T_{OP}^2} [T - T_{OP}]\right) R_0 e^{b\left(\frac{1}{T_{OP}} - \frac{1}{T_0}\right)} \end{aligned}$$

Advantages of linear/affine characteristics:

- easy to understand and handle
- described by only one (linear) or two (affine) parameters
- identical sensitivities (slopes) in all operating points (behaviour is the same for different operating points)

Distinction between linear (proportional relationship) and affine (proportional relationship + offset) → explanation of life and dead zero!

**Solution 6.2 Tank with Inflow and Outflow**

- a) Derive the differential equation of the fill level  $h(t)$   
Dependencies on time are neglected in the following.

$$\rho gh = \frac{1}{2}\rho v^2 \quad (1)$$

$$av + A\dot{h}(t) = \dot{V} \quad (2)$$

From equation 1 we get the formula to calculate the velocity  $v$ :

$$v = \sqrt{2gh} \quad (3)$$

Put equation 3 in equation 2:

$$\begin{aligned} a\sqrt{2gh} + A\dot{h} &= \dot{V} && | : A; -\frac{a}{A}\sqrt{2gh} \\ \Leftrightarrow \dot{h} &= \frac{1}{A}\dot{V} - \frac{a}{A}\sqrt{2gh} \end{aligned}$$

- b) Derive the differential equation of the fill level for a stationary operating point.  
At a stationary operating point the change of the fill level is zero and the volume flow  $\dot{V}$  is approximately constant

$$\begin{aligned} \Rightarrow 0 &= \frac{1}{A}\dot{V}_0 - \frac{a}{A}\sqrt{2gh_0} && | + \frac{a}{A}\sqrt{2gh_0} \\ \Leftrightarrow \frac{a}{A}\sqrt{2gh_0} &= \frac{1}{A}\dot{V}_0 && | \cdot \frac{A}{a} \\ \Leftrightarrow \sqrt{2gh_0} &= \frac{\dot{V}_0}{a} && | ()^2 \\ \Leftrightarrow 2gh_0 &= \frac{\dot{V}_0^2}{a^2} && | : 2g \\ \Leftrightarrow h_0 &= \frac{\dot{V}_0^2}{2ga^2} \end{aligned}$$

- c) Linearize the differential equation around a stationary operating point  $(h_0, \dot{V}_0)$ .  
For simplicity and avoiding confusion we substitute  
 $h(t) \rightarrow y$ , which is our system output  
 $\dot{V}(t) \rightarrow u$ , which is our system input

$$\begin{aligned} \dot{h} &= \frac{1}{A}\dot{V} - \frac{a}{A}\sqrt{2gh} \\ \Leftrightarrow \dot{h} + \frac{a}{A}\sqrt{2gh} - \frac{1}{A}\dot{V} &= 0 \\ \Leftrightarrow \dot{y} + \frac{a}{A}\sqrt{2gy} - \frac{1}{A}u &= 0 \end{aligned}$$

$\rightarrow$  this kind of equation is needed to use the generalized equation to linearize a nonlinear dynamic system with 1 input and 1 output

$$\begin{aligned}
& y_0 + \left. \frac{\partial f}{\partial y} \right|_{OP} (y - y_0) + \left. \frac{\partial f}{\partial y^{(1)}} \right|_{OP} (y^{(1)} - \cancel{y_0^{(1)}}) + \left. \frac{\partial f}{\partial y^{(2)}} \right|_{OP} (y^{(2)} - \cancel{y_0^{(2)}}) + \dots + \dots \\
& \dots + \left. \frac{\partial f}{\partial y^{(n)}} \right|_{OP} (y^{(n)} - \cancel{y_0^{(n)}}) + \left. \frac{\partial f}{\partial u} \right|_{OP} (u - u_0) + \left. \frac{\partial f}{\partial u^{(1)}} \right|_{OP} (u^{(1)} - \cancel{u_0^{(1)}}) + \dots + \dots \\
& \dots + \left. \frac{\partial f}{\partial u^{(m)}} \right|_{OP} (u^{(m)} - \cancel{u_0^{(m)}}) = 0
\end{aligned}$$

Following derivatives have to be calculated:

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{a2g}{A2\sqrt{2gy}} & \left. \frac{\partial f}{\partial y} \right|_{OP} &= \frac{ag}{A\sqrt{2gy_0}} \\
\frac{\partial f}{\partial \dot{y}} &= 1 & \left. \frac{\partial f}{\partial \dot{y}} \right|_{OP} &= 1 \\
\frac{\partial f}{\partial u} &= -\frac{1}{A} & \left. \frac{\partial f}{\partial u} \right|_{OP} &= -\frac{1}{A}
\end{aligned}$$

$$\Rightarrow y_0 + \frac{ag}{A\sqrt{2gy_0}} (y - y_0) + 1 (\dot{y} - \cancel{\dot{y}_0}) - \frac{1}{A} (u - u_0) = 0$$

## 7 Introduction to Signal Processing

No exercises for this chapter yet.

## 8 Time-Discrete Systems and Signals

### Solution 8.1 Relationship Between Unit Impulse and Unit Step

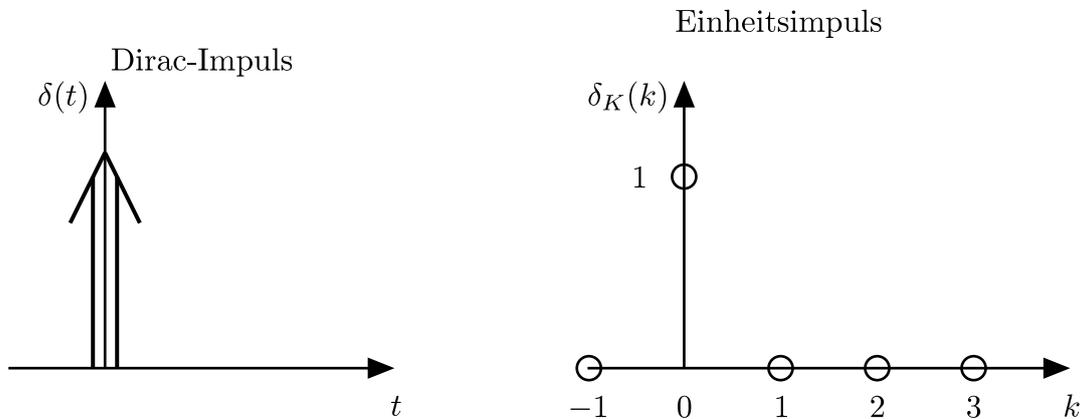


Figure 22: Comparison Dirac-Impuls and Kronecker-Delta.

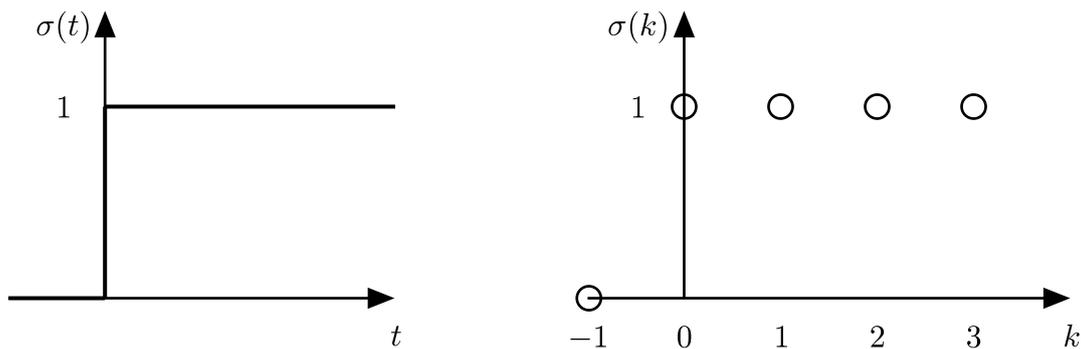


Figure 23: Comparison of the continuous time and discrete time unit-step.

Relationship:

$$\delta_t = \frac{d\sigma}{dt}$$

Relationship:

$$\delta_K(k) = \underbrace{\sigma(k) - \sigma(k-1)}_{\text{Approximation of the time derivative in the discrete World}}$$

### Solution 8.2 Transforming a Differential Equation into a Difference Equation

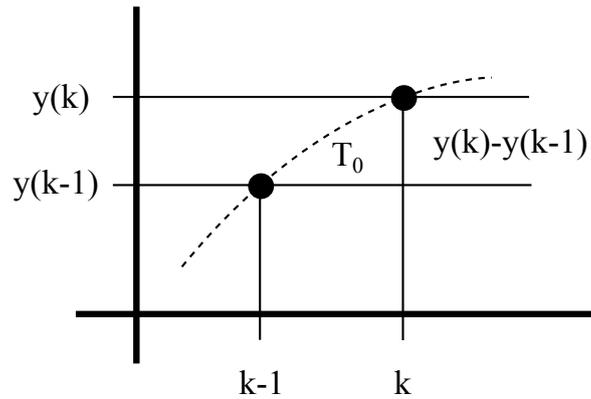


Figure 24: Derivative approximation.

In the continuous time case, the derivative with respect to time can be approximated as follows:

$$\dot{y}(t) \approx \frac{y(t) - y(t - \Delta t)}{\Delta t} .$$

In the discrete time case this corresponds to:

$$\dot{y}(t) \approx \frac{y(kT_0) - y(kT_0 - 1 \cdot T_0)}{T_0} .$$

For our given equation this leads to:

$$\begin{aligned} \dot{y}(t) + ay(t) &= bu(t) && \text{Transformation into the continuous time} \\ \frac{y(kT_0) - y(kT_0 - 1T_0)}{T_0} + ay(kT_0) &= bu(kT_0) && | \cdot T_0 \\ y(kT_0) - y(kT_0 - 1T_0) + T_0ay(kT_0) &= T_0bu(kT_0) && \text{Equation transformation} \\ y(kT_0)[1 + T_0a] - y(kT_0 - 1T_0) &= T_0bu(kT_0) && | + y(kT_0 - 1T_0); : (1 + T_0a) \\ y(kT_0) &= \underbrace{\frac{T_0b}{1 + T_0a}}_{b_0} u(kT_0) + \underbrace{\frac{1}{1 + T_0a}}_{a_1} y(kT_0 - 1T_0) \end{aligned}$$

We can see that the coefficients  $b_0$  and  $a_1$  of the resulting difference equation both depend on the sampling time  $T_0$ .

### Solution 8.3 Gain of a Time Discrete System

The gain  $K$  of the system is the end value of the step response  $h(t)$ :

$$\begin{aligned} K &= \lim_{t \rightarrow \infty} h(t) \\ &= \lim_{s \rightarrow 0} sH(s) \end{aligned}$$

In principle there are two ways to determine the gain of the system:

1st way: For  $k \rightarrow \infty$   $y(k)$  and  $y(k-1)$  both equal the end value of the step response:

$$y(k) = y(k \rightarrow \infty) = \lim_{k \rightarrow \infty} h(k) = K$$

The unit step is one for all time steps equal or greater than  $k \rightarrow u(k \rightarrow \infty) = 1$

This leads us to the following gain of the time discrete system:

$$\begin{aligned} y(k) &= 0,2u(k) + 0,9y(k-1) && | k \rightarrow \infty \\ K &= 0,2 \cdot 1 + 0,9K && | - 0,9K \\ K [1 - 0,9] &= 0,2 && | : [1 - 0,9] \\ K &= \frac{0,2}{1 - 0,9} \\ &= \underline{\underline{2}} \end{aligned}$$

2nd way: Using the relationship between time  $t$ , the Laplace variable  $s$  and the time shift operator  $z$ :

$$t \sim \frac{1}{s}$$

To determine the gain of the system we are searching for the value if  $t$  reaches infinity:  $t \rightarrow \infty$  this is equivalent to  $k \rightarrow \infty$  in the time discrete case.

