Iterative and other methods for linear ill-posed equations

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Preface

It is my purpose to introduce a variety of iterative as well as parameter methods for solving linear ill-posed (symmetric and nonsymmetric) equations $Au = f$ in Hilbert and Banach spaces.

In Chapter 1, operators fulfilling resolvent condtions over certain sectors are considered and classified as weakly sectorial and strictly sectorial operators. Moreover, we define fractional powers of weakly sectorial operators and derive their basic properties in order to show that Abel integral operators in L^p-spaces and spaces of continuous functions are strictly sectorial, see Section 1.3 on that. The corresponding Abel integral equations are weakly singular (nonsymmetric) Volterra integral equations of the first kind that arise in applications like spectroscopy and seismic imaging.

In Chapter 2 iterative as well as parameter methods are presented which are designed for solving those equations considered in the first chapter, and any kind of normalization of the underlying equation is avoided in order to keep computational efforts at the lowest possible level. For any described method we provide a convergence analysis for precisely given data.

In Chapter 3 noise-level-dependent as well noise-level-free parameter choices and stopping rules are discussed, and here always disturbed right-hand sides for the underlying ill-posed equation are admitted. Computational experiments are provided, and in Chapter 4 we shall see that the convergence rates cannot be improved, in general.

The most efficient algorithms for solving symmetric ill-posed problems in Hilbert spaces are conjugate gradient type methods, and thus the last chapter is devoted to of conjugate residuals (for semidefinite problems).

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Kapitel 1

Weakly sectorial and strictly sectorial operators

We consider equations

$$Au = f_* \tag{1.1}$$

for $A \in \mathcal{L}(X)$ and (maybe only approximately known) right-hand side $f_* \in \mathcal{R}(A)$, where $\mathcal{L}(X)$ denote the space of bounded linear operators in the underlying real or complex Banach space X, and $\mathcal{R}(A)$ denote the range of A. Our main subject are equations (1.1) which are ill-posed in sense that arbitrary small perturbations of the right-hand side in (1.1) can lead to arbitrary large deviations in the solution of the problem (a precise definition is given in Definition 1.1.14), and therefore the numerical solution of those equations is crucial. In this chapter we introduce the class of weakly sectorial operators where the (iterated) method of Lavrentiev (to be introduced in Section 2.1) can be used as a stable solver for (1.1), and moreover the (smaller) class of strictly sectorial operators A is introduced where the corresponding equations $Au = f_*$ can be solved by certain iterative methods (see Chapters 2 and 3 for more on these algorithms). Section 1.3 is devoted to (strictly sectorial) Abel integral operators, and two applications are given.

1.1 Weakly sectorial operator

Throughout this Section 1.1 let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, if not further specified.

1.1.1 Some basic properties

We first introduce weakly sectorial operators. For technical reasons in the following definition unbounded operators are admitted, although our main subject are bounded operators (with unbounded inverse).

Definition 1.1.1 *We call a (possibly unbounded) linear operator $B : X \supset \mathcal{D}(B) \to X$ weakly sectorial, if $(0, \infty) \subset \rho(-B)$ and*

$$\|(tI + B)^{-1}\| \le M_0/t, \qquad t > 0, \tag{1.2}$$

(with some $M_0 \ge 1$). Here, $\rho(-B)$ is the resolvent set of $-B$,

$$\rho(-B) = \{ \lambda \in \mathbb{K} : \lambda I + B \text{ is one-to-one and onto, } (\lambda I + B)^{-1} \in \mathcal{L}(X) \},$$

and $\|\cdot\|$ denotes the corresponding operator norm. Frequently we use the notation $M_0(B)$ instead of M_0.

Example 1.1.2 *Let X be a real or complex Hilbert space and $A \in \mathcal{L}(X)$. If $A = A^* \ge 0$, i.e., if A is selfadjoint and positive semidefinite, then A is weakly sectorial (with $M_0 = 1$ in (1.2)).*

Weakly sectorial operators A fulfill a resolvent condition over a (small) sector, see in Proposition 1.1.5 (this justifies the terminology 'sectorial'). First we introduce the sector $\Sigma_\theta \subset \mathbb{C}$,

$$\Sigma_\theta := \{ \lambda = re^{i\varphi} : r > 0, |\varphi| \le \theta \}, \qquad 0 \le \theta \le \pi,$$

and moreover we introduce the notation 'sectorial of angle θ'.

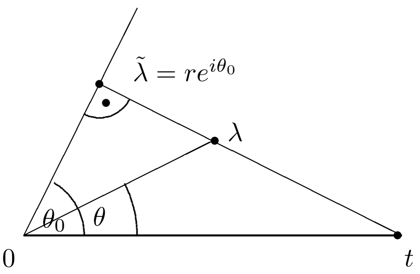


Abbildung 1.1: Sketch of the situation in the proof of Theorem 1.1.5

Definition 1.1.3 *We say that a linear operator $B : X \supset \mathcal{D}(B) \to X$ is sectorial of angle $\theta_0 \in [0, \pi]$, if the resolvent set $\rho(-B)$ of $-B$ contains for any $0 \le \theta < \theta_0$ the sector Σ_θ,*

$$\rho(-B) \;\supset\; \Sigma_\theta, \tag{1.3}$$

and the following estimate for the resolvent operator of $-B$ is satisfied,

$$\|(\lambda I + B)^{-1}\| \;\le\; \frac{M_\theta}{|\lambda|}, \qquad \lambda \in \Sigma_\theta, \tag{1.4}$$

with some $M_\theta \ge 1$. Frequently we use the notation $M_\theta(B)$ instead of M_θ.

Remark 1.1.4 *Let $A \in \mathcal{L}(X)$ (with X being a complex Banach space) be sectorial of angle $\theta_0 \in [0, \pi]$, and moreover let A be similar to $B \in \mathcal{L}(X)$, i.e., there is an $S \in \mathcal{L}(X)$ which is ono-to-one and onto, such that $S^{-1}AS = B$. Then B is sectorial of angle θ_0.*

As mentioned above, weakly sectorial operators are sectorial of some angle:

Theorem 1.1.5 *Let X be a complex Banach space, let $A \in \mathcal{L}(X)$ be weakly sectorial, and $\theta_0 := \arcsin(1/M_0)$. Then A is sectorial of angle θ_0, and more specifically, for any $0 < \theta < \theta_0$, (1.4) holds with $M_\theta = 1/\sin(\theta_0 - \theta)$, where M_0 is as in (1.2).*

Proof. Take any $\lambda \in \Sigma_\theta$ and let $r \ge 0$ such that for $\tilde\lambda := re^{i\theta_0}$ we have the situation as described in Figure 1.1. Here, $t := r/\cos\theta_0$. Obviously

$$|\lambda - t| \;<\; |\tilde\lambda - t| \;=\; t\sin\theta_0 \;=\; t/M_0 \;\le\; \|(tI + A)^{-1}\|^{-1},$$

hence $\lambda \in \rho(-A)$, and due to

$$|t - \lambda| \;+\; |\lambda - \tilde\lambda| \;=\; t/M_0,$$

we get

$$\begin{aligned}\|(\lambda I + A)^{-1}\| \;&\le\; \frac{1}{\|(tI + A)^{-1}\|^{-1} - |\lambda - t|} \;\le\; \frac{1}{t/M_0 - |\lambda - t|} \\ &\le\; \frac{1}{|\lambda - \tilde\lambda|} \;=\; \frac{|\lambda|}{|\lambda - \tilde\lambda|} \cdot \frac{1}{|\lambda|} \\ &=\; \frac{1}{\sin(\theta_0 - \theta)} \cdot \frac{1}{|\lambda|}. \quad \Box\end{aligned}$$

The following lemma shall be used to derive resolvent conditions for fractional powers of weakly sectorial operators.

Lemma 1.1.6 *If X is a complex Banach space and if $A \in \mathcal{L}(X)$ is sectorial of angle θ_0, then cA is weakly sectorial for any $c \in \mathbb{C}$ with $|\theta| < \theta_0$, $\theta := \arg(c)$ (with $M_0(cA) = M_\theta(A)$).*

This lemma can be derived directly from the definition, and its proof thus is omitted. The following result can be used to show that the integration operator $Vu(\xi) \;=\; \int_0^\xi u(\eta)\, d\eta$, $\xi \in [0, a]$, (with respect to $X = C[0, a]$ or $X = L^p[0, a]$, $p \in [1, \infty]$) and modifications of V are weakly sectorial, see Theorems 1.3.1 and 1.3.2.

Proposition 1.1.7 *Let $A \in \mathcal{L}(X)$ have a trivial nullspace $\mathcal{N}(A)$. A is weakly sectorial if and only if the (possibly unbounded) inverse A^{-1} is weakly sectorial, and then (1.2) holds with $M_0(A) = M_0(A^{-1}) + 1$.*

Proof. Let $t > 0$. It is easy to show that $t \in \rho(-A)$ if and only if $t \in \rho(-A^{-1})$, and then

$$(tI + A)^{-1} \;=\; \frac{1}{t}I - \frac{1}{t^2}(A^{-1} + \frac{1}{t}I)^{-1}. \tag{1.5}$$

The desired estimate for $\|(tI + A)^{-1}\|$ then is an immediate consequence of (1.5). □

The next lemma is a preparation for Theorem 1.1.9 and will be used also at other places. We use the notation $A^0 := I$.

Lemma 1.1.8 *For weakly sectorial $A \in \mathcal{L}(X)$ and integers $\alpha,\ m$ with $0 \le \alpha \le m$ one has*

$$\|(I + tA)^{-m}A^{\alpha}\| \;\le\; \gamma_m t^{-\alpha}, \qquad t > 0, \tag{1.6}$$

where, e.g., $\gamma_m = (M_0 + 1)^m$, with M_0 as in (1.2).

Proof. From (1.2) we easily obtain for any $n \ge 0$

$$\|(tI + A)^{-n}A^n\| \;\le\; (M_0 + 1)^n \qquad \text{for } t > 0,$$

and this leads to

$$\|(I + tA)^{-m}A^{\alpha}\| \;\le\; \|(I + tA)^{-\alpha}A^{\alpha}\| \cdot \|(I + tA)^{-(m-\alpha)}\| \;\le\; (M_0 + 1)^{\alpha}M_0^{m-\alpha}t^{-\alpha}. \quad □$$

Remark. Positive semidefinite operators in Hilbert spaces fulfill (1.6) with $\gamma_0 = \gamma_1 = 1$.

For weakly sectorial operators we have the following interpolation inequality, which provides an alternative way to prove convergence of the discrepancy principle (that is introduced in Chapter 3).

Theorem 1.1.9 *(First interpolation inequality) For weakly sectorial $A \in \mathcal{L}(X)$ and integer $\alpha \ge 0$ we have*

$$\|A^{\alpha}u\| \;\le\; C\|A^{\alpha+1}u\|^{\alpha/(\alpha+1)}\|u\|^{1/(\alpha+1)}, \qquad u \in X,$$

with $C = C(M_0, \alpha) := (\alpha + 1)(M_0^{\alpha}\gamma_1)^{\alpha/(\alpha+1)}$, where M_0 is as in (1.2) and γ_1 is as in (1.6).

Proof. We first observe that

$$I - (I + tA)^{-\alpha} \;=\; t\Big(\sum_{j=1}^{\alpha}(I + tA)^{-j}\Big)A,$$

hence

$$A^{\alpha} \;=\; (I + tA)^{-\alpha}A^{\alpha} \;+\; t\Big(\sum_{j=1}^{\alpha}(I + tA)^{-j}\Big)A^{\alpha+1},$$

and (see Lemma 1.1.8)

$$\|A^{\alpha}u\| \;\;\le\;\; \gamma_1^{\alpha}t^{-\alpha}\|u\| \;+\; \alpha M_0^{\alpha}t\|A^{\alpha+1}u\|, \tag{1.7}$$

and take

$$t \;=\; \Big(\frac{\gamma_1^{\alpha}}{M_0^{\alpha}} \cdot \frac{\|u\|}{\|A^{\alpha+1}u\|}\Big)^{1/(\alpha+1)}$$

(if $A^{\alpha+1}u \ne 0$; in the case $A^{\alpha+1}u = 0$ consider $t \to \infty$ in (1.7)) to obtain the desired result. □

1.1.2 Generalized inverses of weakly sectorial operators

The following geometrical observations are helpful to introduce generalized inverses of weakly sectorial operators A. Let $\mathcal{R}(B)$, $\mathcal{N}(B)$ and $\overline{\mathcal{R}(B)}$ denote the range, the nullspace, and the closure of the range, respectively, of a linear operator $B : X \supset \mathcal{D}(B) \to X$.

Theorem 1.1.10 *Let $A \in \mathcal{L}(X)$ be weakly sectorial, and let M_0 as in (1.2).*

(a) $\|\psi\| \le M_0\|u + \psi\|$ for any $u \in \overline{\mathcal{R}(A)},\ \psi \in \mathcal{N}(A)$.

(b) (i) $\overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$.

(ii) The linear subspace

$$X_0 := \overline{\mathcal{R}(A)} \oplus \mathcal{N}(A)$$

is closed in X.

(c) The linear projection $P : X_0 \to X_0$ onto $\overline{\mathcal{R}(A)}$ with nullspace $\mathcal{N}(A)$ is continuous,

$$\|I - P\| \ \le\ M_0.$$

(d) $\mathcal{N}(A) = \mathcal{N}(A^2)$.

(e) $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^2)}$.

(f) $X_0 = X$ if and only if $A(\overline{\mathcal{R}(A)}) = \mathcal{R}(A)$.

(g) If X is a reflexive Banach space then $X_0 = X$.

Proof. We introduce the operator

$$H_t \ =\ (I + tA)^{-1} \tag{1.8}$$

and first observe that from Lemma 1.1.8 with $m = 1$ and $\alpha = 0,\ \alpha = 1$ and the Banach-Steinhaus theorem imply

$$\|H_t u\| \to 0 \ \ (t \to \infty) \ \text{ for any } u \in \overline{\mathcal{R}(A)}.$$

(a) For arbitrary $u \in \overline{\mathcal{R}(A)}$ and $\psi \in \mathcal{N}(A)$ we have, due to $H_t\psi = \psi$,

$$\|\psi\| = \lim_{t\to\infty} \|H_t(u + \psi)\| \ \le\ \limsup_{t\to\infty} \|H_t\| \cdot \|u + \psi\| \ \le\ M_0\|u + \psi\|.$$

(b) (i) Let $\psi \in \overline{\mathcal{R}(A)} \cap \mathcal{N}(A)$. $-\psi \in \overline{\mathcal{R}(A)}$ and (a) implies $\|\psi\| \le M_0\| - \psi + \psi\| = 0$.
(b) (ii) Let $\{u_n\} \subset \overline{\mathcal{R}(A)},\ \{\psi_n\} \subset \mathcal{N}(A)$ and $z \in X$ with $u_n + \psi_n \to z$ as $n \to \infty$. (a) implies

$$\|\psi_n - \psi_m\| \ \le\ M_0\|u_n - u_m + \psi_n - \psi_m\| \to 0 \qquad \text{as } n,\ m \to \infty,$$

hence $\psi_n \to \psi$ as $n \to \infty$ for some $\psi \in \mathcal{N}(A)$, and then necessarily $u_n \to u$ as $n \to \infty$ for some $u \in \overline{\mathcal{R}(A)}$. Hence $z = u + \psi \in X_0$, and this shows that X_0 is closed.
(c) Let again and $u \in \overline{\mathcal{R}(A)}$ and $\psi \in \mathcal{N}(A)$. Then $\|(I - P)(u + \psi)\| = \|\psi\| \le M_0\|u + \psi\|$ which shows $\|I - P\| \le M_0$.
(d) We only have to show $\mathcal{N}(A^2) \subset \mathcal{N}(A)$. $\psi \in \mathcal{N}(A^2)$ then (b) (i) implies $A\psi \in \overline{\mathcal{R}(A)} \cap \mathcal{N}(A) = \{0\}$, and this means $\psi \in \mathcal{N}(A)$.
(e) We only have to show $\overline{\mathcal{R}(A)} \subset \overline{\mathcal{R}(A^2)}$. For that we observe that $A - t(I+tA)^{-1}A^2 \ =\ (I+tA)^{-1}A \to 0$ as $t \to \infty$ (in $\mathcal{L}(X)$), thus $\mathcal{R}(A) \subset \overline{\mathcal{R}(A^2)}$, and we obtain the assertion.
(f) If $X_0 = X$, then $\mathcal{R}(A) = A(X_0) = A(\overline{\mathcal{R}(A)})$, and '$\Leftarrow$' remains to show. For that let $z \in X$. Then $Az \in \mathcal{R}(A) = A(\overline{\mathcal{R}(A)})$, hence $Az = Au$ for some $u \in \overline{\mathcal{R}(A)}$. From $z - u \in \mathcal{N}(A)$ it follows $z \in X_0$.
(g) Let $z \in X$. Since $\{H_t z\}_{t\ge0}$ (see (1.8) for the definition of H_t) is bounded in X, we have weak convergence,

$$H_n z \rightharpoonup \psi \qquad \text{as } n \to \infty,\ n \in N,$$

for some $\psi \in X$ and some infinite set of integers N. Lemma 1.1.8 implies that $AH_t z \to 0$ as $t \to \infty$, hence $\psi \in \mathcal{N}(A)$. The equality $I - H_t = H_t A$ yields $z - H_t z \in \overline{\mathcal{R}(A)},\ t \ge 0$, and then $z - H_n z \rightharpoonup z - \psi$ as $n \to \infty,\ n \in N$, finally shows $z - \psi \in \overline{\mathcal{R}(A)}$, and this yields $z \in X_0$. □

Remarks. 1. Part (a) of Theorem 1.1.10 shows that if A is weakly sectorial with $M_0 = 1$ in (1.2), then $\overline{\mathcal{R}(A)}$ and $\mathcal{N}(A)$ are orthogonal subspaces in a generalized sense.

2. From (d) it is obvious that $\mathcal{N}(A) = \mathcal{N}(A^k)$ for any integer $k \geq 1$, and from (e) it follows that $\overline{\mathcal{R}(A)} = \overline{\mathcal{R}(A^k)}$ for any integer $k \geq 1$.

We now present an example where all the mentioned subspaces are explicitly given.

Example 1.1.11 *On the real space $X = C[-1,1]$ of real-valued continuous functions on [-1,1], supplied with the max-norm $\|\cdot\|_\infty$, we consider the multiplication operator,*

$$Au(s) \;=\; a(s)u(s), \qquad s \in [-1,1],$$

with

$$a(s) \;=\; \begin{cases} 0, & \textit{if } s \leq 0, \\ s, & \textit{if } s > 0. \end{cases}$$

Then A is weakly sectorial (see also Example 1.4.3), and

$$\begin{aligned} \mathcal{R}(A) &= \Big\{ \, v \in X : \; v(s) = 0 \;\; \textit{for} \;\; s \in [-1,0], \; \lim_{s \searrow 0} v(s)/s \;\textit{exists}\;, \\ &\qquad\qquad u(s) = v(s)/s, \; s \geq 0, \; \textit{is continuous in } 0 \, \Big\}, \\ \overline{\mathcal{R}(A)} &= \Big\{ \, v \in X : \; v(s) = 0 \; \textit{for} \;\; s \in [-1,0] \, \Big\}, \\ \mathcal{N}(A) &= \Big\{ \, v \in X : \; v(s) = 0 \;\; \textit{for} \;\; s \in [0,1] \, \Big\}, \\ X_0 &= \Big\{ \, v \in X : \; v(0) = 0 \, \Big\}. \end{aligned}$$

Definition 1.1.12 *Let $A \in \mathcal{L}(X)$ be weakly sectorial. The* <u>*generalized inverse*</u>

$$A^\dagger : X \supset \mathcal{D}(A^\dagger) \to X$$

of A then is defined by

$$\begin{aligned} \mathcal{D}(A^\dagger) &:= A(\overline{\mathcal{R}(A)}) \oplus \mathcal{N}(A), \\ A^\dagger(Au + \psi) &:= u, \qquad u \in \overline{\mathcal{R}(A)}, \quad \psi \in \mathcal{N}(A). \end{aligned}$$

$A^\dagger$ then obviously is well-defined and linear. Note that in the case $X_0 = X$ one has $\mathcal{D}(A^\dagger) = \mathcal{R}(A) \oplus \mathcal{N}(A)$. We present some elementary properties.

Proposition 1.1.13 *For weakly sectorial $A \in \mathcal{L}(X)$ we have*

$$\mathcal{R}(A^\dagger) = \overline{\mathcal{R}(A)}, \tag{1.9}$$

$$\mathcal{N}(A^\dagger) = \mathcal{N}(A), \tag{1.10}$$

$$AA^\dagger = P \qquad \textit{on } \mathcal{D}(A^\dagger), \tag{1.11}$$

$$A^\dagger A = P \qquad \textit{on } X_0, \tag{1.12}$$

and $A^\dagger$ is a closed operator. P in (1.11) and (1.12) again is the projection onto $\overline{\mathcal{R}(A)}$ with nullspace $\mathcal{N}(A)$.

Proof. (1.9) is obvious, and also '$\supset$' in (1.10). Now let $u \in \overline{\mathcal{R}(A)}$, $\psi \in \mathcal{N}(A)$. Then $z := Au + \psi \in \mathcal{N}(A^\dagger)$ means that $0 = A^\dagger(Au + \psi) = u$, hence $z = \psi$, and this is '$\subset$' in (1.10). (1.11) and (1.12) are immediate consequences from the definition of $A^\dagger$.

We now give the proof that $A^\dagger$ is a closed operator. To this end, let $\{f_n\} \subset A(\overline{\mathcal{R}(A)})$, $\{\psi_n\} \subset \mathcal{N}(A)$ and $z, \; u \in X$ with

$$\begin{aligned} f_n + \psi_n &\to z \qquad \text{as } n \to \infty, \\ u_n := A^\dagger(f_n + \psi_n) &\to u \qquad \text{as } n \to \infty, \end{aligned}$$

whence

$$\{u_n\} \subset \overline{\mathcal{R}(A)} \qquad \text{and} \qquad Au_n \;=\; f_n, \quad n = 0, 1, \ldots \,.$$

The continuity of the projection P implies that $\{f_n\}$ converges (in $\overline{\mathcal{R}(A)}$), and hence $\{\psi_n\}$ converges in $\mathcal{N}(A)$, i.e., there are $f \in \overline{\mathcal{R}(A)}$ and $\psi \in \mathcal{N}(A)$ with

$$\begin{aligned} z &= f + \psi, \\ \|f_n - f\| &\to 0, \quad \|\psi_n - \psi\| \to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

A is a bounded operator, hence $Au = f$, which shows

$$z \in \mathcal{D}(A^\dagger), \qquad A^\dagger z = u,$$

this completes the proof. □

Note that the closedness of $A^\dagger$ follows already from the topological properties of X_0 and its decomposition, therefore it is not necessary to assume at this point that A is weakly sectorial.

We now can define what ill-posedness means in terms of the generalized inverse of A.

Definition 1.1.14 *Let $A \in \mathcal{L}(X)$ be weakly sectorial. If the generalized inverse $A^\dagger$ of A is bounded, then the equation is called $Au = f_*$ well-posed. If on the other side $A^\dagger$ is unbounded, then $Au = f_*$ is called ill-posed.*

For example, if A is a compact operator with infinite-dimensional range, then $Au = f_*$ is ill-posed in the sense of Definition 1.1.14. To be more general, let $A_{X_0} : X_0 \to X_0$ be the restriction of A to X_0. The closed graph theorem implies that $Au = f_*$ is ill-posed if and only if $\mathcal{R}(A_{X_0})$ is non-closed, and this again is equivalent to the case that $0 \in \sigma_{ap}(A_{X_0})$, that is, the set of approximate eigenvalues of A_{X_0}. (For an operator $B \in \mathcal{L}(X)$, $\lambda \in \mathbb{K}$ is called an approximate eigenvalue of B, if there is a sequence $\{u_n\} \subset X$ with $\|u_n\| = 1$, $n = 1, 2, ...$, and $(\lambda I - B)u_n \to 0$ as $n \to \infty$.)

1.1.3 Fractional powers of weakly sectorial operators

We introduce fractional powers A^α, $\alpha \geq 0$, of weakly sectorial operators A for the following two reasons: for the approximation methods introduced in the next chapter we can consider then a fractional degree of smoothness for the initial error $u_0 - u_*$. More importantly, however, in applications (e.g., Abel integral equations, see Chapter 1.3), equations $A^\alpha u = f_*$ arise (for some $0 < \alpha < 1$); and we shall see in Chapter 2 and 3 that iterative methods can be used to solve those equations for certain $0 < \alpha$, consult Theorem 1.2.5 and Corollary 1.2.6 for the details.

First properties of fractional powers

Definition 1.1.15 *Let $A \in \mathcal{L}(X)$ be weakly sectorial.*

(a) For $0 < \alpha < 1$ we introduce fractional powers A^α by

$$A^\alpha u := \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{\alpha-1}(sI + A)^{-1} Au \, ds, \qquad u \in X. \tag{1.13}$$

(b) For arbitrary $\alpha > 0$ we define A^α by

$$A^\alpha := A^{\alpha - \lfloor \alpha \rfloor} A^{\lfloor \alpha \rfloor},$$

where $\lfloor \alpha \rfloor$ denotes the greatest integer $\leq \alpha$.

Note that (1.2) implies the existence of the integral in (1.13) as well as $A^\alpha \in \mathcal{L}(X)$.

Example 1.1.16 *Let X be a real or complex Hilbert space, let $A \in \mathcal{L}(X)$ be selfadjoint, positive semidefinite and $\|A\| \leq a$, and let $\{E_\lambda\}_\lambda$ be the resolution of the identity corresponding to A. Then*

$$A^\alpha u = \int_{0-0}^{a} \lambda^\alpha \, dE_\lambda u, \qquad u \in X.$$

If A is not selfadjoint (on a Hilbert space X), however, then $A^\alpha \neq (A^*A)^{\alpha/2}$, in general.

Lemma 1.1.17 *If $c \in \mathbb{K}$, and if A, $cA \in \mathcal{L}(X)$ are weakly sectorial, then*

$$(cA)^\alpha = c^\alpha A^\alpha.$$

This can be derived directly from the definition and the proof thus is omitted.

Interpolation inequalities for fractional powers of operators

The interpolation inequality for fractional powers of operators as it is stated in Corollary 1.1.19 can be used to prove convergence (with speed) for stopping rules and parameter choices (as stated in Chapter 3), if the initial approximation error has some fractional degree of smoothness. We first state a more general result.

Theorem 1.1.18 *For weakly sectorial $A,\ B \in \mathcal{L}(X)$ with $AB = BA$ we have, for any $0 < \alpha < 1$,*

$$\|A^\alpha u - B^\alpha u\| \ \le\ C\|(A-B)u\|^\alpha \|u\|^{1-\alpha}, \qquad u \in X,$$

with some $C > 0$.

