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On the fast and fully discretized solution of integral and pseudo-differential equations on smooth curves

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Abstract. For the fast numerical solution of a fully discrete variant of the trigonometric Galerkin equations associated with periodic integral equations, in this paper we consider approximations with small residuals and provide order-optimal estimates for the associated error. The CGNR method is considered as a method with a simple iteration scheme where those approximations can be obtained by a total

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number of $\mathcal{O}\big(N \log N\big)$ arithmetical operations, with N denoting the dimension of the space of trigonometric trial polynomials associated with the Galerkin method. Noise in the model of the problem as well as in the right-hand side are admitted.

1. Introduction

1.1. A class of operators

In this paper we consider equations of the following form,

$$\mathcal{A}u \;=\; f, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{C}$ is a 1-periodic function, and the operator $\mathcal{A}$ has the form

$$\mathcal{A} \;=\; \mathcal{D} \;+\; \sum_{p=0}^{q} A_p, \;\; \big(\mathcal{D}u\big)(t) \;=\; \int_0^1 \kappa_0(t-s)\, u(s)\; ds, \tag{2}$$

$$\big(A_p u\big)(t) \;=\; \int_0^1 \kappa_p(t-s)\, a_p(t,s)\, u(s)\; ds, \tag{3}$$

$$t \in \big[\,0,1\,\big], \qquad p \;=\; 0,1,\ldots,q,$$

where $q \geq -1$, and $a_p : \mathbb{R}^2 \to \mathbb{C}$ are 1-biperiodic C^∞-smooth functions, and κ_p are 1-periodic functions or distributions with known Fourier coefficients

$$\hat{\kappa}_p(n) \;:=\; \int_0^1 \kappa_p(t)\, e^{-\mathrm{i}n2\pi t}\; dt, \qquad\qquad n \in \mathbb{Z}.$$

It is supposed that

$$a_0(t,t) \;=\; 0 \qquad \big(t \in \mathbb{R}\big), \tag{4}$$

and moreover for certain parameters $\alpha \in \mathbb{R}$ and integers $0 < \beta_1 \leq \beta_2 \leq \ldots \leq \beta_q$, the following estimates are supposed to hold,

$$\big|\triangle^j \hat{\kappa}_0(n)\big| \leq c_j \underline{n}^{\alpha-j} \quad \Big(n \in \mathbb{Z}, \quad j \;=\; 0,1,\ldots\Big), \tag{5}$$

$$c_{00}|n|^\alpha \leq |\hat{\kappa}_0(n)| \quad \Big(0 \neq n \in \mathbb{Z}\Big), \tag{6}$$

$$\big|\triangle^j \hat{\kappa}_p(n)\big| \leq c_j \underline{n}^{\alpha-\beta_p-j} \; \Big(n \in \mathbb{Z},\; j = 0,1,\ldots,\; p = 1,\ldots,q\Big). \tag{7}$$

Here c_j and c_{00} are some positive constants,

$$\underline{n} \;=\; \begin{cases} |n|, & \text{if } n \neq 0 \\ 1\,, & \text{if } n = 0 \end{cases} \qquad \big(n \in \mathbb{Z}\big)$$

and $\triangle$ denotes the forward difference operator, i.e.,

$$\triangle \hat{v}(n) = \hat{v}(n+1) \;-\; \hat{v}(n), \qquad n \in \mathbb{Z}.$$

Equations of the form (1) with operators $\mathcal{A}$ satisfying (2)–(7) arise, e.g., if the boundary integral method is applied to a boundary value problem on a two-dimensional bounded and simply connected domain with a smooth boundary. Two associated examples will be presented in section 1.2 but first the basic mapping properties of the operator (2), (3) are stated, cf. [8] or [18] for a similar situation. As a preparation we consider for any $\lambda \in \mathbb{R}$ the Sobolev space H^λ of those functions or distributions u which satisfy

$$\|u\|_\lambda := \Big(\sum_{n \in \mathbb{Z}} \underline{n}^{2\lambda} |\hat{u}(n)|^2 \Big)^{1/2} < \infty,$$
$$\text{where} \qquad \hat{u}(n) := \int_0^1 u(t)\, e^{-\mathrm{i}n2\pi t}\, dt, \qquad n \in \mathbb{Z},$$

and $\mathcal{L}\big(H^{\lambda_1}, H^{\lambda_2}\big)$ denotes the space of linear bounded operators from H^{λ_1} into H^{λ_2} $\big(\lambda_1,\ \lambda_2 \in \mathbb{R}\big)$. If we decompose $\mathcal{A}$ into its main part and the corresponding remainder,

$$\mathcal{A} \;=\; \mathcal{D} \;+\; \mathcal{B}, \qquad\qquad \mathcal{B} \;=\; \sum_{p=0}^{q} A_p,$$

then the properties (5)–(7) yield that $\mathcal{D} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha}\big)$ is a Fredholm operator of index 0 and that $\mathcal{B} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha+\beta}\big)$, where

$$\beta \;:=\; \begin{cases} \beta_1, & \text{if } a_0 \equiv 0, \\ \min\{\, \beta_1,\, 1 \,\}, & \text{otherwise}, \end{cases} \tag{8}$$

and this means that $\mathcal{B} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha}\big)$ is a compact operator. We thus have:

Proposition 1. *Suppose that an operator of the form (2)–(3) satisfies the conditions (4)–(7). Then* $\mathcal{A} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha}\big)$ *is a Fredholm operator of index* 0 *for each* $\lambda \in \mathbb{R}$.

We recall that a linear bounded operator $T : H^\lambda \to H^{\lambda-\alpha}$ is a Fredholm operator of index 0 if the range $\mathcal{R}(T)$ of T is closed in $H^{\lambda-\alpha}$ and moreover $\operatorname{codim} \mathcal{R}(T) = \dim \mathcal{N}(T) < \infty$ is satisfied, where

$\mathcal{N}(T) \subset H^\lambda$ denotes the nullspace of T. It follows from conditions (4)–(7) that $\mathcal{N}(A) \subset C^\infty$. Thus, if the condition

$$v \text{ 1–periodic } C^\infty\text{–function}, \quad \mathcal{A}v = 0 \quad \Longrightarrow \quad v = 0 \tag{9}$$

is satisfied, then Proposition 1 yields that $\mathcal{A} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha}\big)$ is an isomorphism for each $\lambda \in \mathbb{R}$.

Let us characterize the class of integral operators (2)–(3) in terms of periodic pseudodifferential and classical pseudodifferential operators (see e.g. [20] for these notions and relations to periodic integral operators). A periodic classical pseudodifferential operator of order α has the Agranovich representation

$$A \;=\; \big(a_+(t)P_+ \;+\; a_-(t)P_-\big)\Lambda^\alpha \;+\; K,$$

where $a_\pm(t)$ are C^∞-smooth 1-periodic functions,

$$P_+u = \sum_{n\geq 0} \hat{u}(n)e^{in2\pi t}, \qquad P_-u = \sum_{n<0} \hat{u}(n)e^{in2\pi t},$$
$$\Lambda^\alpha u = \sum_{n\in\mathbb{N}} \underline{n}^\alpha \hat{u}(n)e^{in2\pi t},$$

and K is a periodic classical pseudodifferential operator of order $\alpha-1$. Further A is elliptic if $a_\pm(t) \neq 0$ for $t \in \mathbb{R}$. Our class of operators defined by (2)–(7) covers the class of classical elliptic pseudodifferential operators $(a_+P_+ \;+\; a_-P_-)\Lambda^\alpha$ with constant coefficients $a_\pm \neq 0$. This corresponds to the case $\hat{\kappa}_0(n) \;=\; (a_+ + a_-\mathrm{sign}\, n)\underline{n}^\alpha$. On the other hand, our conditions on $\hat{\kappa}_0(n)$ are more general, and conditions (2)–(7) define a class of integral operators which are periodic pseudodifferential operators but need not to be classical ones.

1.2. Examples

In the sequel we consider two prominent examples, cf. [18].