In the Laplace domain this means  $s \rightarrow 0$

With the definition of  $z = e^{sT_0}$  we are searching the value of the system for  $z=1!$

$$\begin{aligned} K &= \lim_{s \rightarrow 0} sH(s) = \lim_{z \rightarrow 1} (z-1)H(z) && \text{(see script page 232)} \\ H(z) &= G(z) \cdot \underbrace{U(z)}_{\text{Unit step}} \\ y(k) &= 0,2u(k) + 0,9y(k-1) \\ & \circ \\ & \bullet \\ Y(z) &= 0,2U(z) + 0,9z^{-1}Y(z) && | - 0,9z^{-1}Y(z) \\ Y(z) [1 - 0,9z^{-1}] &= 0,2U(z) && | : [1 - 0,9z^{-1}] \\ \Leftrightarrow G(z) &= \frac{0,2}{1 - 0,9z^{-1}} \end{aligned}$$

z-Transform of a unit step:

$$u(k) = \sigma(k) \circ \bullet U(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

So the gain can be calculated:

$$\begin{aligned} K &= \lim_{z \rightarrow 1} (z - 1) \cdot \frac{0,2}{1 - 0,9z^{-1}} \cdot \frac{z}{z - 1} \\ &= \frac{0,2}{1 - 0,9} = \frac{0,2}{0,1} = \underline{\underline{2}} \end{aligned}$$

**Solution 8.4 Difference Equation and z-Transform**

a) Derivation of the difference equation from the following block diagram

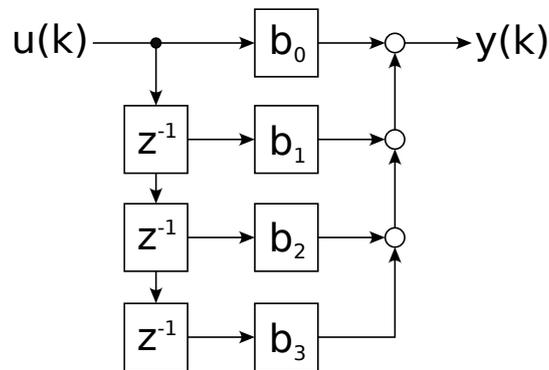


Figure 25: Block diagram of a third order system.

$$y(k) = b_0u(k) + b_1u(k-1) + b_2u(k-2) + b_3u(k-3)$$

b) Because  $y(k)$  only depends on the input  $u$  the system's response to an impulse is of finite length!

c) Definition of a transfer function  $G(z) = \frac{Y(z)}{U(z)}$

$$y(k) = b_0u(k) + b_1u(k-1) + b_2u(k-2) + b_3u(k-3)$$

$$\begin{aligned} Y(z) &= b_0U(z) + b_1z^{-1}U(z) + b_2z^{-2}U(z) + b_3z^{-3}U(z) \\ \Rightarrow G(z) &= \frac{Y(z)}{U(z)} = b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3} && |T \\ \Leftrightarrow G(z) &= \frac{b_0z^3 + b_1z^2 + b_2z + b_3}{\underbrace{z^3}_{3 \text{ poles at } z=0 \text{ are typical for FIR-System}}} \end{aligned}$$

d) Step response  $H(z) = G(z) \cdot U(z)$  with  $U(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

$$\begin{aligned} H(z) &= \frac{b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3}}{1 - z^{-1}} \\ &= b_0 \frac{1}{1 - z^{-1}} + b_1 \frac{1}{1 - z^{-1}} z^{-1} + b_2 \frac{1}{1 - z^{-1}} z^{-2} + b_3 \frac{1}{1 - z^{-1}} z^{-3} \\ h(k) &= b_0\sigma(k) + b_1\sigma(k-1) + b_2\sigma(k-2) + b_3\sigma(k-3) \end{aligned}$$

- e) If there is no information about former input or output signals we always assume these values to be zero!

$$\begin{aligned}h(0) &= b_0 &&= 0,25 \\h(1) &= b_0 + b_1 &&= 0,5 \\h(2) &= b_0 + b_1 + b_2 &&= 0,75 \\h(3) &= b_0 + b_1 + b_2 + b_3 &&= 1 \\h(4) &= h(3) &&= 1 \\h(5) &= h(3) &&= 1\end{aligned}$$

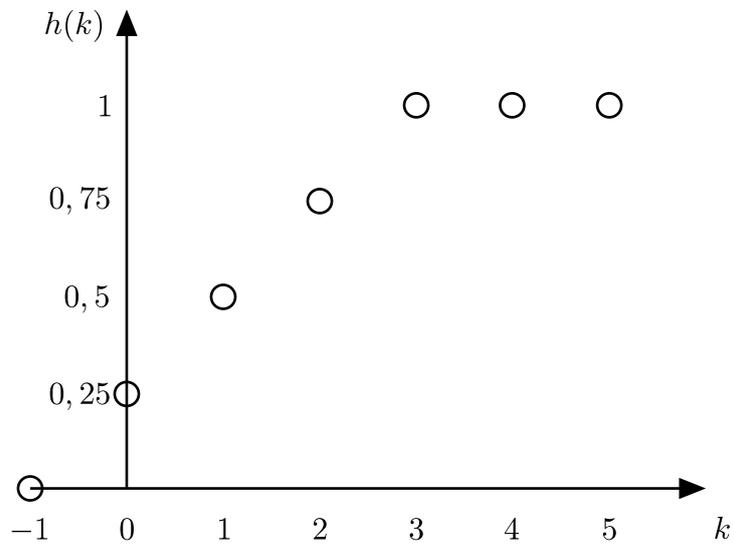


Figure 26: step response of the time discrete system

**Solution 8.5 Block Diagram and z-Transform**

a) The block diagram of the given difference equation:

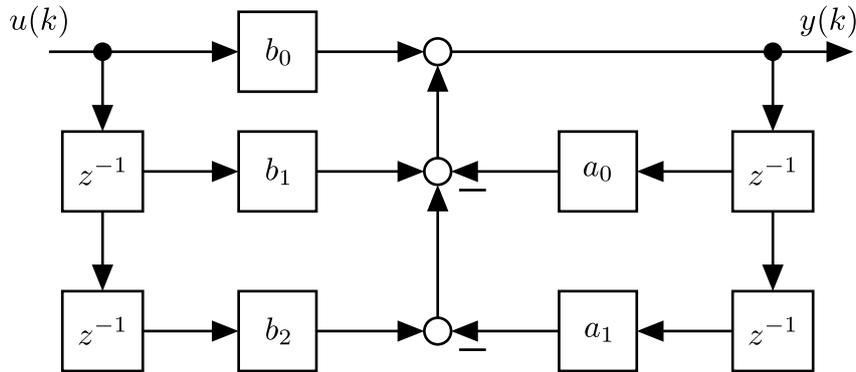


Figure 27: Blockschaltbild

b) Here we have an IIR-System because the current output depends on former outputs!

c) Transfer function  $G(z) = \frac{Y(z)}{U(z)}$

$$\begin{aligned}
 y(k) + a_1 \cdot y(k-1) + a_2 \cdot y(k-2) &= b_0 \cdot u(k) + b_1 \cdot u(k-1) + b_2 \cdot u(k-2) \\
 \Rightarrow Y(z) + a_1 \cdot z^{-1} \cdot Y(z) + a_2 \cdot z^{-2} \cdot Y(z) &= b_0 \cdot U(z) + b_1 \cdot z^{-1} \cdot U(z) + b_2 \cdot z^{-2} \cdot U(z) \\
 \Leftrightarrow Y(z) \cdot (1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}) &= U(z) \cdot (b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}) \\
 \Rightarrow G(z) = \frac{Y(z)}{U(z)} &= \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}}
 \end{aligned}$$

d) With  $a_2 = b_2 = 0$  we get the equation:

$$y(k) = -a_1 \cdot y(k-1) + b_0 \cdot u(k) + b_1 \cdot u(k-1)$$

with

$$u(k) = \delta_k(k) = \begin{cases} 1 & k = 0 \\ 0 & \text{else} \end{cases}$$

For  $u(k) = \delta_k(k)$  we get the impulse response:

$$\begin{aligned}
 y(0) &= g(0) = -a_1 \underbrace{y(-1)}_{=0} + b_0 \underbrace{u(0)}_{=1} + b_1 \underbrace{u(-1)}_{=0} &&= b_0 \\
 y(1) &= g(1) = -a_1 \underbrace{y(0)}_{=b_0} + b_0 \underbrace{u(1)}_{=0} + b_1 \underbrace{u(0)}_{=1} &&= -a_1 b_0 + b_1 \\
 y(2) &= g(2) = -a_1 y(1) + b_0 \underbrace{u(2)}_{=0} + b_1 \underbrace{u(1)}_{=1} &&= -a_1 (-a_1 b_0 + b_1) = (-a_1)^2 b_0 + (-a_1)^1 b_0 \\
 y(3) &= g(3) = -a_1 y(2) + b_0 \underbrace{u(3)}_{=0} + b_1 \underbrace{u(2)}_{=0} &&= (-a_1)^2 (-a_1 b_0 + b_1) = (-a_1)^3 b_0 + (-a_1)^2 b_0 \\
 y(4) &= g(4) = -a_1 y(3) + b_0 \underbrace{u(4)}_{=0} + b_1 \underbrace{u(3)}_{=0} &&= (-a_1)^4 b_0 + (-a_1)^3 b_0 \\
 &\vdots &&
 \end{aligned}$$

$$\Rightarrow g(k) = \begin{cases} b_0 & \text{for } k = 0 \\ b_0 \cdot (-a_1)^k + b_1 \cdot (-a_1)^{k-1} & \text{for } k \geq 1 \end{cases}$$

**Solution 8.6 z-Transform of the Unit Step**

Definition of the unit step:

$$\sigma(k) = \begin{cases} 1 & k \geq 0 \\ 0 & \text{else} \end{cases}$$

The unit step can be interpreted as a train of impulses:

$$u(k) = \delta_k(k) + \delta_k(k-1) + \delta_k(k-2) + \dots$$

If we transform the discrete signal  $u(k)$  into the z-domain, the equation becomes:

$$\begin{aligned} U(z) &= 1 \cdot z^0 + 1 \cdot z^{-1} + 1 \cdot z^{-2} + \dots \\ &= \sum_{i=0}^{\infty} (z^{-1})^i \end{aligned}$$

This geometrical series can easily be transformed (see script on p. 223) into:

$$\begin{aligned} U(z) &= \frac{1}{1 - z^{-1}} \\ &= \frac{z}{z - 1} \end{aligned}$$

If we now use the z-transform of the unit step as a transfer function and apply an unit step to this 'system', we will see the integrating behavior. At first we transform the

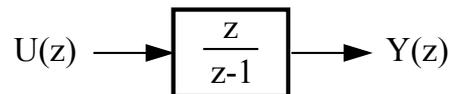


Figure 28: z-Transform of the unit step as a transfer function.

system from the z-domain back to the discrete time domain:

$$\begin{aligned} Y(z) &= \frac{1}{1 - z^{-1}} U(z) && | \cdot (1 - z^{-1}) \\ Y(z)[1 - z^{-1}] &= U(z) && | \text{Equation transformation} \\ Y(z) &= U(z) + z^{-1}Y(z) && | \text{Transformation into the discrete time domain} \\ y(k) &= u(k) + y(k-1) \end{aligned}$$

Now we apply the unit step to our discrete time system:

$$\begin{aligned} y(0) &= u(0) + y(-1) \\ &= 1 + 0 = 1 \\ y(1) &= u(1) + y(0) \\ &= 1 + 1 = 2 \\ y(2) &= u(2) + y(1) \\ &= 1 + 2 = 3 \\ &\vdots \end{aligned}$$

We can see, that our transfer function sums up all input values. In discrete time this is analogous to an integrator in the continuous time.