Proof. We first consider the case $0 < \alpha < 1$. Obviously,

$$\begin{aligned} A^\alpha u - B^\alpha u &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{\alpha-1}\big((sI+A)^{-1}Au \ - \ (sI+B)^{-1}Bu\big)\, ds \\ &= J_1 + J_2, \end{aligned}$$

with

$$\begin{aligned} J_1 &:= \frac{\sin \pi\alpha}{\pi} \int_0^\eta s^{\alpha-1}\big((sI+A)^{-1}Au \ - \ (sI+B)^{-1}Bu\big)\, ds, \qquad (1.14) \\ J_2 &:= \frac{\sin \pi\alpha}{\pi} \int_\eta^\infty s^{\alpha-1}\big((sI+A)^{-1}Au \ - \ (sI+B)^{-1}Bu\big)\, ds, \end{aligned}$$

where

$$\eta := \frac{\|(A-B)u\|}{\|u\|}$$

(if $u \neq 0$; in the case $u = 0$ the assertion is trivially true). We first estimate J_1:

$$\begin{aligned} \|(sI+A)^{-1}Au \ - \ (sI+B)^{-1}Bu\| &\le \big(\|(sI+A)^{-1}A\| \ + \ \|(sI+B)^{-1}Bu\|\big) \cdot \|u\| \\ &\le \big(\gamma_1(A) + \gamma_1(B)\big)\|u\|, \end{aligned}$$

where the constants $\gamma_1(A)$ and $\gamma_1(B)$ are taken from (1.6) for A and B, respectively. Then,

$$\begin{aligned} \|J_1\| &\le \big(\gamma_1(A) \ + \ \gamma_1(B)\big)\frac{\sin \pi\alpha}{\pi}\int_0^\eta s^{\alpha-1}\, ds\ \|u\| \\ &\le \big(\gamma_1(A) \ + \ \gamma_1(B)\big)\eta^\alpha\|u\| \\ &\le \big(\gamma_1(A) \ + \ \gamma_1(B)\big)\|(A-B)u\|^\alpha\|u\|^{1-\alpha}. \end{aligned}$$

We now estimate J_2. To this end, we observe that

$$(sI+A)^{-1}Au \ - \ (sI+B)^{-1}Bu \ = \ s(sI+A)^{-1}(sI+B)^{-1}(Au - Bu),$$

and thus

$$\begin{aligned} \|J_2\| &\le M_0(A)M_0(B)\frac{\sin \pi(1-\alpha)}{\pi}\int_\eta^\infty s^{\alpha-2}\, ds\ \|Au - Bu\| \\ &\le M_0(A)M_0(B)\eta^{\alpha-1}\|Au - Bu\| \\ &\le M_0(A)M_0(B)\|(A-B)u\|^\alpha\|u\|^{1-\alpha}, \end{aligned}$$

which yields the desired result. □

Taking $B = 0$ in Theorem 1.1.18 yields

Corollary 1.1.19 *(Second interpolation inequality) For weakly sectorial $A \in \mathcal{L}(X)$ and $0 < \alpha < 1$ we have*

$$\|A^\alpha u\| \ \le\ C\|Au\|^\alpha\|u\|^{1-\alpha}, \qquad u \in X, \qquad (1.15)$$

with $C = C(M_0) := 2(M_0 + 1)$.

Remarks. 1. The constants in the proof of Theorem 1.1.21 are carefully checked out and '$\gamma_1(B) = 0$' is taken there in order to get the constant $C = 2(M_0 + 1)$ in Corollary 1.1.19.
2. One more remark concerning the minimization of the constants: C in (1.15) can be reduced to $C = 2M_0$, if A is inverse to an (unbounded) operator L, which is weakly sectorial with $M_0(L) \leq M_0(A)$, since then we can write J_1 in (1.14) as $J_1 = \frac{\sin \pi\alpha}{\pi} \int_0^\eta s^{\alpha-1}(sL + I)^{-1}u\ ds$. Moreover, for positive semidefinite operators in Hilbert spaces, the constant in (1.15) can be reduced to $C = 1$.
3. An immediate consequence of Corollary 1.1.19 is $\|A\|^\alpha \leq \|A\|^\alpha$ for any $\alpha > 0$.

Taking $B = \lambda I$ for $\lambda > 0$ in Theorem 1.1.18 gives the following result.

Corollary 1.1.20 *For weakly sectorial $A \in \mathcal{L}(X)$, any $0 < \alpha$ and $0 < \lambda$ with $\lambda \in \sigma_{ap}(A)$ one has $\lambda^\alpha \in \sigma_{ap}(A^\alpha)$. More specifically, if $z_k \in X$, $\|z_k\| \leq \varrho$, $k = 0, 1, ...,$ and $Az_k - \lambda z_k \to 0$ as $k \to \infty$, then $A^\alpha z_k - \lambda^\alpha z_k \to 0$ as $k \to \infty$.*

Proof. For integer $\alpha = m$ this follows immediately from the equality

$$A^m - B^m \ = \ \big(\sum_{k=0}^{m-1} A^k B^{m-1-k}\big)(A - B), \tag{1.16}$$

(with $B = \lambda I$), and the general case for α follows with $m := \lfloor \alpha \rfloor$ and $\beta := \alpha - m$ from the relation

$$A^\alpha - B^\alpha \ = \ A^\beta(A^m - B^m) \ + \ B^m(A^\beta - B^\beta). \tag{1.17}$$

□

We now extend the first interpolation inequality (see Theorem 1.1.9) to fractional powers of weakly sectorial operators.

Theorem 1.1.21 *(First interpolation inequality, revisited) For weakly sectorial $A \in \mathcal{L}(X)$ and any (fractional) $\alpha \geq 0$ we have*

$$\|A^\alpha u\| \ \leq \ C\|A^{\alpha+1}u\|^{\alpha/(\alpha+1)}\|u\|^{1/(\alpha+1)}, \qquad u \in X,$$

with

$$C \ = \ C(M, \alpha) := \gamma_1^{m/(\alpha+1)} M_0^{\alpha m/(\alpha+1)} \cdot \Big(\big(\frac{\alpha}{m}\big)^{-\alpha} + (\alpha m^\alpha)^{1/(\alpha+1)}\Big),$$

where $m = \lceil \alpha \rceil$.

Proof. As in the proof for Theorem 1.1.9 we get for $m = \lceil \alpha \rceil$, the smallest integer bigger than or equal to α,

$$A^\alpha \ = \ (I + tA)^{-m} A^\alpha \ + \ t\Big(\sum_{j=1}^{m} (I + tA)^{-j}\Big) A^{\alpha+1},$$

with Lemma 1.1.8 and Corollary 1.1.23 we find

$$\begin{aligned} \|A^\alpha u\| \ &\leq \ \|(I + tA)^{-1} A^{\alpha/m}\|^m \cdot \|u\| \ + \ mM_0^m t\|A^{\alpha+1}u\| \\ &\leq \ \gamma_1^m t^{-\alpha}\|u\| \ + \ mM_0^m t\|A^{\alpha+1}u\|, \end{aligned}$$

and take, e.g.,

$$t \ = \ \Big(\frac{\alpha}{m}\big(\frac{\gamma_1}{M_0}\big)^m \cdot \frac{\|u\|}{\|A^{\alpha+1}u\|}\Big)^{1/(\alpha+1)}$$

(if $Au \neq 0$; in the case $Au = 0$ the assertion is trivially true) to get the desired result. □

Resolvent conditons for fractional powers

The next lemma is a preparation for the proof of Theorem 1.1.24.

Lemma 1.1.22 *If X is a complex Banach space, if $A \in \mathcal{L}(X)$ is weakly sectorial, and if $0 \in \rho(A)$, then for $0 < \alpha \leq 1$,*

$$A^\alpha \ = \ \frac{1}{2\pi i} \int_\Gamma z^\alpha (zI - A)^{-1}\ dz, \tag{1.18}$$

where Γ is a smooth simple closed curve which surrounds counterclockwise the spectrum $\sigma(A)$ of A and not does intersect the negative real axis $(-\infty, 0]$.

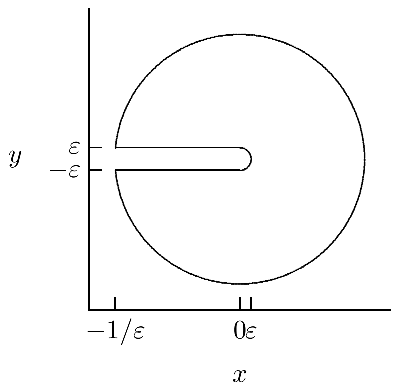


Abbildung 1.2: Path Γ_ε

Proof. We denote with $\mathcal{I}_\alpha$ the right-hand side in (1.18), which is well-defined for any $\alpha \in \mathbb{R}$. The operational calculus implies that for $\alpha_1,\ \alpha_2 \in \mathbb{R}$,

$$\mathcal{I}_{\alpha_1+\alpha_1} \;=\; \mathcal{I}_{\alpha_1}\mathcal{I}_{\alpha_2}, \tag{1.19}$$

and moreover $\mathcal{I}_1 = A$. The integrand in 1.18 is analytical in $\mathbb{C}\backslash(-\infty,0]$, hence we can deform the contour Γ into Γ_ε (for small $\varepsilon > 0$, see Figure 1.2) without changing the value of the integral. In a second step we deform Γ_ε into the negative real axis by letting $\varepsilon \to 0$ and obtain

$$\mathcal{I}_{-\alpha} \;=\; \frac{\sin \pi\alpha}{\pi}\int_0^\infty s^{-\alpha}(sI+A)^{-1}\,ds,$$

hence

$$\mathcal{I}_{1-\alpha} \;=\; \mathcal{I}_{-\alpha}A \;=\; \frac{\sin \pi\alpha}{\pi}\int_0^\infty s^{-\alpha}(sI+A)^{-1}A\,ds, \tag{1.20}$$

and exchanging the roles of α and $1-\alpha$ in (1.20) yields the assertion. □

Corollary 1.1.23 *If $A \in \mathcal{L}(X)$ is weakly sectorial, and if $\alpha_1,\ \alpha_2 \geq 0$, then*

$$A^{\alpha_1+\alpha_2} \;=\; A^{\alpha_1}A^{\alpha_2}, \tag{1.21}$$

$$\mathcal{R}(A^{\alpha_2}) \;\subset\; \mathcal{R}(A^{\alpha_1}), \qquad \text{if } \alpha_1 \leq \alpha_2. \tag{1.22}$$

Proof. To show (1.21), let us assume first that X is complex and $0 \in \rho(A)$. Then Lemma 1.1.23 and (1.19) in its proof imply immediately the assertion. Let us now assume that X is complex but let us drop the assumption $0 \in \rho(A)$. Then $A_\delta := A + \delta I,\ \delta > 0$, is weakly sectorial, and it follows from Theorem 1.1.18 and relations (1.16) and (1.17) that $A_\delta^\alpha \to A^\alpha$ as $\delta \to 0$ (in $\mathcal{L}(X)$), hence one finally has (1.21) for arbitrary weakly sectorial A in complex spaces.
If X is a real space, then (1.21) holds for the complexification $X_\mathbb{C}$ and $A_\mathbb{C}$ of X and A, respectively and restricting again both sides in (1.21) to X shows that (1.21) holds in the general case. (1.22) then is an immediate consequence. □

Remark. Corollary 1.1.23 implies that $\mathcal{N}(A^\alpha) = \mathcal{N}(A),\ \alpha > 0$.

The next theorem states that fractional powers of weakly sectorial operators (for $0 < \alpha \leq 1$) are again weakly sectorial, and a formula for the resolvent operator is given. (As was mentioned, this result can be improved, if α is small enough, see Theorem 1.2.5 and Corollary 1.2.6.)

Theorem 1.1.24 *If $A \in \mathcal{L}(X)$ is weakly sectorial, then A^α is weakly sectorial for any $0 < \alpha \leq 1$ (with $M_0(A^\alpha) = M_0(A)$), and one has the representation*

$$(tI + A^\alpha)^{-1} \;=\; \frac{\sin \pi\alpha}{\pi}\int_0^\infty \frac{s^\alpha(sI+A)^{-1}}{s^{2\alpha} + 2ts^\alpha\cos\pi\alpha + t^2}\,ds, \qquad t > 0. \tag{1.23}$$

Proof. The right-hand side in (1.23), we shall denote it with $\mathcal{J}_\alpha(t)$, exists, since A is weakly sectorial, and moreover $\mathcal{J}_\alpha(t) \in \mathcal{L}(X)$, with

$$\begin{aligned} \|\mathcal{J}_\alpha(t)\| &\leq M_0 \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{s^{\alpha-1}}{s^{2\alpha} + 2ts^\alpha \cos \pi\alpha + t^2}\, ds \\ &= M_0/t. \end{aligned}$$

We now sketch the proof that $\mathcal{J}_\alpha(t)$ in fact inverts $tI + A^\alpha$. To this end we first assume that X is complex and $0 \in \rho(A)$. Then the operational calculus and Lemma 1.1.22 imply that $t \in \rho(-A)$ for $t > 0$, and one has the representation

$$(tI + A^\alpha)^{-1} = \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{1}{z^\alpha + t} (zI - A)^{-1}\, dz.$$

where the path Γ_ε is the same as in Figure 1.2, with $\varepsilon > 0$ small. Similar to the proof of Lemma 1.1.22 we contract Γ_ε to the negative real axis and obtain (1.23), for invertible A.
We now proceed as in the proof for Corollary 1.1.23 and consider the general case for A, in complex spaces. $A_\delta := A + \delta I,\ \delta > 0$, is weakly sectorial, hence one has (1.23) with A_δ instead of A, and it is easy to see that the corresponding integral in (1.23) converges to $\mathcal{J}_\alpha(t)$ (in $\mathcal{L}(X)$) as $\delta \to 0$. Since $tI + A_\delta^\alpha \to tI + A^\alpha$ as $\delta \to 0$ (in $\mathcal{L}(X)$), one finally has (1.23) for arbitrary weakly sectorial A.
If X is a real space, then (1.23) holds for the complexification $X_\mathbb{C}$ and $A_\mathbb{C}$ of X and A, respectively (with M_0 not changing), and restricting again both sides in (1.23) to X shows that (1.23) is valid in real spaces X. □

1.2 Strictly sectorial operators

In Subsection 1.2.1 we introduce the class of strictly sectorial operators (which in fact is smaller than the class of weakly sectorial operators) and provide sufficient conditions for fractional powers of operators to be strictly sectorial (these results shall be used for Abel integral operators, see Section 1.3 for more about these operators and their applications).

In Subsection 1.2.2 a specific integral equation of the first kind is introduced which we shall use then in our numerical experiments; here it is computed directly that the corresponding integral operator is strictly sectorial. In Subsection 1.2.3 we state results that are stronger than those for weakly sectorial operators and that are basical for the iterative methods to be considered in Chapter 2. Throughout this Section 1.2 let X be a complex Banach space, if not further specified.

1.2.1 Introductory remarks

Definition 1.2.1 *A linear operator $B : X \supset \mathcal{D}(B) \to X$ is called strictly sectorial, if there is an $0 < \varepsilon \leq \pi/2$ such that B is sectorial of angle $\pi/2 + \varepsilon$.*

In Hilbert spaces one has the following obvious result.

Example 1.2.2 *Let X be a complex Hilbert space and $A \in \mathcal{L}(X)$. If $A = A^* \geq 0$, i.e., if A is selfadjoint and positive semidefinite, then A is strictly sectorial. More specifically, A is sectorial of angle π, and for any $0 < \varepsilon < \pi/2$, A is sectorial of angle $\pi/2 + \varepsilon$, and (1.4) holds with $M_{\pi/2+\varepsilon} = 1/\cos\varepsilon$.*

Proposition 1.2.3 *Let $A \in \mathcal{L}(X)$ have a trivial nullspace $\mathcal{N}(A)$, and let $0 \leq \theta_0 \leq \pi$. A is sectorial of angle θ_0 if and only if A^{-1} is sectorial of angle θ_0, and then $M_\theta(A) = M_\theta(A^{-1}) + 1,\ 0 \leq \theta < \theta_0$.*

Proof. The proof is similar to the proof of Proposition in Subsection 1.1.2; for $\lambda \in \theta,\ \ \lambda I + A$ is one-to-one and onto, and

$$(\lambda I + A)^{-1} = \frac{1}{\lambda} I - \frac{1}{\lambda^2} (A^{-1} + \frac{1}{\lambda} I)^{-1}, \tag{1.24}$$

and then the estimate for $\|(\lambda I + A)^{-1}\|$ is an immediate consequence of (1.24). □

Corollary 1.2.4 *Let $A \in \mathcal{L}(X)$ have a trivial nullspace $\mathcal{N}(A)$. Then A is strictly sectorial if and only if A^{-1} is strictly sectorial.*

The following theorem provides sufficient conditions for operators to be strictly sectorial, see Corollary 1.2.6; this has (already mentioned) applications to Abel integral equations.

Theorem 1.2.5 *Let A be sectorial of angle $\theta_0 \in [0, \pi]$. Then for any $0 < \alpha < 1$, A^α is sectorial of angle $(1-\alpha)\pi + \alpha\theta_0$. On a smaller sector one has a representation for $(\lambda I + A^\alpha)^{-1}$,*

$$(\lambda I + A^\alpha)^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{s^\alpha (sI + A)^{-1}}{s^{2\alpha} + 2\lambda s^\alpha \cos \pi\alpha + \lambda^2}\, ds, \qquad \lambda \in \operatorname{int} \Sigma_{(1-\alpha)\pi}. \tag{1.25}$$

Here $\operatorname{int} \Sigma_{(1-\alpha)\pi}$ denote the interior of $\Sigma_{(1-\alpha)\pi}$. Note, that the opening angle $(1-\alpha)\pi + \alpha\theta_0$ for the sector $\Sigma_{(1-\alpha)\pi+\alpha\theta_0}$ corresponding to A^α is a convex combination of π and the opening angle θ_0 for the sector Σ_{θ_0} corresponding to A. Before giving the proof we state the following important and nice result which is an immediate consequence of Theorem 1.2.5. (Part (a) follows with Theorem 1.1.5.)

Corollary 1.2.6 *(a) If $A \in \mathcal{L}(X)$ is weakly sectorial and $0 < \alpha \le 1/2$, then A^α is strictly sectorial.*

(b) If $A \in \mathcal{L}(X)$ is sectorial of angle $\pi/2$, then A^α is strictly sectorial for any $0 < \alpha < 1$.

Proof of Theorem 1.2.5. The proof consists of two parts.

(a) We show first that for any weakly sectorial A, A^α is sectorial of angle $(1-\alpha)\pi$, and that moreover one has (1.25); to this end, we take any

$$\lambda = re^{i\varphi}, \qquad |\varphi| < (1-\alpha)\pi, \quad r > 0,$$

and denote the right-hand side in (1.25) with $\mathcal{J}_\alpha(\lambda)$. We find then

$$\begin{aligned} \frac{s^{\alpha-1}}{s^{2\alpha} + 2\lambda s^\alpha \cos \pi\alpha + \lambda^2} &= \frac{s^{\alpha-1}}{(s^\alpha + \lambda e^{i\pi\alpha}) \cdot (s^\alpha + \lambda e^{-i\pi\alpha})} \\ &= \frac{s^{\alpha-1}}{|s^\alpha/r + e^{i(\varphi+\pi\alpha)}| \cdot |s^\alpha/r + e^{i(\varphi-\pi\alpha)}|} \cdot \frac{1}{r^2}, \end{aligned} \tag{1.26}$$

hence $\mathcal{J}_\alpha(\lambda)$ exists (in $\mathcal{L}(X)$), and one shows as in Theorem 1.1.24 that $\mathcal{J}_\alpha(\lambda)$ in fact inverts $\lambda I + A^\alpha$. Moreover, substituting $\tau = s^\alpha/r$ in (1.26) yields

$$\begin{aligned} \|\mathcal{J}_\alpha(\lambda)\| &\le M_0 \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{r/\alpha\, d\tau}{|\tau + e^{i(\varphi+\pi\alpha)}| \cdot |\tau + e^{i(\varphi-\pi\alpha)}|} \cdot \frac{1}{r^2} \\ &\le \Big(M_0 \frac{\sin \pi\alpha}{\pi\alpha} \int_0^\infty \frac{d\tau}{|\tau + e^{i(\varphi+\pi\alpha)}| \cdot |\tau + e^{i(\varphi-\pi\alpha)}|} \Big) \cdot \frac{1}{|\lambda|}, \end{aligned}$$

which shows that A^α is sectorial of angle $(1-\alpha)\pi$.

(b) Assume now that A is sectorial of angle $\theta_0 \in [0, \pi]$. Then for any $0 \le \theta < \theta_0$, $e^{\mp i\theta} A$ is weakly sectorial (with $M_0(e^{\mp i\theta} A) = M_\theta(A)$), and we can apply part (a) then with $e^{\mp i\theta} A$ instead of A and obtain for $\varepsilon > 0$ arbitrary small

$$\|(\lambda e^{\pm i\theta\alpha} I + A^\alpha)^{-1}\| \le \frac{M_{(1-\alpha)\pi-\varepsilon}(e^{\mp i\theta\alpha} A^\alpha)}{|\lambda|}, \qquad \lambda \in \Sigma_{(1-\alpha)\pi-\varepsilon},$$

which yields the desired result. □

1.2.2 An integral equation of the first kind

In the previous subsection we have seen that (small) fractional powers provide one class of strictly sectorial operators, and we shall see in Section 1.3 some applications for that. In this subsection we introduce an integral operator A where one can directly compute that A is strictly sectorial, and we shall use this example in our numerical experiments, see Subsection 3.1.3. We start with a lemma on the differential operator $Lf = -f''$.

Lemma 1.2.7 *Let X be the space of continuous complex-valued functions which are periodic,*

$$X = C_p[0,1] := \{\ u \in C[0,1] : \quad u(0) = u(1)\ \},$$

supplied with the maximum norm $\|\cdot\|_\infty$. We consider the differential operator

$$L : X \supset \mathcal{D}(L) \to X,$$

defined by

$$\begin{aligned} Lf &:= -f'', \qquad f \in \mathcal{D}(L), \\ \mathcal{D}(L) &:= \left\{ f \in C^2[0,1]: \ f, \ f', \ f'' \in C_p[0,1] \right\}. \end{aligned}$$

Then L is strictly sectorial. More specifically, L is sectorial of angle π, and for any $0 < \varepsilon < \pi/2$ one has $M_{\pi/2+\varepsilon}(L) \le 1/\cos(\pi/4+\varepsilon/2)$, i.e.,

$$\|(\lambda I + L)^{-1}\|_\infty \le \frac{1}{\cos(\pi/4+\varepsilon/2)} \cdot \frac{1}{|\lambda|}, \qquad \lambda \in \Sigma_{\pi/2+\varepsilon}.$$

Proof. Let $u \in X$ and $\mu = \varrho e^{i\theta}$ with $\varrho > 0$ and $-\pi/2 < \theta < \pi/2$. Then

$$(\mu^2 I + L) f = u \tag{1.27}$$

has a unique solution $f \in \mathcal{D}(L)$ which is given by

$$f(\xi) := \int_0^1 k_\mu(\xi,\eta) u(\eta) \, d\eta, \qquad \xi \in [0,1],$$

with kernel

$$k_\mu(\xi,\eta) := \begin{cases} a \cdot \cosh(\mu(\xi - \eta - 0.5)), & \text{if } \eta \le \xi, \\ a \cdot \cosh(\mu(\xi - \eta + 0.5)), & \text{if } \eta > \xi, \end{cases} \tag{1.28}$$

where

$$a = \frac{1}{2\mu \sinh \frac{\mu}{2}}.$$

Denoting

$$x := \operatorname{Re} \mu = r\cos\theta > 0,$$

and using the inequalities

$$\begin{aligned} |\sinh(\mu/2)| &\ge \sinh(x/2), \\ |\cosh(\mu(\xi-\eta \pm 0.5))| &\le \cosh(x(\xi - \eta \pm 0.5)), \end{aligned}$$

we find, for arbitrary $\xi \in [0,1]$,

$$\begin{aligned} |f(\xi)| &\le \frac{1}{2|\mu| \sinh \frac{x}{2}} \cdot \left[\int_0^\xi \cosh(x(\xi-\eta-0.5)) \, d\eta + \int_\xi^1 \cosh(x(\xi - \eta + 0.5)) \, d\eta \right] \cdot \|u\|_\infty \\ &= \frac{1}{|\mu|^2 \cos\theta} \|u\|_\infty, \end{aligned}$$

thus (see (1.27))

$$\|f\|_\infty \le \frac{1}{|\mu|^2 \cos\theta} \|(\mu^2 I + L) f\|_\infty.$$

For $\mu^2 = \lambda = re^{i\varphi}$, $|\varphi| \le \pi/2 + \varepsilon$, we find

$$\|(\lambda I + L)^{-1}\|_\infty \le \frac{1}{\cos(\varphi/2)} \cdot \frac{1}{|\lambda|} \le \frac{1}{\cos(\pi/4+\varepsilon/2)} \cdot \frac{1}{|\lambda|},$$

and that completes the proof. □

The following integral operator is taken in our numerical calculations.

Corollary 1.2.8 *Let $X = C_p[0,1]$ and L as in the Lemma 1.2.7, and let $\omega > 0$. Then*

$$A := (L + \omega^2 I)^{-1} \in \mathcal{L}(X) \tag{1.29}$$

is a compact Fredholm integral operator,

$$Au(\xi) = \int_0^1 k_\omega(\xi,\eta) u(\eta) \, d\eta, \qquad \xi \in [0,1],$$

with non-degenerated continuous kernel k_ω as in (1.28), and A is strictly sectorial. More specifically, A is sectorial of angle π, and for any $0 < \varepsilon < \pi/2$ one has

$$\|(\lambda I + A)^{-1}\|_\infty \;\le\; \Big(1 + \frac{1}{\cos(\pi/4 + \varepsilon/2)\cos\varepsilon}\Big) \cdot \frac{1}{|\lambda|}, \qquad \lambda \in \Sigma_{\pi/2+\varepsilon}, \tag{1.30}$$

i.e., $M_{\pi/2+\varepsilon}(A) \;\le\; 1 \;+\; 1/\big(\cos(\pi/4+\varepsilon/2)\cos\varepsilon\big)$.

Note that the underlying equation $Au = f$ then is an (ill-posed) Fredholm integral equation of the first kind.

Proof. It remains to show that (1.30) is fulfilled (for any $0 < \varepsilon < \pi/2$). To this end we observe that Lemma 1.2.7 implies

$$\begin{aligned} \rho(-A^{-1}) \;&\supset\; \Sigma_{\pi/2+\varepsilon}, \\ \|(\lambda I + A^{-1})^{-1}\|_\infty \;&\le\; \frac{1}{\cos(\pi/4+\varepsilon/2)} \cdot \frac{1}{|\lambda + \omega^2|} \\ &\le\; \frac{1}{\cos(\pi/4+\varepsilon/2)\cos\varepsilon} \cdot \frac{1}{|\lambda|}, \qquad \lambda \in \Sigma_{\pi/2+\varepsilon}, \end{aligned}$$

and then Proposition 1.2.3 implies that for any $0 < \varepsilon < \pi/2$, the desired resolvent condition for A is fulfilled. □

1.2.3 Asymptotic behaviour of uniformly bounded semigroups, and discrete versions

The results of this subsection provide basic results for the Richardson iteration, the implicit methods as well as Cauchy's method that are introduced in the next chapter.

Definition 1.2.9 *Let $A \in \mathcal{L}(X)$ and*

$$T(t) := e^{-tA} \;=\; \sum_{k=0}^{\infty} \frac{(-tA)^k}{k!}, \qquad t \ge 0.$$

Then $-A$ is called <u>*infinitesimal generator*</u> *of the semigroup $\{T(t)\}_{t\ge 0}$.*

The spectral theorem implies that

$$\|T(t)\| \;\ge\; 1, \qquad t \ge 0,$$

if $0 \in \sigma(A)$, the spectrum of A. A well-known decay behavior holds, however, if the additional 'weight' A^α is introduced.