Example 1. Symm's integral equation for closed C^∞–smooth boundaries in parametrized form looks as follows,

$$\big(\mathcal{A}u\big)(t) \quad = \quad -\int_0^1 \log|x(t)-x(s)|\,u(s)\;ds \quad = \quad f(t), \qquad t \in \big[\,0,1\,\big],$$

where $x:\mathbb{R}\to\mathbb{R}^2$ is a C^∞–smooth 1-periodic parametrization of the corresponding boundary with $x'(t)\neq 0$ for $t\in\mathbb{R}$. In order to apply Proposition 1 we consider the following decomposition,

$$\big(\mathcal{A}u\big)(t) = \int_0^1 \kappa_0(t-s)\,u(s)\;ds \;+\; \int_0^1 a_1(t,s)\,u(s)\;ds, \qquad t\in\big[\,0,1\,\big],$$

with $\kappa_0(t) = -\log|\sin\pi t|$ and

$$a_1(t,s) \quad = \quad \begin{cases} -\log\dfrac{|x(t)-x(s)|}{|\sin\pi(t-s)|}, & \text{if} \quad t\neq s, \\ -\log\dfrac{|x'(t)|}{\pi}, & \text{if} \quad t=s. \end{cases}$$

Here $|\cdot|$ also denotes the Euclidian norm in $\mathbb{R}^2$. Note that a_1 is a 1-biperiodic C^∞-function, and the Fourier coefficients of κ_0 have the following form,

$$\hat{\kappa}_0(n) \;=\; \begin{cases} \dfrac{1}{2|n|}, & \text{if } \; 0 \neq n \in \mathbb{Z}, \\[2ex] \log 2, & \text{if } \; n = 0. \end{cases}$$

Thus conditions (5)–(7) are satisfied $\big($with $\kappa_1 \equiv 1\big)$ for $\alpha = -1$ and any $\beta_1 > 0$, and it follows from Proposition 1 that $\mathcal{A} \in \mathcal{L}\big(H^\lambda, H^{\lambda+1}\big)$ is a Fredholm operator of index 0 for each $\lambda \in \mathbb{R}$. △

Example 2. One of the boundary integral equations for the homogeneous two-dimensional biharmonic equation in a bounded region with a C^∞–smooth boundary, in parametrized and normalized form, looks as follows,

$$\big(\mathcal{A}u\big)(t) = \frac{\pi^2}{|x'(t)|^2} \int_0^1 |x(t) - x(s)|^2 \log|x(t) - x(s)| u(s)\, ds = f(t), \qquad t \in \big[\,0, 1\,\big],$$

where $x : \mathbb{R} \to \mathbb{R}^2$ is a C^∞–smooth 1-periodic parametrization of the boundary with $x'(t) \neq 0$ for $t \in \mathbb{R}$. In order to apply Proposition 1 we consider the following decomposition,

$$\big(\mathcal{A}u\big)(t) = \int_0^1 \kappa_0(t-s)\, u(s)\; ds \;+\; \int_0^1 \kappa_0(t-s)\, a_0(t,s)\, u(s)\; ds \\ + \int_0^1 a_1(t,s)\, u(s)\; ds, \quad t \in \big[\,0, 1\,\big],$$

with $\kappa_0(t) = \big(\sin \pi t\big)^2 \log|\sin \pi t|$ and

$$a_0(t,s) = \begin{cases} \dfrac{\pi^2}{|x'(t)|^2} \dfrac{|x(t)-x(s)|^2}{\sin^2 \pi(t-s)} - 1, & \text{if } t \neq s, \\ 0, & \text{if } t = s, \end{cases}$$

$$a_1(t,s) = \begin{cases} \dfrac{\pi^2}{|x'(t)|^2} |x(t)-x(s)|^2 \log \dfrac{|x(t)-x(s)|}{|\sin \pi(t-s)|}, & \text{if } t \neq s, \\ 0, & \text{if } t = s. \end{cases}$$

Here a_0 and a_1 are 1-biperiodic C^∞-functions, and the Fourier coefficients of κ_0 have the following form,

$$\hat{\kappa}_0(n) = \begin{cases} \big(1 - 2\log 2\big)/4, & \text{if } n = 0, \\ \big(\log 2 - 3/4\big)/4, & \text{if } n = \pm 1, \\ \dfrac{1}{4|n|(n^2-1)}, & \text{if } |n| \geq 2. \end{cases}$$

Thus, the conditions (5)–(7) are satisfied $\big($with $\kappa_1 \equiv 1\big)$ for $\alpha = -3$ and any $\beta_1 > 0$, and it follows from Proposition 1 that $\mathcal{A} \in \mathcal{L}\big(H^\lambda, H^{\lambda+3}\big)$ is a Fredholm operator of index 0 for each $\lambda \in \mathbb{R}$. $\triangle$

Further examples are, e.g., the Cauchy integral operator, the Hilbert integral operator and the hypersingular integral operator, cf. [18]. We mention also Hackbusch [7], Kress [10], Prössdorf and Silbermann [12] and Sloan [16] as general references that contain material on periodic integral equations and their numerical treatment.

1.3. The trigonometric Galerkin method

As a preparation for the subsequent considerations we recall some basic facts on spaces of trigonometric trial polynomials which are defined as follows,

$$\mathcal{T}_N := \Big\{ \sum_{n\in\mathbb{Z}_N} b_n e^{\mathrm{i}n2\pi t} \;:\; b_n \in \mathbb{C} \text{ for } \; n \in \mathbb{Z}_N \Big\},$$
$$\text{where } \; \mathbb{Z}_N := \Big\{ \; n \in \mathbb{Z} : \; -\frac{N}{2} < n \le \frac{N}{2} \Big\}, \quad N \in \mathbb{N},$$

and the associated Fourier projectors are given by

$$P_N u \;=\; \sum_{n\in\mathbb{Z}_N} \hat{u}(n) e^{\mathrm{i}n2\pi t} \qquad \big(u \in H^\lambda \; \text{ for some } \; \lambda \in \mathbb{R} \big).$$

Two basic and elementary estimates associated with $\mathcal{T}_N$ and P_N are the approximation property and the inverse property,

$$\big\| (I - P_N) u \big\|_\lambda \le \Big(\frac{N}{2} \Big)^{\lambda-\mu} \|u\|_\mu, \quad u \in H^\mu \qquad \big(\lambda \le \mu \big), \tag{10}$$
$$\|u_N\|_\lambda \le \Big(\frac{N}{2} \Big)^{\lambda-\mu} \|u_N\|_\mu, \quad u_N \in \mathcal{T}_N \qquad \big(\lambda \ge \mu \big). \tag{11}$$

The trigonometric Galerkin method for equation (1) is to determine an element $u_N \in \mathcal{T}_N$ that satisfies

$$P_N \mathcal{A} u_N = P_N f. \tag{12}$$

It is easy to show that if an operator $\mathcal{A}$ of the form (2)–(3) satisfies the conditions (4)–(7) and (9), then there exists an integer N_0 such that for each $N \ge N_0$ and each $f \in H^{\mu-\alpha}$ with some $\mu \in \mathbb{R}$, the

Galerkin equations (12) have a unique solution $u_N \in \mathcal{T}_N$, and the following error estimate is satisfied

$$\|u_N - u\|_\lambda \le c_{\lambda,\mu} N^{\lambda-\mu} \|u\|_\mu \qquad \big(\lambda \le \mu\big), \tag{13}$$

where $u \in H^\mu$ denotes the solution of the equation $\mathcal{A}u = f$. Note that the estimate (13) has the optimal order of accuracy with respect to $\mathcal{T}_N$, cf. (10).

We shall need also the interpolation projector Q_N onto the space $\mathcal{T}_N$ which is defined as follows,

$$Q_N u \;\in\; \mathcal{T}_N, \qquad \big(Q_N u\big)\big(\tfrac{j}{N}\big) \;=\; u\big(\tfrac{j}{N}\big), \qquad j = 1, \dots, N$$
$$\Big(u \in H^\lambda \;\text{ for some }\; \lambda > \tfrac{1}{2}\Big).$$

The following error estimate is satisfied, cf. [18], section 7.5, or Saranen and Vainikko [14]:

$$\big\|(I - Q_N)u\big\|_\lambda \;\le\; \gamma_\mu \Big(\frac{N}{2}\Big)^{\lambda-\mu} \|u\|_\mu, \qquad u \in H^\mu \tag{14}$$
$$\big(0 \le \lambda \le \mu, \quad \mu > \tfrac{1}{2}\big),$$

where $\gamma_\mu \;=\; \big(1 + 2\sum_{j=1}^\infty \frac{1}{j^{2\mu}}\big)^{1/2}$.

1.4. Outline of the paper

In this paper for the approximate solution of (1) we consider conjugate gradient-type methods since simple iteration schemes exist for them. In order to obtain fully discrete iteration schemes we apply those iteration methods to a fully discrete Galerkin scheme which is obtained if in (12) the right-hand side f is replaced by its trigonometric interpolant $Q_N f$, and $\mathcal{A}$ is replaced by an approximation that is a specification of that considered in [19] and [18], section 7.5; the basic facts on this specific approximation for the operator $\mathcal{A}$ are considered in section 2. Since matrix compression is involved, the entries of the stiffness matrix for the discrete Galerkin equations can be computed by $\mathcal{O}\big(N \log N\big)$ arithmetical operations, and matrix-vector multiplications can be carried out by $\mathcal{O}\big(N\big)$ arithmetical operations in our situation.