**Solution 8.7 First Order IIR-System**

The following difference equation is given:  $y(k) = 0,2 \cdot u(k) + 0,6 \cdot y(k-1)$ .

a) Impulse Response:

$$u(k) = \delta_K(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} g(0) &= 0,2 \cdot \underbrace{1}_{u(0)} + 0,6 \cdot \underbrace{0}_{y(-1)} = 0,2 \\ g(1) &= 0,2 \cdot \underbrace{0}_{u(1)} + 0,6 \cdot \underbrace{0,2}_{y(0)} = 0,2 \cdot 0,6 \\ g(2) &= 0,2 \cdot \underbrace{0}_{u(2)} + 0,6 \cdot \underbrace{0,6 \cdot 0,2}_{y(1)} = 0,2 \cdot 0,6^2 \\ g(3) &= 0,2 \cdot \underbrace{0}_{u(3)} + 0,6 \cdot \underbrace{0,6^2 \cdot 0,2}_{y(2)} = 0,2 \cdot 0,6^3 \\ &\vdots \\ \Rightarrow y(k) &= 0,2 \cdot 0,6^k \end{aligned}$$

Step Response:

$$u(k) = \sigma(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} h(0) &= 0,2 \cdot \underbrace{1}_{u(0)} + 0,6 \cdot \underbrace{0}_{y(-1)} = 0,2 \\ h(1) &= 0,2 \cdot \underbrace{1}_{u(1)} + 0,6 \cdot \underbrace{0,2}_{y(0)} = 0,2 \cdot (1 + 0,6) \\ h(2) &= 0,2 \cdot \underbrace{1}_{u(2)} + 0,6 \cdot \underbrace{0,2 \cdot (1 + 0,6)}_{y(1)} = 0,2 \cdot (1 + 0,6 + 0,6^2) \\ h(3) &= 0,2 \cdot \underbrace{1}_{u(3)} + 0,6 \cdot \underbrace{0,2 \cdot (1 + 0,6 + 0,6^2)}_{y(2)} = 0,2 \cdot (1 + 0,6 + 0,6^2 + 0,6^3) \\ &\vdots \\ \Rightarrow y(k) &= 0,2 \cdot \sum_{i=0}^k (0,6)^i \end{aligned}$$

b) Transformation into the time domain

The geometrical series has already been determined in a)!

$$\begin{aligned} g(k) &= 0,2 \cdot (0,6)^k \\ &\Downarrow \\ \Rightarrow G(z) &= g(0) \cdot z^0 + g(1) \cdot z^{-1} + g(2) \cdot z^{-2} + \dots \\ &= 0,2 \cdot z^0 + 0,2 \cdot 0,6 \cdot z^{-1} + 0,2 \cdot 0,6^2 \cdot z^{-2} + \dots \\ &= 0,2 \cdot \sum_{k=0}^{\infty} (0,6)^k \cdot z^{-k} = 0,2 \cdot \sum_{k=0}^{\infty} (0,6 \cdot z^{-1})^k \end{aligned}$$

With the help of the following relationship

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x},$$

we obtain the transfer function:

$$\Rightarrow G(z) = \frac{Y(z)}{U(z)} = \frac{0.2}{1-0.6 \cdot z^{-1}} = \frac{0.2 \cdot z}{z-0.6}$$

c) Transfer function:

$$G(z) = \frac{0.2}{1-0.6 \cdot z^{-1}}$$

- Biproper property:  
Because the numerator and denominator have the same order the biproper property is given!  
 $b_0 = 0.2 \neq 0 \rightarrow u(k)$  has a direct influence on  $y(k)$ !
- Dead Time:  
This system has no dead time because no term of kind  $z^{-d}$  can be factored out (in the writing with only negative exponents!)
- Causality:  
Because the degree of the numerator and denominator are equal (in the writing with only positive exponents) the system is causal! (No future values must be predicted!)
- Stability:  
System is stable  
One pole at  $0.6 = z < 1$ !

Gain of the system:

The value of step response after infinite time!

$$t \sim \frac{1}{s}; z = e^{sT_0}$$

$$h(k \rightarrow \infty) = G(z = 1)$$

$$\text{because: } h(k \rightarrow \infty) = \lim_{z \rightarrow 1} (z-1) \cdot G(z) \cdot \underbrace{\frac{z}{z-1}}_{\text{step } U(z)} = \lim_{z \rightarrow 1} G(z) \cdot z = \underline{G(z = 1)}$$

$$\underbrace{\hspace{10em}}_{\text{step response } H(z)}$$

$$\Rightarrow h(k \rightarrow \infty) = \frac{0.2}{1-0.6} = \frac{0.2}{0.4} = \underline{\underline{0.5}}$$

**Solution 8.8 Step Response of a PT1-System**

Transfer function of the  $PT_1$ -system:

$$G(z) = \frac{1}{1 - 0.8z^{-1}} .$$

To calculate the end value of the step response, we need to know the z-transform of the unit step, which is:

$$u(k) = \sigma(k) \circ \bullet U(z) = \frac{z}{z - 1} .$$

Now we can calculate the end value as follows:

$$\begin{aligned} h(k \rightarrow \infty) &= \lim_{z \rightarrow 1} (z - 1) \frac{1}{1 - 0.8z^{-1}} \cdot \frac{z}{z - 1} \\ &= \frac{1}{1 - 0.8} \cdot 1 \\ &= \frac{1}{0.2} = 5 \end{aligned}$$

The end value is 5.

### Solution 8.9 Comparison of the Invariance of the Impulse Response and the Step Response

d) The relationship between the impulse response  $g(t)$  and the step response  $h(t)$  is given by (see script page 218):

$$h(t) = \int_0^t g(\tau) d\tau = \int_0^t \frac{K}{T} \cdot e^{-\frac{\tau}{T}} d\tau$$

Hint:  $\int e^{a \cdot x} dx = \frac{1}{a} \cdot e^{a \cdot x}$

$$\begin{aligned}
 h(t) &= \frac{K}{T} \int_0^t e^{-\frac{\tau}{T}} d\tau = \frac{K}{T} \cdot [-T \cdot e^{-\frac{\tau}{T}}]_0^t = \frac{K}{T} \cdot \left[ -T \cdot e^{-\frac{t}{T}} - \underbrace{(-T \cdot e^0)}_{=T} \right] \\
 &\Rightarrow h(t) = \underline{\underline{K \left( 1 - e^{-\frac{t}{T}} \right)}} \\
 &\text{with } K = 5; T = 5 \Rightarrow h(t) = 5 \cdot \left( 1 - e^{-\frac{t}{5}} \right)
 \end{aligned}$$

Because of sampling the continuous time  $t$  becomes  $kT_0$ :

$$\begin{aligned}
 t &= kT_0 \\
 \Rightarrow h(kT_0) &= h(k) = 5 \cdot \left( 1 - e^{-\frac{k}{5}} \right)
 \end{aligned}$$

Transformation (every value at a discrete time is multiplied by a time shifted impulse):

$$\begin{aligned}
 H(z) &= \sum_{k=0}^{\infty} h(k) \cdot z^{-k} = \sum_{k=0}^{\infty} 5 \cdot \left( 1 - e^{-\frac{k}{5}} \right) \cdot z^{-k} \\
 \Rightarrow H(z) &= 5 \cdot \left[ \sum_{k=0}^{\infty} z^{-k} - \sum_{k=0}^{\infty} e^{-\frac{k}{5}} \cdot z^{-k} \right] \\
 &= 5 \cdot \left[ \sum_{k=0}^{\infty} z^{-k} - \sum_{k=0}^{\infty} \left( e^{-\frac{1}{5}} \cdot z^{-1} \right)^k \right]
 \end{aligned}$$

Geometrical series:  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$$\text{here: } x^k = z^{-k} = (z^{-1})^k \Rightarrow x = z^{-1} = \frac{1}{z} \Rightarrow \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}}$$

$$\text{and: } x^k = \left( e^{-\frac{1}{5}} \cdot z^{-1} \right)^k \Rightarrow x = e^{-\frac{1}{5}} \cdot z^{-1} \Rightarrow \sum_{k=0}^{\infty} \left( e^{-\frac{1}{5}} \cdot z^{-1} \right)^k = \frac{1}{1 - e^{-\frac{1}{5}} \cdot z^{-1}}$$

$$\Rightarrow H(z) = 5 \cdot \left[ \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-\frac{1}{5}} \cdot z^{-1}} \right] \rightarrow \text{step response in the } z\text{-domain!}$$

To get the transfer function  $G(z)$  of the system we use the following relationship:

$$H(z) = G(z) \cdot \frac{1}{1 - z^{-1}}$$

with  $\sigma(k) \circ \bullet \frac{1}{1 - z^{-1}}$

$$\Leftrightarrow G_1(z) = H(z) \cdot (1 - z^{-1}) \quad | \text{Transfer function via the invariance of the step response!}$$

$$G_1(z) = 5 \cdot \left[ \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-\frac{1}{5}} z^{-1}} \right] (1 - z^{-1}) = 5 \cdot \left( 1 - \frac{1 - z^{-1}}{1 - e^{-\frac{1}{5}} z^{-1}} \right)$$

$$G_1(z) = 5 \cdot \frac{\cancel{1} - e^{-\frac{1}{5}} z^{-1} \cancel{1} + z^{-1}}{1 - e^{-\frac{1}{5}} z^{-1}}$$

$$\boxed{G_1(z) = 5 \cdot \frac{\left(1 - e^{-\frac{1}{5}}\right) z^{-1}}{1 - e^{-\frac{1}{5}} z^{-1}} \quad \text{mit } e^{-\frac{1}{5}} \approx 0.82}$$

$$\Rightarrow \boxed{G_1(z) = \frac{0.9 \cdot z^{-1}}{1 - 0.82 \cdot z^{-1}} = \frac{0.9}{z - 0.82}}$$

b) The following impulse response is given:  $g(t) = \frac{K}{T} \cdot e^{-\frac{t}{T}}$ .

$$g(t) = \frac{K}{T} \cdot e^{-\frac{t}{T}} \quad \text{with: } K = 5, T = 5$$

$$\Rightarrow \boxed{g(t) = e^{-\frac{t}{5}}}$$

Sampling:

$$t = k \cdot T_0 \rightarrow g(k) = e^{-\frac{kT_0}{5}}$$

$$T_0 = 1 \rightarrow \boxed{g(k) = e^{-\frac{k}{5}}}$$

z-Transformation:

$$G_2(z) = \sum_{k=0}^{\infty} g(k) \cdot z^{-k} = \sum_{k=0}^{\infty} e^{-\frac{k}{5}} \cdot z^{-k} \Rightarrow G_2(z) = \sum_{k=0}^{\infty} e^{-\frac{k}{5}} \cdot z^{-k} = \sum_{k=0}^{\infty} \left( e^{-\frac{1}{5}} \cdot z^{-1} \right)^k$$

Geometrical series:  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$$\text{with: } x^k = \left( e^{-\frac{1}{5}} \cdot z^{-1} \right)^k \Rightarrow x = e^{-\frac{1}{5}} \cdot z^{-1} \Rightarrow \sum_{k=0}^{\infty} \left( e^{-\frac{1}{5}} \cdot z^{-1} \right)^k = \frac{1}{1 - e^{-\frac{1}{5}} \cdot z^{-1}}$$

$$\Rightarrow \boxed{G_2(z) = \frac{1}{1 - e^{-\frac{1}{5}} \cdot z^{-1}} \quad \text{mit } e^{-\frac{1}{5}} \approx 0.82}$$

$$\Rightarrow \boxed{G_2(z) = \frac{1}{1 - 0.82 \cdot z^{-1}} \neq G_1(z) = \frac{0.9 \cdot z^{-1}}{1 - 0.82 \cdot z^{-1}}}$$

Comparison of the stability of  $G(z)$  and  $G_2(z)$ :

→ Same denominator, same stability properties! Both transfer functions have one pole at  $p = 0.82$ , so both are stable!