Theorem 1.2.10 *Let $A \in \mathcal{L}(X)$ be strictly sectorial, and let $\{T(t)\}_{t\ge0}$ be the semigroup with infinitesimal generator $-A$. Then we have for $\alpha \ge 0$,*

$$\|T(t)A^\alpha\| \;\le\; c_\alpha t^{-\alpha} \qquad \textit{for} \;\; t > 0, \tag{1.31}$$

with postive constants $c_\alpha > 0$. More specifically, if A is sectorial of angle $\pi + \varepsilon_0$, then (1.31) holds with

$$c_0 := M_{\pi/2+\varepsilon}(A)\Big(2\int_{\sin\varepsilon}^{\infty} e^{-s}/s\; ds \;+\; \int_{-\pi/2-\varepsilon}^{\pi/2+\varepsilon} e^{\cos\varphi} d\varphi\Big)/(2\pi), \tag{1.32}$$

and for integer $\alpha \ge 1$ we may take $c_\alpha = \Big(M_{\pi/2+\varepsilon}(A) \cdot \alpha/(\pi\sin\varepsilon)\Big)^\alpha$, with any $0 < \varepsilon < \varepsilon_0$.

Proof. We first prove the assertion for integer α. (1.31) for uniformly bounded analytical semigroups is well-known, hence we give only the sketch of the proof. For $\alpha = 0$, (1.32) can be proved with the Cauchy integral,

$$T(t) \;=\; \frac{1}{2\pi i}\sum_{j=1}^{3} \int_{\Gamma_j^{(t)}} e^{\lambda t}(\lambda I + A)^{-1}\; d\lambda,$$

with paths of integration taken to be

$$\Gamma_{1/3}^{(t)} = \{\ re^{\mp i(\pi/2+\varepsilon)} : \ 1/t \le r < \infty\ \},$$
$$\Gamma_2^{(t)} = \{\ t^{-1}e^{i\varphi} : \ |\varphi| \le \pi/2 + \varepsilon\ \},$$

and orientation is chosen so that Im λ increases along the integration path.

For $\alpha = 1$, (1.31) follows from an estimation of the Cauchy integral

$$-AT(t) = \frac{d}{dt}T(t) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda t}(\lambda I + A)^{-1}\, d\lambda,$$

with contour

$$\Gamma = \Gamma_1 \cup \Gamma_2, \quad \Gamma_{1/2} = \{\ re^{\mp i(\pi/2+\varepsilon)} : \ 0 \le r < \infty\ \},$$

and where the orientation is chosen as above. (1.31) for integer $\alpha \ge 1$ follows easily,

$$\|A^\alpha T(t)\| = \|(AT(t/\alpha))^\alpha\| \le \|AT(t/\alpha)\|^\alpha \le \big(c_1\alpha/t\big)^\alpha = c_\alpha t^{-\alpha}.$$

Finally, (1.31) for fractional α follows with the second interpolation inequality. □

For strictly sectorial operators we can improve Lemma 1.1.8 such that the arising constants does not depend on n. This is stated in the following theorem which also provide a first discrete analogue for Theorem 1.2.10.

Theorem 1.2.11 *If $A \in \mathcal{L}(X)$ is strictly sectorial, then for arbitrary $\mu > 0$,*

$$\|(I + \mu A)^{-n}\| \le \gamma_0 \qquad \text{for } n = 0, 1, 2, ..., \tag{1.33}$$

with $\gamma_0 = c_0$ (c_0 as in (1.32)). Moreover, for $\alpha > 0$ there is a $\gamma_\alpha > 0$ such that

$$\|(I + \mu A)^{-n}A^\alpha\| \le \gamma_\alpha n^{-\alpha} \qquad \text{for } n = 1, 2, ... \tag{1.34}$$

Proof. The function $t \mapsto T(t)$ is differentiable in norm, and one can show as in the proof of the Hille-Yosida theorem that for $\lambda > 0$,

$$(\lambda I + A)^{-(n+1)} = \frac{1}{n!}\int_0^\infty t^n e^{-\lambda t}T(t)\, dt.$$

Then

$$(\lambda I + A)^{-(n+1)}A^\alpha = \frac{1}{n!}\int_0^\infty t^n e^{-\lambda t}T(t)A^\alpha\, dt,$$

and (1.31) yield for $n \ge \alpha$, with c_α as above,

$$\begin{aligned}\|(\lambda I + A)^{-(n+1)}A^\alpha\| &\le \frac{c_\alpha}{n!}\int_0^\infty t^{n-\alpha}e^{-\lambda t}\, dt \\ &= c_\alpha\frac{(n-\alpha)!}{n!}\cdot\frac{1}{\lambda^{n+1-\alpha}},\end{aligned}$$

in other terms,

$$\begin{aligned}\|(I + \mu A)^{-n}A^\alpha\| &\le \frac{c_\alpha}{\mu^\alpha}\frac{1}{(n-\alpha)...(n-1)} \\ &\le \frac{c_\alpha}{\mu^\alpha}(\alpha+1)^\alpha n^{-\alpha}, \qquad n > \alpha.\end{aligned} \tag{1.35}$$

For $\alpha = 0$ this is (1.33), and (1.35) also implies (1.34) for integer α. Finally, (1.34) for fractional α follows with the second interpolation inequality. □

Remark. 1. A further estimation of (1.32) shows that we can take

$$\gamma_0 = M\Big((\pi/2 + \varepsilon)e + e^{-1} - \log(\sin\varepsilon)\Big)/\pi \tag{1.36}$$

in (1.33) which of course is not as sharp as (1.32) but more easy to calculate.
2. If A is a selfadjoint, positive semidefinite operator in a real or complex Hilbert space X, then (1.33) holds with $\gamma_0 = 1$.

The following theorem provides a second discrete analogue for Theorem 1.2.10 and enables us to state some results for the Richardson iteration, see the next chapter. Conditions (1.38), (1.39) say that the spectrum $\sigma(L)$ of L approximates 1 within a Stolz angle, and the resolvent operator does not increase too fast as it approximates 1 inside the prescribed sector.

Theorem 1.2.12 *We assume that for $T \in \mathcal{L}(X)$ and some $0 < \varepsilon < \pi/2$ and $C > 0$,*

$$\sigma(T) \subset \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \cup \{ 1 \}, \tag{1.37}$$

$$\rho(T) \supset 1 + \Sigma_{\pi/2+\varepsilon}, \tag{1.38}$$

$$\|(\lambda I - T)^{-1}\| \leq \frac{C}{|\lambda - 1|}, \qquad \lambda \in 1 + \Sigma_{\pi/2+\varepsilon}, \tag{1.39}$$

hold. Then T is power bounded,

$$\|T^n\| \leq a_0 \qquad \text{for } n = 0, 1, 2, ...,$$

and for $\alpha > 0$ there exist some constant a_α such that

$$\|T^n (I - T)^\alpha\| \leq a_\alpha n^{-\alpha} \qquad \text{for } n = 1, 2, ... \tag{1.40}$$

Proof. For reader's convenience we give the line of the proof. The power boundedness of T can be obtained with the Cauchy integral

$$T^n = \frac{1}{2\pi i} \sum_{j=1}^{3} \int_{\Gamma_j^{(n)}} \lambda^n (\lambda I - T)^{-1} \, d\lambda, \tag{1.41}$$

with contours

$$\Gamma_{1/3}^{(n)} = \{ 1 + \frac{1}{n} + t e^{\pm i(\pi/2+\varepsilon)} : \ 0 \leq t \leq t_n \},$$
$$\Gamma_2^{(n)} = \{ r e^{i\varphi} : \qquad \pi/2 + \varepsilon_n \leq \varphi \leq 3\pi/2 - \varepsilon_n \},$$

with n large enough and r, t_n and ε_n taken such that $\cos\varepsilon < r < 1$ and

$$\Gamma_2 = \{ r e^{i\varphi} : \ \pi/2 + \varepsilon \leq \varphi \leq 3\pi/2 - \varepsilon \} \subset \rho(T),$$

and such that the composition of these contours yields a closed curve. To obtain (1.40) for integer α first we may consider the corresponding Cauchy integral and take the integration path $\Gamma_{1/3} = \{ 1 + t e^{\pm i(\pi/2+\varepsilon)} : \ 0 \leq t \leq t_* \}$ (with an appropriate t_*) and Γ_2 (with $r < 1$ sufficiently large). The assertion for fractional $\alpha > 0$ then follows with the second interpolation inequality (1.15). □

Remarks 1. Since in Theorem 1.2.12 requirements on the behaviour of $\|(\lambda I - L)^{-1}\|$ are made near 1 only, one cannot give any concrete estimate for $\sup_{n \geq 0} \|T^n\|$. It is, however, necessary to have one, if it comes to the implementation of the stopping rules and parameter choices for our methods.
2. One can treat the case in Theorem 1.2.11 as a special case of Theorem 1.2.12. Theorem 1.2.11 provides, however, an estimate for $\sup_{n \geq 0} \|(I + \mu A)^{-n}\|$.
3. The estimates obtained in this subsection cannot be improved, in general; this is discussed in Section 1.4.

1.3 Fractional integration, Abel integral equations

1.3.1 (Fractional) integration

For $0 < a < \infty$ let $X = C[0, a]$ be the space of $\mathbb{K}$-valued continuous functions on [0,a], supplied with the maximum norm $\| \cdot \|_\infty$. In the first proposition we introduce the Volterra integral operators V_1 and V_2 (which we consider throughout the whole section) and state elementary properties of them.

Theorem 1.3.1 *Let $\beta > 0$ be real and $X = C[0,a]$. The Volterra integral operators $V_1,\ V_2 \in \mathcal{L}(X)$, defined by*

$$(V_1 u)(\xi) \ := \ \int_0^\xi \eta^{\beta-1} u(\eta)\ d\eta, \qquad \xi \in [0,a], \tag{1.42}$$

$$(V_2 u)(\xi) \ := \ \int_\xi^a \eta^{\beta-1} u(\eta)\ d\eta, \qquad \xi \in [0,a], \tag{1.43}$$

are weakly sectorial (with $M_0(V_j) = 2, \ \ j = 1,2$), and in fact

$$\lim_{t\to\infty} \|(I + tV_j)^{-1}\|_\infty \ = \ 2, \qquad j = 1,2. \tag{1.44}$$

Moreover, for $j = 1,2$, $\rho(V_j) = \mathbb{K}\backslash\{0\}$. If $\mathbb{K} = \mathbb{C}$, then V_j is sectorial of angle $\pi/2$, and this angle $\pi/2$ is best possible, i.e., V_j is not strictly sectorial.

Proof. We present the proof for V_1 only, since the same technique applies to prove the assertion for V_2. V_1 obviously is well-defined and in $\mathcal{L}(X)$, with $\|V_1\|_\infty = a^\beta/\beta$. Moreover V_1 is inverse to the (unbounded) operator $L : X \supset \mathcal{D}(L) \to X$ defined by

$$\begin{aligned} (Lf)(\xi) \ &= \ \xi^{-(\beta-1)} f'(\xi), \qquad f \in \mathcal{D}(L), \ \ \xi \in [0,a], \\ \mathcal{D}(L) \ &:= \ \big\{ \ f \in X : \ f \in C^1(0,a], \ \xi \mapsto \xi^{-(\beta-1)} f'(\xi) \ \in \ X, \ f(0) = 0 \ \big\}. \end{aligned} \tag{1.45}$$

We observe first that $\rho(-L) = \mathbb{K}$, since for $\lambda \in \mathbb{K}$ and $u \in X$, the equation

$$(\lambda I + L) f \ = \ u \tag{1.46}$$

has the unique solution

$$f(\xi) \ = \ \int_0^\xi \eta^{\beta-1} e^{-\lambda(\xi^\beta - \eta^\beta)/\beta} u(\eta)\ d\eta, \qquad \xi \in [0,a], \tag{1.47}$$

and thus $\rho(V_1) = \mathbb{K}\backslash\{0\}$. We shall show that L is weakly sectorial with $M_0(L) = 1$, i.e.,

$$\|(tI + L)^{-1}\|_\infty \ \le \ 1/t, \qquad t > 0, \tag{1.48}$$

(which in fact means that L is dissipative). We find then from (1.48) and Proposition 1.1.7 that V_1 is weakly sectorial (with $M_0(V_1) = 2$). For $\mathbb{K} = \mathbb{C}$, (1.48) and Theorem 1.1.5 imply that L is sectorial of angle $\pi/2$, and then also $V_1 = L^{-1}$ is sectorial of angle $\pi/2$, see Proposition 1.2.3.

To show (1.48), let $t = \lambda > 0$ in (1.47) and $\xi \in [0,a]$. Then

$$\begin{aligned} t|f(\xi)| \ &\le \ \Big(t \int_0^\xi \eta^{\beta-1} e^{-t(\xi^\beta - \eta^\beta)/\beta}\ d\eta \Big) \cdot \|u\|_\infty \\ &= \ \big(1 - e^{-t\xi^\beta/\beta}\big) \cdot \|u\|_\infty \ \le \ \|u\|_\infty \ = \ \|(tI + L)f\|_\infty, \end{aligned}$$

and taking the supremum over $\xi \in [0,a]$ yields (1.48).

We show that $M_0(V_1) = 2$ cannot be reduced. To this end, we observe that $(I + tV_1)u = f$ if and only if (see (1.5) with t^{-1} for t, (1.46) and (1.47))

$$u(\xi) \ = \ f(\xi) \ - \ t \int_0^\xi \eta^{\beta-1} e^{-t(\xi^\beta - \eta^\beta)/\beta} f(\eta)\ d\eta.$$

Now take for small $\varepsilon > 0$ some $f \in C[0,a], \ \|f\|_\infty = 1$, such that $f(a) = 1$ and $f(\eta) = -1, \ \ \eta \in [0, a-\varepsilon]$, and then

$$\begin{aligned} |u(a)| \ &\ge \ 1 \ + \ t \int_0^{a-\varepsilon} \eta^{\beta-1} e^{-t(a^\beta - \eta^\beta)/\beta}\ d\eta \ - \ t \int_{a-\varepsilon}^a \eta^{\beta-1} e^{-t(a^\beta - \eta^\beta)/\beta}\ d\eta \\ &= \ 1 \ + \ e^{-t\left(a^\beta \ - \ (a-\varepsilon)^\beta\right)/\beta} \ - \ e^{-ta^\beta/\beta} \ - \ \Big[1 - e^{-t\left(a^\beta - (a-\varepsilon)^\beta\right)/\beta}\Big] \\ &= \ 2e^{-t\left(a^\beta \ - \ (a-\varepsilon)^\beta\right)/\beta} \ - \ e^{-ta^\beta/\beta}, \end{aligned}$$

and we find (1.44) by taking $\varepsilon = \varepsilon(t)$ sufficiently small.

We finally show that L (and then also V_1) is not strictly sectorial. To this end, let $t > 0$ and $u(\eta) = e^{-it\eta^\beta/\beta}$, $\eta \in [0, a]$. Then the equation

$$(itI + L)f \;=\; u$$

has the unique solution

$$\begin{aligned} f(\xi) &= \int_0^\xi \eta^{\beta-1} e^{-it(\xi^\beta - \eta^\beta)/\beta} u(\eta)\, d\eta \\ &= e^{-it\xi^\beta/\beta} \int_0^\xi \eta^{\beta-1}\, d\eta \;=\; e^{-it\xi^\beta/\beta} \cdot \frac{\xi^\beta}{\beta}, \end{aligned}$$

hence $|f(\xi)| = \xi^\beta/\beta$, and thus

$$\|f\|_\infty \;=\; \frac{a^\beta}{\beta}, \qquad \|u\|_\infty = 1,$$

and this shows that L (and then also V_1) is not strictly sectorial. □

For real $\beta > 0$ let $L^p\big([0, a],\ \xi^{\beta-1} d\xi\big)$ be the space of $\mathbb{K}$-valued, measurable functions u on [0,a], such that $|u|^p$ is integrable with respect to the measure $\xi^{\beta-1} d\xi$, and this space is supplied with the norm

$$\|u\|_{L^p} \;=\; \Big(\int_0^a |u(\xi)|^p\, \xi^{\beta-1} d\xi\Big)^{1/p}, \qquad u \in L^p\big([0, a],\ \xi^{\beta-1} d\xi\big).$$

By $L^\infty\big([0, a],\ \xi^{\beta-1} d\xi\big)$ we denote the space of $\mathbb{K}$-valued, measurable functions u on [0,a] which are essentially bounded with respect to the measure $\xi^{\beta-1} d\xi$, and this space is supplied with the norm

$$\|u\|_{L^\infty} \;=\; \operatorname*{ess\,sup}_{\xi \in [0,a]} |u(\xi)| \xi^{\beta-1}, \qquad u \in L^\infty\big([0, a],\ \xi^{\beta-1} d\xi\big).$$

Theorem 1.3.2 *(Integration in $L^p\big([0, a],\ \xi^{\beta-1} d\xi\big)$) On $X = L^p\big([0, a],\ \xi^{\beta-1} d\xi\big)$ (for some $1 \le p \le \infty$) the operators $V_1,\ V_2$, defined by (1.42), (1.43), respectively, are weakly sectorial with $M_0(V_j) = 2,\ j = 1, 2$. For $\mathbb{K} = \mathbb{C}$, V_1 and V_2 are sectorial of angle $\pi/2$.*

Proof. We again give the proof for V_1 only and shall show that V_1 is well-defined and in $\mathcal{L}(X)$, and that it is sectorial of angle $\pi/2$. To this end we consider

$$\begin{aligned} Lf(\xi) &= \xi^{-(\beta-1)} f'(\xi), \qquad f \in \mathcal{D}(L),\quad \xi \in [0, a], \\ \mathcal{D}(L) &:= \{\ f \in X : f \text{ is absolutely continuous},\ \xi \mapsto \xi^{-(\beta-1)} f'(\xi) \in X,\quad f(0) = 0\ \}. \end{aligned} \tag{1.49}$$

Similar to the proof of Theorem 1.3.1, for $t \ge 0$,

$$(tI + L)f \;=\; u$$

has the unique solution

$$f(\xi) \;=\; \int_0^\xi \eta^{\beta-1} e^{-t(\xi^\beta - \eta^\beta)/\beta} u(\eta)\, d\eta, \qquad \xi \in [0, a].$$

To show that L is weakly sectorial with $M_0(L) = 1$, we substitute $\tilde\eta = \eta^\beta/\beta,\ \tilde\xi = \xi^\beta/\beta$, and define

$$\begin{aligned} \tilde f(\tilde\xi) &:= f\big((\beta\tilde\xi)^{1/\beta}\big), \\ \tilde u(\tilde\eta) &:= u\big((\beta\tilde\eta)^{1/\beta}\big), \end{aligned}$$

and $\tilde a := a^\beta/\beta$. Note that

$$\|\tilde f\|_{L^p([0,\tilde a], d\xi)} \;=\; \|f\|_{L^p([0,a],\ \xi^{\beta-1} d\xi)}, \qquad \|\tilde u\|_{L^p([0,\tilde a], d\xi)} \;=\; \|u\|_{L^p([0,a],\ \xi^{\beta-1} d\xi)}. \tag{1.50}$$

We then obtain

$$\begin{aligned}\tilde f(\tilde\xi) &= \int_0^\xi \eta^{\beta-1} e^{-t(\xi^\beta-\eta^\beta)/\beta} u(\eta)\, d\eta\\ &= \int_0^{\tilde\xi} e^{-t(\tilde\xi-\tilde\eta)}\tilde u(\tilde\eta)\, d\tilde\eta,\end{aligned}$$

and applying Young's inequality for convolutions,

$$\|k*\psi\|_{L^p(\mathbb{R},d\xi)} \le \|k\|_{L^1(\mathbb{R},d\xi)}\cdot\|\psi\|_{L^p(\mathbb{R},d\xi)}, \qquad k\in L^1(\mathbb{R},d\xi),\quad \psi\in L^p(\mathbb{R},d\xi),$$

with

$$k(s) = \begin{cases} e^{-ts}, & \text{if } s\in[0,\tilde a],\\ 0, & \text{if } s\notin[0,\tilde a],\end{cases}$$

and

$$\psi(\tilde\eta) = \begin{cases} \tilde u(\tilde\eta), & \text{if } \tilde\eta\in[0,\tilde a],\\ 0, & \text{if } \tilde\eta\notin[0,\tilde a],\end{cases}$$

yields for $t=0$

$$\|\tilde f\|_{L^p([0,\tilde a],d\xi)} \le \tilde a\|\tilde u\|_{L^p([0,\tilde a],d\xi)},$$

and this implies that V_1 is well-defined and in $\mathcal{L}(X)$, and (1.50) implies

$$\|V_1\|_{L^p([0,a],\xi^{\beta-1}\ d\xi)} \le \tilde a,$$

and obviously L inverts V_1. For $t>0$ we find

$$t\|\tilde f\|_{L^p([0,\tilde a],d\xi)} \le \|\tilde u\|_{L^p([0,\tilde a],d\xi)},$$

which in conjunction with (1.50) shows that that L is weakly sectorial with $M_0(L)=1$. The rest of the proof is similar to that of Theorem 1.3.1. □

It follows immediately from Theorems 1.3.1 and 1.3.2 together with Corollary 1.2.6 that V_1^α and V_2^α are strictly sectorial for any $0<\alpha<1$ (with respect to $X=C[0,a]$ or $X=L^p\big([0,a],\ \xi^{\beta-1}d\xi\big),\ 1\le p\le\infty$). In the following theorem these fractional powers are explicitly given, and in fact they are (generalized) Abel integral operators, with the classical case obtained for $\alpha=1/2,\ \beta=1$.

Theorem 1.3.3 *Let $\beta>0$. In $X=C[0,a]$ or $X=L^p\big([0,a],\ \xi^{\beta-1}d\xi\big),\ 1\le p\le\infty$, for the operators V_1,V_2, defined by (1.42), (1.43), respectively, one has for $0<\alpha<1$,*

$$\begin{aligned}(V_1^\alpha u)(\xi) &= \frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^\xi \frac{\eta^{\beta-1}u(\eta)}{(\xi^\beta-\eta^\beta)^{1-\alpha}}\, d\eta, \qquad u\in X,\quad \xi\in[0,a],\\ (V_2^\alpha u)(\xi) &= \frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_\xi^a \frac{\eta^{\beta-1}u(\eta)}{(\xi^\beta-\eta^\beta)^{1-\alpha}}\, d\eta, \qquad u\in X,\quad \xi\in[0,a],\end{aligned}$$

where Γ denotes the Gamma function. V_1^α and V_2^α are strictly sectorial (for $\mathbb{K}=\mathbb{C}$).

Proof. The inverse operator of V_1 is $(Lf)(\xi)=\xi^{-(\beta-1)}f'(\xi),\ f\in\mathcal{D}(L)$ (for the domain of definition of L see (1.45) and (1.49), respectively), hence one has

$$\begin{aligned}V_1^\alpha u &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{\alpha-1}(I+tL)^{-1}u\ dt\\ &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty s^{-\alpha}(L+sI)^{-1}u\ ds,\end{aligned}$$

therefore (see (1.48))

$$(V_1^\alpha u)(\xi) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty s^{-\alpha}\int_0^\xi e^{-s(\xi^\beta-\eta^\beta)/\beta}\cdot\eta^{\beta-1}u(\eta)\ d\eta\ ds$$

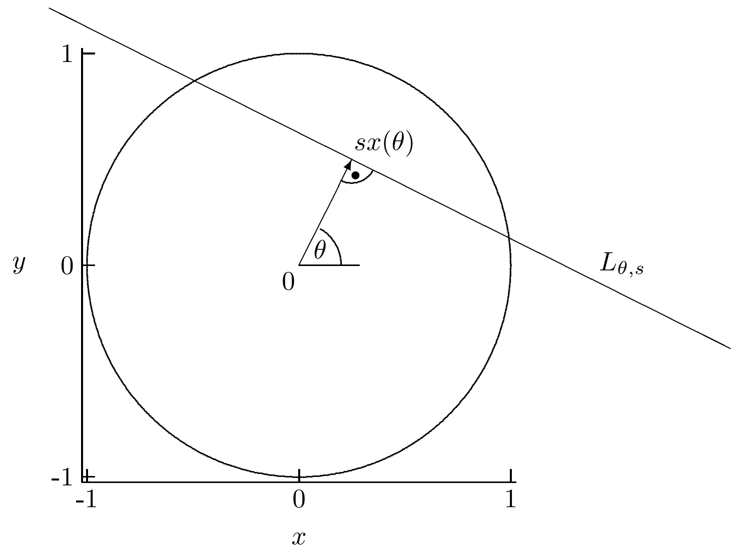


Abbildung 1.3: Illustration for the Radon transform

$$
\begin{aligned}
&= \frac{\sin\pi\alpha}{\pi}\int_0^\xi \Big(\int_0^\infty s^{-\alpha}e^{-s(\xi^\beta-\eta^\beta)/\beta}\,ds\Big)\cdot\eta^{\beta-1}u(\eta)\,d\eta\\
&= \frac{\sin\pi\alpha}{\pi}\int_0^\xi \Big(\frac{\beta}{\xi^\beta-\eta^\beta}\Big)^{1-\alpha}\Big(\int_0^\infty t^{-\alpha}e^{-t}\,dt\Big)\cdot\eta^{\beta-1}u(\eta)\,d\eta\\
&= \beta^{1-\alpha}\Big(\frac{\sin\pi\alpha}{\pi}\int_0^\infty t^{-\alpha}e^{-t}\,dt\Big)\cdot\int_0^\xi \frac{\eta^{\beta-1}u(\eta)}{(\xi^\beta-\eta^\beta)^{1-\alpha}}\,d\eta\\
&= \beta^{1-\alpha}\Big(\frac{\sin\pi\alpha}{\pi}\Gamma(1-\alpha)\Big)\int_0^\xi \frac{\eta^{\beta-1}u(\eta)}{(\xi^\beta-\eta^\beta)^{1-\alpha}}\,d\eta\\
&= \frac{\beta^{1-\alpha}}{\Gamma(\alpha)}\int_0^\xi \frac{\eta^{\beta-1}u(\eta)}{(\xi^\beta-\eta^\beta)^{1-\alpha}}\,d\eta,
\end{aligned}
$$

and exchanging the order of integration is justified by the integrability conditions on u. Finally, it follows from Theorems 1.3.1 and 1.3.2 as well as Corollary 1.2.6 that V_1^α and V_2^α are strictly sectorial. □

1.3.2 Two applications

The Radon transform for radial functions

The two-dimensional Radon transform $\mathbf{R}$ maps a function $\psi:\mathbb{R}^2\to\mathbb{R}$ into the set of integrals of ψ over the lines $L_{\theta,s},\ \theta\in[0,2\pi],\ s\ge 0$. Here

$$
\begin{aligned}
L_{\theta,s} &:= \big\{\ sx(\theta)+tx(\theta)^\perp:\ t\in\mathbb{R}\ \big\},\\
x(\theta) &:= (\cos\theta,\sin\theta)^T, \qquad x(\theta)^\perp := (-\sin\theta,\cos\theta)^T,
\end{aligned}
$$

and we have the situation as described in Figure 1.3. $\mathbf{R}f$ thus can be written in the form

$$
(\mathbf{R}\psi)(\theta,s) \;=\; \int_{L_{\varphi,s}}\psi(x)\ S(dx), \qquad \theta\in[0,2\pi],\quad s\ge 0,
$$

and the task is to recover ψ from $g=\mathbf{R}\psi$. If ψ has support in the closed unit disk $D:=\big\{\ x\in\mathbb{R}^2:\ |x|_2\le 1\ \big\}$ and it is moreover a radial function, i.e., for some function $u:[0,1]\to\mathbb{R}$ one has (with $|\cdot|_2$ denoting the Euklidian norm in $\mathbb{R}^2$)

$$
\psi(x) \;=\; u(|x|_2), \qquad x\in\mathbb{R}^2,
$$

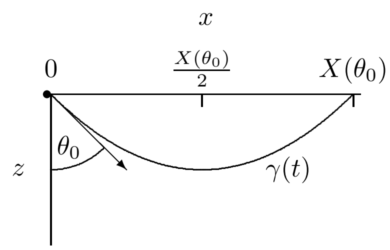


Abbildung 1.4: Situation for seismic travel times

(this is a realistic assumption for the spectroscopy of cylindrical gas discharges) then

$$\begin{aligned}(\mathbf{R}\psi)(\theta,s) &= \int_{L_{\theta,s}} \psi(x)\ S(dx) \ = \ \int_{L_{\theta,s}} u(|x|)\ S(dx) \\ &= \int_{-\infty}^{\infty} u\big((s^2+t^2)^{1/2}\big)\ dt \ = \ 2\int_0^{\infty} u\big((s^2+t^2)^{1/2}\big)\ dt \\ &= 2\int_s^{\infty} \frac{ru(r)}{(r^2-s^2)^{1/2}}\ dr \ = \ 2\int_s^1 \frac{ru(r)}{(r^2-s^2)^{1/2}}\ dr,\end{aligned}$$

and this implies that $g = \mathbf{R}\psi$ is also radial and has support in D, and if

$$f(s) := g(\theta,s), \qquad s \in [0,1],$$

then the resulting equation $Au = f$ is an Abel integral equation of the first kind (up to some constant factor).