For the discrete Galerkin equations, in section 3 *approximate* solutions with small residuals are considered in a general setting, and associated error estimates are provided that have an optimal order of accuracy. In section 4 we then recall some well-known facts on the CGNR method, this is the classical conjugate gradient method of Hestenes and Stiefel applied to the normal equations which in our situation originates from the discrete Galerkin equations. A residual-based stopping rule is considered that yields approximate solutions so that the results from section 3 are applicable. We shall see also that this stopping rule leads to a termination of the iteration after

$\mathcal{O}\big(\log N\big)$ steps so that the approximations can be obtained by a total number of $\mathcal{O}\big(N \log N\big)$ arithmetical operations, i.e., a fast solver is obtained.

In section 5, we examine the influence of noise in the model of the problem as well in the right-hand side, and in the final section 6 numerical illustrations are presented.

1.5. Bibliographical remarks

On the basis of fully discrete Galerkin schemes and collocation methods, fast solvers, i.e., algorithms where approximations can be computed by $\mathcal{O}\big(N \log N\big)$ arithmetical operations, can be generated also by two-grid iteration schemes, see [18], [19] or Saranen and Vainikko ([13], [14]). Compared with those, the CGNR method has essentially a simpler computational algorithm – only matrix-vector computations are involved. For other fast solvers for periodic integral equations see e.g., Amosov [1], Berthold, Hoppe and Silbermann [2] or Schneider [15], and for other results concerning fully discrete schemes we refer, e.g., to Elschner and Stephan [5], McLean, Prössdorf and Wendland [11] or Kieser, Kleemann and Rathsfeld [9].

2. A specific approximation of $\mathcal{A}$

We again suppose that $\mathcal{A}$ is an operator of the form (2)–(3) that satisfies the conditions (4)–(7). Following the lines mentioned in subsection 1.4, in the sequel we recall the basic features of [19] and [18], section 7.5, on a specific approximation to $\mathcal{A}$. Note that in our situation the main part $\mathcal{D}$ is simple and needs no further approximation, thus in the sequel the considerations are restricted to the remainder $\mathcal{B} = \sum_{p=0}^{q} A_p$.

2.1. A specific approximation of a_p for $p \geq 0$

In the sequel we keep $p \in \{\, 0, \ldots, q \,\}$ fixed, and for $L \in \mathbb{N}$ we consider the function

$$a_{p,L} \;=\; P_{D_L} Q_{L,L} a_p, \tag{15}$$

where it is supposed that the set $D_L \subset \mathbb{Z}^2$ satisfies

$$\begin{aligned}
&D_L^0 \;\subset\; D_L \;\subset\; D_L^1,\\
&D_L^0 = \Big\{ \, (j_1, j_2) \in \mathbb{Z}^2 \; : \; \llcorner j_1 \lrcorner, \llcorner j_2 \lrcorner \leq L/2, \quad |j_1| + |j_2| \leq L/2 \, \Big\},\\
&D_L^1 = \Big\{ \, (j_1, j_2) \in \mathbb{Z}^2 \; : \; |j_1| \;+\; |j_2| \;\leq\; L/2 \, \Big\}.
\end{aligned} \tag{16}$$

Furthermore, $Q_{L,L}$ and P_{D_L} denote the two-dimensional interpolation and Fourier projection operators, respectively, i.e.,

$$Q_{L,L}\psi \in \mathcal{T}_L \otimes \mathcal{T}_L, \ \big(Q_{L,L}\psi\big)\big(\tfrac{j_1}{L}, \tfrac{j_2}{L}\big) = \psi\big(\tfrac{j_1}{L}, \tfrac{j_2}{L}\big), \ j_1, j_2 = 1, \ldots, L,$$

$$P_{D_L} v = \sum_{(k_1,k_2)\in D_L} \hat{v}\big(k_1, k_2\big) e^{\mathrm{i}k_1 2\pi t} e^{\mathrm{i}k_2 2\pi s},$$

where $v, \ \psi$ are 1-biperiodic, and ψ is C^∞-smooth; moreover,

$$\mathcal{T}_L \otimes \mathcal{T}_L = \Big\{ \sum_{k_1,k_2\in\mathbb{Z}_L} b_{k_1,k_2} e^{\mathrm{i}k_1 2\pi t} e^{\mathrm{i}k_2 2\pi s} \ : \ b_{k_1,k_2} \in \mathbb{C} \ \big(k_1, \ k_2 \in \mathbb{Z}_L\big) \Big\},$$

$$\hat{v}(k_1, k_2) = \int_0^1 \int_0^1 v(t,s)\, e^{-\mathrm{i}k_1\pi t}\, e^{-\mathrm{i}k_2\pi s}\ ds\ dt, \qquad k_1, \ k_2 \in \mathbb{Z}.$$

Note that for the computation of $a_{p,L}$, only the values of the function a_p at the grid points $\big(\frac{j_1}{L}, \frac{j_2}{L}\big), \ j_1, \ j_2 = 1, \ldots, L$, are needed.

In the sequel specific approximations of A_p are considered; throughout it is supposed that

$$L \sim N^\sigma \qquad \text{with } 0 < \sigma \le 1, \tag{17}$$

holds, i.e., $c_1 N^\sigma \le L \le c_2 N^\sigma$ as $L, \ N \to \infty$. Later, cf. section 2.4.2 below, the restriction $\sigma \le 1/2$ is introduced to keep the number of arithmetical operations sufficiently small.

2.2. Approximation of A_p on $\mathcal{T}_L$

On $\mathcal{T}_L$, an approximation of the operator A_p is obtained replacing the kernel a_p by $a_{p,L}$ as defined in (15),

$$\big(A_{p,L}u\big)(t) \;=\; \int_0^1 \kappa_p(t-s)\, a_{p,L}(t,s)\, u(s)\; ds, \qquad\qquad t \in \big[\,0,1\,\big].$$

One has $A_{p,L} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha+\beta_p}\big)$ for each $\lambda \in \mathbb{R}$, with $\beta_0 := 0$, and

$$\big\|A_{p,L} - A_p\big\|_{\mathcal{L}(H^\lambda, H^{\lambda-\alpha})} \quad\leq\quad c_{\lambda,r} N^{-r} \qquad\qquad \big(\lambda \in \mathbb{R}\big) \qquad\qquad \forall\; r \geq 0.$$

This estimate is a straightforward consequence of (17) and the following two estimates. First, for an integral operator $(Au)(t) = \int_0^1 \kappa(t-s)a(t,s)u(s)ds$ with a $1-$periodic $\kappa(t)$ satisfying $|\hat{\kappa}(n)| \leq c\underline{n}^\alpha$ $(n \in \mathbb{Z})$ and an $1-$periodic C^∞ smooth $a(t,s)$ we have $A \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha}\big)$ for all $\lambda \in \mathbb{R}$, and

$$\|A\|_{\mathcal{L}(H^\lambda, H^{\lambda-\alpha})} \leq c_{\lambda,\nu} \|a\|_{\lambda-\alpha|+\nu,|\lambda|+\nu} \text{ with any } \;\nu > \tfrac{1}{2}$$

(this estimate can be improved, see [8]; we presented just the simplest result from [19]). Here

$$\|a\|_{\lambda_1,\lambda_2} = \Big(\sum_{n_1,n_2 \in \mathbb{Z}} \underline{n_1}^{2\lambda_1}\, \underline{n_2}^{2\lambda_2} |\hat{a}(n_1,n_2)|^2\Big)^{1/2}.$$

Secondly, the following two-dimensional counterpart of (14) holds true (see [18]):

$$\big\|(I - P_{D_L} Q_{L,L})a\big\|_{\lambda_1,\lambda_2} \leq \big(\gamma_{\lambda_1+r} + \gamma_{\lambda_2+r}\big)\Big(\frac{L}{2}\Big)^{-r} \|a\|_{\lambda_1+r,\lambda_2+r},$$

$$L \geq L_0,$$

with any $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $r > 0, L_0 > 0$ such that $\lambda_1 + r > 1/2$, $\lambda_2 + r > 1/2$, $\big(\gamma_{\lambda_1+r} + \gamma_{\lambda_2+r}\big)(L_0/2)^{-r} \leq 1$.