Gain-Comparison:

$$\begin{aligned}
 h_1(k \rightarrow \infty) &= \lim_{z \rightarrow 1} (z-1)H_1(z) \\
 &= \lim_{z \rightarrow 1} \cancel{(z-1)} \frac{0.9}{1-0.82} \cdot \frac{z}{\cancel{z-1}} \\
 &= \frac{0.9}{1-0.82} = 5 \\
 h_2(k \rightarrow \infty) &= \lim_{z \rightarrow 1} (z-1)H_2(z) \\
 &= \lim_{z \rightarrow 1} \cancel{(z-1)} \frac{1}{1-0.82} \cdot \frac{z}{\cancel{z-1}} \\
 &= \frac{1}{1-0.82} = 5,56 \\
 &\Rightarrow 5 \neq 5,56 \Rightarrow h_1(k \rightarrow \infty) \neq h_2(k \rightarrow \infty)
 \end{aligned}$$

Biproper Comparison:

$$G_1(z) = \frac{Y(z)}{U(z)} = \frac{0.9z^{-1}}{1-0.82z^{-1}} \Rightarrow y_1(k) = 0.82y_1(k-1) + 0.9u(k-1)$$

⇒ System 1 is not biproper because there is no feedthrough ( $b_0 = 0$ ).

$$G_2(z) = \frac{Y_2(z)}{U(z)} = \frac{1}{1-0.82z^{-1}} \Rightarrow y_2(k) = 0.82y_2(k-1) + u(k)$$

⇒ System 2 is biproper because of the feedthrough ( $b_0 \neq 0$ ).

c) Step responses of the time continuous system, of  $G_1(z)$  and of  $G_2(z)$

At first we calculate the systems output, where we demanded the invariance of the step response:

$$G_1(z) = \frac{Y(z)}{U(z)} = \frac{0,9 \cdot z^{-1}}{1-0,82 \cdot z^{-1}}$$

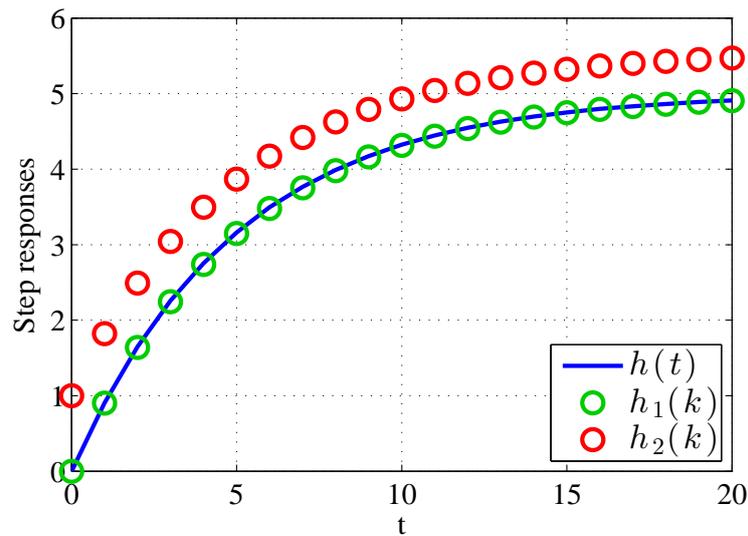


Figure 29: Step response of the time continuous system (blue), of the time discrete system with invariance of the step response (green) and of the time discrete system with invariance of the impulse response (red).

$$\begin{aligned} \rightarrow y_1(k) &= 0,82y_1(k-1) + 0,9u(k-1) \\ y_1(k=0) &= 0 \\ y_1(k=1) &= 0,9 \\ y_1(k=2) &= 0,82 \cdot 0,9 + 0,9 \cdot 1 = 1,638 \\ y_1(k=3) &= 2,24 \\ y_1(k=4) &= 2,73 \\ y_1(k=5) &= 3,14 \\ y_1(k=6) &= \dots \\ &\vdots \\ y_1(k=10) &= 4,31 \\ y_1(k=20) &= 4,91 \\ y_1(k=30) &= 4,99 \end{aligned}$$

Then we calculate the system output, where we demanded the invariance of the impulse response:

$$G_2(z) = \frac{Y_2(z)}{U(z)} = \frac{1}{1 - 0,82 \cdot z^{-1}}$$

$$\begin{aligned} \rightarrow y_2(k) &= 0.82 \cdot y_2(k-1) + u(k) \\ y_2(k=0) &= 1 \\ y_2(k=1) &= 0.82y(0) + 1 = 0.82 + 1 = 1.82 \\ y_2(k=2) &= 1 + 0.82y(1) = 1 + 0.82(1 + 0.82) = 1 + 0.82 + 0.82^2 \approx 2.5 \\ y_2(k=3) &= 1 + 0.82y(2) = 1 + 0.82(1 + 0.82 + 0.82^2) = 1 + 0.82 + 0.82^2 + 0.82^3 \approx 3.04 \\ y_2(k=4) &= 3,5 \\ y_2(k=5) &= 3,9 \end{aligned}$$

d) Impulse response values  $G_1(z)$

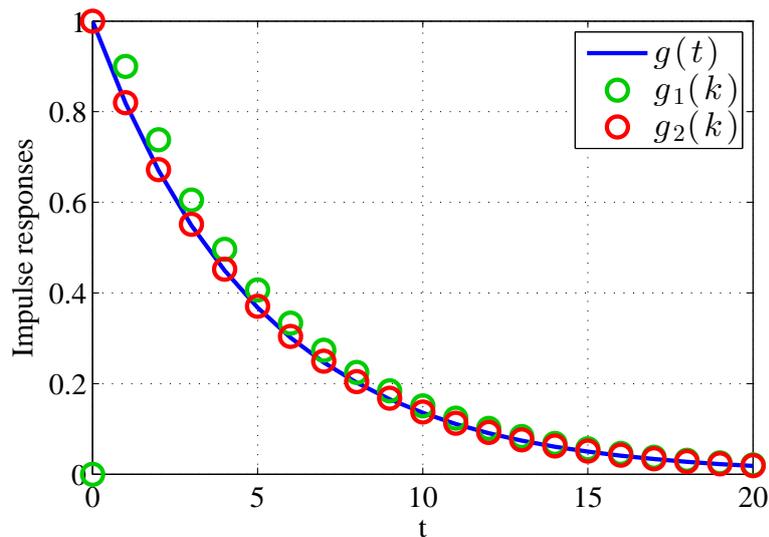


Figure 30: Impulse response of the time continuous system (blue), of the time discrete system with invariance of the step response (green) and of the time discrete system with invariance of the impulse response (red).

$$\begin{aligned} g_1(k) &= 0.82g_1(k-1) + 0.9u(k-1) \\ g_1(k=0) &= 0.82 \cdot 0 + 0.9 \cdot 0 = 0 \\ g_1(k=1) &= 0.82 \cdot 0 + 0.9 \cdot 1 = 0.9 \\ g_1(k=2) &= 0.82 \cdot 0.9 + 0.9 \cdot 0 = 0.738 \\ g_1(k) &= 0.9 \cdot 0.82^{k-1} \\ g_1(k) &= \begin{cases} 0 & \text{for } k \leq 0 \\ 0.9 \cdot 0.82^{k-1} & \text{else} \end{cases} \\ g_1(k=3) &= 0.605 \\ g_1(k=4) &= 0.496 \\ g_1(k=5) &= 0.407 \\ g_1(k=6) &= 0.334 \end{aligned}$$

Impulse response values  $G_2(z)$

$$g_2(k) = 0.82g_2(k-1) + u(k)$$

here you could also use the sampled impulse response  $g_2(k) = e^{-\frac{k}{5}}$  because the transfer function  $G_2$  is derived via impulse response invariance!

$$g_2(k) = 0.82^k$$

$$g_2(k = 0) = 1$$

$$g_2(k = 1) = 0.82$$

$$g_2(k = 2) = 0.672$$

$$g_2(k = 3) = 0.551$$

$$g_2(k = 4) = 0.452$$

$$g_2(k = 5) = 0.371$$

$$g_2(k = 6) = 0.304$$

**Solution 8.10 Converting an IIR-System to a FIR-System**

To convert the given system

$$G(z) = \frac{Y(z)}{U(z)} = \frac{0.5 + 0.5z^{-1}}{1 - 0.6z^{-1}}$$

to an FIR-system of order 8 we simply calculate the first eight values of the impulse response in the time domain. These values are used as coefficients for the FIR-system. At first we convert from the z-domain to the discrete time domain:

$$\begin{aligned} Y(z) - 0.6z^{-1}Y(z) &= 0.5U(z) + 0.5z^{-1}U(z) \quad |\text{Transformation to discrete time domain} \\ y(k) - 0.6y(k-1) &= 0.5u(k) + 0.5u(k-1) \end{aligned}$$

Now we calculate the value sequence of the impulse response

$(u(k) = \delta_k(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{else} \end{cases})$ . To determine a general equation to calculate the value sequence of the impulse response we use the coefficients  $a_1 = 0.6$ ,  $b_0 = 0.5$  and  $b_1 = 0.5$ .

$$\begin{aligned} y(0) &= a_1y(-1) + b_0u(0) + b_1u(-1) \\ &= a_1 \cdot 0 + b_0 \cdot 1 + b_1 \cdot 0 \\ &= b_0 \\ y(1) &= a_1y(0) + b_0u(1) + b_1u(0) \\ &= a_1 \cdot b_0 + b_0 \cdot 0 + b_1 \cdot 1 \\ &= a_1b_0 + b_1 \\ y(2) &= a_1y(1) + b_0u(2) + b_1u(1) \\ &= a_1 \cdot (a_1b_0 + b_1) + b_0 \cdot 0 + b_1 \cdot 0 \\ &= a_1 \cdot (a_1b_0 + b_1) \\ y(3) &= a_1y(2) + b_0u(3) + b_1u(2) \\ &= a_1 \cdot a_1 \cdot (a_1b_0 + b_1) + b_0 \cdot 0 + b_1 \cdot 0 \\ &= a_1^2 \cdot (a_1b_0 + b_1) \\ y(4) &= a_1^3 \cdot (a_1b_0 + b_1) \\ &\vdots \end{aligned}$$

The value of the impulse response for an arbitrary value of k, can be calculated as follows:

$$y(k) = \begin{cases} 0 & \text{for } k < 0 \\ b_0 = 0.5 & \text{for } k = 0 \\ (a_1b_0 + b_1) \cdot a_1^{k-1} = 0.8 \cdot 0.6^{k-1} & \text{else.} \end{cases}$$

Now we can determine the coefficients of the FIR of order 8:

$$\hat{y}(k) = \tilde{b}_0u(k) + \tilde{b}_1u(k-1) + \tilde{b}_2u(k-2) + \dots + \tilde{b}_8u(k-8)$$

with  $\tilde{b}_i = y(k=i)$ . The whole FIR equation becomes:

$$\begin{aligned} \hat{y}(k) &= 0.5u(k) + 0.8u(k-1) + 0.8 \cdot 0.6^1u(k-2) + 0.8 \cdot 0.6^2u(k-3) + 0.8 \cdot 0.6^3u(k-4) + \dots \\ &\quad + 0.8 \cdot 0.6^4u(k-5) + 0.8 \cdot 0.6^5u(k-6) + 0.8 \cdot 0.6^6u(k-7) + 0.8 \cdot 0.6^7u(k-8) \end{aligned}$$

### Comparison of the impulse responses

This is really easy because of the way we constructed the FIR-system. The value sequence will be the same until  $k$  becomes 9. From that point on the values of the FIR-system will be zero. For smaller values of  $k$  the values of the impulse responses will exactly be the same!