Seismic imaging

We present a model for recovering characteristics of the sub-surface medium from seismic travel times: Here artificial seismic waves are sent with fixed initial veloscity v_0, and it is assumed that its scalar veloscity does not depend on the incident angle θ_0, i.e., it depends on the depth z only. If we denote the trajectory of the seismic wave by $\gamma(t) = \big(x(t), z(t)\big)$, then this assumption on v can be written as

$$v(z(t)) \ = \ |\dot\gamma(t)|_2 \ = \ \Big(\dot x(t)^2 + \dot z(t)^2\Big)^{1/2},$$

and the incident angle then is

$$\theta_0 \ = \ \measuredangle\Big((0,-1)^T, \dot\gamma(0)\Big),$$

see Figure 1.4 for an illustration of the situation. We introduce the ray parameter

$$p \ = \ \frac{\sin\theta_0}{v_0},$$

and denote with $X = X(p)$ and $T = T(p)$ location and time where and when the seismic ray reaches the surface again, respectively. In fact, this case happens only for $p \in [p_*, 1/v_0]$ with an appropriate $p_* > 0$. If w is reciprocal to v,

$$w(z) \ = \ 1/v(z),$$

if u is inverse to w, and if

$$f(p) := T(p) - pX(p), \qquad p \in [p_*, w_0],$$

then one can show that f and u are related via an Abel integral equation of the first kind,

$$\int_p^{w_0} \frac{wu(w)}{(w^2-p^2)^{1/2}}\ dw \ = \ \frac{1}{2} f(p), \qquad p \in [p_*, w_0],$$

where

$$w_0 = 1/v_0.$$

The formulation is slightly different from that in Section 1.3 since $p_* \neq 0$, but all assertions in that section remain valid if the origin $0 \in \mathbb{R}$ is substituted by some $c > 0$.

1.4 Appendix: Some converse results

1.4.1 Lower bounds for the speed, and saturation

If the origin 0 is accumulation point of the spectrum of A, then Lemma 1.1.8, Theorem 1.2.10 and Theorem 1.2.11 cannot be improved. We state this in the following proposition. Estimate (1.51) provides a saturation effect, i.e., speed cannot increase, if α exceed m; this holds also if 0 is not an accumulation point of the origin 0.

Proposition 1.4.1 *Assume that $A \in \mathcal{L}(X)$ is weakly sectorial and that there exist $0 \neq \lambda_j \in \sigma(A)$, $j = 0, 1, ...,$ with $\lambda_j \to 0$ as $j \to \infty$.*

(a) (Cf. Lemma 1.1.8) For integer α we have

$$\begin{aligned} \limsup_{t\to\infty} \|(I+tA)^{-m}A^\alpha\| \cdot t^\alpha &\geq \alpha^\alpha \frac{(m-\alpha)^{m-\alpha}}{m^m}, && \text{if } 1 \leq \alpha \leq m-1, \\ \limsup_{t\to\infty} \|(I+tA)^{-m}A^\alpha\| \cdot t^m &\geq r_\sigma(A)^{\alpha-m}, && \text{if } \alpha \geq m. \end{aligned} \tag{1.51}$$

Here,

$$r_\sigma(A) := \sup\{\ |\lambda| :\ \lambda \in \sigma(A)\ \}$$

is the spectral radius of A.

(b) (Cf. Theorem 1.2.10) We additionally assume that $\rho(-A) \supset \Sigma_{\pi/2+\varepsilon}$ for some $0 < \varepsilon < \pi/2$. For integers $\alpha \geq 1$,

$$\limsup_{t\to\infty} \|T(t)A^\alpha\| \cdot t^\alpha \ \geq\ \Big(\frac{\alpha}{e\sin\varepsilon}\Big)^\alpha$$

holds, where $\{T(t)\}_{t\geq 0}$ is the semigroup with infinitesimal generator $-A$.

(c) (Cf. Theorem 1.2.11) For fixed $\mu > 0$ and $\alpha \geq 1$,

$$\limsup_{n\to\infty} \|(I+\mu A)^{-n}A^\alpha\| \cdot n^\alpha \ \geq\ \Big(\frac{\alpha}{\mu e}\Big)^\alpha.$$

Proof. (a) The spectral theorem implies

$$\|(I+tA)^{-m}A^\alpha\| \cdot t^\alpha \ \geq\ \sup_{\lambda\in\sigma(A)} \Big|\frac{(\lambda t)^\alpha}{(1+t\lambda)^m}\Big| \ \geq\ \sup_{\lambda\in\sigma(A)} \frac{(|\lambda| t)^\alpha}{(1+t|\lambda|)^m}.$$

Take

$$t_j \ =\ \frac{\alpha}{(m-\alpha)|\lambda_j|}, \qquad j = 0, 1, ...,$$

to obtain

$$\|(I+t_jA)^{-m}A^\alpha\| \cdot t_j^\alpha \ \geq\ \alpha^\alpha \frac{(m-\alpha)^{m-\alpha}}{m^m}.$$

Similarly we obtain the second part of (a):

$$\begin{aligned} \|(I+tA)^{-m}A^\alpha\| \cdot t^m &\geq \sup_{\lambda\in\sigma(A)} \Big|\frac{\lambda^\alpha t^m}{(1+t\lambda)^m}\Big| \ \geq\ \sup_{\lambda\in\sigma(A)} \frac{|\lambda|^\alpha}{(t^{-1}+|\lambda|)^m} \\ &= \frac{r_\sigma(A)^\alpha}{(t^{-1}+r_\sigma(A))^m} \ \to\ r_\sigma(A)^{\alpha-m} \quad \text{as } t \to \infty. \end{aligned}$$

(b) We have $\sigma(A) \subset \Sigma_{\pi/2-\varepsilon} \cup \{0\}$, and the spectral theorem again implies

$$\begin{aligned} \|T(t)A^\alpha\| \cdot t^\alpha &\geq \sup_{\lambda\in\sigma(A)} e^{-t\mathrm{Re}\lambda}(|\lambda| t)^\alpha, \\ &\geq \sup_{\lambda\in\sigma(A)} e^{-t|\lambda|\sin\varepsilon}(|\lambda| t)^\alpha, \end{aligned}$$

and for

$$t_j \ =\ \frac{\alpha}{|\lambda_j|\sin\varepsilon}, \qquad j = 0, 1, ...,$$

we get

$$\|T(t_j)A^\alpha\| \cdot t_j^\alpha \;\geq\; \Big(\frac{\alpha}{e\sin\varepsilon}\Big)^\alpha.$$

(c) Here,

$$\|(I+\mu A)^{-n}A^\alpha\| \cdot n^\alpha \;\geq\; \sup_{\lambda\in\sigma(A)} \mu^{-\alpha}\frac{(n\mu|\lambda|)^\alpha}{(1+\mu|\lambda|)^n},$$

and for large j we choose n_j such that

$$\frac{\alpha}{n_j-\alpha+1} \;\leq\; \mu|\lambda_j| \;\leq\; \frac{\alpha}{n_j-\alpha}.$$

Then

$$\mu^{-\alpha}\frac{(n_j\mu|\lambda_j|)^\alpha}{(1+\mu|\lambda_j|)^{n_j}}\Big| \;\geq\; \mu^{-\alpha}\frac{\Big(\frac{n_j}{n_j-\alpha+1}\Big)^\alpha \alpha^\alpha}{\Big(1+\frac{\alpha}{n_j-\alpha}\Big)^{n_j}} \;\to\; \Big(\frac{\alpha}{\mu e}\Big)^\alpha \quad \text{as } j\to\infty. \quad \square$$

The situation in Proposition 1.4.1 does not apply to the Abel integral operators, since their spectrum consists of the the origin 0 only. For that case we state without proof the following result which provide (with $T = I - \mu A$ and $T = (I+\mu A)^{-1}$) converse results for the iterative methods, in the case $\alpha = 1$.

Theorem 1.4.2 *Let $T \in \mathcal{L}(X)$, $T \neq I$, with $\sigma(T) = \{1\}$. Then*

$$\liminf_{n\to\infty}\; n\|T^n - T^{n+1}\| \;\geq\; 1/96.$$

1.4.2 Converse results for the angle conditions

For operators A being not strictly sectorial we cannot expect the speed as stated in Theorems 1.2.10 and 1.2.11. This and more can be derived from the following example.

Example 1.4.3 On $X = C[0,1]$, the space of complex-valued continuous functions on [0,1], supplied with the max-norm $\|\cdot\|_\infty$, we consider the multiplication operator,

$$Au(s) \;=\; a(s)u(s), \qquad s\in[0,1],$$

where a is assumed to be a complex-valued continuous function. In the first part of this example we derive necessary and sufficient conditions for A to be weakly sectorial or strictly sectorial. For that we note that for $0 < \varepsilon \leq \pi/2$,

$$\begin{aligned} \operatorname{dist}(0, 1+\Sigma_{\pi/2+\varepsilon}) &= \cos\varepsilon, \\ \operatorname{dist}(0, 1+\Sigma_{\pi/2-\varepsilon}) &= 1. \end{aligned} \tag{1.52}$$

It is easy to see that the following properties are valid.

(a) A is weakly sectorial if and only if there is an $0 < \varepsilon \leq \pi/2$ with

$$a(s) \;\in\; \Sigma_{\pi/2+\varepsilon}\cup\{0\} \qquad \text{for } \; s\in[0,1]. \tag{1.53}$$

To this end, take any $M_0 \geq 1$, and define $\varepsilon = \arccos(1/M_0)$. A is weakly sectorial with bound M_0 in (1.2), if and only if

$$|1+ta(s)| \;\geq\; \cos\varepsilon \;=\; 1/M_0 \qquad \text{for } s\in[0,1], \quad t>0.$$

and it follows from (1.52) that this is equivalent to (1.53).

(b) Let $0 < \varepsilon_0 < \pi/2$ be fixed. Then: A is strictly sectorial if and only if

$$a(s) \;\in\; \sigma_{\pi/2-\varepsilon_0}\cup\{0\} \qquad \text{for } s\in[0,1],$$

and then (1.3)-(1.4) holds for any $0 < \varepsilon_1 < \varepsilon_0$ (with $M_0 = 1/\sin(\varepsilon_0-\varepsilon_1)$).

In the second part of this example we consider several concrete choices for the function a.

(a) Take e.g. $a(s) = s$ to see that the constants in Proposition 1.4.1 cannot be improved.

(b) Here is an example showing that (for fixed α) of an operator being not strictly sectorial and where we do not have the rates $\mathcal{O}(n^{-\alpha})$ as in Theorem 1.2.11. For that we consider an operator A

$$\rho(-A) \supset \Sigma_{\pi/2},$$

but which is not strictly sectorial since for any $0 < \varepsilon \le \pi/2$,

$$\rho(-A) \not\supset \Sigma_{\pi/2+\varepsilon}.$$

Take e.g. for arbitrary $1/2 \le \tau < 1$,

$$a(s) \;=\; s^\tau e^{i\beta(s)}, \qquad s \in [0,1],$$

where β is continuous with $0 \le \beta(s) \le \pi/2$. $\beta(s)$ is uniquely determined if we moreover require that

$$|1 + \mu s^\tau e^{i\beta(s)}| \;=\; 1 + \mu s, \qquad s \in [0,1],$$

(here we need that $1/2 \le \tau$). Note that here

$$\rho(-A) \supset \Sigma_{\pi/2},$$

but for any $0 < \varepsilon \le \pi/2$,

$$\rho(-A) \not\supset \Sigma_{\pi/2+\varepsilon},$$

hence A cannot be strictly sectorial. For n large we have,

$$\|(I + \mu A)^{-n} A^\alpha\| \;=\; \sup_{0 \le s \le 1} s^{\tau\alpha} |1 + \mu a(s)|^{-n} \;\sim\; \Big(\frac{\tau\alpha}{\mu e}\Big)^{\tau\alpha} \cdot n^{-\tau\alpha} \qquad \text{as } \; n \to \infty.$$

(c) Note, however, that the exponent for n^{-1} cannot be smaller then $\alpha/2$ for this example. This observation on the speed of convergence is no accident, however; to demonstrate this we consider the multiplication operator A with

$$a(s) \;=\; is, \qquad s \in [0,1],$$

and then even

$$\rho(-A) \not\supset \Sigma_{\pi/2}.$$

We observe that for large n,

$$\|(I + \mu A)^{-n} A^\alpha\| \;=\; \sup_{0 \le s \le 1} s^{\alpha} (1 + \mu^2 s^2)^{-n/2} \;\sim\; \Big(\frac{\alpha}{\mu^2 e}\Big)^{\alpha/2} \cdot n^{-\alpha/2} \qquad \text{as } \; n \to \infty,$$

so that the speed of convergence again is not arbitrarily slow (for fixed α). The reason for that is the following: $T = (I + \mu A)^{-1}$ can be written as nontrivial convex combination of the identity operator I and the power bounded operator $(I + \mu A)^{-1}(I - \mu A)$, such that lower speed then is not possible.

Bibliographical notes and remarks.

Chapter 1.1 Weakly sectorial operaters are also called 'weakly positive' in Pustyl'nik [61], and they are close to the 'sectorial operaters' defined in Prüß [60]. Theorem 1.1.5 can be found e.g. in Fattorini [16], our presentation is very similar to that in Blank [8]. The interpolation inequality 1.1.9 for unbounded operators having a possibly unbounded inverse can be found in Schock and Phong [66], our proof of 1.1.9 is similar to that in Komatsu [35]. Theorem 1.1.10 is an extension of a result in Plato [55], similar results can be found in Shaw [67]. Orthogonal subspaces in Banach spaces are studied by Birkhoff [7] and James [31]. There exist a wide literatur on generalized inverses, we refer to Nashed [48] as a general reference. Our approach to introduce a generalized inverse is slightly different, and it is taken from Plato and Hämarik [58].

Chapter 1.2 Balakrishnan ([4],[5]) and Kato ([33]), were one of the first who studied fractional powers of (unbounded) operators, and Theorems 1.1.24 and 1.2.5 in fact can be found in [33]. Monographs containing these two theorems are, e.g., Krein [36] and Tanabe [69].

Lemma 1.2.7 is Pazy [53], Lemma 8.2.1 and its proof. Theorem 1.2.10 and its proof also can be found in [53]: The case $\alpha = 0$ is [53], Theorem 1.7.7 and its proof, and the case $\alpha = 1$ is [53], Theorem 2.5.2 and its proof. For the Hille-Yosida theorem and its proof see e.g. [53], Theorem 1.5.3. (1.33) in Theorem 1.2.11 can be derived from the proof of [53], Theorem 1.7.7 (the assumption $0 \in \rho(A)$ there is not needed in our case), and (1.33) (1.34) in Theorem 1.2.11 generalizes (for bounded operators) [53], Theorem 2.5.5, where one has the restriction $\alpha = 1$. For possible generalizations of Theorem 1.2.11 to rational functions see Lubich and Nevanlinna [44].

Theorem 1.2.12 is due to Nevanlinna [52], Theorems 4.5.4 and 4.9.3. Related results are the Kreiss Matrix theorem, see, e.g., J.L. van Dorsselaer, J.F. Kraaijevanger and M.N. Spijker [74] for a recent survey on that and other topics, as well as the Katznelson-Tzafriri theorem [34] which provide under weaker resolvent assumptions weaker results.

Theorem 1.4.2 is due to Esterle [14], Corollary 9.5, and Zemánek [75] mentions that 1/96 in fact can be reduced to 1/12 (due to Berkani [6]).

Chapter 1.3 Many problems in applied physics, where Abel integral equations of the first kind arise can be found in Gorenflo and Vessella [19]. There also range conditions for Abel integral operators are given. Monographs including results on the numerical solution of Volterra integral equations of the first kind via discretization are Linz [41], Brunner and Houwen [11], Gripenberg, Londen and Staffans [21], to mention some of them. Resolvent conditions for Abel integral operators I only found in Gohberg and Krein [18], for the L^2 case and in terms of the numerical range. For Young's inequality for convolutions see Reed and Simon [64]. The examples in Subsection 1.3.2 are taken from Gorenflo and Vessella [19]. Finally, Malamud [45] consider operators which are similar to Abel integral operators (in L^p-spaces) and this provides a larger class of strictly sectorial operators (compare Remark 1.1.4)

Chapter 1.4 For the last observation on the lower speed see Nevanlinna [52], Theorem 4.5.3.

Kapitel 2

Linear approximation methods for ill-posed problems

2.1 A class of parameter dependent methods

Let X be a real or complex Banach space, and let $A \in \mathcal{B}(X)$ and $f_* \in \mathcal{R}(A)$. We again consider the equation (1.1),

$$Au = f_*$$

which may be ill-posed or ill-conditioned in general so that small perturbations of the right-hand side can lead to large deviations in the searched-for solution of the problem. We now look at some of parameter dependent methods, which for some approximation $f \in X$ for the right-hand side f_* in (1.1),

$$f \approx f_*,$$

generates

$$u_t := u_0 - G_t(Au_0 - f), \qquad t \geq 0, \tag{2.1}$$

with some initial guess $u_0 \in X$, and with $G_t \in \mathcal{L}(X)$ to be specified. Those methods are called linear, since $u_t - u_*$ depends linear on $Au_0 - f$ (which is not the case for conjugate gradient type methods in Hilbert spaces; one of these methods is considered in Chapter 5).

Any of these methods is designed to approximate a solution $f_* \in X$ of (1.1) as $t \to \infty$, if the exact right-hand side is available, i.e., $u_t^* \to f_*$ as $t \to \infty$, where u_t^* denote the family generated by (2.1) with $f = f_*$. With the notation

$$H_t := I - G_t A$$

this means that we expect $u_* - u_t^* = H_t(f_* - u_0) \to 0$ as $t \to \infty$ to hold. For all the methods to be considered we shall see that, under appropriate assumptions on A, for a typical smoothness assumption like '$u_0 - u_* \in \mathcal{R}(A^\alpha)$' we can expect some speed for the decay of $u_* - u_t^*$ as $t \to \infty$, since for some $0 < \alpha_0 \leq \infty$,

$$\|H_t A^\alpha\| \;\leq\; \gamma_\alpha t^{-\alpha} \qquad \text{for} \;\; t > 0, \;\; 0 < \alpha \leq \alpha_0 \tag{2.2}$$

($0 < \alpha < \infty$, if $\alpha_0 = \infty$) does hold.

Definition 2.1.1 *For any method of type (2.1), $0 < \alpha_0 \leq \infty$ is called the qualification of the method, if it is the largest number such that (2.2) does hold.*

Any method to be considered has qualification $\alpha_0 \geq 1$, and then (2.2) and the principle of uniform boundedness implies that for $u \in \overline{\mathcal{R}(A)}$,

$$\|H_t u\| \to 0 \qquad \text{as} \;\; t \to \infty.$$

This means in particular that

$$\lim_{t\to\infty} G_t \psi \;=\; A^{-1}\psi \qquad \text{for all} \;\; \psi \in A(\overline{\mathcal{R}(A)}),$$

if $\mathcal{N}(A) = \{0\}$. If (1.1) is ill-posed, then $\|G_t\| \to \infty$ as $t \to \infty$, and then

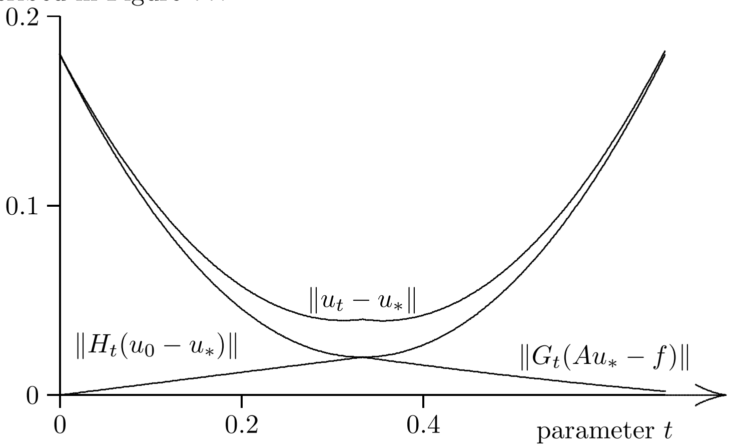
$$\{ \ \psi \in X : \ \|G_t\psi\| \to \infty \quad \text{as } t \to \infty \ \}$$

is a set of second category in X.

We return to the case where perturbed right-hand side is permitted, i.e., assume that $f \approx f_*$. Observe that if u_* solves (1.1), then

$$u_t - u_* \ = \ H_t(u_0 - u_*) - G_t(Au_* - f).$$

Since we assume the underlying problem to be ill-posed we can expect $\|G_t(Au_* - f)\| \to \infty$ as $t \to \infty$ and then we have have only *semiconvergence* in general, this is, the error $\|u_t - u_*\|$ decreases as t increases up to some point, and then the error begins to to increase after t exceed this point. This typical situation is described in Figure **??**.



We will see that for any method to be considered, $\|G_t\|$ does not grow too fast,

$$\|G_t\| \ \le \ \gamma_* t, \qquad t > 0, \tag{2.3}$$

with some constant $\gamma_* > 0$.

For the discrepancy principle to be defined in the following section, it is shown by means of (2.2) and (2.3) that the chosen parameter is not too large. On the other hand, the parameter t_δ found by the parameter choice strategies to be defined can be very small. In that case we use the following condition,

$$\|H_t u\| \ \le \ \kappa\Big(\|H_{t_*}u\| + t_*\|AH_t u\|\Big) \qquad \text{for } \ 0 \le t \le t_*, \ \ \text{and for } \ u \in X, \tag{2.4}$$

with some constant $\kappa > 0$, and which is fulfilled by all examples to be considered. Actually, we can show the stronger estimate

$$\|H_t u - H_{t_*}u\| \ \le \ \kappa(t_* - t)\|AH_t u\| \qquad \text{for } 0 \le t \le t_*, \ \ \text{and for } u \in X,$$

but (2.4) is sufficiently strong to prove convergence results.

These three conditions (2.2)-(2.4) are the ingredients for the main Theorem 3.2.3. We will also see that

$$G_t A = AG_t, \qquad t \ge 0, \tag{2.5}$$

is also fulfilled for any method to be considered.

2.1.1 The iterated method of Lavrentiev

For weakly sectorial operators A (see (1.2) for the definition) we first consider the (iterated) method of Lavrentiev,

$$(I + tA)v_{n+1} \ = \ v_n + tf, \qquad n = 0, 1, 2, ..., m - 1, \tag{2.6}$$

$$u_t := \ v_m, \tag{2.7}$$

with $v_0 = u_0$ and fixed integer $m \ge 1$.

Theorem 2.1.2 *Let $A \in \mathcal{L}(X)$ be weakly sectorial. u_t, defined by (2.6), (2.7), is of the form (2.1), with*

$$G_t \;=\; t\sum_{j=1}^{m}(I+tA)^{-j},$$

and fulfill (2.2)-(2.5)with $H_t = (I+tA)^{-m}$, and with finite qualification $\alpha_0 = m$.

Proof. Elementary calculations show that u_t in (2.7) is of the form (2.1) with G_t and H_t as claimed. We then obviously have (2.3) with

$$\gamma_* := \begin{cases} (M_0^m - 1)/(1 - M_0^{-1}), & \text{if } M_0 > 1, \\ m, & \text{if } M_0 = 1, \end{cases}$$

and it follows with (1.6) that (2.2) holds with $\alpha_0 = m$, and we obviously can choose

$$\gamma_0 = M_0^m$$

there. We show finally (2.8) and (2.9) what implies (2.4) for the (iterated) method of Lavrentiev. For t and t_* with $0 < t \le t_*$ we have

$$\begin{aligned} H_t - H_{t_*} &= (I+tA)^{-m} - (I+t_*A)^{-m} \\ &= H_t\Big((I+t_*A)^m - (I+tA)^m\Big)(I+t_*A)^{-m} \\ &= (t_*-t)H_tA\Big(\sum_{j=0}^{m-1}(I+t_*A)^{m-1-j}(I+tA)^j\Big)(I+t_*A)^{-m} \\ &= (t_*-t)H_tA\Big(\sum_{j=0}^{m-1}(I+t_*A)^{-j-1}(I+tA)^j\Big) \\ &= (t_*-t)H_tA(I+t_*A)^{-1}\sum_{j=0}^{m-1}\Big((I+t_*A)^{-1}(I+tA)\Big)^j. \end{aligned} \tag{2.8}$$

Moreover, $(I+t_*A)^{-1}(I+tA)$ is uniformly bounded for t and t_* with $0 < t \le t_*$, since for $\omega := t/t_*$,

$$(I+t_*A)^{-1}(I+tA) \;=\; \omega I \;+\; (1-\omega)(I+t_*A)^{-1}$$

and thus

$$\|(I+t_*A)^{-1}(I+tA)\| \;\le\; M_0. \tag{2.9}$$

As already mentioned, (2.8), (2.9) imply that (2.4) holds. □

We can see from the relations (2.6), (2.7) that for the iterated method of Lavrentiev, instead of (2.4) we have a stronger estimate. Since this observation and (2.7) will be used to prove results for a parameter choice strategy, we state the following corollary.

Corollary 2.1.3 *If A is weakly sectorial (with M_0 in (1.2)), then with $H_t = (I+tA)^{-m}$, and for $0 \le t \le t_*$, we have*

$$\begin{aligned} \|H_{t_*}H_t^{-1}\| &\le M_0^m, \\ \|H_tu - H_{t_*}u\| &\le C(t_*-t)\|(I+tA)^{-1}AH_tu\|, \qquad u \in X, \end{aligned}$$

where

$$C := \begin{cases} (M_0^m - 1)/(1 - M_0^{-1}), & \text{if } M_0 > 1, \\ m, & \text{if } M_0 = 1. \end{cases}$$

We conclude this subsection with the observation that here we may consider also real spaces X. If X is a complex space and A is strictly sectorial, however, then we can use Theorem 1.2.11 to find a γ_0 which may be significantly smaller than M_0^m.