2.3. Approximation of A_p on $\mathcal{T}_N \ominus \mathcal{T}_L$

In order to obtain an approximation of the operator A_p that allows to keep the number of arithmetical operations sufficiently small, on the subspace $\mathcal{T}_N \ominus \mathcal{T}_L = \operatorname{span}\{e^{\mathrm{i}n2\pi t} : n \in \mathbb{Z}_N \backslash \mathbb{Z}_L\}$ the following asymptotic approximation $A_{p,L,d}$ to A_p is considered,

$$A_{p,L,d} = \sum_{j=0}^{d-\lfloor\beta_p\rfloor-1} B_{p,L,j},$$

$$\big(B_{p,L,j}u\big)(t) \;=\; b_{p,L,j}(t) \sum_{n\in\mathbb{Z}} \Big[\triangle^j \hat{\kappa}_p(n)\Big] \hat{u}(n)\, e^{in2\pi t},$$

$$b_{p,L,j}(t) = \frac{1}{j!}\, \partial_s^{[j]} a_{p,L}(t,s)\Big|_{s=t}, \qquad t \in \big[\,0,1\,\big],$$

$$j \;=\; 0,\ldots,d-\lfloor\beta_p\rfloor-1,$$

where $d \geq \beta$ is an integer, and $\lfloor x \rfloor$ denotes the biggest integer smaller or equal to a real number $x \in \mathbb{R}$. Moreover,

$$\partial_s^{[0]} = 1, \qquad \partial_s^{[1]} \;=\; \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial s},$$

$$\partial_s^{[j]} = \Big(\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial s} \;-\; j+1\Big)\cdots\Big(\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial s} \;-\; 1\Big)\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial s}, \quad j \;=\; 2,3,\ldots\;.$$

The Fourier coefficients of the functions $b_{p,L,j}$ can be obtained recursively for $j = 0,\ldots,d-\lfloor\beta_p\rfloor-1$, cf. [19] or [18] for the details. One has $A_{p,L,d} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha+\beta_p}\big)$; the difference $A_{p,L,d} - A_p$ is not small, but nevertheless due to (10) and (17) the following estimate holds,

$$\big\| \big(A_{p,L,d} - A_p\big)(I - P_L)\big\|_{\mathcal{L}(H^\lambda, H^{\lambda-\alpha})} \quad \leq \quad c_\lambda N^{-\sigma d} \qquad \big(\lambda \in \mathbb{R}\big),$$

cf. [19] or [18], section 7.5, for the details. In the following remark two special situations are considered which are of practical relevance.

Remark 1. 1. In the situation $d \leq \lfloor \beta_p \rfloor$ one has $A_{p,L,d} = 0$; this arises, e.g., if $\kappa_p \equiv 1$ holds.

2. If the function $a_p(t,s)$ does not depend on s $\big($e.g., $a_p(t,s) \equiv \text{const}\big)$ and $d \geq \lfloor \beta_p \rfloor + 1$, then we obtain $A_{p,L,d} \;=\; A_{p,L}$. △

2.4. Putting together the approximations

2.4.1. Error estimates We define the following approximation for the remainder $\mathcal{B} = \sum_{p=0}^{q} A_p$,

$$\mathcal{B}_{L,d} \quad = \quad \Bigg(\sum_{p=0}^{q} A_{p,L}\Bigg) P_L \; + \; \Bigg(\sum_{p=0}^{q} A_{p,L,d}\Bigg)\big(I - P_L\big). \qquad (18)$$

Then $\mathcal{B}_{L,d} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha}\big)$ for $\lambda \in \mathbb{R}$, and

$$\big\|\mathcal{B}_{L,d} - \mathcal{B}\big\|_{\mathcal{L}(H^\mu, H^{\lambda-\alpha})} \leq c_{\lambda,\mu} N^{-\sigma(d+\mu-\lambda)} \qquad \big(\lambda \leq \mu\big), \qquad (19)$$

where the constants can be chosen boundedly in λ on any bounded interval $\big[\,\lambda_0, \mu\,\big], \lambda_0 < \mu$.

In the sequel on $\mathcal{T}_N$ we consider $\mathcal{D} + P_M \mathcal{B}_{L,d} P_M$ as approximation to $\mathcal{A}$, with some appropriate $M \leq N$; the corresponding approximation properties are presented in section 3.

2.4.2. On the complexity This section is concluded by recalling from [19] or [18], section 7.6, the considerations on the complexity, with the notation adapted to our situation where the main part $\mathcal{D}$ of $\mathcal{A}$ is supposed to be simple and the remainder $\mathcal{B}_{L,d}$ is compressed to the subspace $\mathcal{T}_M$.

In fact, if $L \sim N^\sigma$ holds for some $0 < \sigma \le 1/2$ as well as $M \sim N^\tau$ for some $0 \le \tau < 1$, then the stiffness matrix associated with $\mathcal{D} + P_M\mathcal{B}_{L,d}P_M : \mathcal{T}_N \to \mathcal{T}_N$ has fully discrete entries that can be computed by $\mathcal{O}\big(N \log N\big)$ arithmetical operations. Moreover, for each $v_N \in \mathcal{T}_N$ with known Fourier coefficients, the computation of the vector $\mathcal{D}v_N$ requires N arithmetical operations, and the vector $P_M\mathcal{B}_{L,d}P_M v_N$ can be computed by a fully discrete scheme that requires $\mathcal{O}\big(M \log M\big) = \mathcal{O}\big(N\big)$ arithmetical operations, if the FFT is applied. One can show that a similar statement holds for the vector $\big(\mathcal{D} + P_M\mathcal{B}_{L,d}P_M\big)^* v_N$, where $\big(\mathcal{D}+P_M\mathcal{B}_{L,d}P_M\big)^* \in \mathcal{L}\big(H^{\lambda-\alpha}, H^\lambda\big)$ denotes the Hilbert adjoint operator of $\mathcal{D} + P_M\mathcal{B}_{L,d}P_M \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha}\big)$.

3. Approximate solution of the discrete Galerkin equations

We suppose again that $\mathcal{A}$ is an operator of the form (2)–(3) that satisfies the conditions (4)–(7). The following lemma presents the approximating and stability properties of the operator $\mathcal{D}+P_M\mathcal{B}_{L,d}P_M$ considered in section 2.

Lemma 1. *Suppose that an operator $\mathcal{A}$ satisfies (2)–(7), and moreover suppose that*

$$L \sim N^{\sigma}, \qquad M \sim N^{\tau}, \qquad \sigma, \tau \in \big(0, 1\big),$$

and let the operator $\mathcal{B}_{L,d}$ be as considered in (18). Then the following estimate is satisfied,

$$\begin{aligned}\big\|\mathcal{D} + P_M\mathcal{B}_{L,d}P_M \; - \; \mathcal{A}\big\|_{\mathcal{L}(H^{\mu},H^{\lambda-\alpha})} \\ \leq \; c_{\lambda,\mu}\big(N^{-\sigma(d+\mu-\lambda)} \; + \; N^{-\tau(\beta+\mu-\lambda)}\big) \qquad \big(\lambda \leq \mu\big). \quad (20)\end{aligned}$$

If additionally (9) holds, then there exists an N_0 such that for each $N \geq N_0$ the operator $\mathcal{D} + P_M\mathcal{B}_{L,d}P_M \in \mathcal{L}\big(H^{\lambda}, H^{\lambda-\alpha}\big)$ is an isomorphism for any $\lambda \in \mathbb{R}$, and

$$\Big\|\big(\mathcal{D} + P_M\mathcal{B}_{L,d}P_M\big)^{-1}\Big\|_{\mathcal{L}(H^{\lambda-\alpha},H^{\lambda})} \; \leq \; c_{\lambda} \qquad \big(\lambda \in \mathbb{R}\big). \quad (21)$$

Proof. The estimate (20) follows from (19) and the following calculations,

$$\begin{aligned}
&\big\| P_M \mathcal{B} P_M u \ - \ \mathcal{B} u \big\|_{\lambda-\alpha} \\
&\quad \le \big\| (I - P_M) \mathcal{B} P_M u \big\|_{\lambda-\alpha} \ + \ \big\| \mathcal{B}(I - P_M) u \big\|_{\lambda-\alpha} \\
&\quad \le \Big(\frac{M}{2} \Big)^{\lambda-\mu-\beta} \big\| \mathcal{B} P_M u \big\|_{\mu-\alpha+\beta} \ + \ c_\lambda \big\| (I - P_M) u \big\|_{\lambda-\beta} \\
&\quad \le c_{\lambda,\mu} N^{\tau(\lambda-\mu-\beta)} \big\| u \big\|_{\mu}, \qquad u \in H^\mu.
\end{aligned}$$

The inverse stability (21) then follows from estimate (20) for $\mu = \lambda$ and from Proposition 1. □

In the sequel we suppose that the right-hand side f in equation (1) satisfies

$$f \ \in \ H^{\mu-\alpha} \qquad \text{for some} \ \ \mu > \alpha + 1/2, \tag{22}$$

and consider approximate solutions of the discrete Galerkin equations

$$\big(\mathcal{D} + P_M \mathcal{B}_{L,d} P_M \big) u_N = Q_N f \tag{23}$$

in a general setting: for $\eta > 0$ let $u_{N,L,d}^{[\eta]} \in \mathcal{T}_N$ satisfy

$$\Big\| \big(\mathcal{D} + P_M \mathcal{B}_{L,d} P_M \big) u_{N,L,d}^{[\eta]} \ - \ Q_N f \Big\|_0 \quad \le \quad \eta \big\| Q_N f \big\|_0. \tag{24}$$

The following proposition provides a basic error estimate. As a preparation we specify the conditions on L, M and d:

$$L \ \sim \ N^\sigma, \quad 0 < \sigma < 1, \qquad M \ \sim \ N^\tau, \quad \tfrac{\mu-\alpha}{\mu-\alpha+\beta} \ \le \ \tau \ < \ 1, \tag{25}$$

$$d \ge \tfrac{1-\sigma}{\sigma} (\mu - \alpha). \tag{26}$$

Proposition 2. *Suppose that an operator $\mathcal{A}$ satisfies (2)–(7), (9), and let (22), (25), (26) be satisfied. Then there exists an N_0 such that for each integer N with $N \geq N_0$ and for any $u_{N,L,d}^{[\eta]} \in \mathcal{T}_N$ satisfying (24) the following error estimate holds,*

$$\left\| u_{N,L,d}^{[\eta]} - u \right\|_\lambda \quad \leq \quad c_{\lambda,\mu} \big(N^{\lambda-\mu} \ + \ N^{\lambda-\alpha} \eta \big) \| u \|_\mu \qquad \big(\alpha \leq \lambda \leq \mu \big),$$

where $u \in H^\mu$ denotes the solution of equation (1).