### Comparison of the step responses

We calculate the sequence of values for a unit step for both systems (IIR and FIR). At first the values for IIR-system.

$$\begin{aligned}
 y(0) &= 0.6y(-1) + 0.5u(0) + 0.5u(-1) \\
 &= 0.5 \\
 y(1) &= 0.6y(0) + 0.5u(1) + 0.5u(0) \\
 &= 0.6 \cdot 0.5 + \underbrace{0.5 + 0.5}_{=1 \text{ will maintain in all following steps}} \\
 &= 1.3 \\
 y(2) &= 0.6 \cdot 1.3 + 1 = 1.78 \\
 y(3) &= 0.6 \cdot 1.78 + 1 = 2.068 \\
 &\vdots \\
 y(8) &= 2.4664 \\
 y(9) &= 2.4798
 \end{aligned}$$

The end value of the IIR-system's steps response is:

$$y(k \rightarrow \infty) = \lim_{z \rightarrow 1} (z-1)G_{IIR}(z) \frac{z}{z-1} = 2.5 .$$

If we look at the FIR-system's response, we discover that all values up to the 8th equals the ones from the IIR-system. In every step one more summand comes into play (is no longer being zero).

$$\hat{y}(k) = \sum_{i=0}^k \tilde{b}_i$$

Only a few values are shown below:

$$\begin{aligned}
 \hat{y}(0) &= \tilde{b}_0 = 0.5 \\
 \hat{y}(1) &= \tilde{b}_0 + \tilde{b}_1 = 0.5 + 0.8 = 1.3 \\
 &\vdots \\
 \hat{y}(8) &= 2.4664 \\
 \hat{y}(9) &= 2.4664
 \end{aligned}$$

Because after 8 discrete time steps no more coefficients are left to sum up, the value keeps the same from the 8th value on. So the end value of the IIR-system can not be reached.

## 9 Transformation into the Frequency Domain (Discrete Fourier Transform)

### Solution 9.1 Relationship between the Fourier-Transformation and the Discrete Fourier-Transformation

Fourier-Transform in the continuous case:

$$X(i\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t} dt .$$

At first we discretize the time  $t \rightarrow kT_0$ :

$$X(i\omega) = \sum_{k=-\infty}^{\infty} x(kT_0)e^{-i\omega kT_0} .$$

Then we discretize the frequencies  $\omega \rightarrow n\frac{\omega_0}{N}$  with the number of samples  $N$  and  $n = 0, 1, \dots, N-1$ . After the discretization of the continuous time and the frequencies, we obtain the following equation:

$$X(i\omega) \rightarrow X(n) = \sum_{k=0}^{N-1} x(kT_0)e^{-in\frac{\omega_0}{N}kT_0} .$$

With the knowledge about the definition of the sampling frequency  $\omega_0 = 2\pi f_0 = 2\pi\frac{1}{T_0}$ , we see that the sampling time  $T_0$  in the exponent cancels out:

$$X(n) = \sum_{k=0}^{N-1} x(kT_0)e^{-in\frac{2\pi}{NT_0}kT_0} .$$

The equation for the discrete Fourier-Transform finally becomes:

$$X(n) = \sum_{k=0}^{N-1} x(kT_0)e^{\frac{-in2\pi k}{N}} .$$

With the abbreviation  $W_N = e^{\frac{-i2\pi}{N}}$  we get:

$$X(n) = \sum_{k=0}^{N-1} x(kT_0)W_N^{nk} .$$

### Solution 9.2 Superposition Principle of the FFT

MATLAB!

### Solution 9.3 DFT of an Impulse

At first we can sketch the signal  $x(k)$  over the discrete time  $k$ . For the calculation of the

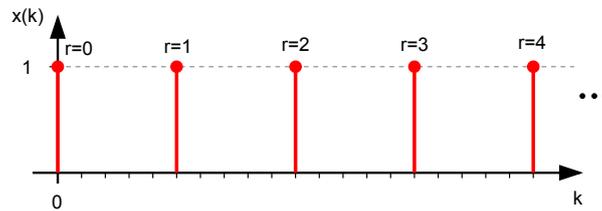


Figure 31: Signal  $x(k)$  over the discrete time  $k$ .

DFT we look at one period of the whole signal:

$$\text{DFT}\{x(k)\} = X(n) = \sum_{k=0}^{N-1} x(k)W_N^{nk} .$$

Here  $N = 5$ , which leads to:

$$X(n) = \sum_{k=0}^4 x(k)W_5^{nk} \text{ with}$$

$$W_5^{nk} = e^{-i\frac{2\pi nk}{5}} .$$

Now we calculate at the values for the discrete frequencies  $n$ :

$$X(0) = \underbrace{x(0)W_5^{1 \cdot 0}}_{=1} + \underbrace{x(1)W_5^0 + x(2)W_5^0 + x(3)W_5^0 + x(4)W_5^0}_{=0}$$

$$X(1) = \underbrace{x(0)W_5^{1 \cdot 0}}_{=1} + \underbrace{x(1)W_5^{1 \cdot 1} + x(2)W_5^{1 \cdot 2} + x(3)W_5^{1 \cdot 3} + x(4)W_5^{1 \cdot 4}}_{=0}$$

$$\vdots$$

$$X(4) = 1 .$$

Because all values except for  $x(0)$  are zero, the DFT equals:

$$X(n) = 1 .$$

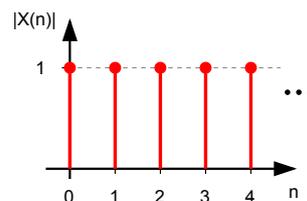


Figure 32: Signal  $|X(n)|$  over the discrete frequency  $n$ .

**Solution 9.4 DFT of a Periodic Sequence of Values**

a) At first determination of the periodicity N

Here: N=4

sequence of values that are repeated:  $\vec{x}$

$$\vec{x} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \quad \text{mit } k = 0, 1, 2, 3, \quad n = 0, 1, 2, 3$$

In general:

$$\text{DFT}\{x(k)\} = X(n) = \sum_{k=0}^3 x(k) \cdot W_4^{nk}, \quad \text{mit } W_4 = e^{-i\frac{2\pi}{4}} = e^{-i\frac{\pi}{2}}$$

DFT: As system of equations:

$$n = 0 : X(0) = x(0)W_4^{0\cdot0} + x(1)W_4^{0\cdot1} + x(2)W_4^{0\cdot2} + x(3)W_4^{0\cdot3}$$

$$n = 1 : X(1) = x(0)W_4^{1\cdot0} + x(1)W_4^{1\cdot1} + x(2)W_4^{1\cdot2} + x(3)W_4^{1\cdot3}$$

$$n = 2 : X(2) = x(0)W_4^{2\cdot0} + x(1)W_4^{2\cdot1} + x(2)W_4^{2\cdot2} + x(3)W_4^{2\cdot3}$$

$$n = 3 : X(3) = x(0)W_4^{3\cdot0} + x(1)W_4^{3\cdot1} + x(2)W_4^{3\cdot2} + x(3)W_4^{3\cdot3}$$

In matrix-vector notation:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

b) In general  $Z(\varphi) = r \cdot e^{i\varphi}$

Calculation of the Fourier-coefficients  $W_N^{nk}$ :

$$W_N = W_4 = e^{-i\frac{2\pi}{4}} = e^{-i\frac{\pi}{2}} \hat{=} -90^\circ\text{-Drehung des komplexen Zeigers}$$

$$W_4^0 = 1 \quad = W_4^4 = W_4^8 \dots$$

$$W_4^1 = e^{-i\frac{\pi}{2}} = \underbrace{\cos\left(-\frac{\pi}{2}\right)}_{=0} + i \underbrace{\sin\left(-\frac{\pi}{2}\right)}_{=-1} = -i \quad = W_4^5 = W_4^9 \dots$$

$$W_4^2 = e^{-i\frac{\pi}{2} \cdot 2} = \underbrace{\cos(-\pi)}_{=-1} + i \underbrace{\sin(-\pi)}_{=0} = -1 \quad = W_4^6 = W_4^{10} \dots$$

$$W_4^3 = e^{-i\frac{\pi}{2} \cdot 3} = \underbrace{\cos\left(-\frac{3}{2}\pi\right)}_{=-0} + i \underbrace{\sin\left(-\frac{3}{2}\pi\right)}_{=1} = i \quad = W_4^7 = W_4^{11} \dots$$

$$W_4^4 = e^{-i\frac{\pi}{2} \cdot 4} = \cos(-2\pi) + i \sin(-2\pi) = 1$$

⋮

Depicted in the complex plane, Bild 34.

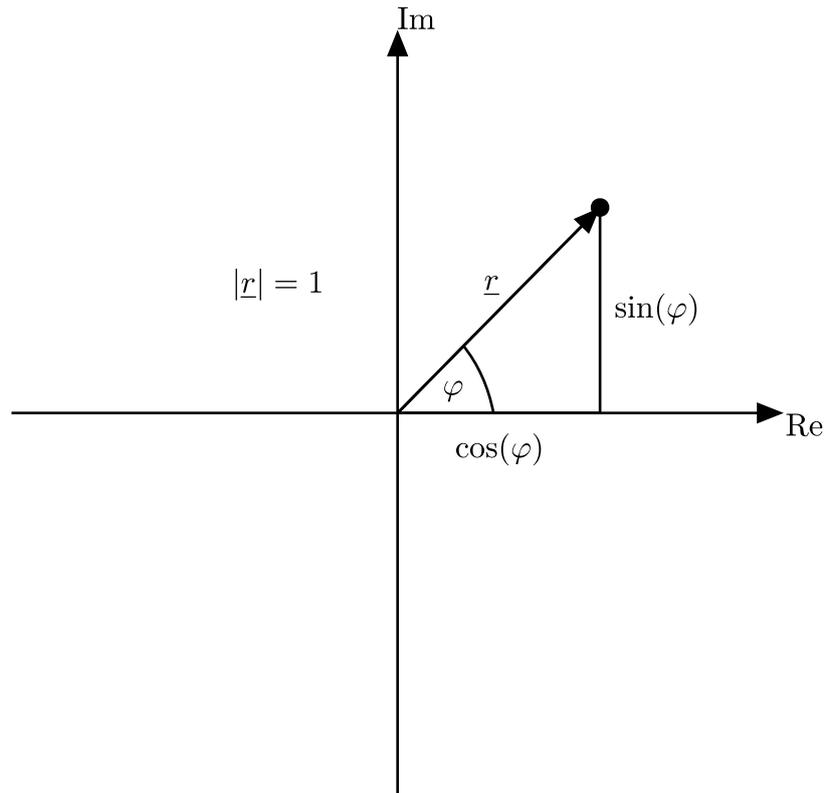


Figure 33: Relation complex plane

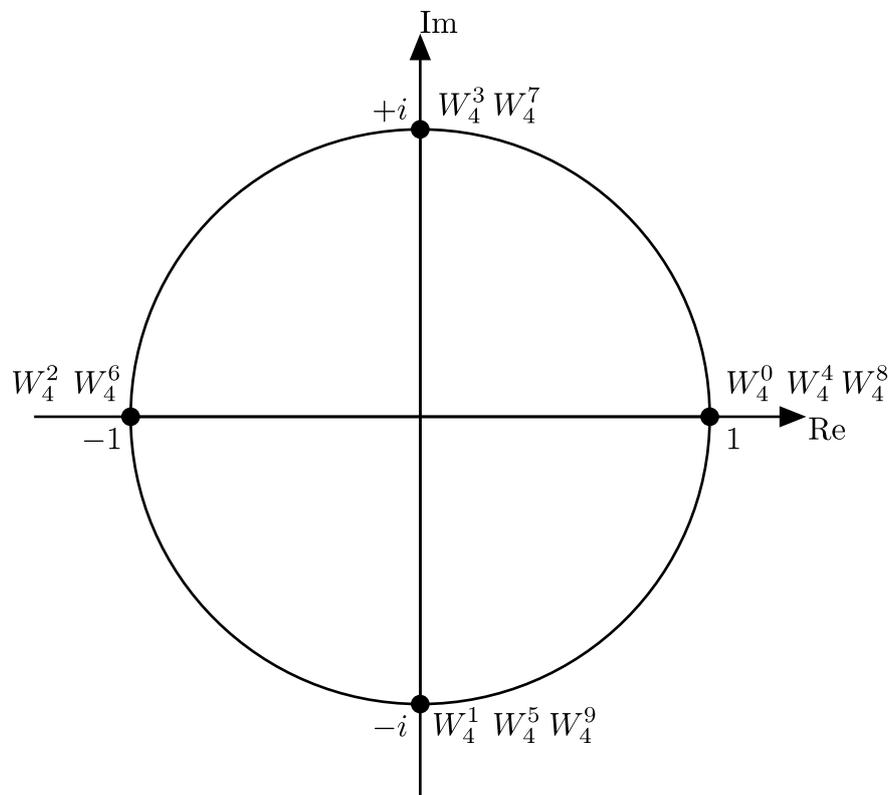


Figure 34: Fourier-coefficients in the complex plane

c) Discrete amplitude spectrum:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & +i \\ 1 & -1 & 1 & -1 \\ 1 & +i & -1 & -i \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

$$\rightarrow X(0) = 3 - 2 + 0 + 1 = 2$$

$$X(1) = 3 + 2i + 0 + i = 3 + 3i$$

$$X(2) = 3 + 2 + 0 - 1 = 4$$

$$X(3) = 3 - 2i + 0 - i = 3 - 3i$$

$$\Rightarrow \vec{X} = \begin{bmatrix} 2 \\ 3 + 3i \\ 4 \\ 3 - 3i \end{bmatrix} \Rightarrow |\vec{X}| = \begin{bmatrix} 2 \\ \sqrt{3^2 + 3^2} \\ 4 \\ \sqrt{3^2 + (-3)^2} \end{bmatrix} = \begin{bmatrix} 2 \\ 4.24 \\ 4 \\ 4.24 \end{bmatrix}$$

**Solution 9.5 Another DFT of a Periodic Sequence of Values**

a) Fourier-Transform in matrix-vector notation.