2.1.2 Cauchy's method

For an operator fulfilling (1.3) and (1.4), and for initial approximation $u_0 \in X$ let us consider the abstract Cauchy problem,

$$\begin{aligned} u'(t) + Au(t) &= f, \qquad t > 0, && (2.10)\\ u(0) &= u_0. && (2.11) \end{aligned}$$

Its solution $u(t),\ t \geq 0$, exists for all times $t \geq 0$, and can be written as (2.1) if we use the notation

$$u_t = u(t), \tag{2.12}$$

see the following theorem.

Theorem 2.1.4 *Let $A \in \mathcal{L}(X)$ be weakly sectorial. u_t, defined by (2.10)-(2.12), is of the form (2.1), with*

$$G_t\psi = \int_0^t e^{-sA}\psi\, ds, \qquad \psi \in X, \tag{2.13}$$

and fulfill (2.2)-(2.5)with $H_t = e^{-tA}$, the uniformly continuous semigroup with infinitesimal generator $-A$, and with qualification $\alpha_0 = \infty$.

Proof. Elementary calculations show that u_t is of the form (2.1) with G_t and H_t as above. The decay property (2.2) is a consequence of Theorem 1.2.10. The semigroup e^{-sA} is uniformly bounded by γ_0, in particular, and then $\|G_t\| \leq \gamma_0 t$, i.e., (2.3) holds with $\gamma_* = \gamma_0$, which follows immediately from (2.13). To see that the third condition (2.4) is fulfilled, we observe that for $0 < t < t_*,\ u \in X$,

$$(H_t - H_{t_*})u \;=\; -\int_t^{t^*} (\frac{d}{ds}H_s)u\, ds \;=\; \int_t^{t^*} AH_s u\, ds \;=\; \int_t^{t^*} H_{s-t}AH_t u\, ds,$$

hence

$$\|(H_t - H_{t_*})u\| \;\leq\; \gamma_0(t_* - t)\|AH_t u\|,$$

which is the desired result. $\Box$

We call this method Cauchy's method. It is of interest since the iterated method of Lavrentiev can be understood as the backward Euler scheme, with fixed number of timesteps m and varying stepsize; moreover, also the iteration methods considered in the following Section 2.2 can be understood as discretized variant of Cauchy' method.

Finally we give an equivalent formulation for this method. To this end, let $N > 0$ be a fixed real number, consider for (large) parameter $t > 0$ and $\epsilon := 1/t$ the solution of the problem

$$\begin{aligned} \epsilon u'(s) \;+\; Au(s) &= f, \qquad s > 0,\\ u(0) &= u_0, \end{aligned}$$

and take $u_t = u(N)$ as approximation for the solution of (1.1). It is easy to see that

$$u_t := u_0 - \tilde{G}_t(Au_0 - f),$$

with

$$\begin{aligned} \tilde{G}_t\psi \;&:=\; t\int_0^N e^{-stA}\psi\, ds \;=\; \int_0^{Nt} e^{-sA}\psi\, ds\\ &=\; G_{Nt}\psi, \end{aligned}$$

with G_t as for Cauchy's method.

2.2 A class of iterative methods

Let again X be a real or complex Banach space. In order to solve equation (1.1) with only approximately given right-hand side $f \in X$, we consider methods which for inital guess $u_0 \in X$ generate sequences

$$u_n := u_0 - G_n(Au_0 - f) \qquad \text{for } n = 0, 1, 2, ..., \tag{2.14}$$

(with $G_n \in \mathcal{L}(X)$). Our main subject of methods of type (2.14) are of iterative type. When it comes to the implementation of Lavrentiev's (iterated) method, however, then the corresponding family $\{u_t\}$ will be evaluated for discrete values of t only, and we shall see in Section 2.6 how this does fit into our framework (2.14). For that it is convenient to state most general assumptions on $\{G_n\}$; let $F(n),\ n = 0, 1, ...$ be a real-valued function with the following properties:

$$\begin{aligned} F(0) &= 0, \\ F(n) &\geq F(n-1), \qquad n \geq 1, \\ F(n) &\leq \varkappa F(n-1), \qquad n \geq 2, \\ F(n) &\to \infty \qquad \text{as } n \to \infty, \end{aligned}$$

for some $\varkappa > 0$. Analogously to the previous subsection we define

$$H_n := I - G_n A$$

and assume that for some fixed $\alpha_0 > 0$

$$\begin{aligned} \|H_n A^\alpha\| &\leq \gamma_\alpha F(n)^{-\alpha} \qquad \text{for } n = 1, 2, ..., \quad 0 \leq \alpha \leq \alpha_0, & (2.15) \\ \|G_n\| &\leq \gamma_* F(n) \qquad \text{for } n = 0, 1, 2, ..., & (2.16) \\ \|H_n u\| &\leq \kappa\big(\|H_{n_*} u\| + F(n_*)\|AH_n u\|\big) & \\ & \qquad \text{for } n,\ n_* = 0, 1, 2, ...,\ n \leq n_*, \quad \text{and for } u \in X. & (2.17) \\ G_n A &= AG_n \qquad \text{for } n = 0, 1, 2, ..., & (2.18) \end{aligned}$$

(with $0 < \alpha$ in (2.15), if $\alpha_0 = \infty$). The function F is responsible for the speed of the method (in the case of exact data); for the stationary methods to be considered in this section one always has $F(n) = n$, and then (2.15)-(2.18) are identical with the four conditions (2.2)-(2.5) for the parameter methods (with $t = n$).

In order to look at some concrete methods belonging to the class above (with $F(n) = n,\ n = 0, 1, ...$), in the following three subsections we assume that $A \in \mathcal{B}(X)$ is strictly sectorial. The terminolgy 'qualification of a method' is used like for parameter methods.

2.2.1 The Richardson iteration

We now consider the Richardson iteration,

$$u_{n+1} = u_n - \mu(Au_n - f), \qquad n = 0, 1, 2, ..., \tag{2.19}$$

which is also called Landweber iteration if X is a Hilbert space and the equation $Au = f$ results from a normalization.

Theorem 1.2.12 enables us to show that the Richardson iteration (2.19), for $\mu > 0$ small enough, belongs to the general class of methods (with $F(n) = n,\ n = 0, 1, ...,$ and $\alpha_0 = \infty$).

Theorem 2.2.1 *For strictly sectorial $A \in \mathcal{L}(X)$, take any*

$$0 < \mu \leq (2 \sin \epsilon)/\|A\|, \tag{2.20}$$

and let $L := I - \mu A$. Then u_n, defined by (2.19), is of the form (2.14), with

$$G_n = \mu \sum_{j=0}^{n-1} L^j,$$

and fulfill (2.15)-(2.18)with $H_n = (I - \mu L)^n$, with $F(n) = n,\ n = 0, 1, ...,$ and qualification $\alpha_0 = \infty$.

Proof. A is sectorial of angle $\pi/2+\epsilon$ for some $0<\epsilon$, and then one can see that (1.39) is fulfilled by $I-\mu A$. The condition $0<\mu\le(2\sin\epsilon)/\|A\|$, guarantees $\sigma(I-\mu A)\subset\{\lambda\in\mathbb{C}: \ |\lambda|<1\}\cup\{1\}$, since we have $r_\sigma(T)\le\|T\|$ for any $T\in\mathcal{L}(X)$; thus (1.38) in Theorem 1.2.12 is fullfilled by $T=I-\mu A$, and then (2.15) holds with $\gamma_*=\mu a_0$, and (2.16) holds for $F(n)=n,\ n=0,1,...,$ and qualification $\alpha_0=\infty$ (with $\gamma_\alpha=a_\alpha\mu^{-\alpha}$ for integer α). The general case for fractional $\alpha>0$ follows with the second interpolation inequality (1.15). Finally, one has for $n\le n_*$

$$H_n \;=\; \mu\sum_{j=n}^{n_*-1}L^jA+H_{n_*}$$

which implies

$$\|H_nu-H_{n_*}u\| \;\le\; \kappa(n_*-n)\|H_nAu\|$$

with $\kappa=\mu a_0$, i.e., (2.17) holds with $F(n)=n,\ n=0,1,\dots$. □

Remark. For Abel integral operators $V_j^\alpha,\ j\in\{0,1\},\ 0<\alpha<1$, we do not need the restriction (2.20) on μ: $\sigma(V_j^\alpha)=\{0\}$ and thus, somewhat surprising, $I-\mu V_j^\alpha$ is power bounded for any $\mu>0$ (and any $j\in\{0,1\},\ 0<\alpha<1$).

2.2.2 An implicit iteration method

For strictly sectorial operators A we now consider the implicit method (2.21) which for $u_0\in X$ generates iteratively a sequence $u_n\in X,\quad n=0,1,2,...,$ by

$$(I+\mu A)u_{n+1} \;=\; u_n+\mu f,\qquad n=0,1,2,\dots . \tag{2.21}$$

Theorem 2.2.2 *Let $A\in\mathcal{L}(X)$ be strictly sectorial and define $L:=(I+\mu A)^{-1}$. Then u_n, defined by (2.21), is of the form (2.14), with*

$$G_n=\mu\sum_{j=1}^{n}L^j,$$

and fulfill (2.15)-(2.18)with $H_n=L^{-n}$, with $F(n)=n,\ n=0,1,...,$ and qualification $\alpha_0=\infty$.

Proof. It is possible to obtain (2.15) and (2.16) for the implicit method by applying again Theorem 1.2.12. More natural, however, is to use Theorem 1.2.11. It enables us also to give a reasonable estimate for $\sup_{n\ge0}\|L^n\|$. Finally, properties (2.15) and (2.16) for the implicit method are immediate consequences of Theorem 1.2.11, and condition (2.17) is verified as for the Richardson iteration. □

We conclude this subsection with the following observation: 'Stability' of the Richardson iteration with respect to some $\mu_0>0$ implies stability of the implicit method.

Proposition 2.2.3 *If $A\in\mathcal{L}(X)$ and for given $\mu_0>0$ the operator $I-\mu_0A$ is nonexpansive,*

$$\|I-\mu_0A\| \;\le\; 1,$$

then for any $0\le\mu\le\mu_0$ the operator $I-\mu A$ is nonexpansive and furthermore A is weakly sectorial with bound $M=1$,

$$\|(I+tA)^{-1}\| \;\le\; 1 \qquad for\ all\ t\ge0.$$

Proof. Let

$$T_\omega:=I-\omega\mu_0A.$$

Then $T_\omega=(1-\omega)I+T_1$ and hence $\|T_\omega\|\le1$ for $0\le\omega\le1$. Furthermore, for any $t>0$ and $u\in X$ with $s=\frac{t}{\mu_0}$ the inequality

$$\begin{aligned}\|(I+tA)u\| \;&=\; \|(1+s)u-s(I-\mu_0A)u\| \;\ge\; (1+s)\|u\|-s\|(I-\mu_0A)u\|\\ &\ge\; (1+s)\|u\|-s\|u\| \;=\; \|u\|\end{aligned} \tag{2.22}$$

holds. Hence, for arbitrary $\lambda>\|A\|$ (then $\lambda\in\rho(-A)$) one has $\|(\lambda I+A)^{-1}\|\le1/\lambda$. Then, however, one has for any $\mu\in\mathbb{K}$ with

$$|\mu-\lambda| \;<\; \lambda \;\le\; \|(\lambda I+A)^{-1}\|^{-1}$$

that $\mu\in\rho(-A)$, and thus in particular for any $0<\mu\le\lambda$ one has $\mu\in\rho(-A)$, and (2.22) yields the assertion. □

Corollary 2.2.4 *If $A \in \mathcal{L}(X)$ and for given $\mu_0 > 0$ the operator $I - \mu_0 A$ is power bounded,*

$$\sup_{n\geq 0} \|(I-\mu_0 A)^n\| \ \leq \ M < \infty,$$

then for any $0 \leq \mu \leq \mu_0$ the operator $I-\mu A$ is power bounded. Moreover, $(0,\infty) \subset \rho(-A)$ and $(I+tA)^{-1}$ is power bounded for any $t > 0$.

Proof. We define

$$\|u\|' := \sup_{n\geq 0} \|(I-\mu_0 A)^n u\|, \qquad u \in X,$$

and this norm is equivalent to the original one,

$$\|u\| \ \leq \ \|u\|' \ \leq \ M\|u\|, \qquad u \in X,$$

and $I - \mu_0 A$ is nonexpansive with respect to $\|\cdot\|'$. The rest follows with Proposition 2.2.3. □

Remark. The converse direction in Corollary 2.2.4 does not hold; consider e.g., multiplication operators with purely imaginary spectrum.

2.2.3 An iteration method with alternating directions (ADI)

This method is defined by

$$u_{n+1/2} \ = \ u_n - \frac{\mu}{2}(Au_n - f), \tag{2.23}$$

$$(I + \frac{\mu}{2}A)u_{n+1} \ = \ u_{n+1/2} \ + \ \frac{\mu}{2}f, \qquad n = 0,1,2,... \tag{2.24}$$

Theorem 2.2.5 *Let $A \in \mathcal{L}(X)$ be strictly sectorial. Then, u_n, defined by (2.23), (2.24), is of the form (2.14), with*

$$G_n \ = \ \mu \sum_{j=0}^{n-1} L^j (I + \frac{\mu}{2}A)^{-1},$$

and fulfill (2.15)-(2.18)with $H_n = L^n$, with

$$L := (I + \frac{\mu}{2}A)^{-1}(I - \frac{\mu}{2}A),$$

and with $F(n) = n, \ n = 0,1,...,$ and qualification $\alpha_0 = \infty$.

Proof. We again apply Theorem 1.2.12 to obtain (2.15), (2.16). For $\lambda \in \mathbb{C}, \ \lambda \neq -1$, we have

$$\lambda I - L \ = \ (\lambda+1)\frac{\mu}{2}(I + \frac{\mu}{2}A)^{-1}\Big(\frac{2}{\mu}\frac{\lambda-1}{\lambda+1}I + A\Big). \tag{2.25}$$

Elementary calculations show that $\lambda \in 1 + \Sigma_\epsilon$ implies $2(\lambda-1)/(\mu(\lambda+1)) \in \Sigma_\epsilon$, with (2.25) we obtain (1.38), (1.39) with $C = \|I + (\mu/2)A\|M$. Since the transformation $\lambda \mapsto (\lambda-1)/(\lambda+1) = \omega$ maps the exterior of the open unit disk in the λ-plane into Re $\omega \geq 0$, also (1.37) holds and L fulfils the conditions of Theorem 1.2.12. We obtain (2.15) with $\gamma_* = \mu a_0 M$, and moreover (2.16) hold (with $\gamma_\alpha = a_\alpha \mu^{-\alpha}\|(I + (\mu/2)A)^\alpha\|$ for integer α. Condition (2.17) is verified as for the Richardson iteration. □

Note that the Richardson iteration and the implicit method as well as the alternating direction method considered in this section can be conceived as forward and backward Euler schemes as well as the Crank-Nicolson scheme for Cauchy's method, the stepsize μ being fixed for each iteration method.

2.2.4 On faster linear methods

There exist linear semiiterative methods which are faster in the sense that they fulfill (2.15)-(2.18) with $F(n) = n^\tau, \ n = 0,1,...,$ with some $\tau > 1$. In Hilbert spaces, (selfadjoint) ν-methods fulfill these four conditions with $\tau = 2$ and finite qualification $\alpha_0 = \nu$.

2.2.5 Practical implementations of Lavrentiev's (iterated) method.

When it comes to computational implementations of Lavrentiev's (iterated) method, then the family of type (2.1), i.e.,

$$u_t \;=\; u_0 - G_t(Au_0 - f), \qquad t \geq 0,$$

$\big($with $G_t = t\sum_{j=1}^m (I + tA)^{-j}$ $\big)$ will be certainly evaluated only for a finite numbers of parameters t, e.g.,

$$\begin{aligned} t_n &= n^\tau h, \qquad n \geq 1, \qquad \text{or} \\ t_n &= h\theta^{-n}, \qquad n \geq 1, \end{aligned}$$

respectively (with certain $h,\ \tau > 0$ and $0 < \theta < 1$), and u_{t_n} then can be written in the form

$$\begin{aligned} u_{t_n} &= u_0 - \tilde{G}_n(Au_0 - f) \\ &=: \tilde{u}_n, \end{aligned}$$

with

$$\tilde{G}_n := G_{t_n},$$

and it is obvious then that $\tilde{u}_n$ is of the form (2.14), and that the main conditions (2.15)-(2.18)are fulfilled with $\tilde{G}_n$ in place of G_n, with $\alpha_0 = m$, and with

$$\begin{aligned} F(n) &= n,^\tau \qquad n \geq 1, \qquad \text{or} \\ F(n) &= \theta^{-n}, \qquad n \geq 1, \end{aligned}$$

respectively

Bibliographical notes and remarks

Most of the material in this chapter is taken from Plato [57]. The first two conditions (2.2), (2.5) for the parameter dependent methods and (2.15), (2.18) for the iterative methods, respectively, are standard assumptions, see e.g. Louis [43] (for the non-selfadjoint case). The third conditions (2.4) and (2.17), respectively, are generalizations of an approach in Plato and Vainikko [59].

Early results on solvers for ill-posed problems in Banach spaces (with exact given right-hand side) using resolvent conditions can be found in Bakushinskiĭ [2]. Other appraoches for regularization methods with respect to maximum norms can be found in Groetsch [22], Speckert [68], and in Engl and Hodina [13],

Chapter 2.1 The terminology 'qualification of a method' is due to Vainikko and Veretennikov [73]. The semiconvergence effect is described in Natterer [49].

Chapter 2.2 Proposition 2.2.3 is similar to Browder [10]. The implicit method (2.21) is considered e.g. by Riley [65] and by Fakeev [15] for the finite-dimensional case and more general by Kryanev [38]. For ν-methods see Brakhage [9] and Hanke [23].

Kapitel 3

Parameter choices and stopping rules for linear methods

3.1 Introduction

Throughout this chapter let X be a real or complex Banach space, and let $A \in \mathcal{L}(X)$. We assume that some approximation $f^\delta \in X$ for the exact right-hand side f_* in (1.1) is given, with some known level of noise $\delta \geq 0$,

$$\| f_* - f^\delta \| \;\leq\; \delta, \qquad \delta > 0. \tag{3.1}$$

Let $u_0 \in X$ be some initial guess for a solution of equation (1.1). In this section parameter choices for parameter methods

$$u_t^\delta := u_0 - G_t(Au_0 - f^\delta), \qquad t \geq 0, \tag{3.2}$$

as well as stopping rules for iteration methods

$$u_n^\delta := u_0 - G_n(Au_0 - f^\delta), \qquad n = 0, 1, ..., \tag{3.3}$$

are introduced and discussed. These parameter methods and stopping rules are designed to yield some

$$t = t(f^\delta, \delta) \quad \text{and} \quad n = n(f^\delta, \delta), \tag{3.4}$$

respectively, in order to provide good approximations $u^\delta_{t(\delta, f^\delta)}$ and $u^\delta_{n(\delta, f^\delta)}$, respectively, for a solution u_* of $Au = f_*$. We will classify them:

(a) **a priori choices** of the parameter or the stopping index are of type

$$t = t(\delta) \quad \text{and} \quad n = n(\delta),$$

i.e, here one has dependence on the noise level and not on the data. However, no natural choices of this type exist, and additional knowledge of the solution is necessary in order to obtain convergence rates. Therefore, they are not discussed further here.

(b) **a posteriori choices** of the parameter or the stopping index are of the general form (3.4), i.e, they depend explicitly on the data f^δ and the noise-level δ. The discrepancy principle and a modification for the iterated method of Lavrentiev are of this type; they are introduced in this chapter, and convergence is proved for them.

(c) We discuss also **noise-level-free choices**; they also of type (3.4) but here we drop the assumption (3.1), and instead assume that $\| f_* - f^\delta \| \to 0$ as $\delta \to 0$. Then δ becomes an additional independent parameter. Popular noise-level-free choices of the parameter or the stopping index, however, are independent of δ, i.e., they are of type

$$t = t(f^\delta) \quad \text{and} \quad n = n(f^\delta),$$

respectively.

We first discuss noise-level-dependent choicesi, and the crucial question is whether the rules are convergent in the sense of the following definition. At this point we also introduce the classical notation of a regularization method.

Definition 3.1.1 *(a) A family $\mathcal{P}_\delta : X \to X,\ \delta > 0$, is called regularization method, if for any $u_* \in \overline{\mathcal{R}(A)}$ and $f^\delta \in X$ with $\|Au_* - f^\delta\| \le \delta$ one has*

$$\|\mathcal{P}_\delta f^\delta - u_*\| \to 0 \qquad \text{as} \ \ \delta \to 0.$$

(b) Let $u_0 = 0$.

- *Either a parameter choice for (3.2), supplying some $t = t(f^\delta, \delta) \ge 0$,*
- *or a stopping rule for iterates (3.3), supplying some $n = n(f^\delta, \delta) \ge 0$,*

is called convergent, if

$$\mathcal{P}_\delta f^\delta := u^\delta_{t(\delta, f^\delta)} \qquad \big(\mathcal{P}_\delta f^\delta := u^\delta_{n(\delta, f^\delta)}\big),$$

respectively, defines a regularization method. Otherwise the parameter choice or stopping rule is called divergent.

In the following two sections, the discrepancy principle as a specific parameter choice and stopping rule is introduced and its convergence is proven, and under additional assumptions we obtain convergence rates (that cannot be improved, in general, see Chapter 4). Finally, Section 3.4 is devoted to noiselevel-free choices of the parameter and the stopping index since they seem to be quite popular.

3.2 The discrepancy principle

3.2.1 The discrepancy principle for parameter methods

We first consider approximations of type (3.2),

$$u^\delta_t := u_0 - G_t(Au_0 - f^\delta), \qquad t \ge 0.$$

As was mentioned in the preceding chapter, the parameter t has to be chosen appropriately. We shall do this by values of the norm of the defect

$$r^\delta_t := Au^\delta_t - f^\delta$$

and the error level δ. To this end, let again $H_t = I - G_t A$ and let the main conditions (2.2)-(2.5) are fulfilled. Then we have

$$r^\delta_t \ = \ H_t(Au_0 - f^\delta) \ = \ H_t A(u_0 - u_*) \ + \ H_t(Au_* - f^\delta)$$

and (2.2) (with $\alpha = 0$) and implies

$$\big|\ \|r^\delta_t\| - \|H_t A(u_0 - u_*)\|\ \big| \le \ \gamma_0 \delta. \tag{3.5}$$

This together with (2.2) for $\alpha = 1$ yields

$$\limsup_{t \to \infty} \ \|r^\delta_t\| \ \le \ \gamma_0 \delta,$$

such that the following parameter choice is applicable. For that we require additionally that $t \mapsto H_t u$ is a continuous function on $\{\ 0 \le t < \infty\ \}$ for any $u \in X$ (that is the case for Lavrentiev's iterated method as well as for Cauchy' method) in order to ensure that the set of $t_\delta \ge 0$ fulfilling this assumption is non-void.

Parameter Choice 3.2.1 *(Discrepancy principle)*
Assume u_t^δ is given by (3.2) and that the main conditions (2.2)-(2.5) are fulfilled by $\{G_t\}$. Fix positive constants $b_0,\ b_1$ with $b_1 \geq b_0 > \gamma_0$ (with γ_0 as in (2.2)).
(a) If $\|r_0^\delta\| \leq b_1\delta$ then choose $t_\delta = 0$.
(b) If $\|r_0^\delta\| > b_1\delta$ then choose t_δ such that

$$b_0\delta \ \leq\ \|r_{t_\delta}^\delta\| \ \leq\ b_1\delta.$$

t_δ also depend on f^δ and hence is an a posteriori parameter choice; for notational reasons this dependence is not further indicated, however. Before we state the main result of this section, we introduce α-norms on $\mathcal{R}(A^\alpha)$.

Definition 3.2.2 *1. We define α-norms on $\mathcal{R}(A^\alpha)$ by*

$$\|u\|_\alpha := \inf\left\{\ \|z\| : z \in X,\ A^\alpha z = u\ \right\}, \qquad u \in \mathcal{R}(A^\alpha). \tag{3.6}$$

(3.7) in the following theorem shows that Parameter Choice 3.2.1 for the parameter methods fulfilling (2.2)-(2.5) are convergent in the sense of Definition 3.1.1. In (3.9), under additional smoothness assumptions we obtain convergence rates which are optimal in a sense to be precised in Chapter 4.

Theorem 3.2.3 *Assume that $A \in \mathcal{L}(X)$ is weakly sectorial. Let $u_t^\delta,\ t \geq 0$, be defined by (3.2), and assume that (3.1) and (2.2)-(2.5) hold, with qualification $\alpha_0 > 1$. Let t_δ be chosen by Parameter Choice 3.2.1.*
1. If u_* solves (1.1) with $u_0 - u_* \in \overline{\mathcal{R}(A)}$ then

$$\|u_{t_\delta}^\delta - u_*\| \ \rightarrow\ 0 \qquad \text{as}\ \ \delta \to 0, \tag{3.7}$$

$$t_\delta\delta \ \rightarrow\ 0 \qquad \text{as}\ \ \delta \to 0. \tag{3.8}$$

2. If moreover for $0 < \alpha \leq \alpha_0 - 1$ and $\varrho \geq 0$,

$$u_0 - u_* \in \mathcal{R}(A^\alpha), \qquad \rho := \|u_0 - u_*\|_\alpha,$$

then with some constants $c_\alpha,\ e_\alpha > 0$ we have the estimates

$$\|u_{t_\delta}^\delta - u_*\| \ \leq\ c_\alpha(\varrho\delta^\alpha)^{1/(\alpha+1)}, \tag{3.9}$$

$$t_\delta \ \leq\ e_\alpha(\varrho\delta^{-1})^{1/(\alpha+1)}. \tag{3.10}$$

c_α and e_α depend also on b_0 and b_1 which is not further indicated.

Proof of Theorem 3.2.3. We first prove the assertions for the parameter t_δ. First we observe that (3.5) implies

$$(b - \gamma_0)\delta \ \leq\ \|H_{t_\delta}A(u_0 - u_*)\|, \qquad \text{if } t_\delta \neq 0. \tag{3.11}$$

To prove (3.8), let $\delta_k > 0,\ k = 0, 1, 2, ...,$ such that $\delta_k \to 0$ as $k \to \infty$. If $\{t_{\delta_k}\}_k$ is bounded then $t_{\delta_k}\delta_k \to 0$ as $k \to \infty$ holds trivially. If $t_{\delta_k} \to \infty$ as $k \to \infty$, then (3.11), with δ_k instead of δ, and

$$t\|H_tA(u_0 - u_*)\| \to 0 \qquad \text{as}\ \ t \to \infty$$

(this follows from (2.2) and the principle of uniform boundedness) imply again $t_{\delta_k}\delta_k \to 0$ as $k \to \infty$. We obtain (3.8) by arguing for subsequences. From (3.11) and (2.2) we find

$$(b_0 - \gamma_0)\delta \ \leq\ \gamma_{\alpha+1}\varrho t_\delta^{-(\alpha+1)},$$

which gives (3.10).