Proof. We estimate as follows,

$$\begin{aligned} \left\| u_{N,L,d}^{[\eta]} - u \right\|_\lambda &\leq c_\lambda \left\| \big(\mathcal{D} + P_M \mathcal{B}_{L,d} P_M \big) \big(u_{N,L,d}^{[\eta]} - u \big) \right\|_{\lambda-\alpha} \\ &\leq c_\lambda \Bigg(\eta \big(\tfrac{N}{2} \big)^{\lambda-\alpha} \left\| Q_N f \right\|_0 \ + \ \left\| (\mathcal{D} + P_M \mathcal{B}_{L,d} P_M) u - \mathcal{A} u \right\|_{\lambda-\alpha} \\ &\qquad\qquad + \left\| (I - Q_N) f \right\|_{\lambda-\alpha} \Bigg) \\ &\leq c_{\lambda,\mu} \big(N^{\lambda-\alpha} \eta + N^{\lambda-\mu} \big) \| u \|_\mu, \end{aligned}$$

where the first estimate follows from the inverse stability (21), and the second estimate follows from (24) as well as from the inverse property (11) applied to $\big(\mathcal{D} + P_M \mathcal{B}_{L,d} P_M \big) u_{N,L,d}^{[\eta]} \ - \ Q_N f \in \mathcal{T}_N$. Finally, the third estimate is obtained from (20) and the following calculations,

$$\begin{aligned} &\| Q_N f \|_0 \leq \| Q_N f \|_{\mu-\alpha} \leq \big(1 + \gamma_{\mu-\alpha} \big) \| f \|_{\mu-\alpha} \leq c_\mu \big(1 + \gamma_{\mu-\alpha} \big) \| u \|_\mu, \\ &\left\| (I - Q_N) f \right\|_{\lambda-\alpha} \quad \leq \quad \gamma_\mu \big(\tfrac{N}{2} \big)^{\lambda-\mu} \| f \|_{\mu-\alpha} \quad \leq \quad c'_\mu N^{\lambda-\mu} \| u \|_\mu, \end{aligned}$$

cf. estimate (14) for the approximation properties of the interpolation projector Q_N. This completes the proof. ∎

As an immediate consequence of Proposition 2 we obtain the following result.

Corollary 1. *In the situation of Proposition 2 with* $\eta = cN^{\alpha-\mu}$*, one has the optimal estimate*

$$\big\| u_{N,L,d}^{[\eta]} - u \big\|_\lambda \quad \le \quad c_{\lambda,\mu} N^{\lambda-\mu} \|u\|_\mu \qquad \big(\alpha \le \lambda \le \mu\big).$$

4. The CGNR method

In the sequel we suppose that the assumptions of Proposition 2 are satisfied and that the values of $N \ge N_0,\ L,\ M$ and d are fixed. For notational convenience we introduce

$$\mathcal{M} \ := \ \mathcal{D} + P_M \mathcal{B}_{L,d} P_M \ : \ \mathcal{T}_N \to \mathcal{T}_N, \qquad y := Q_N f,$$

and the discrete Galerkin method (23) then takes the following form,

$$\mathcal{M}x \ = \ y. \tag{27}$$

We next recall the basic facts about the CGNR method for solving (27) with $\mathcal{M}$ being conceived as a linear bounded mapping with respect to the weakest possible Sobolev norms,

$$\mathcal{M} \ : \ \big(\mathcal{T}_N, \|\cdot\|_\alpha\big) \ \to \ \big(\mathcal{T}_N, \|\cdot\|_0\big). \tag{28}$$

4.1. The CGNR method for (27), (28)

As a preparation we denote by $\mathcal{M}^* : \big(\mathcal{T}_N, \|\cdot\|_0\big) \to \big(\mathcal{T}_N, \|\cdot\|_\alpha\big)$ the adjoint operator of $\mathcal{M}$ and consider Krylov subspaces with respect to $\mathcal{M}^*\mathcal{M}$ and a vector $r \in \mathcal{T}_N$,

$$\mathcal{K}_\nu\big(\mathcal{M}^*\mathcal{M}, r\big) = \operatorname{span}\Big\{r, \mathcal{M}^*\mathcal{M}r, \ldots, (\mathcal{M}^*\mathcal{M})^{\nu-1}r\Big\} \subset \mathcal{T}_N, \qquad \nu = 0, 1, \ldots .$$

We are now in a position to consider the CGNR method for (27), (28): let the (terminating) sequence $x_\nu \in \mathcal{T}_N,\ \nu = 0, 1, \ldots$, be given by

$$x_\nu \in \mathcal{K}_\nu\big(\mathcal{M}^*\mathcal{M}, \mathcal{M}^*y\big), \tag{29-a}$$

$$\big\|\mathcal{M}x_\nu - y\big\|_0 = \min_{x \in \mathcal{K}_\nu(\mathcal{M}^*\mathcal{M}, \mathcal{M}^*y)} \big\|\mathcal{M}x - y\big\|_0. \tag{29-b}$$

The sequence formally terminates when the residual

$$r_\nu = \mathcal{M}^*\big(\mathcal{M}x_\nu - y\big) \in \mathcal{T}_N, \qquad \nu = 0, 1, \ldots, \tag{30}$$

vanishes for some ν.

Remark 2. 1. For notational convenience in the definition of the CGNR method we take $x_0 = 0$ as initial vector.

2. The CGNR method applied to our setting coincides with the classical conjugate gradient method applied to the normal equations $\mathcal{M}^*\mathcal{M}x = \mathcal{M}^*y$. As general references for conjugate gradient-type

methods we refer to Elman [4], Freund, Golub and Nachtigal [6] as well as to Trefethen and Bau [17] for matrix formulations, and see Daniel [3], chapter 5, for a Hilbert space setting.

3. We note that in Remark 3 below a stopping criterion is considered that provides a stopping index which typically is much smaller then the formal termination index considered above. △

The iteration scheme for the computation of x_ν is as follows, cf. Daniel [3], chapter 5.4:

Algorithm 1 Step 0: Let $x_0 = 0,\ r_0 = -\mathcal{M}^* y$.

For $\nu = 0, 1, \ldots$:

(1) If $r_\nu = 0$ then terminate;

(2) If otherwise $r_\nu \neq 0$, then proceed with step $\nu + 1$, and compute:

$$d_\nu = \begin{cases} -r_\nu + \theta_{\nu-1} d_{\nu-1}, & \theta_{\nu-1} = \dfrac{\|r_\nu\|_\alpha^2}{\|r_{\nu-1}\|_\alpha^2}, \quad \text{if} \quad \nu \geq 1, \\ -r_0, & \text{if} \quad \nu = 0, \end{cases}$$

$$x_{\nu+1} = x_\nu + \omega_\nu d_\nu, \qquad \omega_\nu = \frac{\|r_\nu\|_\alpha^2}{\|\mathcal{M} d_\nu\|_0^2},$$

$$r_{\nu+1} = r_\nu + \omega_\nu \mathcal{M}^* \mathcal{M} d_\nu.$$

△

We note that at each iteration step two matrix-vector multiplications, $\mathcal{M} d_\nu$ and $\mathcal{M}^* \mathcal{M} d_\nu$ in fact, have to be employed.

4.2. The basic properties of the CGNR method

The following result on the decay of the residuals associated with the CGNR method is well-known, cf. Daniel [3], Proposition 5.4.2.