$x(k)$  is a periodic sequence, that repeats after  $N = 10$  values. For the discrete Fourier-Transform we can focus on only one period of the signal (in this case  $k \in [0, 9]$ ).

$$x(k) = \begin{cases} 1 & 0 \leq k \leq 4 \\ 0 & 4 < k \leq 9 \end{cases}$$

The discrete Fourier-Transform is:

$$\text{DFT}\{x(k)\} = X(n) = \sum_{k=0}^9 x(k)W_{10}^{nk} .$$

Because all summands where  $k > 4$  are zero, the equation can further be simplified:

$$\text{DFT}\{x(k)\} = X(n) = \sum_{k=0}^4 x(k)W_{10}^{nk} .$$

All values  $x(k)$  equal one, such that:

$$\text{DFT}\{x(k)\} = X(n) = \sum_{k=0}^4 1 \cdot W_{10}^{nk} ,$$

with  $W_{10} = e^{i2\pi/10}$ . The following equation system arises:

$$\begin{aligned} X(0) &= W_{10}^0 + W_{10}^0 + W_{10}^0 + W_{10}^0 + W_{10}^0 \\ X(1) &= W_{10}^0 + W_{10}^1 + W_{10}^2 + W_{10}^3 + W_{10}^4 \\ X(2) &= W_{10}^0 + W_{10}^{2 \cdot 1} + W_{10}^{2 \cdot 2} + W_{10}^{2 \cdot 3} + W_{10}^{2 \cdot 4} \\ X(3) &= W_{10}^0 + W_{10}^{3 \cdot 1} + W_{10}^{3 \cdot 2} + W_{10}^{3 \cdot 3} + W_{10}^{3 \cdot 4} \\ X(4) &= W_{10}^0 + W_{10}^{4 \cdot 1} + W_{10}^{4 \cdot 2} + W_{10}^{4 \cdot 3} + W_{10}^{4 \cdot 4} \\ X(5) &= W_{10}^0 + W_{10}^{5 \cdot 1} + W_{10}^{5 \cdot 2} + W_{10}^{5 \cdot 3} + W_{10}^{5 \cdot 4} \\ X(6) &= W_{10}^0 + W_{10}^{6 \cdot 1} + W_{10}^{6 \cdot 2} + W_{10}^{6 \cdot 3} + W_{10}^{6 \cdot 4} \\ X(7) &= W_{10}^0 + W_{10}^{7 \cdot 1} + W_{10}^{7 \cdot 2} + W_{10}^{7 \cdot 3} + W_{10}^{7 \cdot 4} \\ X(8) &= W_{10}^0 + W_{10}^{8 \cdot 1} + W_{10}^{8 \cdot 2} + W_{10}^{8 \cdot 3} + W_{10}^{8 \cdot 4} \\ X(9) &= W_{10}^0 + W_{10}^{9 \cdot 1} + W_{10}^{9 \cdot 2} + W_{10}^{9 \cdot 3} + W_{10}^{9 \cdot 4} . \end{aligned}$$

In matrix-vector notation, this equation system becomes:

$$\vec{X}(n) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_{10}^1 & W_{10}^2 & W_{10}^3 & W_{10}^4 \\ 1 & W_{10}^2 & W_{10}^4 & W_{10}^6 & W_{10}^8 \\ 1 & W_{10}^3 & W_{10}^6 & W_{10}^9 & W_{10}^{12} \\ 1 & W_{10}^4 & W_{10}^8 & W_{10}^{12} & W_{10}^{16} \\ 1 & W_{10}^5 & W_{10}^{10} & W_{10}^{15} & W_{10}^{20} \\ 1 & W_{10}^6 & W_{10}^{12} & W_{10}^{18} & W_{10}^{24} \\ 1 & W_{10}^7 & W_{10}^{14} & W_{10}^{21} & W_{10}^{28} \\ 1 & W_{10}^8 & W_{10}^{16} & W_{10}^{24} & W_{10}^{32} \\ 1 & W_{10}^9 & W_{10}^{18} & W_{10}^{27} & W_{10}^{36} \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \end{bmatrix} .$$

b) Simplification of the result.

For one discrete frequency  $n$ , the corresponding discrete Fourier-Transform value can be calculated as follows:

$$X(n) = \sum_{k=0}^4 1 \cdot (W_{10}^n)^k .$$

In more detail, the sum that has to be calculated looks like:

$$X(n) = (W_{10}^n)^0 + (W_{10}^n)^1 + (W_{10}^n)^2 + (W_{10}^n)^3 + (W_{10}^n)^4 . \quad (4)$$

If we multiply  $X(n)$  with  $W_{10}^n$  we get:

$$W_{10}^n X(n) = (W_{10}^n)^1 + (W_{10}^n)^2 + (W_{10}^n)^3 + (W_{10}^n)^4 + (W_{10}^n)^5 . \quad (5)$$

Now we subtract equation 5 from 4:

$$\begin{aligned} X(n) - W_{10}^n X(n) &= (W_{10}^n)^0 - (W_{10}^n)^5 && \text{|Transformation} \\ X(n)[1 - W_{10}^n] &= 1 - (W_{10}^n)^5 && \text{| : [1 - W_{10}^n]} \\ X(n) &= \frac{1 - (W_{10}^n)^5}{1 - W_{10}^n} . \end{aligned}$$

Now we have to proof the following statement:

$$\frac{1 - (W_{10}^n)^5}{1 - W_{10}^n} = e^{-i4\pi \frac{n}{10}} \frac{\sin(\frac{n}{2}\pi)}{\sin(\frac{n}{10}\pi)} .$$

At first we use the Euler-Equation for the sine functions:

$$\begin{aligned} \sin(\frac{n}{2}\pi) &= \sin(\frac{5n}{10}\pi) = \frac{1}{2i} \left( e^{i\pi \frac{5n}{10}} - e^{-i\pi \frac{5n}{10}} \right) , \\ \sin(\frac{n}{10}\pi) &= \frac{1}{2i} \left( e^{i\pi \frac{n}{10}} - e^{-i\pi \frac{n}{10}} \right) , \\ \frac{\sin(\frac{n}{2}\pi)}{\sin(\frac{n}{10}\pi)} &= \frac{e^{i\pi \frac{5n}{10}} - e^{-i\pi \frac{5n}{10}}}{e^{i\pi \frac{n}{10}} - e^{-i\pi \frac{n}{10}}} . \end{aligned}$$

Now we do some transformations:

$$\begin{aligned} \frac{1 - (W_{10}^n)^5}{1 - W_{10}^n} &= \frac{1 - e^{-i2\pi \frac{5n}{10}}}{1 - e^{-i2\pi \frac{n}{10}}} && \cdot \frac{e^{i\pi \frac{n}{10}}}{e^{i\pi \frac{n}{10}}} \\ &= \frac{e^{i\pi \frac{n}{10}} - e^{-i\pi \frac{9n}{10}}}{e^{i\pi \frac{n}{10}} - e^{-i\pi \frac{n}{10}}} && \text{|Transformation} \\ &= e^{-i4\pi \frac{n}{10}} \frac{e^{i\pi \frac{5n}{10}} - e^{-i\pi \frac{5n}{10}}}{e^{i\pi \frac{n}{10}} - e^{-i\pi \frac{n}{10}}} \end{aligned}$$

Through some transformations, we see, that the expressions are equivalent.

### Solution 9.6 Leakage Effect and Picket Fence Effect

MATLAB!

### Solution 9.7 Leakage Effect

MATLAB!

## 10 Filters

### Solution 10.1 Types of Filters

a) Frequency responses:

- top left: band-stop  
 $\omega \rightarrow 0, \omega \rightarrow \frac{\omega_0}{2}$  (Nyquist-Frequency) :  $|G(i\omega)| = 0 \text{ dB} = 1 \Rightarrow$  pass band  
 $0 < \omega < \frac{\omega_0}{2}$  :  $|G(i\omega)| = -80 \text{ dB} \approx 0.0001 \Rightarrow$  stop band
- top right: low-pass  
 $\omega \rightarrow 0$  :  $|G(i\omega)| = 0 \text{ dB} = 1 \Rightarrow$  pass band  
 $\omega \rightarrow \frac{\omega_0}{2}$  :  $|G(i\omega)| \rightarrow -\infty \text{ dB} = 0 \Rightarrow$  stop band
- bottom left: high pass  
 $\omega \rightarrow 0$  :  $|G(i\omega)| \rightarrow -\infty \text{ dB} = 0 \Rightarrow$  stop band  
 $\omega \rightarrow \frac{\omega_0}{2}$  :  $|G(i\omega)| \rightarrow 0 \text{ dB} = 1 \Rightarrow$  pass band
- bottom right: band-pass  
 $\omega \rightarrow 0, \omega \rightarrow \frac{\omega_0}{2}$  :  $|G(i\omega)| \rightarrow -\infty \text{ dB} = 0 \Rightarrow$  stop band  
 $0 < \omega < \frac{\omega_0}{2}$  :  $|G(i\omega)| \approx 1 \text{ dB} = 1 \Rightarrow$  pass band

Sampling frequency:

Amplitude responses end at  $\omega \approx 30 \frac{\text{rad}}{\text{s}}$ . This is the Nyquist-Frequency. The Nyquist-Frequency equals

$$\omega_{Ny} = \frac{1}{2}\omega_0 \Rightarrow 60 \frac{\text{rad}}{\text{s}} = \omega_0$$

$$\omega_0 = 2\pi f_0 \Rightarrow f_0 = \frac{\omega_0}{2\pi} = \frac{30}{\pi} \text{ Hz} \approx 9,55 \text{ Hz}$$

- b)
- top left: high-pass  
ramp very low frequency
  - top right: low-pass  
only the high frequency is suppressed
  - bottom left: band-pass  
very low frequencies and high frequencies are suppressed
  - bottom right: band-stop  
low and high frequencies are not suppressed

**Solution 10.2 Properties of an Ideal Filter**

- a) stop-band:  $0 / -\infty$  dB  
pass-band:  $1 / 0$  dB
- b) Phase shifts are not desirable  $\rightarrow 0^\circ$
- c) Steepness:  $-\infty$  (no transition range, just one frequency where the amplitude response suddenly changes!)