2. To prove the assertions for $\|u_{t_\delta}^\delta - u_*\|$ we observe that for any t we have the basic estimate

$$\|u_t^\delta - u_*\| \ \leq\ \|H_t(u_0 - u_*)\| \ +\ \gamma_* t\delta. \tag{3.12}$$

Having in mind (3.8) and (3.10) we have to deal with the first term in the right-hand side of (3.12) only. (3.5) yields $\|H_{t_\delta}A(u_0 - u_*)\| \le (\gamma_0 + b_1)\delta$, and hence, by assumption (2.4),

$$\|H_{t_\delta}(u_0 - u_*)\| \;\le\; \kappa\Big(\|H_{t_*}(u_0 - u_*)\| \;+\; (\gamma_0 + b_1)t_*\delta\Big) \qquad \text{for all} \;\; t_* \ge t_\delta, \tag{3.13}$$

which we shall need if the algorithm breaks up early. In order to show

$$H_{t_\delta}(u_0 - u_*) \to 0 \qquad \text{as} \;\; \delta \to 0, \tag{3.14}$$

let again $\delta_k > 0, \;\; k = 1, 2, 3, ...,$ with $\delta_k \to 0$ as $k \to \infty$. If $t_{\delta_k} \to \infty$ as $k \to \infty$ then again with the Banach-Steinhaus theorem we obtain

$$H_{t_{\delta_k}}(u_0 - u_*) \to 0 \qquad \text{as} \;\; k \to \infty, \tag{3.15}$$

and if $\{t_{\delta_k}\}_k$ is bounded then we obtain (3.15) by applying (3.13) with $\delta = \delta_k$ and $t_* = \delta_k^{-1/2}$. We get (3.14) by arguing for subsequences. Finally we prove (3.9). We have

$$\|H_t(u_0 - u_*)\| \;\le\; \gamma_\alpha \varrho t^{-\alpha}, \qquad t > 0,$$

hence for $t_\delta \ge (\varrho\delta^{-1})^{1/(\alpha+1)}$ we already obtain

$$\|H_{t_\delta}(u_0 - u_*)\| \;\le\; \gamma_\alpha \varrho t_\delta^{-\alpha} \;\le\; \gamma_\alpha (\varrho\delta^\alpha)^{1/(\alpha+1)}.$$

If on the other side $t_\delta \le (\varrho\delta^{-1})^{1/(\alpha+1)} =: t_*$ then we apply (3.13) and obtain

$$\begin{aligned}
\|H_{t_\delta}(u_0 - u_*)\| \;&\le\; \kappa\Big(\|H_{t_*}(u_0 - u_*)\| \;+\; (\gamma_0 + b_1)t_*\delta\Big) \\
&\le\; \kappa\Big(\gamma_\alpha \varrho t_*^{-\alpha} \;+\; (\gamma_0 + b_1)t_*\delta\Big) \\
&\le\; \kappa\Big(\gamma_\alpha \;+\; (\gamma_0 + b_1)\Big)(\varrho\delta^\alpha)^{1/(\alpha+1)},
\end{aligned}$$

which completes the proof. □

Remarks. 1. A different technique can be used to prove Theorem 3.2.3. It uses the first interpolation inequality (see Theorem 1.1.21), and we then do not need the third assumption (2.4) on our methods. (2.4) is fulfilled by any of our presented examples, however, and it gives additional insight on the behaviour of these methods.

2. Let us assume that $u_0 = 0$.The condition '$u_* \in \overline{\mathcal{R}(A)}$' in the first part of Theorem 3.2.3 can be fulfilled for a solution u_* of (1.1) if and only if $f_* \in A(\overline{\mathcal{R}(A)})$ (which can be more restrictive than '$f_* \in \mathcal{R}(A)$'), and then $u_* = A^\dagger f_*$.

3. The constraint $\alpha_0 > 1$ in Theorem 3.2.3 is necessary, since for the iterated method of Lavrentiev, the discrepancy principle works only if $m \ge 2$: see Proposition 3.2.4 for the case $m = 1$. If $\alpha_0 = 1$, then we obtain (3.11) with $\alpha = 0$ only, i.e.,

$$t_\delta \delta \;\le\; \gamma_1 \|u_0 - u_*\|/(b_0 - \gamma_0),$$

and by means of (3.12),

$$\|u_{t_\delta}^\delta - u_*\| \;\le\; \big(\gamma_0 + \gamma_* \gamma_1/(b_0 - \gamma_0)\big)\|u_0 - u_*\|.$$

Hence, if X is a reflexive Banach space and if $\mathcal{N}(A) = \{0\}$, then we have weak convergence,

$$u_{t_\delta}^\delta \rightharpoonup u_* \qquad \text{as} \;\; \delta \to 0,$$

since $\|Au_{t_\delta}^\delta - f_*\| \to 0$ as $\delta \to 0$.

4. A simple strategy for choosing the parameter t is to calculate the defect $r_t^\delta = Au_t^\delta - f^\delta$ for a finite numbers of t, say

$$\|r_{t_n}^\delta\|, \qquad t_n = nh, \qquad n = 0, 1, 2..., \quad \text{or} \tag{3.16}$$

$$\|r_{t_n}^\delta\|, \qquad t_n = \theta^{-n}h, \qquad n = 1, 2..., \tag{3.17}$$

(with some fixed $0 < \theta < 1,\ h > 0$) until it falls under the level $b\delta$ for the first time, with b chosen such that $b > \gamma_0$, where γ_0 as in (2.2). We then can apply Theorem 3.2.6 on the discrepancy principle for iteration methods to obtain convergence results for parameter methods (although these methods have a non-iterative character).

5. For Lavrentiev's iterated method with $m > 1$ one has the qualification $\alpha_0 = m$, and the best rate we can expect to obtain with the discrepancy principle is $\|u^\delta_{t_\delta} - u_*\| = \mathcal{O}(\delta^{(m-1)/m})$ as $\delta \to 0$ (for $u_0 - u_* \in \mathcal{R}(A^{m-1})$.

As mentioned in the preceding remark, the discrepancy principle for Lavrentiev's classical method, this is the case $m = 1$, is divergent:

Proposition 3.2.4 *Let $A \in \mathcal{L}(X)$ be weakly sectorial and assume that $0 \in \sigma_{ap}(A)$. Then Parameter Choice 3.2.1 for Lavrentiev's method, this is (2.6), (2.7) with $m = 1$, is divergent.*

Proof. Assume that $0 \neq u_* \in \overline{\mathcal{R}(A)}$, and define $f_* = Au_*$. Take for $\delta > 0$ a $\psi^\delta \in X$ such that

$$\begin{aligned} \|\psi^\delta\| &= \delta, \\ \|A\psi^\delta\| &\leq \delta^3. \end{aligned}$$

(This is possible since by the assumption $0 \in \sigma_{ap}(A)$, there are $v^\delta \in X$ with $\|v^\delta\| = 1,\ \|Av^\delta\| \leq \delta^2$, and take then $\psi^\delta = \delta v^\delta$). We consider

$$f^\delta := f_* + \psi^\delta, \qquad \delta > 0.$$

To start our analysis, observe that (for $u_0 = 0$)

$$u^\delta_t \ = \ t(I + tA)^{-1} f^\delta \ = \ -tr^\delta_t,$$

hence, for $\delta > 0$ not too large (then $t_\delta \neq 0$, with t_δ obtained by the discrepancy principle),

$$b_0 t_\delta \delta \ \leq \ \|u^\delta_{t_\delta}\| \ \leq \ b_1 t_\delta \delta. \tag{3.18}$$

We will show that $\|u^\delta_{t_\delta} - u_*\| \not\to 0$ as $\delta \to 0$, or even stronger,

$$\liminf_{\delta \to 0} \|u^\delta_{t_\delta} - u_*\| \ > \ 0. \tag{3.19}$$

To this end we assume contradictory that there is a set countable $H \subset (0, \infty)$ with accumulation point $0 \in \mathbb{R}$ and

$$\|u^\delta_{t_\delta} - u_*\| \to 0 \qquad \text{as } H \ni \delta \to 0,$$

and then (3.18) implies

$$\liminf_{H \ni \delta \to 0} t_\delta \delta \ \geq \ \lim_{H \ni \delta \to 0} \|u^\delta_{t_\delta}\|/b_1 \ = \ \|u_*\|/b_1 \ > \ 0, \tag{3.20}$$

$$\limsup_{H \ni \delta \to 0} t_\delta \delta \ \leq \ \lim_{H \ni \delta \to 0} \|u^\delta_{t_\delta}\|/b_0 \ = \ \|u_*\|/b_0 \ < \ \infty, \tag{3.21}$$

and (3.20) yields

$$t_\delta \to \infty \qquad \text{as } H \ni \delta \to 0. \tag{3.22}$$

We have

$$\begin{aligned} u^\delta_{t_\delta} - u_* &= -(I + t_\delta A)^{-1} u_* \ + \ t_\delta (I + t_\delta A)^{-1} \psi^\delta \\ &= -(I + t_\delta A)^{-1} u_* \ + \ t_\delta \psi^\delta \ - \ t_\delta^2 (I + t_\delta A)^{-1} A\psi^\delta, \end{aligned}$$

hence

$$\begin{aligned} \|u^\delta_{t_\delta} - u_*\| &\geq t_\delta \|\psi^\delta\| \ - \ \|(I + t_\delta A)^{-1} u_*\| \ - \ M_0 t_\delta^2 \delta^3 \\ &= t_\delta \delta \ - \ \|(I + t_\delta A)^{-1} u_*\| \ - \ M_0 t_\delta^2 \delta^3, \end{aligned}$$

and this estimate together with (3.19)-(3.22) yields the contradiction

$$0 \ = \ \lim_{H \ni \delta \to 0} \|u^\delta_{t_\delta} - u_*\| \ \geq \ \|u_*\|/b_1 > 0. \qquad \square$$

Remark. Proposition 3.2.4 shows that in the case $\mathcal{N}(A) \neq \{0\}$, the discrepancy principle for Lavrentiev's method (with $m = 1$) fails even in the well-posed case.

3.2.2 The discrepancy principle for iterative methods

We again assume that there is given some noise level for the approximations of the exact right-hand side: let $f^\delta \in X$ and $\delta > 0$ with known error level (3.1), and solve equation (1.1) approximately with iterates of type (3.3),

$$u_n^\delta := u_0 - G_n(Au_0 - f^\delta) \qquad \text{for} \;\; n = 0, 1, 2, ...,$$

where we again the four main conditions (2.15)-(2.18) are supposed to hold for $\{G_n\}$. As was mentioned in the preceding chapter, iteration has to be stopped at appropriate time, it is the purpose of this paper to do this by values of the norm of the defect

$$r_n^\delta := Au_n^\delta - f^\delta$$

and the error level δ. Similar to the parameter methods one has

$$\big|\, \|r_n^\delta\| - \|H_n A(u_0 - u_*)\| \,\big| \leq \; \gamma_0 \delta. \tag{3.23}$$

This together with (2.15) implies

$$\limsup_{n\to\infty} \; \|r_n^\delta\| \; \leq \; \gamma_0\delta,$$

such that the computation of u_n^δ terminates after a finite number of iteration steps n, if the following stopping rule is applied:

Stopping Rule 3.2.5 *(Discrepancy principle) Fix a real $b > \gamma_0$. Stop process of calculating u_n^δ, $n = 0, 1, 2, ...$ if for the first time*

$$\|r_n^\delta\| \; \leq \; b\delta,$$

and let $n_\delta := n$.

n_δ depends also on f^δ which is not further indicated. We state the following main result for iterative methods, which in fact is the analogue to Theorem 3.2.3 for parameter methods. (3.24) shows that the iteration methods fulfilling (2.15)-(2.18) are defining regularization methods in the sense of Definition 3.1.1, if stopped according to Stopping Rule 3.2.5. (3.26) provides, under additional smoothness assumptions, convergence rates which are optimal in a sense to be precised in Chapter 4. Estimates (3.25) and (3.27) give some information about the efficiency of the underlying algorithm.

Theorem 3.2.6 *Assume that $A \in \mathcal{L}(X)$ is weakly sectorial. Let u_n^δ, $n = 0, 1, 2, ...$, be defined by (3.3), and assume that (3.1) and (2.15)-(2.18) hold, with qualification $\alpha_0 > 1$. Let the stopping index n_δ be obtained by Stopping Rule 3.2.5.*
1. If u_* is a solution of (1.1) with $u_0 - u_* \in \overline{\mathcal{R}(A)}$ then

$$\|u_{n_\delta}^\delta - u_*\| \;\; \to \;\; 0 \qquad \text{as} \;\; \delta \to 0, \tag{3.24}$$
$$F(n_\delta)\delta \;\; \to \;\; 0 \qquad \text{as} \;\; \delta \to 0. \tag{3.25}$$

2. If moreover for $0 < \alpha \leq \alpha_0 - 1$ ($0 < \alpha < \infty$, if $\alpha_0 = \infty$) and $\varrho \geq 0$,

$$u_0 - u_* \in \mathcal{R}(A^\alpha), \qquad \rho := \|u_0 - u_*\|_\alpha,$$

then we have, with certain positive constants $c_{\alpha,b}$ and $e_{\alpha,b}$, the estimates

$$\|u_{n_\delta}^\delta - u_*\| \;\; \leq \;\; c_{\alpha,b}(\varrho\delta^\alpha)^{1/(\alpha+1)}, \tag{3.26}$$
$$F(n_\delta) \;\; \leq \;\; e_{\alpha,b}(\varrho\delta^{-1})^{1/(\alpha+1)}. \tag{3.27}$$

The proof of this theorem is similar to the proof of Theorem 3.2.3 and will be given for convenience of the reader. We have to consider the case $n_\delta - 1$ instead of t_δ which complicates the proof to some extent.

Proof of Theorem 3.2.6. We first prove the assertions for the stopping index n_δ. We first observe that

$$(b - \gamma_0)\delta \;\leq\; \|H_{n_\delta - 1}A(u_0 - u_*)\|, \qquad \text{if} \;\; n_\delta \geq 1. \tag{3.28}$$

This follows from (3.23) and $\|r_{n_\delta-1}\| > b\delta$. To prove (3.25), let $\delta_k > 0,\ k = 0,1,2,...$, such that $\delta_k \to 0$ as $k \to \infty$. If $\{n_{\delta_k}\}_k$ is bounded then $F(n_{\delta_k})\delta_k \to 0$ as $k \to \infty$ holds trivially. If $n_{\delta_k} \to \infty$ as $k \to \infty$, then (3.28), with $\delta = \delta_k$, and

$$F(n)\|H_n A(u_0 - u_*)\| \to 0 \qquad \text{as} \;\; n \to \infty$$

(this follows from (2.15) and the principle of uniform boundedness) together with (3.28) imply again $F(n_{\delta_k})\delta_k \to 0$ as $k \to \infty$. We obtain (3.25) by arguing for subsequences. For proving (3.27) we first observe that $\varrho\delta^{-1} \le (b-1)/\|A\|^{\alpha+1}$ implies $n_\delta = 0$, and in the case $\varrho\delta^{-1} \ge (b-1)/\|A\|^{\alpha+1}$ we may assume $n_\delta \ge 2$ (take e.g.

$$e_\alpha \;\ge\; F(1)\|A\|/(b-1)^{1/(\alpha+1)} \tag{3.29}$$

and then for $n_\delta = 1$, (3.27) is fulfilled automatically). An application of (2.15) and (3.28) implies

$$\begin{aligned}(b-\gamma_0)\delta \;&\le\; \gamma_{\alpha+1}\varrho F(n_\delta - 1)^{-(\alpha+1)} \\ &\le\; \gamma_{\alpha+1}\varrho\varkappa^{\alpha+1} F(n_\delta)^{-(\alpha+1)},\end{aligned} \tag{3.30}$$

this gives (3.27).

2. To prove the assertions for $\|u^\delta_{n_\delta} - u_*\|$ we observe again that for any n,

$$u^\delta_n - u_* \;=\; H_n(u_0 - u_*) - G_n(Au_* - f^\delta)$$

holds, and then by (2.16)

$$\|u^\delta_n - u_*\| \;\le\; \|H_n(u_0 - u_*)\| \;+\; \gamma_* F(n)\delta. \tag{3.31}$$

Having in mind (3.25) and (3.27) we have to deal with the first term in the right-hand side of (3.31) only. By definition $\|r_{n_\delta}\| \le b\delta$, and (3.23) implies $\|H_{n_\delta} A(u_0 - u_*)\| \le (\gamma_0 + b)\delta$, and hence, by assumption (2.17),

$$\|H_{n_\delta}(u_0 - u_*)\| \;\le\; \kappa\Big(\|H_{n_*}(u_0 - u_*)\| \;+\; (\gamma_0 + b)F(n_*)\delta\Big) \qquad \text{for all} \;\; n_* \ge n_\delta, \tag{3.32}$$

which we shall need for the case that iteration stops early. In order to show

$$H_{n_\delta}(u_0 - u_*) \to 0 \qquad \text{as} \;\; \delta \to 0, \tag{3.33}$$

let again $\delta_k > 0,\ \ k = 1,2,3,...$, with $\delta_k \to 0$ as $k \to \infty$. If $n_{\delta_k} \to \infty$ as $k \to \infty$ then (2.15) and the Banach-Steinhaus theorem implies

$$H_{n_{\delta_k}}(u_0 - u_*) \to 0 \qquad \text{as} \;\; k \to \infty. \tag{3.34}$$

If $\{n_{\delta_k}\}_k$ is bounded then choose $n_* = n_*(\delta_k)$ such that $n_* \to \infty$ as $k \to \infty$, and $F(n_*) \le \delta_k^{-1/2},\ k = 1,2,....$ We then obtain (3.34) by applying (3.32) with $\delta = \delta_k$ and those n_*. We get (3.33) by arguing for subsequences. Finally, to prove (3.26) we first observe again that in the case

$$\varrho\delta^{-1} \;\le\; (b-1)/\|A\|^{\alpha+1}$$

we have $n_\delta = 0$, and then $\|u_0 - u_*\| \le \|A\|^\alpha \varrho \le (b-1)^{\alpha/(\alpha+1)}(\varrho\delta^\alpha)^{1/(\alpha+1)}$. We now assume that

$$(\varrho\delta^{-1})^{1/(\alpha+1)} \;\ge\; (b-1)^{1/(\alpha+1)}/\|A\| =: c.$$

We have

$$\|H_n(u_0 - u_*)\| \;\le\; \gamma_\alpha \varrho F(n)^{-\alpha} \qquad \text{for all} \;\; n \ge 1,$$

hence for the case $F(n_\delta) \ge c^{-1}(\varrho\delta^{-1})^{1/(\alpha+1)}$ we already obtain

$$\|H_{n_\delta}(u_0 - u_*)\| \;\le\; \gamma_\alpha \varrho F(n_\delta)^{-\alpha} \;\le\; \gamma_\alpha c^\alpha (\varrho\delta^\alpha)^{1/(\alpha+1)}.$$

If on the other side $F(n_\delta) \le c^{-1}(\varrho\delta^{-1})^{1/(\alpha+1)}$, then let $n_* \ge 2$ with

$$F(n_* - 1) \;\le\; c^{-1}(\varrho\delta^{-1})^{1/(\alpha+1)} \;<\; F(n_*),$$

and applying (3.32) with this n_* yields

$$\begin{aligned} \|H_{n_\delta}(u_0 - u_*)\| &\leq \kappa\Big(\|H_{n_*}(u_0 - u_*)\| \;+\; (\gamma_0 + b)F(n_*)\delta\Big) \\ &\leq \kappa\Big(\gamma_\alpha \varrho F(n_*)^{-\alpha} \;+\; \varkappa(\gamma_0 + b)F(n_* - 1)\delta\Big) \\ &\leq \kappa\Big(\gamma_\alpha c^\alpha \;+\; \varkappa c^{-1}(\gamma_0 + b)\Big)(\varrho\delta^\alpha)^{1/(\alpha+1)}. \end{aligned}$$

This completes the proof. □

Remark. 1. The convergence rates cannot be improved, in general, see Chapter 4 for more on that.
2. Consideration of (3.29) and (3.30) in the proof of Theorem 3.2.6 shows that estimate (3.27) for n_δ holds with

$$e_\alpha \;=\; \max\big\{\; \varkappa\gamma_{\alpha+1}^{1/(\alpha+1)}, F(1)\|A\| \;\big\} \cdot (b - \gamma_0)^{-1/(\alpha+1)},$$

(and thus e_α depends also on $\varkappa$ and $F(1)$).

3.2.3 Computational experiments

Our model equation is

$$Au(t) := \int_0^1 k_\omega(t,s)u(s)\; ds \;=\; f(t), \qquad 0 \;\leq\; t \;\leq\; 1$$

with kernel

$$k_\omega(t,s) \;=\; \begin{cases} a \cdot \cosh(\omega(t - s \; - 0.5)), & \text{if } s \leq t \\ a \cdot \cosh(\omega(t - s + 0.5)), & \text{if } s > t \end{cases}.$$

as in Subsection 1.2.2. Here, $\omega > 0$ is some constant and $a = 1/(2\omega \sinh(\omega/2))$. As underlying space X we consider the space of periodic continuous complex-valued functions $X = C_p[0,1]$ with the sup-norm.

A can be characterized by (1.29): for $u,\; f \in C_p[0,1]$ we have

$$Au = f \quad\Longleftrightarrow\quad \begin{aligned} & f \in C^2[0,1],\; f',\; f'' \in C_p[0,1], \\ & \omega^2 f - f'' \;=\; u, \end{aligned}$$

see again Subsection 1.2.2 for the details. In our numerical experiments we consider the test equation

$$Au = f_*$$

with

$$\begin{aligned} f_*(t) &= \Big(-\frac{6}{\omega^2} + (\frac{12}{\omega^2} + 1)t - 3t^2 + 2t^3\Big)/\omega^2 + ce^{\omega t} + de^{-\omega t}, \qquad 0 \leq t \leq 1, \\ c &= 6e^\omega/N, \qquad d \;=\; -6e^{2\omega}/N, \\ N &= (e^\omega - e^{2\omega})\omega^4, \end{aligned}$$

therefore

$$u_*(s) \;=\; s(2s^2 - 3s + 1), \qquad 0 \leq s \leq 1,$$

and thus $u_* \in \mathcal{R}(A)$ but $u_* \notin \mathcal{R}(A^2)$. (In fact, the symbolic programming language Maple V did the inconvenient part of the work and computed f_* from our prescribed u_*.) We choose perturbed right-hand side $f^\delta = f_* + \delta \cdot v$, where the values of $v(t)$ are randomly chosen such that $\|v\|_\infty \leq 1$, and where

$$\delta \;=\; \|f_*\|_\infty \cdot \%/100,$$

with $\% = 0.33, 1, 3$.

We choose

$$\omega = 1$$

in the definition of the kernel k_ω, and it follows from Lemma 1.2.7 that $\|A\|_\infty \leq 1$, hence we can again take $\mu = 0.2$ for the Richardson iteration (and, of course, for the other iteration methods).

We now compute bounds γ_0 for the implicit iterative method and the iterated version of Lavrentiev's method; this is necessary to determine a set of admissible b, which appears in the definitions of the discrepancy principles (3.2.5) and (3.2.1). From Corollary 1.2.8 and numerical estimates of (1.32) in Theorem 1.2.11 we obtain

$$\gamma_0 \approx 4.02$$

(obtained for $\epsilon \approx 0.30$), and it follows that we can take

$$b = 4.1$$

in our implementations. That b is chosen for the Richardson iteration as well as the ADI method where we do not get estimates for γ_0 (since we are not able to estimate properly the integral (1.41) near the negative real axis).

A rectangular rule and collocation (with collocation points $s_j = jh, \ j = 1, ..., N, \ h = 1/N$, with $N = 128$) is used to discretize the problem. The same tests as for the first example are repeated. The last column always include the number of flops for computing the entries of the underlying matrix (which in fact is 0.5e+06).

Here are the results for the Richardson iteration, the implicit method and the iteration method with alternating directions. All computations where performed in MATLAB on an IBM RISC/6000. Note that the approximation errors and stopping indeces are very similar for all methods, for each noise level.

Richardson iteration			
% noise	$\|u_{n_\delta}^\delta - u_*\|_\infty$	n_δ	♯ flops
0.33	0.0069	1203	4.1e+07
1.00	0.0112	757	2.6e+07
3.00	0.0194	492	1.7e+07

Implicit iteration method			
% noise	$\|u_{n_\delta}^\delta - u_*\|_\infty$	n_δ	♯ flops
0.33	0.0061	1272	8.6e+07
1.00	0.0112	744	5.1e+07
3.00	0.0195	494	3.4e+07

ADI method			
% noise	$\|u_{n_\delta}^\delta - u_*\|_\infty$	n_δ	♯ flops
0.33	0.0068	1216	8.3e+07
1.00	0.0112	750	5.1e+07
3.00	0.0211	468	3.3e+07

The following table contains the results for the iterated method of Lavrentiev which are surprisingly good. Note that for any parameter $t_n = nh, \ \ n = 1, 2, ..., \ \ h = 1.0$, a cholesky decomposition has to be calculated so that slightly more computational effort is necessary to obtain the approximations.

Lavrentiev's (iterated) method			
% noise	$\|u_{t_{n_\delta}}^\delta - u_*\|_\infty$	n_δ	♯ flops
0.33	0.0064	166	1.5e+08
1.00	0.0129	91	8.0e+07
3.00	0.0221	49	4.3e+07

We conclude with two remarks concerning the constant γ_0. A further possibility to obtain such a bound for the implicit method as well as Lavrentiev's iterated method is to use (1.36). Numerical experiments show that the best estimate which we can obtain in this case is $\gamma_0 \approx 5.581$ (obtained for $\epsilon \approx 0.189$) but further experiments show that iteration breaks up too early for this choice of γ_0. Finally, computation of the powers of the iteration matrices T and their norms indicate that we have $\sup_n \|T^n\| \approx 2.0$ (for any of the considered iterative methods), and the same bound holds for the iterated version of Lavrentiev's method.

3.3 Quasioptimal methods

3.3.1 A class of regularization methods, quasioptimality

Let again X be a Banach space and $A \in \mathcal{L}(X)$. In order to solve the ill-posed equation $Au = f^*$ with only approximately given right-hand side $f^\delta \in X$ and noise level $\delta > 0$, we again consider parameter methods $u_t^\delta \; = \; u_0 - G_t(Au_0 - f^\delta)$ for $t \geq 0$, where $u_0 \in X$ is some initial guess, and $G_t \in \mathcal{L}(X)$. We use again the notation $H_t := I - G_tA$, and assume that (2.3) is fulfilled. If u_* solves $Au = f_*$, then we have the basic estimate (3.12), i.e.,

$$\|u_t^\delta - u_*\| \;\leq\; \max\{1, \gamma_*\}\Big(\|H_t(u_0 - u_*)\| \;+\; t\delta\Big),$$

and this gives rise to the following definition.

Definition 3.3.1 *Let $\delta > 0,\ u_*,\ f^\delta \in X$ with $\|Au_* - f^\delta\| \leq \delta$, and let $\{u_t^\delta\}_t$ be defined by (3.2), and let (2.3) hold. A parameter choice supplying some $t_\delta \geq 0$, is called quasioptimal, if there exist some constant K, independently of $\delta > 0,\ u_*,\ f^\delta \in X$ and the initial guess $u_0 \in X$, such that*

$$\|H_{t_\delta}(u_0 - u_*)\| \;+\; t_\delta\delta \;\leq\; K \inf_{t\geq 0}\Big(\|H_t(u_0 - u_*)\| \;+\; t\delta\Big). \tag{3.35}$$

Quasioptimality of stopping rules for iteration methods of type (3.3) that fulfill (2.16) with $F(n) = n$, is defined similar, i.e., under the same assumptions on K, $\delta > 0,\ u_*,\ u_0$ and $f^\delta \in X$ we require

$$\|H_{n_\delta}(u_0 - u_*)\| \;+\; n_\delta\delta \;\leq\; K \inf_{n\geq 0}\Big(\|H_n(u_0 - u_*)\| \;+\; n\delta\Big). \tag{3.36}$$

We first observe that quasioptimal parameter choices and stopping rules are as least as good as the discrepancy principle. Note that here we replace the restriction '$0 < \alpha \leq \alpha_0 - 1$' by the weaker condition '$0 < \alpha \leq \alpha_0$'.