Theorem 2. *Let $x_\nu \in \mathcal{T}_N,\ \nu = 0, 1, \ldots,$ be generated by the CGNR method. Then*

$$\big\|\mathcal{M}x_\nu - y\big\|_0 \le 2q^\nu \big\|y\big\|_0, \qquad \nu = 0, 1, \ldots;$$
$$q \;=\; \frac{\gamma - 1}{\gamma + 1}, \quad \gamma \;=\; \frac{\sup_{\|x\|_\alpha = 1} \|\mathcal{M}x\|_0}{\inf_{\|x\|_\alpha = 1} \|\mathcal{M}x\|_0}.$$

Remark 3. (a) The constant γ in Theorem 2 is bounded in N which is a consequence of Lemma 1. Moreover, it follows from Theorem 2 and from Corollary 1 that the following a priori error estimate is satisfied,

$$\|x_\nu - u\|_\lambda \quad\le\quad c_{\lambda,\mu} N^{\lambda-\mu} \|u\|_\mu \qquad \text{for} \quad \nu \;\ge\; \tfrac{\gamma}{2} \log\big(2N^{\mu-\alpha}/c\big)$$
$$\big(\alpha \le \lambda \le \mu\big),$$

where c has the same meaning as in Corollary 1.

(b) In practical implementations the following a posteriori stopping criterion is considered: terminate the iteration at step $\nu =: \nu_*$ when

$$\big\|\mathcal{M}x_{\nu_*} - y\big\|_0 \quad\le\quad cN^{\alpha-\mu}\|y\|_0 \tag{31}$$

is satisfied for the first time, where c denotes some positive constant. It follows from Corollary 1 and Theorem 2 that

$$\|x_{\nu_*} - u\|_\lambda \le c_{\lambda,\mu} N^{\lambda-\mu} \|u\|_\mu,$$

$$\nu_* \le \tfrac{\gamma}{2} \log\big(2N^{\mu-\lambda}/c\big) + 1 \;=\; \mathcal{O}\big(\log N\big) \quad \big(\alpha \le \lambda \le \mu\big),$$

and the total number of matrix-vector multiplications in the course of iteration finally is $\mathcal{O}\big(\log N\big)$. △

5. Noise in the parameters a_p and in the right-hand side

5.1. Introductory remarks

Throughout this section we again suppose that $\mathcal{A}$ is an operator of the form (2)–(3) that satisfies the conditions (4)–(7). Following the lines mentioned in subsection 1.4, similar to [18], section 7.10, in the sequel we admit perturbations of the parameters $a_p,\ p = 0, \ldots, q$, as well as in the right-hand side.

5.2. A specific approximation of $\mathcal{A}$ with perturbed parameters

We start with the consideration of the perturbation of the parameters $a_p,\ p = 0, \ldots, q$: it is supposed that instead of the functions a_p only 1-biperiodic C^∞-smooth functions $a_{p,\varepsilon} : \mathbb{R}^2 \to \mathbb{C}$ are available with

$$\frac{1}{L^2}\Bigg(\sum_{j_1,j_2=1}^{L} \Big| a_{p,\varepsilon}\big(\tfrac{j_1}{L}, \tfrac{j_2}{L}\big) \;-\; a_p\big(\tfrac{j_1}{L}, \tfrac{j_2}{L}\big) \Big|^2 \Bigg)^{1/2} \le \varepsilon, \quad p \;=\; 0, 1, \ldots, q,$$

where $L = L(N) \in \mathbb{N}$ satisfies (17); since $a_0(t,t) = 0$ $(t \in \mathbb{R})$ we assume that $a_{0,\varepsilon}(t,t) = 0$ $(t \in \mathbb{R})$.

5.2.1. The specific approximation of a_p for $p \geq 0$ In the sequel we keep $p \in \{0, 1, \ldots, q\}$ fixed and consider the function

$$a_{p,L,\varepsilon} \;=\; P_{D_L^0} Q_{L,L} a_{p,\varepsilon},$$

for the definition of the set $D_L^0 \subset \mathbb{Z}^2$ see (16). Recall that in the case of non-perturbed parameters one has more possibilities for the choice of $a_{p,L} = P_{D_L} Q_{L,L} a_p$. In the sequel for perturbed parameters the specific approximations to A_p are considered.

5.2.2. Perturbed approximation of A_p on $\mathcal{T}_L$ Similar to section 2.2 we consider the following perturbed approximation to the operator A_p,

$$\big(A_{p,L,\varepsilon} u\big)(t) \;=\; \int_0^1 \kappa_p(t-s)\, a_{p,L,\varepsilon}(t,s)\, u(s)\; ds, \qquad t \in \big[\,0,1\,\big].$$

One has $A_{p,L,\varepsilon} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha+\beta_p}\big)$ for each $\lambda \in \mathbb{R}$, with $\beta_0 := 0$, and for L satisfying (17) the following estimate holds,

$$\big\| A_{p,L,\varepsilon} - A_{p,L} \big\|_{\mathcal{L}(H^\lambda, H^{\lambda-\alpha})} \;\leq\; c_{\lambda,\nu} N^{\sigma \max\{\lambda-\alpha, |\lambda|, \nu\}} \varepsilon$$
$$\big(\lambda \geq \alpha, \quad \nu > \tfrac{1}{2}\big),$$

cf. [18], section 7.10 for details. This approximation $A_{p,L,\varepsilon}$ in fact is used only on $\mathcal{T}_L$.

5.2.3. Perturbed approximation of A_p on $\mathcal{T}_N \ominus \mathcal{T}_L$ Similar to section 2.3 we next consider the following perturbed asymptotic approximation to the operator A_p,

$$\begin{aligned} A_{p,L,d,\varepsilon} &= \sum_{j=0}^{d-\lfloor\beta_p\rfloor-1} B_{p,L,j,\varepsilon}, \\ \big(B_{p,L,j,\varepsilon}u\big)(t) &= b_{p,L,j,\varepsilon}(t) \sum_{n\in\mathbb{Z}} \Big[\triangle^j \hat{\kappa}_p(n)\Big] \hat{u}(n)\, e^{\mathrm{i}n2\pi t}, \\ b_{p,L,j,\varepsilon}(t) &= \frac{1}{j!}\, \partial_s^{[j]} a_{p,L,\varepsilon}(t,s)\Big|_{s=t}, \quad t \in \big[0,1\big], \\ & \qquad j = 0,\ldots,d-\lfloor\beta_p\rfloor-1, \end{aligned}$$

where $d \geq \beta$ is an integer. Here one has $A_{p,L,d,\varepsilon} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha+\beta_p}\big)$, and with L satisfying (17) the following estimate holds,

$$\Big\|\big(A_{p,L,d,\varepsilon} - A_{p,L,d}\big)(I-P_L)\Big\|_{\mathcal{L}(H^\lambda,H^{\lambda-\alpha})} \leq c_{\lambda,\nu} N^{\sigma\max\{\lambda-\alpha,\nu\}}\varepsilon \qquad \big(\lambda \geq \alpha, \quad \nu > \tfrac{1}{2}\big),$$

cf. again [18], section 7.10 for more details.

5.2.4. Putting together the perturbed approximations Finally we obtain the following perturbed approximation for $\mathcal{B} = \sum_{p=0}^q A_p$,

$$\mathcal{B}_{L,d,\varepsilon} = \Bigg(\sum_{p=0}^q A_{p,L,\varepsilon}\Bigg) P_L + \Bigg(\sum_{p=0}^q A_{p,L,d,\varepsilon}\Bigg)\big(I-P_L\big) \tag{32}$$

with the following properties, $\mathcal{B}_{L,d,\varepsilon} \in \mathcal{L}\big(H^\lambda, H^{\lambda-\alpha}\big)$ for $\lambda \geq \alpha$, and

$$\big\|\mathcal{B}_{L,d,\varepsilon} - \mathcal{B}_{L,d}\big\|_{\mathcal{L}(H^\lambda,H^{\lambda-\alpha})} \leq c_{\lambda,\nu} N^{\sigma\max\{\lambda-\alpha,|\lambda|,\nu\}}\varepsilon \qquad \big(\lambda \geq \alpha, \quad \nu > \tfrac{1}{2}\big), \tag{33}$$

where the constant $c_{\lambda,\nu}$ can be chosen boundedly in λ on any bounded interval $\big[\,\alpha, \lambda_0\,\big],\ \alpha \le \lambda_0$. The considerations in section 2.4.2 on the complexity remain valid, with the notation adapted to the present situation with perturbed data.