**Solution 10.3 Requirements for Filters**

a) In contrast to FIR-Filters, IIR-Filters can become unstable!

b) Filter with linear phase:

'System with pure dead time' or 'group propagation delay':  $\tau_g = -\frac{d\varphi}{d\omega}$

Can exactly only be achieved through FIR-Filters!

Filter with linear phase means that the phase is shifted by a function of the following type:

$$\varphi(\omega) = \alpha \cdot \omega \quad (\text{linear equation!}) \quad (1)$$

Mathematically a phase shift looks as follows:

$$y = A \cdot \sin(\omega t + \underbrace{\varphi}_{\text{phase shift}}) \quad (2)$$

(1) in (2):

$$\begin{aligned} y &= A \cdot \sin(\omega t + \alpha\omega) \\ &= A \cdot \sin(\omega(t + \alpha)) \end{aligned}$$

As can be seen from this equation all frequencies are shifted in the same manner!

→ very important in acoustic environments because our ears are very sensitive to frequency-dependent phase differences!

c) An acausal filter needs values from the "future" to calculate the current system/ filter output.

→ This is only possible for offline applications or systems where buffers can be used!

Advantage: The phase shift of a filter can be eliminated! (see script page 316)

**Solution 10.4 Properties of IIR- and FIR-Filters**

- a) IIR
- b) FIR
- c) FIR
- d) IIR
- e) FIR

**Solution 10.5 Impulse Response and Step Response of IIR- and FIR-Filters**

a) Filter order: 3  $\Rightarrow u(k-3)$  most delayed signal

$\Rightarrow$  4 terms (from 0...3)  $\Rightarrow$  all terms have the same coefficient  $\frac{1}{4}$  (averaging!)

$$y(k) = \frac{1}{4}u(k) + \frac{1}{4}u(k-1) + \frac{1}{4}u(k-2) + \frac{1}{4}u(k-3)$$

Impulse Response:

$$u(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{else} \end{cases}$$

$$\rightarrow y(0) = \frac{1}{4} = \frac{1}{4}u(0) + \underbrace{\frac{1}{4}u(-1)}_{=0} + \underbrace{\frac{1}{4}u(-2)}_{=0} + \underbrace{\frac{1}{4}u(-3)}_{=0}$$

$$y(1) = \frac{1}{4} = \underbrace{\frac{1}{4}u(1)}_{=0} + \frac{1}{4}u(0) + \underbrace{\frac{1}{4}u(-1)}_{=0} + \underbrace{\frac{1}{4}u(-2)}_{=0}$$

$$y(2) = \frac{1}{4}$$

$$y(3) = \frac{1}{4}$$

$$y(4) = 0$$

$\vdots$

$$y(k > 3) = 0$$

$\Rightarrow$  FIR; after 3 time steps the impulse response reaches exactly 0 and keeps being zero!

Step- Response:

$$u(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{else} \end{cases}$$

$$y(0) = \frac{1}{4}$$

$$y(1) = \frac{2}{4} = \frac{1}{2}$$

$$y(2) = \frac{3}{4}$$

$$y(3) = \frac{4}{4} = 1$$

$$y(k > 3) = 1$$

$\rightarrow$  In every future step all delayed inputs maintain their value of 1!

b) New difference equation:

$$y(k) = 0.5u(k) + 0.25u(k-1) + 0.25u(k-2) + 0.25u(k-3)$$

Problem: The gain changes!

Step- Response:

$$u(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{else} \end{cases}$$

$$y(0) = 0,5$$

$$y(1) = 0,75$$

$$y(2) = 1$$

$$y(3) = 1,25$$

$$y(k \geq 3) = 1,25 \neq 1 \quad (\text{value before the change of the first coefficient})$$

Solution: Decrease of other coefficients such that the sum of all coefficients keeps the same!

For example: 2 last coefficients reduced to  $\frac{1}{8}$

$$\rightarrow y(k) = 0,5u(k) + \frac{1}{4}u(k-1) + \frac{1}{8}u(k-2) + \frac{1}{8}u(k-3)$$

Step-response:  $y(0) = 0,5; y(1) = 0,75; y(2) = 0,875; \underline{\underline{y(k \geq 3) = 1}}$

c)  $y(k) = 0,5u(k) + 0,5y(k-1)$  with  $y(k < 0) = 0$

Impulse- Response:

$$u(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{else} \end{cases}$$

$$y(0) = 0,5u(0) + 0,5y(-1) = 0,5$$

$$y(1) = 0,5u(1) + 0,5y(0) = 0,5 \cdot 0,5 = 0,25$$

$$y(2) = 0 + 0,5y(1) = 0,5^3$$

⋮

$$y(k) = 0,5^{k+1}$$

$y(k \rightarrow \infty) \rightarrow 0$ ; For any integer  $k > 0$   $y(k) > 0 \Rightarrow$  IIR!

Step- Response:

$$u(k) = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{else} \end{cases}$$

$$y(0) = 0,5u(0) + 0,5y(-1) = 0,5$$

$$y(1) = 0,5 + 0,5y(0) = \underbrace{0,5 + 0,5 \cdot 0,5}_{b_0 + b_0 a_1 = b_0(1 + a_1)} = 0,75$$

$$y(2) = b_0 + \underbrace{b_0(1 + a_1)}_{y(1)} a_1 = b_0(1 + a_1 + a_1^2) = 0,5(1 + 0,5 + 0,5^2) = 0,875$$

$$y(k) = b_0 \sum_{i=0}^k a_1^i = 0,5 \sum_{i=0}^k 0,5^i$$

$$\begin{aligned} &\rightarrow s = \sum_{i=0}^k 0,5^i \\ &0,5s = \sum_{i=1}^{k+1} 0,5^i \\ &s - 0,5s = 0,5^0 - 0,5^{k+1} \\ \Rightarrow s &= \frac{1 - 0,5^{k+1}}{1 - 0,5} = 2(1 - 0,5^{k+1}) \end{aligned}$$

$y(k \rightarrow \infty) \rightarrow 1$ ; For every integer of  $k < \infty$ , 1 is not exactly reached!

Approximated FIR-Filter: Impulse response values up to the desired order are used!

$$y(0) = 0,5; y(1) = 0,25 = \frac{1}{4}; y(2) = 0,125 = \frac{1}{8}$$

$$G_{FIR}^{(2)} = 0,5 + \frac{1}{4}z^{-1} + \frac{1}{8}z^{-2} = \frac{Y(z)}{U(z)}$$

$$y(k) = \frac{1}{2}u(k) + \frac{1}{4}u(k-1) + \frac{1}{8}u(k-2)$$

Step- Response:

$$y(0) = \frac{1}{2}; y(1) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}; y(2) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$y(k \geq 2) = \frac{7}{8}$  (values maintain -> FIR)

Step response should at least tend to 1!

Solution: Increasing one or more of the coefficients such that

$$\begin{aligned} &\sum_{i=0}^2 b_i = 1 \\ \text{i.e. } &0,5 + \frac{1}{4} + x = 1 ? \\ &\Rightarrow x = \frac{1}{4} \text{ instead of } \frac{1}{8} \\ \Rightarrow &G_{FIR}^{(2)} = 0,5 + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} \end{aligned}$$

**Solution 10.6 Acausal Filter**

a) Which of the following 3 filters is causal, which is acausal and why?

To determine if a filter is causal or acausal, one way is to compare the maximum degree of the numerator and the denominator. If  $n$  is the maximum degree of the numerator and  $m$  the maximum degree of the denominator, a filter is causal, if the following statement holds:

$$m \geq n .$$

Now we transform the filter transfer functions into the writing with only positive exponents and determine, if they are causal or acausal:

$$G_1(z) = \frac{0.2z}{z - 0.8} \rightarrow \text{causal filter}$$

$$\begin{aligned} G_2(z) &= \frac{0.2z}{1 - 0.8z^{-1}} && | \cdot \frac{z}{z} \\ &= \frac{0.2z^2}{z - 0.8} \rightarrow \text{acausal filter} \end{aligned}$$

$$\begin{aligned} G_3(z) &= \frac{0.2}{z^{-1} - 0.8z^{-2}} && | \cdot \frac{z^2}{z^2} \\ &= \frac{0.2z^2}{z - 0.8} \rightarrow \text{acausal filter} \end{aligned}$$

b) Design an acausal filter of order 2, that has no phase shift. Sketch the response to a time-shifted step  $\sigma(k - 3)$  of the acausal filter and explain why there is no phase shift.

Acausal filter of order 2:

$$y(k) = \frac{1}{3}u(k - 1) + \frac{1}{3}u(k) + \frac{1}{3}u(k + 1) .$$

Now we calculate the step response to a time shifted unit step  $\sigma(k - 3) = \begin{cases} 1 & \text{for } k \geq 3 \\ 0 & \text{else} \end{cases}$ :

$$\begin{aligned} k = 0 : y(0) &= \frac{1}{3}[u(-1) + u(0) + u(1)] \\ &= \frac{1}{3}[0 + 0 + 0] = 0 \end{aligned}$$

$$\begin{aligned} k = 1 : y(1) &= \frac{1}{3}[u(0) + u(1) + u(2)] \\ &= \frac{1}{3}[0 + 0 + 0] = 0 \end{aligned}$$

$$\begin{aligned} k = 2 : y(2) &= \frac{1}{3}[u(1) + u(2) + u(3)] \\ &= \frac{1}{3}[0 + 0 + 1] = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} k = 3 : y(3) &= \frac{1}{3}[u(2) + u(3) + u(4)] \\ &= \frac{1}{3}[0 + 1 + 1] = \frac{2}{3} \end{aligned}$$

$$\begin{aligned}
 k \geq 4 : y(4) &= \frac{1}{3}[u(3) + u(4) + u(5)] \\
 &= \frac{1}{3}[1 + 1 + 1] = 1
 \end{aligned}$$

Why is there no phase shift? To answer this question we take a closer look at the

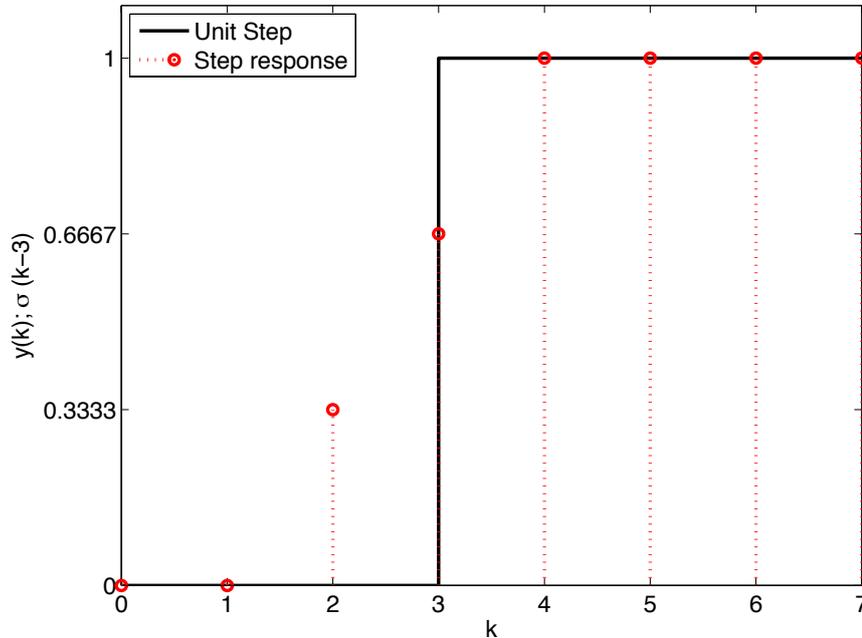


Figure 35: Step response of an acausal filter.

filter's transfer function:

$$\begin{aligned}
 G(z) &= \frac{1}{3} (z^{-1} + 1 + z^{+1}) & |z = e^{sT_0} \text{ with } s = i\omega \\
 &= \frac{1}{3} (1 + e^{-i\omega T_0} + e^{i\omega T_0})
 \end{aligned} \tag{1}$$

Now we use the following relationships:

$$\begin{aligned}
 e^{i\phi} &= \cos(\phi) + i\sin(\phi) \\
 \cos(-\phi) &= \cos(\phi) \\
 \sin(-\phi) &= -\sin(\phi) .
 \end{aligned}$$

With this relationships equation 1 becomes:

$$\begin{aligned}
 G(z) &= \frac{1}{3} (1 + \cos(\omega T_0) - i\sin(\omega T_0) + \cos(\omega T_0) + i\sin(\omega T_0)) \\
 &= \frac{1}{3} (1 + 2\cos(\omega T_0))
 \end{aligned}$$

The equation to calculate the phase shift is:

$$\varphi = \tan^{-1} \left( \frac{\text{Im}(G(z))}{\text{Re}(G(z))} \right) .$$

The imaginary part of our transfer function is exactly zero. So the phase shift becomes zero as well.