Theorem 3.3.2 *(a) Let $A \in \mathcal{L}(X)$ be weakly sectorial, and let $\{u_t^\delta\}_t$ be of type (3.2), such that the conditions (2.2) and (2.3) are fulfilled, and let t_δ be a quasioptimal parameter choice.*
1. If u_* solves (1.1) with $u_0 - u_* \in \overline{\mathcal{R}(A)}$ then

$$\|u_{t_\delta}^\delta - u_*\| \;\rightarrow\; 0 \qquad \textit{as}\ \ \delta \rightarrow 0, \tag{3.37}$$
$$t_\delta\delta \;\rightarrow\; 0 \qquad \textit{as}\ \ \delta \rightarrow 0. \tag{3.38}$$

2. If moreover for $0 < \alpha \leq \alpha_0$,

$$u_0 - u_* \in \mathcal{R}(A^\alpha), \qquad \varrho := \|u_0 - u_*\|_\alpha,$$

then with some constants $d_{\alpha,b},\ e_{\alpha,b} > 0$ we have the estimates

$$\|u_{t_\delta}^\delta - u_*\| \;\leq\; d_{\alpha,b}(\varrho\delta^\alpha)^{1/(\alpha+1)}, \tag{3.39}$$
$$t_\delta \;\leq\; e_{\alpha,b}(\varrho\delta^{-1})^{1/(\alpha+1)}. \tag{3.40}$$

(b) (1) and (2) in (a), with t_δ replaced by n_δ, are also valid for iterative methods $\{u_n^\delta\}$ of type (3.3) that fulfill (2.15) and (2.16) (with $F(n) = n$).

Proof. For the parameter methods take e.g. $t = \delta^{-1/2}$ and $t = (\varrho\delta^{-1})^{1/(\alpha+1)}$, respectively, in (3.35) to get (3.37), (3.38) and (3.39), (3.40), respectively. We now consider iterative methods. Take e.g. $n = \lceil\delta^{-1/2}\rceil$ in (3.36) to get (3.37), (3.38) with n_δ instead of t_δ. In order to get (3.39), (3.40) we distinguish two cases. First, if δ is large,

$$\varrho\delta^{-1} < C, \qquad C := \frac{\gamma_*}{K\|A\|^\alpha},$$

then we take $n = 0$ in (3.36) and obtain $n_\delta = 0$, and moreover

$$\|u_0 - u_*\| \;\leq\; \|A\|^\alpha\varrho \;\leq\; \|A\|^\alpha C^{1/(\alpha+1)}(\varrho\delta^\alpha)^{1/(\alpha+1)};$$

and in the second case

$$\varrho\delta^{-1} \;\geq\; 1/C$$

take $n = \lceil (\varrho\delta^{-1})^{1/(\alpha+1)} \rceil$ in (3.36), since $n\delta \le (1 + C^{1/(\alpha+1)}) \cdot (\varrho\delta^{\alpha})^{1/(\alpha+1)}$, and since (2.15) implies

$$\| H_n(u_0 - u_*) \| \ \le\ \gamma_\alpha \varrho n^{-\alpha} \ \le\ \gamma_\alpha (\varrho\delta^{\alpha})^{1/(\alpha+1)}. \quad \Box$$

It is shown in the following two sections that the discrepancy principle for a class of iteration methods as well as a modified discrepancy principle for the iterated method of Lavrentiev are quasioptimal.

3.3.2 The quasioptimality of the discrepancy principle for iteration methods

Let X be a real or complex Banach space and $A \in \mathcal{L}(X)$. We consider iterative methods of type (3.3) where the main conditions (2.15)-(2.18) are fulfilled (for $\alpha_0 = \infty$, and for integer α only this time) with $F(n) = n,\ n = 0, 1, \ldots$. Now, however, we assume additionally that the semigroup property

$$H_{n_1+n_2} \ =\ H_{n_1} H_{n_2} \qquad \text{for}\ \ n_1,\ n_2 \ =\ 0, 1, 2, \ldots, \tag{3.41}$$

is fulfilled $\big($and in fact for all examples with (3.41) one has $F(n) = n\big)$.

Note that if for $\alpha = 1, 2,$

$$\| H_n A^{\alpha} \| \ \le\ \gamma_\alpha n^{-\alpha} \qquad \text{for}\ \ n \ =\ 1, 2, \ldots, \tag{3.42}$$

(with a certain constant γ_α), then (2.15), (2.18) and (3.41) guarantee that (3.42) holds for all $\alpha \ge 0$; to see this, consider first those n for which n/α is an integer, and then consider the general case for n.

We now state the first main result of this section.

Theorem 3.3.3 *Let $u_n^\delta,\ n = 0, 1, 2, \ldots,$ be defined by (3.3), and assume that (2.15)-(2.18) and (3.41) hold. Then Stopping Rule 3.2.6 is quasioptimal.*

Proof. We take any $n \ge 0$, and due to (3.36) it is sufficient to prove

$$\| H_{n_\delta}(u_0 - u_*) \| \ +\ n_\delta \delta \ \le\ K\Big(\| H_n(u_0 - u_*) \| \ +\ n\delta \Big), \tag{3.43}$$

with some constant K not depending on $n,\ u_*,\ u_0$ and δ. (We indicate in the course of the proof how to choose K).

(i) We first prove (3.43) for $n \ge n_\delta$. If we take $K \ge 1$, then the first term in the left-hand side of (3.43) remains to be estimated. By definition we have $\| r_{n_\delta}^\delta \| \le b\delta$, together with (3.23) we obtain $\| H_{n_\delta} A(u_0 - u_*) \| \le (\gamma_0 + b)\delta$, with property (2.17) we get

$$\| H_{n_\delta}(u_0 - u_*) \| \ \le\ \kappa\Big(\| H_n(u_0 - u_*) \| \ +\ (\gamma_0 + b) n\delta \Big),$$

which yields (3.43), if $K \ge \kappa(\gamma_0 + b)$.

(ii) We now consider the case $0 \le n \le n_\delta - 1$. Then from the semigroup property (3.41) and the boundedness condition (2.15) we obtain

$$\| H_{n_\delta}(u_0 - u_*) \| \ \le\ \gamma_0 \| H_n(u_0 - u_*) \|,$$

and if we choose $K \ge \gamma_0$, then the second term in the left-hand side of (3.43) remains to be estimated. From $b\delta \le \| r_{n_\delta - 1}^\delta \|$ and (3.23) we obtain

$$(b - \gamma_0)\delta \ \le\ \| H_{n_\delta - 1} A(u_0 - u_*) \|. \tag{3.44}$$

We first assume $n_\delta \ge 2$, and then obviously $n_\delta - 2n \le 2(n_\delta - n - 1)$. From the semigroup property (3.41), the decay property (2.15) and from (3.44) we obtain the estimate

$$\begin{aligned} n_\delta \delta \ &=\ 2n\delta \ +\ (n_\delta - 2n)\delta \\ &\le\ 2n\delta \ +\ 2(n_\delta - n - 1)(b - \gamma_0)^{-1} \| H_{n_\delta - n - 1} A H_n(u_0 - u_*) \| \\ &\le\ 2n\delta \ +\ 2(b - \gamma_0)^{-1} \gamma_1 \| H_n(u_0 - u_*) \|, \end{aligned} \tag{3.45}$$

which yield (3.43) for $0 \le n \le n_\delta - 1,\ n_\delta \ge 2$, if we choose $K \ge 2\max\{1, \gamma_1/(b - \gamma_0)\}$.

The case $n = 0,\ n_\delta = 1$ remains to be considered. Here (3.44) yields

$$(b-\gamma_0)n_\delta\delta \;=\; (b-\gamma_0)\delta \;\le\; \|A\|\cdot\|H_{n_\delta-1}(u_0-u_*)\| \;=\; \|A\|\cdot\|H_0(u_0-u_*)\|.$$

Therefore take also $K \ge \|A\|/(b-\gamma_0)$ in (3.43), and the proof is completed. □

For positive semidefinite operators in Hilbert spaces we have the stronger estimate

$$\|u^\delta_{n_\delta} - u_*\| \;\le\; K \sup_{f^\delta\in X:\ \|f_*-f^\delta\|\le\delta}\ \inf_{n\ge 0} \|u^\delta_n - u_*\|. \tag{3.46}$$

(with K not depending on $u_*,\ u_0$ and δ). It seems that one has not such a result without the inner product structure of a Hilbert space.

3.3.3 The quasioptimality of two modified discrepancy principles for the iterated method of Lavrentiev

General considerations

In order to get results on the quasioptimality of parameter choices for general non-iterative methods of type (3.2), i.e.,

$$u^\delta_t = u_0 - G_t(Au_0 - f^\delta), \qquad t \ge 0,$$

with some $G_t \in \mathcal{L}(X)$ we assume that (2.3) holds, and for $H_t = I - G_tA$ and

$$B_t \in \mathcal{L}(X), \qquad t \ge 0,$$

we require:

$$\|B_t\| \;\le\; \kappa_0 \qquad \text{for } t \ge 0, \tag{3.47}$$

$$\|B_tA\| \;\le\; \kappa_1 t^{-1} \qquad \text{for } t > 0, \tag{3.48}$$

$$\|H_tu\| \;\le\; \kappa\Big(\|H_{t_*}u\| + t_*\|B_tH_tAu\|\Big) \quad \text{for } 0 \le t \le t_*, \text{ and for } u \in X, \tag{3.49}$$

$$\|H_{t_*}u\| \;\le\; \gamma_0\|H_tu\| \qquad \text{for } 0 \le t \le t_*, \text{ and for } u \in X. \tag{3.50}$$

with constants $\kappa_0,\ \kappa_1,\ \kappa$ and γ_0. Note that (3.47) and (3.50) imply the existence of a $\tau_0 > 0$ such that

$$\|B_tH_t\| \;\le\; \tau_0 \qquad \text{for } t \ge 0. \tag{3.51}$$

We also again assume that the operators $G_t \in \mathcal{B}(X)$ commute with A. An example of a method of type (2.1) fulfilling all these conditions (with suitables B_t) is the (iterated) method of Lavrentiev's; details are given after the general considerations. Note, that the operators B_t shall have implications for the parameter choices, while $\{u^\delta_t\}_t$ itself does not depend on B_t. We state two parameter choices (a motivation for these choices is given at the end of this subsection) and the main result. In order to ensure applicability of the following rule, we require also that $t \mapsto B_tH_tu$ is continuous on $[0,\infty)$ for any $u \in X$.

Parameter Choice 3.3.4 *Fix real numbers $b_1 \ge b_0 > \tau_0$. If $\|B_0r^\delta_0\| \le b_1\delta$, then take $t_\delta = 0$. Otherwise choose t_δ such that*

$$b_0\delta \;\le\; \|B_{t_\delta}r^\delta_{t_\delta}\| \;\le\; b_1\delta.$$

The following parameter choice is designed for those strategies which are of type $t_j = h\theta^{-j},\ j = 1,2,...,$ or $t_j = hj,\ j = 0,1,...,$ where $0 < \theta < 1$ is independent of that used in the following parameter choice.

Parameter Choice 3.3.5 *Fix $\theta \in (0,1]$ and $b > \tau_0$. If $\|B_0r^\delta_0\| \le b\delta$, then take $t_\delta = 0$. Otherwise choose t_δ such that*

$$\|B_{t_\delta}r^\delta_{t_\delta}\| \;\le\; b\delta,$$

and such that for some $s_\delta \in [\theta t_\delta, t_\delta]$, we have

$$b\delta \;\le\; \|B_{s_\delta}r^\delta_{s_\delta}\|.$$

These choices of t_δ are possible, since we have an analogue to (3.5),

$$\big|\, \|B_t r_t^\delta\| - \|B_t H_t A(u_0 - u_*)\| \,\big| \le \tau_0 \delta \tag{3.52}$$

(this follows from (3.51)), and since the second term in the left-hand side of (3.52) tends to 0 as t tends to ∞,

$$\|B_t H_t A(u_0 - u_*)\| \le \gamma_0 \kappa_1 t^{-1} \|u_0 - u_*\| \to 0 \qquad \text{as } t \to \infty.$$

We now state the second main theorem of this section.

Theorem 3.3.6 *Let $u_t^\delta,\ t \ge 0$, be defined by (3.2), and assume that (2.3), and (3.47)-(3.50) hold. Then, parameter choices 3.3.4 and 3.3.5 are quasioptimal.*

Proof. We first observe that it follows from (3.52) that for t_δ chosen by parameter choices 3.3.4 or 3.3.5, we have for some $s_\delta \in [\theta t_\delta,\ t_\delta]$

$$\|B_{t_\delta} H_{t_\delta} A(u_0 - u_*)\| \le (b_1 + \tau_0)\delta, \tag{3.53}$$

$$\|B_{s_\delta} H_{s_\delta} A(u_0 - u_*)\| \ge (b_0 - \tau_0)\delta, \qquad \text{if } t_\delta \ne 0, \tag{3.54}$$

where $\theta = 1$ for Choice 3.3.4, and $b_0 = b_1 = b$ for Choice 3.3.5.

We take any $t \ge 0$, and it is sufficient to prove that for some K independent of $t,\ u_*,\ u_0$ and δ,

$$\|H_{t_\delta}(u_0 - u_*)\| + t_\delta \delta \le K\Big(\|H_t(u_0 - u_*)\| + t\delta\Big). \tag{3.55}$$

(i) We first prove (3.55) for the case $t \ge t_\delta$. Then $t_\delta \delta \le t\delta$, and we shall estimate the first term in the left-hand side of (3.55): (3.49) and (3.53) imply

$$\begin{aligned}\|H_{t_\delta}(u_0 - u_*)\| &\le \kappa\Big(\|H_t(u_0 - u_*)\| + t\|B_{t_\delta} H_{t_\delta} A(u_0 - u_*)\|\Big)\\ &\le \kappa\Big(\|H_t(u_0 - u_*)\| + (b_1 + \tau_0)t\delta\Big),\end{aligned}$$

which yields (3.55) for an appropriate K.

(ii) We prove (3.55) for the case $t \le t_\delta$. Then (3.50) implies

$$\|H_{t_\delta}(u_0 - u_*)\| \le \gamma_0 \|H_t(u_0 - u_*)\|$$

and in the sequel we estimate the second term in the left-hand side of (3.55). If $t \ge \theta t_\delta$, then $t_\delta \delta \le \theta^{-1} t\delta$, and if on the other side $0 \le t < \theta t_\delta$, then (3.48), (3.50) and (3.54) imply

$$\begin{aligned}(b_0 - \tau_0)\delta &\le \|B_{s_\delta} H_{s_\delta} A(u_0 - u_*)\| \le \kappa_1 s_\delta^{-1} \|H_{s_\delta}(u_0 - u_*)\|\\ &\le \kappa_1 \gamma_0 s_\delta^{-1} \|H_t(u_0 - u_*)\| \le \kappa_1 \gamma_0 \theta^{-1} t_\delta^{-1} \|H_t(u_0 - u_*)\|,\end{aligned}$$

which supplies an estimate for $t_\delta \delta$ being sufficiently good, and this completes the proof. □

The iterated method of Lavrentiev

We recall that Lavrentiev's (iterated) method for weakly sectorial A is defined by

$$\begin{aligned}(I + tA)v_{n+1} &= v_n + tf, \qquad n = 0, 1, 2, ..., m - 1,\\ u_t &:= v_m,\end{aligned}$$

with $v_0 = u_0$ and fixed integer $m \ge 1$ (see (2.6), (2.7)).

Theorem 3.3.7 *Let $A \in \mathcal{L}(X)$ be weakly sectorial (with $M_0 \ge 1$ in (1.2)) Then Parameter choices 3.3.4 and 3.3.5 for Lavrentiev's (iterated) method are quasioptimal if we take $B_t := (I + tA)^{-1}$.*

Proof. It is already shown show that u_t^δ is of the form (3.2) with G_t as in Theorem 2.1.2, and the growth estimate (2.3) for $\|G_t\|$ holds. We obviously have (3.47) and (3.48) with $\kappa_0 = M_0,\ \kappa_1 = M_0 + 1$, and (3.49) and (3.50) are immediate consequences of Corollary 2.1.3, and then the assertion follows with Theorem 3.3.6. □

Remarks

We shall motivate the parameter choice for the iterated method of Lavrentiev. Since here instead of the semigroup property (3.41) we have only the weaker property (3.50), we cannot apply the same techniques as in the preceding section in order to prove the quasioptimality for the common discrepancy principle, namely, estimate (3.45) does not apply to Lavrentiev's iterated method. In fact, the parameter can be too large if it is chosen according to common discrepancy principles. We therefore introduce the operator B_t, and due to (3.48) we can expect a faster decay for $\|B_{t_\delta} r_t^\delta\|$ than for $\|r_{t_\delta}^\delta\|$; hence if t_δ is chosen such that $\|B_{t_\delta} r_{t_\delta}^\delta\| \approx \delta$, then we can hope that the corresponding parameter t_δ is small enough. The proof of the quasioptimality for this choice, however, uses the fact, that instead of (2.17) for iterative methods we have the stronger result (3.49) where again B_t arises. This enables us to show that $\|H_{t_\delta}(u_0 - u_*)\|$ in fact is small enough even for the modified parameter choices.

3.4 On noise-level-free parameter choices and stopping rules

3.4.1 General results

Again we assume that $A \in \mathcal{L}(X)$ is weakly sectorial. A natural question is whether parameter choices $t = t_\delta(f^\delta)$ like

$$\|Au_t^\delta - f^\delta\| \approx \delta,$$

yield good results if the condition '$\|f^\delta - f_*\| \le \delta$' is replaced by the weaker assumption '$\|f^\delta - f^*\| \to 0$ as $\delta \to 0$'. Note that δ then becomes a free parameter. Let us for convenience assume that $\overline{\mathcal{R}(A)} = X$. The answer then is affirmative in the well-posed case: Then A is one-to-one and onto, and obviously

$$\|u_{t_\delta}^\delta - u_*\| \ = \ \mathcal{O}(\delta + \|f^\delta - f_*\|) \qquad \text{as} \ \ \delta \to 0.$$

In the ill-posed case, however, the answer to our question is negative. We put this into a more general frame-work.

Theorem 3.4.1 *Let $A \in \mathcal{L}(X)$ be weakly sectorial and let $\mathcal{P}_\delta : X \to X,\ \delta > 0$, be a stable approximation method for $A^\dagger$, i.e., for any $u_* \in \overline{\mathcal{R}(A)}$ and $f^\delta \in X$ with $\|Au_* - f^\delta\| \to 0$ as $\delta \to 0$ one has*

$$\|\mathcal{P}_\delta f^\delta - u_*\| \to 0 \qquad \textit{as} \ \ \delta \to 0.$$

Then $A^\dagger$ is bounded.

Proof. Note first that the assumption on $\mathcal{P}_\delta$ implies that

$$\mathcal{P}_\delta f \to u \ = \ A^\dagger f \qquad \text{as} \ \ \delta \to 0.$$

if $u \in \overline{\mathcal{R}(A)},\ f = Au$ (just take $f_\delta = f, \delta > 0$, fixed). If $A^\dagger$ is unbounded then take any $f_* \in A(\overline{\mathcal{R}(A)})$, and then there are $\{f_k\} \subset A(\overline{\mathcal{R}(A)})$ with

$$\begin{aligned} \|A^\dagger f_* - A^\dagger f_k\| \ &\to \ \infty \qquad \text{as} \ \ k \to \infty \\ \|f_* - f_k\| \ &\to \ 0 \qquad \text{as} \ \ k \to \infty. \end{aligned}$$

For k then choose δ_k so small that

$$\|\mathcal{P}_{\delta_k} f_k - A^\dagger f_k\| \to 0 \qquad \text{as} \ \ k \to \infty.$$

Then we necessarily have

$$\|\mathcal{P}_{\delta_k} f_k - A^\dagger f_*\| \to \infty \qquad \text{as} \ \ k \to \infty,$$

which contradicts the assumption that $\{\mathcal{P}_\delta\}$ is a stable approximation of $A^\dagger$. □

As an immediate consequence we observe that δ-free stable approximations $\mathcal{P} \equiv \mathcal{P}_\delta,\ \delta > 0$, for $A^\dagger$ only exists in the well-posed case, and then it equals $A^\dagger$.

Corollary 3.4.2 *Let $A \in \mathcal{L}(X)$ be weakly sectorial and let $\mathcal{P} : X \to X$ be a δ-free stable approximation of $A^\dagger$. Then $A^\dagger$ is bounded, and on $A(\overline{\mathcal{R}(A)})$ it equals $\mathcal{P}$,*

$$\mathcal{P} f \ = \ A^\dagger f \qquad \text{for all} \ \ f \in A(\overline{\mathcal{R}(A)}).$$

Hence, if $A^\dagger$ is bounded but ill-conditioned, then there is no regularization effect by this δ-free stable approximation $\mathcal{P}$ of $A^\dagger$.

There exist nevertheless several interesting suggestions for δ-free parameter choices and stopping rules for solving ill-posed problems, and we will present some of them.

3.4.2 Examples for δ-free methods

A first δ-free method

We again consider any method of type (3.2) supplying $\{u_t^\delta\}_t$ and fulfilling (2.2)-(2.5). The first δ-free parameter choice strategy is based on the following observation. Assume that $u_0 - u_* \in \mathcal{R}(A^\alpha)$ for some $0 < \alpha \le \alpha_0 - 1$ $(0 < \alpha < \infty$, if the qualification α_0 is infinite), and let $\varrho := \|u_0 - u_*\|_\alpha$. Then for the error $u_t^\delta - u_*$ and the defect $r_t^\delta = Au_t^\delta - f^\delta$ we have the following elementary estimates, for fixed δ,

$$e(t) \;:=\; \|u_t^\delta - u_*\| \;\le\; \gamma_\alpha \varrho t^{-\alpha} \;+\; \gamma_* t\delta \;=:\; e_1(t), \tag{3.56}$$

$$\phi(t) \;:=\; t\|Au_t^\delta - f^\delta\| \;\le\; \gamma_{\alpha+1}\varrho t^{-\alpha} \;+\; \gamma_0 t\delta \;=:\; \phi_1(t). \tag{3.57}$$

Our aim is to minimize the unknown function $e(t)$, and the typical assumption in ill-posed problems is that $e(t)$ behaves like its estimator $e_1(t)$. We observe that $e_1(t)$ is similar to $\phi_1(t)$, if $\gamma_\alpha \approx \gamma_{\alpha+1}$ and $\gamma_* \approx \gamma_0$, and both functionals e_1 and ϕ_1 are minimized for a unique and finite $t_* \ge 0$. If we make the heuristic assumption that the test functional $\phi(t)$ behaves like its estimator ϕ_1, then the only thing we have to do is to find a minimizer t_* for ϕ, and this then should be close to the minimizer for e.

We take notice of the following facts, however.

- $e(t)$ does not behave like $\phi(t)$ near $t = 0$; we have $\phi(t) = 0$ and $e(0) = \|u_0 - u_*\|$ which is > 0, in general. Hence, the minimization process for the test functional ϕ should be done for $t \ge \eta$ only, with some $\eta > 0$ to be specified.

- If $f^\delta \in \mathcal{R}(A)$ and if the qualification α_0 of the underlying method is > 1, then (2.15) and the Banach-Steinhaus theorem implies

  $$t\|Au_t^\delta - f^\delta\| \to 0 \qquad \text{as } \; t \to \infty,$$

  hence ϕ is minimized at '$t_* = \infty$' and thus does not behave like ϕ_1, and moreover $\mathcal{P}f^\delta = A^\dagger f^\delta$. Hence, as above mentioned, if $A^\dagger$ is bounded but ill-conditioned, then there is no regularization effect by this method.

- Nothing can be said, in general, about the behaviour of $\phi(t)$ as $t \to \infty$, if $f^\delta \in \overline{\mathcal{R}(A)}\backslash\mathcal{R}(A)$. Let us now assume that f^δ is not in closure of the range of A,

  $$f^\delta \notin \overline{\mathcal{R}(A)}.$$

  Then

  $$t\|Au_t^\delta - f^\delta\| \to \infty \qquad \text{as } \; t \to \infty.$$

  Hence, if there is a (possibly nonunique) minimizer $t_* \ge \eta$, then it must be finite, and then

  $$t_*\|Au_{t_*}^\delta - f^\delta\| \;=\; \inf_{t \ge \eta} t\|Au_t^\delta - f^\delta\|,$$

  and we define

  $$\mathcal{P}f^\delta \;=\; u_{t_*}^\delta,$$

  and do hope that this will be an approximation for a solution u_* of $Au = f_*$.

The same approach can be used, of course, for iteration methods. Here we have the iteration index n instead of the parameter t, and it is natural to start the minimization process with $n = 1$, i.e., to require

$$n_*\|Au_{n_*}^\delta - f^\delta\| \;=\; \inf_{n \ge 1} n\|Au_n^\delta - f^\delta\|.$$

The quasioptimality criterion

Here the basic method is Lavrentiev's classical method, this is (2.6), (2.7) with $m = 1$, and the test functional to be minimized is

$$\tilde{\phi}(t) := t\|(I + tA)^{-1} r_t^\delta\|$$

(replacing $\phi(t) = t\|r_t^\delta\|$). The same motivation as above applies, since we have the estimate

$$\tilde{\phi}(t) \;\le\; \tilde{\gamma}_\alpha \varrho t^{-\alpha} \;+\; \tilde{\gamma}_0 t\delta.$$

with $\tilde{\gamma}_\alpha,\ \tilde{\gamma}_0$ as for Lavrentiev's iterated method with $m = 2$. It is proposed to take the largest local minimizer t_* and define $\mathcal{P} f^\delta = u_{t_*}^\delta$.

The L-curve criterion

It is observed by several authors that that the curve $t \mapsto (\|Au_t^\delta - f^\delta\|^q, \|u_t^\delta\|^q)$ has the form of an L (for certain $q > 0$), and it is proposed to take the t_* that is connected to the 'corner', and define $\mathcal{P} f^\delta = u_{t_*}^\delta$. Note, however, that in general we do not have $\mathcal{P} f^\delta \neq A^\dagger f^\delta$, as it should be in view of Corollary 3.4.2.

Bibliographical notes and remarks

Section 3.2 The results are taken from Plato [57], and the main theorems Theorems 3.2.6 and 3.2.3 are generalizations of Vainikko [71]. Proposition 3.2.4 on the divergence of the discrepancy principle for Lavrentiev's classical method generalizes earlier results in Hilbert spaces, see e.g., Leonov [40] for a late result on that.

Section 3.3 These results can be found in Plato and Hämarik [58]. Theorem 3.3.3 in fact generalizes an approach of Raus [62] (see also Raus [63], Gfrerer [17] and Engl, Gfrerer [12] for the non-selfadjoint case) to Banach spaces. In fact they have shown that estimates like (3.46) are valid in Hilbert spaces.