5.3. Approximate solution of noisy discrete Galerkin equations

Lemma 2. *Suppose that an operator $\mathcal{A}$ satisfies (2)–(7), and moreover suppose that*

$$L \sim N^{\sigma}, \qquad M \sim N^{\tau}, \qquad \sigma, \tau \in \big(0, 1\big).$$

Then the following estimate is satisfied,

$$\begin{aligned} &\big\| \mathcal{D} + P_M \mathcal{B}_{L,d,\varepsilon} P_M \ - \ \mathcal{A} \big\|_{\mathcal{L}(H^{\mu}, H^{\lambda-\alpha})} \\ &\quad \le c_{\lambda,\mu}\Big(N^{-\sigma(d+\mu-\lambda)} + N^{-\tau(\beta+\mu-\lambda)} \Big) + c_{\lambda,\nu} N^{\sigma \max\{\lambda-\alpha, |\lambda|, \nu\}} \varepsilon \qquad (34) \\ &\qquad\qquad \big(\alpha \le \lambda \le \mu, \quad \nu > \tfrac{1}{2}\big), \end{aligned}$$

cf. (32) for the definition of $\mathcal{B}_{L,d,\varepsilon}$. If additionally (9) holds, then there exists an N_0 and an $\varepsilon_0 > 0$ such that for each integer N with

$$N \ge N_0, \quad N^{\sigma \max\{\lambda-\alpha, |\lambda|, \nu\}} \varepsilon \le \varepsilon_0 \quad \textit{for any } \alpha \le \lambda \le \mu, \qquad (35)$$

the operator $\mathcal{D} + P_M \mathcal{B}_{L,d,\varepsilon} P_M \in \mathcal{L}\big(H^{\lambda}, H^{\lambda-\alpha}\big)$ is an isomorphism for any $\alpha \le \lambda \le \mu$, and

$$\Big\| \big(\mathcal{D} + P_M \mathcal{B}_{L,d,\varepsilon} P_M \big)^{-1} \Big\|_{\mathcal{L}(H^{\lambda-\alpha}, H^{\lambda})} \ \le \ c_{\lambda} \qquad \big(\alpha \le \lambda \le \mu\big). \quad (36)$$

Proof. Estimate (34) follows both from (33) and Lemma 1. The inverse stability (36) then immediately follows from (34) for $\mu = \lambda$ and from Proposition 1. ∎

In the sequel we suppose that the right-hand side f in equation (1) satisfies (22), and instead of $Q_N f$ only an approximation f_N^δ is available:

$$f_N^\delta \ \in \ \mathcal{T}_N, \quad \big\| f_N^\delta - Q_N f \big\|_0 \ \le \ \delta \|f\|_{\mu-\alpha} \text{ for some } 0 < \delta \le \delta_0. \quad (37)$$

Remark 4. The estimate in (37) is equivalent to

$$\left(\frac{1}{N} \sum_{j=1}^{N} \big| f_N^\delta\big(\tfrac{j}{N}\big) \ - \ f\big(\tfrac{j}{N}\big) \big|^2 \right)^{1/2} \ \le \ \delta \|f\|_{\mu-\alpha}; \qquad (38)$$

thus condition (37) practically means that noisy grid values $f_N^\delta\big(\frac{j}{N}\big)$, $j = 1, \ldots, N$, as well as an estimate (38) for the noise are given. △

Similar to (24) we next consider in a general setting approximate solutions of the discrete Galerkin equations $\big(\mathcal{D} + P_M \mathcal{B}_{L,d,\varepsilon} P_M\big) u_N = f_N^\delta$: for $\eta > 0$ let $u_{N,L,d}^{[\eta,\delta,\varepsilon]} \in \mathcal{T}_N$ satisfy

$$\Big\| \big(\mathcal{D} + P_M \mathcal{B}_{L,d,\varepsilon} P_M\big) u_{N,L,d}^{[\eta,\delta,\varepsilon]} \ - \ f_N^\delta \Big\|_0 \quad \le \quad \eta \big\| f_N^\delta \big\|_0. \qquad (39)$$

The following proposition provides a basic error estimate.

Proposition 3. *Suppose that an operator $\mathcal{A}$ satisfies (2)–(7), (9), and let (22), (25), (26) be satisfied. Then there exists an N_0 and* *an* $\varepsilon_0 > 0$ *such that for each integer* N *satisfying (35) and for any* $u_{N,L,d}^{[\eta,\delta,\varepsilon]} \in \mathcal{T}_N$ *satisfying (39) the following error estimate holds,*

$$\big\|u_{N,L,d}^{[\eta,\delta,\varepsilon]} - u\big\|_\lambda \le c_{\lambda,\mu}\big(N^{\lambda-\mu} + N^{\lambda-\alpha}\big(\eta+\delta\big) + N^{\sigma\max\{\lambda-\alpha,|\lambda|,\nu\}}\varepsilon\big)\|u\|_\mu$$

$$\big(\alpha \le \lambda \le \mu\big),$$

where $u \in H^\mu$ *denotes the solution of equation (1).*

Proof. We estimate as follows,

$$\begin{aligned}
\big\|u_{N,L,d}^{[\eta,\delta,\varepsilon]} - u\big\|_\lambda &\le c_\lambda \Big\|\big(\mathcal{D} + P_M\mathcal{B}_{L,d,\varepsilon}P_M\big)\big(u_{N,L,d}^{[\eta,\delta,\varepsilon]} - u\big)\Big\|_{\lambda-\alpha} \\
&\le c_\lambda\Big(\eta\big(\tfrac{N}{2}\big)^{\lambda-\alpha}\big\|f_N^\delta\big\|_0 \;+\; \big\|\big(\mathcal{D} + P_M\mathcal{B}_{L,d,\varepsilon}P_M\big)u - \mathcal{A}u\big\|_{\lambda-\alpha} \\
&\qquad + \big\|f_N^\delta - f\big\|_{\lambda-\alpha}\Big) \\
&\le c_\mu\Big(\big(\tfrac{N}{2}\big)^{\lambda-\alpha}\big(\eta+\delta\big) + N^{\lambda-\mu} \;+\; N^{\sigma\max\{\lambda-\alpha,|\lambda|,\nu\}}\varepsilon\Big)\|u\|_\mu,
\end{aligned}$$

where the first estimate follows from the inverse stability (36) for $\lambda = 0$, and the second estimate follows from the inverse property (11) and (39). Finally, the third estimate is obtained from (34) and the following calculations,

$$\begin{aligned}
\big\|f_N^\delta - f\big\|_{\lambda-\alpha} &\le \big\|f_N^\delta - Q_N f\big\|_{\lambda-\alpha} \;+\; \big\|\big(I - Q_N\big)f\big\|_{\lambda-\alpha} \\
&\le \big(\tfrac{N}{2}\big)^{\lambda-\alpha}\big\|f_N^\delta - Q_N f\big\|_0 \;+\; \gamma_{\mu-\alpha}\big(\tfrac{N}{2}\big)^{\lambda-\mu}\|f\|_{\mu-\alpha} \\
&\le \Big(\big(\tfrac{N}{2}\big)^{\lambda-\alpha}\delta \;+\; \gamma_{\mu-\alpha}\big(\tfrac{N}{2}\big)^{\lambda-\mu}\Big)\|f\|_{\mu-\alpha} \\
&\le c_\mu\Big(\big(\tfrac{N}{2}\big)^{\lambda-\alpha}\delta \;+\; \gamma_{\mu-\alpha}\big(\tfrac{N}{2}\big)^{\lambda-\mu}\Big)\|u\|_\mu,
\end{aligned}$$

and

$$\|f_N^\delta\|_0 \le \|f_N^\delta - Q_N f\|_0 \; + \; \|Q_N f\|_0 \quad \le \quad \delta\|f\|_{\mu-\alpha} \; + \; \|Q_N f\|_{\mu-\alpha}$$

$$\le \Big(\delta_0 \; + \; \gamma_{\mu-\alpha} \; + \; 1\Big)\|f\|_{\mu-\alpha} \quad \le \quad c_\mu\|u\|_\mu.$$

This completes the proof. □

As an immediate consequence of Proposition 3 we obtain the following result.

Corollary 2. *In the situation of Proposition 3 with* $\eta = cN^{\alpha-\mu}$*, one has*

$$\|u_{N,L,d}^{[\eta,\delta,\varepsilon]} - u\|_\lambda \le c_{\lambda,\mu}\big(N^{\lambda-\mu} \; + \; N^{\lambda-\alpha}\delta \; + \; N^{\sigma\max\{\lambda-\alpha,|\lambda|,\nu\}}\varepsilon\big)\|u\|_\mu$$

$$\big(\alpha \le \lambda \le \mu\big).$$

5.4. The CGNR method for perturbed data

To the perturbed situation $\mathcal{M} = \mathcal{D} + P_M\mathcal{B}_{L,d,\varepsilon}P_M : \mathcal{T}_N \to \mathcal{T}_N$ and $y := f_N^\delta$ considered here, the statements of section 4 can be applied. Only the concluding error estimates in Remark 3 for the iterates have to be modified; for example, the a posteriori stopping criterion (31) in the current situation leads to the following error estimate,

$$\|x_{\nu_*} - u\|_\lambda \le c_{\lambda,\mu,\nu}\big(N^{\lambda-\mu} \; + \; N^{\lambda-\alpha}\delta \; + \; N^{\sigma\max\{\lambda-\alpha,|\lambda|,\nu\}}\varepsilon\big)\|u\|_\mu$$

$$\big(\alpha \le \lambda \le \mu\big).$$

Especially,

$$\|x_{\nu_*} - u\|_\alpha \le c_{\mu,\nu}\big(N^{\alpha-\mu} + \delta + N^{\sigma \max\{|\alpha|,\nu\}}\varepsilon\big)\|u\|_\mu \quad \big(\nu > \tfrac{1}{2}\big). \qquad (40)$$