**Solution 10.7 Bilinear Transformation**

The equation to perform the bilinear transform is:

$$s = \frac{2}{T_0} \frac{1 - z^{-1}}{1 + z^{-1}} .$$

With the sampling time  $T_0 = 1$  we can calculate the bilinear transform:

$$\begin{aligned} G_{BT}(z) &= \frac{5}{1 + 5 \cdot \left(\frac{2}{1} \frac{1 - z^{-1}}{1 + z^{-1}}\right)} && \text{|Transformation} \\ &= \frac{5(1 + z^{-1})}{1 + z^{-1} + 10(1 - z^{-1})} && \text{|Transformation} \\ &= \frac{5 + 5z^{-1}}{11 - 9z^{-1}} && \text{|} \cdot \frac{\frac{1}{11}}{\frac{1}{11}} \\ &= \frac{\frac{5}{11} + \frac{5}{11}z^{-1}}{1 - \frac{9}{11}z^{-1}} \end{aligned}$$

We substitute the fractions, with their rounded values:  $\frac{5}{11} \approx 0.45$  and  $\frac{9}{11} \approx 0.82$ :

$$G_{BT}(z) = \frac{0.45 + 0.45z^{-1}}{1 - 0.82z^{-1}} .$$

Now we compare  $G_{BT}(z)$  with  $G(z)$  with respect to the poles, zeros, gain and properness.

We start with the poles: Because both denominators are equal, there are no differences between the two discrete transfer functions regarding the stability. Both transfer functions have the same pole at  $p_0 = 0.82$ . Because  $|p_0| < 1$ , the transfer functions are stable.

Note: Through the bilinear transform no stability properties are affected, but the exact pole-location may differ (in general).

Comparison of the gain: Starting with the gain of  $G(z)$ :

$$y(k \rightarrow \infty) = \lim_{z \rightarrow 1} (z - 1) \underbrace{\frac{0.9z^{-1}}{1 - 0.82z^{-1}}}_{G(z)} \underbrace{\frac{z}{z - 1}}_{\text{Unit step}} = 5 .$$

Now the gain of the transfer function  $G_{BT}(z)$  is calculated:

$$y(k \rightarrow \infty) = \lim_{z \rightarrow 1} (z - 1) \underbrace{\frac{0.45 + 0.45z^{-1}}{1 - 0.82z^{-1}}}_{G_{BT}(z)} \underbrace{\frac{z}{z - 1}}_{\text{Unit step}} = 5 .$$

→ no difference - the correct gain of the continuous time system is reached.

Comparison of zeros:

$$\begin{aligned} G(z) &= \frac{0.9}{z - 0.82} \rightarrow \text{no zeros!} \\ G_{BT}(z) &= \frac{0.45z + 0.45}{z - 0.82} \rightarrow \text{one zero at } z = -1 \end{aligned}$$

The continuous time system has no zeros.

Properness:

$$G(z) = \frac{0.9}{z - 0.82} \rightarrow \text{strictly proper}$$

Maximum degree of the numerator is smaller than the maximum degree of the denominator.

$$G_{BT}(z) = \frac{0.45z + 0.45}{z - 0.82} \rightarrow \text{proper (not strictly! - system with feed-through)}$$

Maximum degree of the numerator is equal the maximum degree of the denominator.

**Solution 10.8 Properties of Common Filters**

- a) Butterworth-Filter
- b) Chebyshev
  - Type 1 (Ripples in the pass-band)
  - Type 2 (Ripples in the stop-band)
- c) Bessel-Filter
- d) Cauer-Filter
- e) Bessel  $\rightarrow$  Butterworth  $\rightarrow$  Chebyshev  $\rightarrow$  Cauer

**Solution 10.9 Block-Diagram of a Time-Discrete Filter**

- a) Transform the transfer function into the form with only negative powers of  $z$  and evaluate the corresponding difference equation.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{2z^3 + 3z^2}{z^3 + 2z^2 + z + 5} \quad | \cdot \frac{z^{-3}}{z^{-3}}$$

$$\Leftrightarrow \frac{Y(z)}{U(z)} = \frac{2 + 3z^{-1}}{1 + 2z^{-1} + z^{-2} + 5z^{-3}} \quad | \cdot \text{Transformation}$$

$$\Leftrightarrow Y(z) + 2Y(z)z^{-1} + Y(z)z^{-2} + 5Y(z)z^{-3} = 2U(z) + 3U(z)z^{-1}$$

This equation can easily be transformed into the discrete time domain:

$$y(k) + 2y(k-1) + y(k-2) + 5y(k-3) = 2u(k) + 3u(k-1) \quad | \text{Transformation}$$

$$y(k) = 2u(k) + 3u(k-1) - (2y(k-1) + y(k-2) + 5y(k-3))$$

- b) Sketch the corresponding block-diagram.

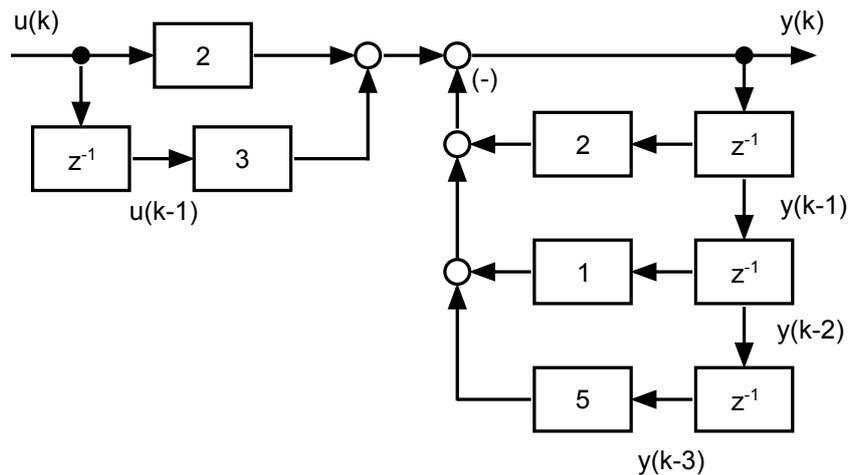
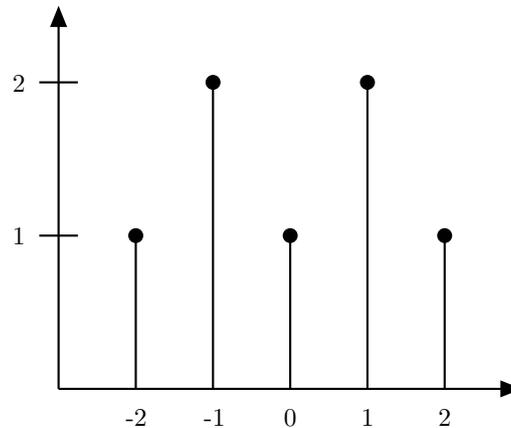


Figure 36: Block diagram of the filter  $G(z)$ .

**Solution 10.10 Symmetric FIR-Filter**

Determine the phase shift for both filters.



For the acausal filter the transfer function is:

$$G(z) = g(-2) \cdot z^2 + g(-1) \cdot z^1 + g(0) \cdot z^0 + g(1) \cdot z^{-1} + g(2) \cdot z^{-2}$$

Now we rearrange some terms:

$$G(z) = g(2) \cdot (z^2 + z^{-2}) + g(1) \cdot (z^1 + z^{-1}) + g(0) .$$

Note:

$$z^{\pm n} = e^{\pm in\omega T_0} = \cos(n\omega T_0) \pm i \sin(n\omega T_0) .$$

Phase shift:

$$\varphi(H(i\omega)) = \arctan\left(\frac{\operatorname{Im}\{H(i\omega)\}}{\operatorname{Re}\{H(i\omega)\}}\right) .$$

For conjugate complex pole pairs the following holds (the cos-function is point-symmetrically):

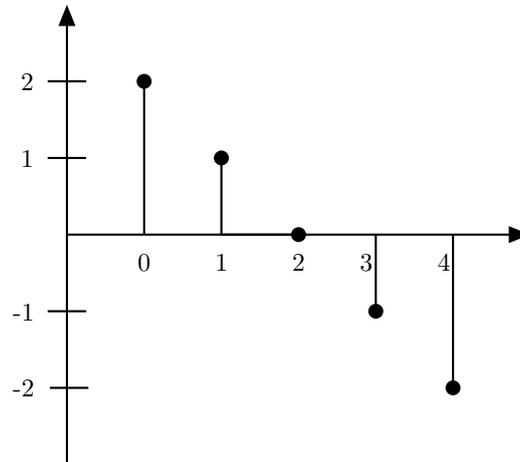
$$\begin{aligned} z^n + z^{-n} &= e^{in\omega T_0} + e^{-in\omega T_0} \\ &= \cos(n\omega T_0) + \cancel{i \sin(n\omega T_0)} + \cos(n\omega T_0) - \cancel{i \sin(n\omega T_0)} \\ &= 2 \cos(n\omega T_0) . \end{aligned}$$

$$\Rightarrow \varphi(z^n + z^{-n}) = \arctan(0) = 0! .$$

It follows, that the phase shift of the acausal filter is exactly zero!

For the causal filter the transfer function is:

$$\begin{aligned} G(z) &= g(0) + g(1) \cdot z^{-1} + g(2) \cdot z^{-2} + g(3) \cdot z^{-3} + g(4) \cdot z^{-4} \\ &= z^{-2} \cdot [g(0) \cdot z^2 + g(1) \cdot z^1 + g(2) \cdot z^0 + g(3) \cdot z^{-1} + g(4) \cdot z^{-2}] \end{aligned}$$



Once again we rearrange some terms and use the following relationships (see figure above):

$$g(0) = -g(4) \text{ and}$$

$$g(1) = -g(3)$$

$$G(z) = z^{-2} \cdot [g(0) \cdot (z^2 - z^{-2}) + g(1) \cdot (z^1 - z^{-1})] \quad (1)$$

Note:

$$\begin{aligned} z^n - z^{-n} &= \cos(n\omega T_0) + i \cdot \sin(n\omega T_0) - (\cos(n\omega T_0) - i \cdot \sin(n\omega T_0)) \\ &= 2i \sin(n\omega T_0) \end{aligned} \quad (2)$$

Combining 1 and 2 leads to the transfer function for the causal filter:

$$\begin{aligned} G(i\omega) &= e^{-2i\omega T_0} (g(1) \cdot 2i \sin(\omega T_0) + g(0) \cdot 2i \sin(2\omega T_0)) \\ &= i \cdot e^{-2i\omega T_0} (2g(1) \sin(\omega T_0) + 2g(0) \sin(2\omega T_0)) \\ &= e^{i\frac{\pi}{2}} \cdot e^{-2i\omega T_0} (2g(1) \sin(\omega T_0) + 2g(0) \sin(2\omega T_0)) \\ &= e^{\underbrace{i\left(\frac{\pi}{2} - 2\omega T_0\right)}_{\text{phase}}} \underbrace{(2g(1) \sin(\omega T_0) + 2g(0) \sin(2\omega T_0))}_{\text{amplitude}} \end{aligned}$$