6. Numerical Experiments

6.1. Introductory remarks

In each of the following two sections 6.2 and 6.3 a specific equation of the form $\mathcal{A}u = f$ is considered where the operator $\mathcal{A}$ fulfills the conditions (2)–(7) for $\alpha = -1$ and some $\beta \ge 1$, and where the solution $u : \mathbb{R} \to \mathbb{R}$ is the 1-periodic extension of the following function,

$$u(t) \;=\; \begin{cases} 1, & \text{if } \; 0.25 \le t \le 0.75, \\ 0, & \text{if } \; 0 \le t < 0.25 \quad \text{or} \quad 0.75 < t \le 1, \end{cases} \qquad (41)$$

and then

$$u \in H^\mu, \; f \in H^{\mu+1} \; \text{ for each } \mu < \tfrac{1}{2}, \;\; \text{but } u \notin H^{1/2}, \; f \notin H^{3/2}. \; (42)$$

For each specific equation, different choices of N are considered, and for each choice of N perturbed right-hand sides $f_N^\delta \in \mathcal{T}_N$ are considered that satisfy the following condition,

$$\big|f_N^\delta\big(\tfrac{j}{N}\big) \;-\; f\big(\tfrac{j}{N}\big)\big| \;\le\; N^{-3/2}\|Q_N f\|_{3/2}, \qquad j \;=\; 1, \ldots, N,$$

i.e., (37)–(38) is valid for each $\mu = 1/2 - \eta$ for $\eta > 0$ arbitrarily small, with corresponding noise level $\delta = c_\mu N^{-1-\mu}$. The values of the function $f = \mathcal{A}u$ at the grid points in fact are computed numerically with a high precision, and the values of the function f_N^δ are chosen such that $f_N^\delta\left(\frac{j}{N}\right) - f\left(\frac{j}{N}\right),\ j = 1, \ldots, N$, are uniformly and randomly distributed in the interval $\left[\ -\eta_1, \eta_1\ \right],\ \eta_1 = N^{-3/2}\|Q_N f\|_{3/2}$. We set $\varepsilon = 0$, i.e., no noise in the model is considered.

We consider the following specific choices of N, M and L,

$$N \ = \ 2^k, \qquad M \ = \ 2^{\lfloor 4k/5 \rfloor}, \qquad L \ = \ 2^{\lceil k/2 \rceil} \tag{43}$$

for $k = 5, 6, 7, 8$, where $\lceil x \rceil$ denotes the smallest integer bigger or equal to a real number x. The relations (43) mean $L \sim N^{1/2}$ and $M \sim N^{4/5}$ as $N \to \infty$ so that M in fact satisfies (25), and in our situation we may choose $d = 2$ for the asymptotical approximation.

The first iteration scheme under consideration is the CGNR method (cf. section 3) applied to the equation $\mathcal{M}v_N = f_N^\delta$, with $\mathcal{M} = \mathcal{D} + P_M \mathcal{B}_{L,d} P_M : \mathcal{T}_N \to \mathcal{T}_N$ as in section 4. According to the general analysis presented in section 3, it is reasonable to terminate the iteration at step $\nu =: \nu_*$ when

$$\left\| \mathcal{M}x_{\nu_*} - f_N^\delta \right\|_0 \quad \le \quad N^{-3/2}\|f_N^\delta\|_0$$

is satisfied for the first time, where x_{ν_*} denotes the ν_*-th iterate of the CGNR method. The estimate (40) yields $\|x_{\nu_*} - u\|_{-1} = \mathcal{O}(N^{-1-\mu})$ for any $\mu < 1/2$. Note that due to the second property in (42) one cannot conclude from (40) that the quotient $\|x_{\nu_*} - u\|_{-1}/N^{-3/2}$ stays bounded for experiments with different and increasingly ordered values of N. On the other hand, however, due to the first properties in (42) it is no surprise that these quotients stay bounded in our experiments.

For comparison we consider also GMRES which in our situation is applied to the equation $\mathcal{M}v_N = f_N^\delta$, and the associated norm is $\|\cdot\|_0$, i.e., the corresponding iterates $x_0, x_1, \ldots$ have the following properties,

$$x_\nu \in \mathcal{K}_\nu\big(\mathcal{M}, f_N^\delta\big),$$
$$\big\|\mathcal{M}x_\nu - y\big\|_0 = \min_{x \in \mathcal{K}_\nu(\mathcal{M}, f_N^\delta)} \big\|\mathcal{M}x - y\big\|_0, \qquad \nu = 0, 1, \ldots.$$

For an introduction to GMRES see, e.g., [4], [6] or [17].

Remark 5. We recall only two basic facts on GMRES: for each step $\nu \to \nu + 1$, GMRES requires only one matrix-vector multiplication (while CGNR method needs two matrix-vector multiplications in each iteration step), and secondly, in general no estimates for the speed of convergence of GMRES are available. The latter means also that no a priori estimates for the stopping criterion considered next are available. △

For the approximations associated with GMRES the same stopping criterion as for the CGNR method is applied, thus according to Corollary 1 we may expect similar error estimates as above which is confirmed by the numerical results to be presented in the sequel. As it turns out, for each experiment with fixed period integral equation and fixed N, GMRES practically needs approximately twice as much steps as the CGNR method needs to to satisfy the stopping criterion, thus the complexity associated with GMRES finally is approximately the same as the complexity of the CGNR method (cf. Remark 5 on the required matrix-vector multiplications in each step). All computations are performed in MATLAB on an IBM RISC/6000.

6.2. Symm's integral equation for an ellipse

In the sequel we present the numerical results for Symm's integral equation, cf. Example 1, which is considered here for $x(t) = \big(\frac{1}{2}\cos 2\pi t, \frac{1}{4}\sin 2\pi t\big)^\top$, $t \in \mathbb{R}$, parametrizing a special ellipse Γ. Here we have $q = 1$, $a_0 \equiv 0$ and $\kappa_1 \equiv 1$, thus remark 1 on the specific form of the asymptotical approximation applies. Tables 1 and 2 contain the results for the CGNR method and GMRES, respectively.

Table 1. Numerical results for Symm's integral equation for an ellipse, CGNR method

N	M	L	$\|x_{\nu_*} - u\|_{-1}$	$\|x_{\nu_*} - u\|_{-1}/N^{-3/2}$	ν_*
32	16	4	4.05e–02	7.33	3
64	32	8	1.80e–02	9.22	3
128	64	8	7.98e–03	11.56	3
256	128	16	2.60e–03	10.66	4

Table 2. Numerical results for Symm's integral equation for an ellipse, GMRES

N	M	L	$\|x_{\nu_*} - u\|_{-1}$	$\|x_{\nu_*} - u\|_{-1}/N^{-3/2}$	ν_*
32	16	4	4.06e–02	7.35	5
64	32	8	1.78e–02	9.11	6
128	64	8	8.03e–03	11.62	8
256	128	16	2.64e–03	10.81	10

6.3. A model problem

In the sequel we consider the following model problem, cf. [8] for a similar example:

$$\int_0^1 \kappa_0(t-s)\, u(s)\, ds \;+\; \int_0^1 \kappa_0(t-s)\, \frac{a(t,s)-a(t,t)}{a(t,t)}\, u(s)\, ds = \frac{f(t)}{a(t,t)}, \qquad t \in \big[\,0,1\,\big],$$

with

$$\hat{\kappa}_0(n) = \begin{cases} \dfrac{4}{\pi}\,\dfrac{1}{4n-1}, & \text{if } \; 0 \neq n \in \mathbb{Z}, \\ \dfrac{4}{3\pi}, & \text{if } \; n = 0, \end{cases}$$

$$a(t,s) = b(t)\,b(s), \qquad b(t) \;=\; 3 \;+ \sum_{0 \neq k \in \mathbb{Z}} 2^{-4|k|} e^{\mathrm{i}k2\pi t}.$$

Here we have $q = 0$, with an asymptotical approximation corresponding to A_0 which is non-trivial. Tables 3 and 4 contain the results for the CGNR method and GMRES, respectively.

Table 3. Numerical results for the model problem, CGNR method

N	M	L	$\Vert x_{\nu_*} - u \Vert_{-1}$	$\Vert x_{\nu_*} - u \Vert_{-1} / N^{-3/2}$	ν_*
32	16	4	3.81e–02	6.90	2
64	32	8	1.83e–02	9.37	2
128	64	8	7.77e–03	11.26	3
256	128	16	2.66e–03	10.89	3

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Table 4. Numerical results for the model problem, GMRES

N	M	L	$\|x_{\nu_*} - u\|_{-1}$	$\|x_{\nu_*} - u\|_{-1}/N^{-3/2}$	ν_*
32	16	4	3.89e–02	7.04	4
64	32	8	1.81e–02	9.29	9
128	64	8	7.73e–03	11.19	7
256	128	16	2.62e–03	10.75	10

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