Section 3.4 The terminology 'regularization method' is due to Tikhonov, see [70]. Corollary 3.4.2 is due to Bakushinskiĭ [3]. In Subsection 3.4.1, the motivation for δ-free methods is borrowed from Hanke and Raus [25], and the quasioptimality criterion is due to Tikhonov, see e.g., [70]. Leonov [39] proposes to take the largest local minimizer of $\tilde{\phi}(t)$, and he in fact regularizes with this approach positive semidefinite problems $Au = f$ with $\mathcal{N}(A) = \{0\}$ in finite-dimensional Hilbert spaces. For the L-curve criterion consult Hansen [26].

Kapitel 4

On the accuracy of algorithms in the presence of noise

4.1 General results

We show that the convergence rates obtained in the previous chapter cannot be improved, in general. To this end we give a short introduction into the theory of the accuracy of algorithms in the presence of noise. Throughout this section we assume that X is a real or complex Banach space and that $A \in \mathcal{L}(X)$.

Definition 4.1.1 *1. Let $M \subset X$ and $\delta > 0$ be fixed.*
(a) The maximal error of an algorithm $\mathcal{P}_\delta : X \to X$ with respect to the error level δ and the set M is defined by

$$E_{\mathcal{P}_\delta}(M,\delta) := \sup \left\{ \|u_* - \mathcal{P}_\delta f^\delta\| : \ u_* \in M, \ f^\delta \in X, \ \|Au_* - f^\delta\| \le \delta \right\}.$$

(b) We define the best possible maximal error with respect to $\delta > 0$ and M by

$$E(M,\delta) := \inf_{\mathcal{P}_\delta : X \to X} E_{\mathcal{P}_\delta}(M,\delta).$$

2. A method $\mathcal{P}_\delta : X \to X$, $0 < \delta \le \delta_0$, is called order optimal with respect to $M \subset X$, if for some $c > 0$ one has

$$E_{\mathcal{P}_\delta}(M,\delta) \ \le \ cE(M,\delta), \qquad 0 < \delta \le \delta_0.$$

Note that in our applications $\mathcal{P}_\delta f^\delta = u^\delta_{t(\delta,f^\delta)}$ or $\mathcal{P}_\delta f^\delta = u^\delta_{n(\delta,f^\delta)}$. The following definitions are useful to obtain upper and lower bounds for $E(M,\delta)$.

Definition 4.1.2 *For $M \subset X$ and $\delta > 0$ we introduce ω and e,*

$$\begin{aligned} \omega(M,\delta) \ &:= \ \sup \left\{ \|u_1 - u_2\| : \ u_1, \ u_2 \in M, \ \|Au_1 - Au_2\| \le \delta \right\}, \\ e(M,\delta) \ &:= \ \sup \left\{ \|u\| : \ u \in M, \ \|Au\| \le \delta \right\}. \end{aligned}$$

If $\mathcal{N}(A) = \{0\}$ then $\omega(M,\delta)$ is the modulus of continuity of A^{-1} on $A(M)$. The connection between $e(M,\delta)$ and $\omega(M,\delta)$ for sets M with additional properties is described in the following lemma.

Proposition 4.1.3 *Let $M \subset X$ be convex and assume further that $M = -M$. Let $\delta > 0$. Then*

$$2e(M,\delta) \ = \ \omega(M,2\delta).$$

Proof. To obtain $2e \le \omega$, take any $u \in M$ with $\|Au\| \le \delta$. Then $-u \in M$ and

$$\|Au - A(-u)\| \ = \ 2\|Au\| \le 2\delta,$$

i.e.,

$$2\|u\| \ = \ \|u - (-u)\| \ \le \ \omega(M,2\delta).$$

To obtain $\omega \leq 2e$, let $u_1,\ u_2 \in M$ with $\|Au_1 - Au_2\| \leq 2\delta$. Then $u := (u_1 - u_2)/2 \in M$ and $\|Au\| \leq \delta$, hence $\|u_1 - u_2\| = 2\|u\| \leq 2e(M, \delta)$. □

Due to the following estimate we can hope to get sharp estimates for $E(M, \delta)$. In the following, B denotes the closed unit ball in X,

$$B := \{\ u \in X : \ \|u\| \leq 1\ \}.$$

Theorem 4.1.4 *For $M \subset X$ and $\delta > 0$ one has*

$$\frac{1}{2}\omega(M, 2\delta) \ \leq\ E(M, \delta) \ \leq\ \omega(M, 2\delta). \tag{4.1}$$

Proof. We first give a proof of the first estimate. To this end, let $\mathcal{P}_\delta : X \to X$ be an arbitrary algorithm and $u_1,\ u_2 \in X$ with $\|Au_1 - Au_2\| \leq 2\delta$. For $f = (Au_1 + Au_2)/2$ we have $\|Au_1 - f\| \leq \delta$ and $\|Au_2 - f\| \leq \delta$, hence

$$\begin{aligned} E_{\mathcal{P}_\delta}(M, \delta) &\geq \max\{\ \|u_1 - \mathcal{P}_\delta f\|,\ \|u_2 - \mathcal{P}_\delta f\|\ \} \ \geq\ \frac{1}{2}\Big(\|u_1 - \mathcal{P}_\delta f\| \ +\ \|u_2 - \mathcal{P}_\delta f\|\Big) \\ &\geq \frac{1}{2}\|u_1 - u_2\|, \end{aligned}$$

and that implies

$$\frac{1}{2}\omega(M, 2\delta) \ \leq\ E_{\mathcal{P}_\delta}(M, \delta).$$

In order to prove the second estimate in (4.1) we consider the sets

$$\begin{aligned} A(M) + \delta \cdot B &= \{\ f \in X : \text{ there is an } u \in M \text{ with } \|Au - f\| \leq \delta\ \}, \\ H_\delta(f) &:= \{\ u \in M : \|Au - f\| \leq \delta\ \}, \qquad \text{for } f \in A(M) + \delta \cdot B. \end{aligned}$$

$H_\delta(f)$ is a nonempty set and thus there exists an algorithm $\mathcal{P}_\delta : X \to X$ with

$$\mathcal{P}_\delta f \ \in\ H_\delta(f) \qquad \text{for all } f \in A(M) + \delta \cdot B.$$

Then

$$\begin{aligned} E_{\mathcal{P}_\delta}(M, \delta) &= \sup\{\ \|u - \mathcal{P}_\delta f\| :\ f \in A(M) + \delta \cdot B,\ u \in H_\delta(f)\ \} \\ &\leq \sup\{\ \|u - \mathcal{P}_\delta f\| :\ f \in A(M) + \delta \cdot B,\ u \in H_\delta(f),\ \|Au - A\mathcal{P}_\delta f\| \leq 2\delta\ \} \\ &\leq \omega(M, 2\delta), \end{aligned}$$

and this completes the proof. □

As an immediate consequence we have the following result.

Corollary 4.1.5 *Let $M \subset X$ be convex with $M = -M$, and let $\delta > 0$. Then*

$$e(M, \delta) \ \leq\ E(M, \delta) \ \leq\ 2e(M, \delta). \tag{4.2}$$

In the following proposition it is shown under some general assumptions on $\mathcal{P}_\delta : X \to X, \quad \delta > 0$, that it is order optimal with respect to M und δ. Note, however, that this result cannot be applied to the algorithms defined through various discrepancy principles and to $M = M_{\alpha,\varrho}$ (to be defined in Definition 4.2.1) since (4.3) cannot be guaranteed for them.

Proposition 4.1.6 *Let $\emptyset \neq M \subset X$ be convex and let $M = -M$. Assume that for*

$$\mathcal{P}_\delta : X \to X, \qquad \delta > 0,$$

there are constants $c_1,\ c_2$ with

$$\begin{aligned} \mathcal{P}_\delta f &\in c_1 M \qquad \textit{for all } f \in A(M) + \delta \cdot B, \\ \|A\mathcal{P}_\delta f - f\| &\leq c_2\delta \qquad \textit{for all } f \in A(M) + \delta \cdot B. \end{aligned} \tag{4.3}$$

Then $\{\mathcal{P}_\delta\}_{\delta>0}$ is order optimal with respect to M, more specifically,

$$E_{\mathcal{P}_\delta}(M, \delta) \ \leq\ \Big(1 + \max\{c_1, c_2\}\Big) \cdot e(M, \delta), \qquad \delta > 0.$$

Proof. For $u \in M,\ f \in X$ with $\|Au - f\| \le \delta$ we have $\mathcal{P}_\delta f \in c_1 M$, therefore $\mathcal{P}_\delta f - u \in c_1 M - M = (1 + c_1)M$, and

$$\|A(\mathcal{P}_\delta f - u)\| \ \le\ \|A\mathcal{P}_\delta f - f\| \ +\ \|f - Au\| \ \le\ (c_2 + 1)\delta.$$

Thus, with

$$m := \max\{c_1, c_2\}$$

we have

$$\frac{1}{1+m}(\mathcal{P}_\delta f - u) \ \in\ M, \qquad \|A\Big(\frac{1}{1+m}(\mathcal{P}_\delta f - u)\Big)\| \ \le\ \delta,$$

that implies

$$\|\frac{1}{1+m}(\mathcal{P}_\delta f - u)\| \ \le\ e(M, \delta),$$

i.e.,

$$\|\mathcal{P}_\delta f - u\| \ \le\ (1 + m)e(M, \delta). \quad \Box$$

4.2 Source sets $M = M_{\alpha,\varrho}$

4.2.1 General results

We again assume in this subsection that X is a Banach space. Now we take a closer look at those sets $M = M_{\alpha,\varrho}$ which arise in our context.

Definition 4.2.1 *Let $A \in \mathcal{L}(X)$ be weakly sectorial. For $\alpha > 0,\ \varrho \ge 0$, the source-set $M_{\alpha,\varrho}$ is defined by*

$$M_{\alpha,\varrho} := \big\{\ u \in \mathcal{R}(A^\alpha) :\ \|u\|_\alpha \le \varrho\ \big\}.$$

The first interpolation inequality enables us to prove part 1 of the following result. For conclusions from part 2 of the next theorem we refer to the remarks following the proof of it.

Theorem 4.2.2 *Let $A \in \mathcal{L}(X)$ be weakly sectorial (with M_0 as in (1.2)), and let $\delta > 0,\ \varrho > 0$.*
1.

$$e(M_{\alpha,\varrho}, \delta) \ \le\ C(\varrho\delta^\alpha)^{1/(\alpha+1)},$$

with some constant $C = C(M_0, \alpha)$.

2. If $(\delta/\varrho)^{1/(\alpha+1)} \in \sigma_{ap}(A)$, then $e(M_{\alpha,\varrho}, \delta) \ \ge\ (\varrho\delta^\alpha)^{1/(\alpha+1)}$.

Proof. We have for $\epsilon > 0$ arbitrary small,

$$\begin{aligned} e(M_{\alpha,\varrho}, \delta) &= \sup\big\{\ \|u\| :\ u \in \mathcal{R}(A^\alpha),\ \|u\|_\alpha \le \varrho,\ \|Au\| \le \delta\ \big\} \\ &\le \sup\big\{\ \|A^\alpha z\| :\ z \in X,\ \|z\| \le \varrho + \epsilon,\ \|A^{\alpha+1} z\| \le \delta\ \big\} \\ &\le C\big((\varrho + \epsilon)\delta^\alpha\big)^{1/(\alpha+1)}, \end{aligned}$$

where the first interpolation inequality has been used. The assertion follows by letting $\epsilon \to 0$.
We prove the second part. To this end, let $\lambda = (\delta/\varrho)^{1/(\alpha+1)}$ be an approximate eigenvalue of A, i.e., there exist $z_n \in X,\ n = 0, 1, ...$, with

$$\|z_n\| \ =\ \varrho, \qquad Az_n - \lambda z_n \to 0 \qquad \text{as}\ \ n \to \infty.$$

Corollary 1.1.20 implies that $A^{\alpha+1} z_n - \lambda^{\alpha+1} z_n \to 0$ as $n \to \infty$, and then

$$\|A^{\alpha+1} z_n\| \to \lambda^{\alpha+1}\varrho \ =\ \delta \qquad \text{as}\ \ n \to \infty.$$

Also $A^\alpha z_n - \lambda^\alpha z_n \to 0$ as $n \to \infty$, i.e.,

$$\|A^\alpha z_n\| \to \lambda^\alpha \varrho \ =\ (\varrho\delta^\alpha)^{1/(\alpha+1)} \qquad \text{as}\ \ n \to \infty.$$

Let

$$a_n := \min\big\{\ 1,\ \delta/\|A^{\alpha+1} z_n\|\ \big\}.$$

We have $A^\alpha z_n \neq 0$ for n sufficiently large, and then a_n is well-defined. Moreover $a_n \nearrow 1$ as $n \to \infty$, and with

$$w_n \;=\; a_n z_n, \qquad n = 0, 1, ...,$$

we have

$$\begin{aligned} \|w_n\| \;&\nearrow\; \varrho \qquad \text{as } \; n \to \infty, \\ \|A^{\alpha+1} w_n\| \;&\nearrow\; \delta \qquad \text{as } \; n \to \infty, \\ \|A^\alpha w_n\| \;&\to\; (\varrho \delta^\alpha)^{1/(\alpha+1)} \qquad \text{as } \; n \to \infty, \end{aligned}$$

and that completes the proof. □

We can therefore conclude that the estimates in Chapter 3 (and the next chapter) cannot be improved, if there is a sequence of positive approximate eigenvalues of A converging to 0, since then

$$\limsup_{\delta \to 0} \; e(M_{\alpha,\varrho}, \delta)/(\varrho \delta^\alpha)^{1/(\alpha+1)} \;\geq\; 1.$$

For a related observation we refer to the end of this chapter.

4.2.2 On positive semidefinite operators

For symmetric positive semidefinite $A \in \mathcal{L}(X)$ in Hilbert spaces X, much more can be stated; we will present some of these results without proofs. For example, one has

$$e(M_{\alpha,\varrho}, \delta) \;=\; E(M_{\alpha,\varrho}, \delta), \tag{4.4}$$

which improves (4.2). Moreover, one can compute $e(M_{\alpha,\varrho}, \delta)$ explicitly, and to state this for compact operators, the following lemma is helpful.

Lemma 4.2.3 *Let the continuous function* $g : [0, \infty) \to \mathbb{R}$ *be twice differentiable on* $(0, \infty)$ *and suppose that* $g(0) = 0$ *and* $g'(s) > 0, \; g''(s) < 0, \; s > 0$. *Let*

$$a_1 > a_2 > ... > 0 \qquad \textit{with} \;\; a_k \to 0 \;\; \textit{as } k \to \infty.$$

Furthermore let

$$\zeta(\delta) := \sup \Big\{ \sum_{l=1}^{\infty} g(a_l)\mu_l : \; \mu_l \geq 0, \; \sum_{l=1}^{\infty} \mu_l \;\leq\; 1, \; \sum_{l=1}^{\infty} a_l \mu_l \;\leq\; \delta \; \Big\}.$$

Then, for $0 < \delta \leq a_1$ *the function* ζ *is the interpolating linear spline for* g *with (an infinite numbers of) nodes* $a_1, a_2, ...$ *. For* $\delta \geq a_1$ *one has* $\zeta(\delta) = g(a_1)$*. In particular,* $\zeta(\delta) \leq g(\delta)$ *holds for any* $\delta > 0$.

Proof. We denote by s the interpolating linear spline for g with (an infinite numbers of) nodes $a_1, a_2, ...$, and for $\delta \geq a_1$ we define $s(\delta) = g(a_1)$. We shall show that $\zeta = s$. To this end, we first show that $\zeta(\delta) \geq s(\delta)$. If $\delta \in [0, a]$, then there is a unique k with

$$a_{k+1} \;\leq\; \delta \;\leq\; a_k,$$

and then take $t \in [0,1]$ such that $\delta = (1-t)a_{k+1} + t a_k$, and define $\mu_{k+1} := 1 - t, \; \mu_k := t$, and $\mu_l := 0$ otherwise. Then

$$\sum_{l=1}^{\infty} \mu_l \;=\; 1, \qquad \sum_{l=1}^{\infty} a_l \mu_l \;=\; \delta,$$

hence

$$\zeta(\delta) \;\geq\; (1-t) g(a_{k+1}) \;+\; t g(a_k) \;=\; s(\delta).$$

Finally, if $\delta \geq a_1$, then take $\mu_1 = 1$ and $\mu_l = 0$ otherwise and proceed in the same way.

We now show that $\zeta(\delta) \leq s(\delta)$. For that we assume first that $\delta = a_k$ for some k and take then $\mu_l \geq 0, \; l \geq 0$, with $\sum_{l=0}^{\infty} \mu_l \leq 1$ and $\sum_{l=0}^{\infty} \mu_l a_l \leq \delta$. g is an increasing, continuous and concave function, hence with $a_0 := 0$ and $\mu_0 := 1 - \sum_{l=1}^{\infty} \mu_l$ we get

$$\sum_{l=0}^{\infty} \mu_l g(a_l) \;\leq\; g\big(\sum_{l=0}^{\infty} \mu_l a_l \big) \;\leq\; g(\delta) \;=\; s(\delta),$$

hence

$$\zeta(a_k) \;\le\; s(a_k) \qquad \text{for all} \;\; k \ge 1.$$

s is also a monotonically increasing, concave and continuous function, hence from $\sum_{l=0}^{\infty} \mu_l a_l \le \delta$ one gets

$$\sum_{l=0}^{\infty} \mu_l g(a_l) \;=\; \sum_{l=0}^{\infty} \mu_l s(a_l) \;\le\; s\big(\sum_{l=0}^{\infty} \mu_l a_l\big) \;\le\; s(\delta). \qquad \square$$

We thus have:

Corollary 4.2.4 *Let X be a Hilbert space and $A \in \mathcal{L}(X)$ be self-adjoint, positive semidefinite and compact with non-closed range, and let $\lambda_1 > \lambda_2 > \dots > 0$ be the nonvanishing eigenvalues of A. For fixed $\alpha > 0$ and $\varrho > 0$ one has*

$$e(M_{\alpha,\varrho}, \delta)^2 \;=\; \varrho^2 \zeta\big((\delta/\varrho)^2\big), \qquad \textit{if} \;\; \delta/\varrho \;\le\; \lambda_1^{\alpha+1}, \tag{4.5}$$

where ζ is the interpolating linear spline for $g(s) = s^{\alpha/(\alpha+1)}$ with knots $\lambda_1^{2\alpha+2}, \lambda_2^{2\alpha+2}, \dots$. For $\delta/\varrho \ge \lambda_1^{\alpha+1}$ one has $e(M_{\alpha,\varrho}, \delta) = \varrho\lambda_1^{\alpha}$.

Proof. If P_l is the orthogonal projection onto $\mathcal{N}(A - \lambda_l I)$, then

$$Au \;=\; \sum_{l=1}^{\infty} \lambda_l P_l u, \qquad u \in X,$$

and thus

$$\begin{aligned} e(M_{\alpha,\varrho}, \delta)^2 \;&=\; \sup\big\{ \sum_{l=1}^{\infty} \lambda_l^{2\alpha} \|P_l z\|^2 : \;\; z \in X, \; \sum_{l=1}^{\infty} \|P_l z\|^2 \;\le\; \varrho^2, \; \sum_{l=1}^{\infty} \lambda_l^{2\alpha+2} \|P_l z\|^2 \;\le\; \delta^2 \;\big\} \\ &=\; \varrho^2 \sup\big\{ \sum_{l=1}^{\infty} \lambda_l^{2\alpha} \|P_l z\|^2 : \;\; z \in X, \; \sum_{l=1}^{\infty} \|P_l z\|^2 \;\le\; 1, \; \sum_{l=1}^{\infty} \lambda_l^{2\alpha+2} \|P_l z\|^2 \;\le\; (\delta/\varrho)^2 \;\big\}, \end{aligned}$$

and then a reformulation of Lemma 4.2.3 yields the assertion. $\square$

This corollary can be applied to show, under the conditions of Corollary 4.2.4, that

$$\liminf_{\delta \to 0} \; e(M_{\alpha,\varrho}, \delta)/(\varrho\delta^{\alpha})^{1/(\alpha+1)} \;=\; 0, \qquad \text{if} \;\; \lim_{k\to\infty} \lambda_{k+1}/\lambda_k \to 0 \text{ as } \; k \to \infty. \tag{4.6}$$

Bibliographical notes and remarks

Section 4.1 This is standard material that can be found, e.g., in Ivanov, Vasin and Tanana [30], except for Proposition 4.1.6 which is taken from Vainikko [72].

Section 4.2 (4.4) is Melkman and Micchelli [46] and Grigorieff and Plato [20]. Lemma 4.2.3 and Corollary 4.2.4 are due to Ivanov and Korolyuk [29] and can be found also in the textbooks by Morozov, e.g., [1], Section 2.11.3. For an extension of Corollary 4.2.4 for non-compact operators see Ivanov [28], and for (4.6) see Hegland [27].

Kapitel 5

The method of conjugate residuals

5.1 Introductory remarks

The most efficient iterative methods for solving symmetric problems in Hilbert spaces are conjugate gradient type methods, and one of them will be introduced in this chapter. Througout this chapter we assume that X is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ and corresponding norm $\|\cdot\|$, and we moreover assume that $A \in \mathcal{B}(X)$ is symmetric and positive semidefinite. Then for $f_* \in \mathcal{R}(A)$ we consider equations $Au = f_*$ which are assumed to be ill-posed, in general. A prominent example is Symm's integral equation.

Example 5.1.1 *Consider Symm's operator*

$$Au(t) := -\int_\Gamma \log|s-t|_2 \; u(s) \; ds \;=\; f(t), \qquad t \in \Gamma,$$

where $G \subset \mathbb{R}^2$ is assumed to have a smooth boundary $\Gamma = \partial G$. If diam$(G) < 1$, then $A : L^2(\Gamma) \to L^2(\Gamma)$ is bounded, symmetric and positive semidefinite, i.e., $A = A^* \geq 0$. Moreover, for $\alpha \geq 0$, A^α can be conceived as a mapping $L^2(\Gamma) \to H^\alpha(\Gamma)$, and it is one-to-one and onto then. Here, $H^\alpha(\Gamma)$ denote the Sobolev space of order α of functions $f : \Gamma \to \mathbb{R}$.

In the next section we introduce the the method of conjugate residuals (cr-method), and as a preparation we define Krylov subspaces.

Definition 5.1.2 *Let $r_0 \in X$. We denote by*

$$\mathcal{K}_n(A, r_0) \;=\; \operatorname{span}\{\; r_0, Ar_0, ..., A^{n-1} r_0 \;\}$$

the <u>n-th Krylov subspace with respect to A and r_0</u>.

Let $f \in X$ be an approximation (not excluding f_* itself) for the right-hand side of $Au = f_*$. Our aim is to generate (with some initial guess $u_0 \in X$) iteratively a possibly terminating sequence $u_1, \; u_2, ...$ with

$$u_n \;\in\; u_0 - \mathcal{K}_n(A, r_0), \tag{5.1}$$

$$\|Au_n - f\| \;=\; \min_{u \in u_0 - \mathcal{K}_n(A, r_0)} \|Au - f\|, \tag{5.2}$$

where $r_0 = Au_0 - f$. Note, that u_n minimize the norm of the defect over $u_0 - \mathcal{K}_n(A, r_0)$ while for the classical method of conjugate gradients by Hestenes and Stiefel, u_n minimizes the energy $J(u) = \frac{1}{2}\langle Au, u \rangle - \langle u, f \rangle$ over the same subspace.

With the notation

$$r_n \;:=\; Au_n - f,$$

(5.2) is equivalent to

$$Ar_n \in \mathcal{K}_n(A, r_0)^\perp, \tag{5.3}$$

(and the process terminates after the n-th step if $Ar_n = 0$). By $M^\perp$ we denote the orthogonal complement of a subset $M \subset X$, i.e., $M^\perp = \{\; u \in X : \; \langle u, v \rangle = 0 \text{ for all } v \in M \;\}$.

Note that here for n fixed, r_n does not depend linear on r_0, in contrast to the methods in the previous chapters. (5.1)-(5.3) are the main ingredients to prove the main results on the discrepancy principle, but a few more properties of the generating polynomials are needed for the proofs, and they are provided in Section 5.3. The reader being interested in the main results can go forward directly to Section 5.4.

For computational implementions we need an algorithm which supply u_n having the properties (5.1)-(5.2); in the following Section 5.2, the cr-method for computing these iterates u_n is presented and explained.

5.2 Introducing the method of conjugate residuals

The basic algorithm for computing u_n is:

Algorithm 5.2.1 *Step 0: Choose* $u_0 \in X$.
For $n = 0, 1, ...$:
1) If $Ar_n = 0$, *then terminate after step* n.
2) If otherwise $Ar_n \neq 0$, *then proceed with step n+1 and compute from* $u_n,\ d_{n-1}$:

$$d_n = -r_n + \beta_{n-1} d_{n-1}, \quad \beta_{n-1} = \frac{\langle Ar_n, r_n\rangle}{\langle Ar_{n-1}, r_{n-1}\rangle}, \tag{5.4}$$

$$u_{n+1} = u_n + \alpha_n d_n, \qquad \alpha_n = \frac{\langle Ar_n, r_n\rangle}{\|Ad_n\|^2}. \tag{5.5}$$

Here we assume $d_{-1} = 0,\ \beta_{-1} = 0$. It follows from (5.4) and (5.5) that

$$Ad_n = -Ar_n + \beta_{n-1} Ad_{n-1}, \tag{5.6}$$

$$r_{n+1} = r_n + \alpha_n Ad_n, \tag{5.7}$$

and in fact in any step for computational reasons Ad_n and r_{n+1} are computet by (5.6) and (5.7) and stored until the next step, so that only one operator-vector multiplication has to be performed in every step (to obtain Ar_n).

It is shown in the course of Section 5.2 that the iterates u_n defined by Algorithm 5.2.1 in fact have the two characteristic properties as described in (5.1) and (5.2).

5.2.1 Krylov subspaces and the termination case

The cr-method uses Krylov subspaces $\mathcal{K}_n(A, r_0)$ as sequence of subspaces in order to create A^2-conjugate search directions d_n and approximations u_n, and the following lemma does apply when iteration terminates after a finite number of steps.

Lemma 5.2.2 *For arbitrary* $r_0 \in X$ *and* $n \geq 0$ *the following assertions are equivalent:*

(a) $r_0, Ar_0, ..., A^n r_0$ *are linearly dependent.*

(b) $\mathcal{K}_n(A, r_0) = \mathcal{K}_{n+1}(A, r_0)$.

(c) $A(\mathcal{K}_n(A, r_0)) \subset \mathcal{K}_n(A, r_0)$.

(d) There is a linear subspace $M \subset X$ *with* $\dim M \leq n,\ r_0 \in M$, *which is invariant under* A, *i.e.,* $A(M) \subset M$.

Proof.
$(a) \Rightarrow (b)$: The condition implies first that there are an $m \leq n$ and constants $\gamma_0, ..., \gamma_{m-1}$, such that

$$A^m r_0 = \sum_{j=0}^{m-1} \gamma_j A^j r_0.$$

Then, however,

$$A^n r_0 \;=\; \sum_{j=n-m}^{n-1} \gamma_{j-(n-m)} A^j r_0,$$

and this gives $A^n r_0 \in \mathcal{K}_n(A, r_0)$.

$(b) \Rightarrow (c)$: $A(\mathcal{K}_n(A, r_0)) \subset \mathcal{K}_{n+1}(A, r_0) = \mathcal{K}_n(A, r_0)$.
$(c) \Rightarrow (d)$: Take $M = \mathcal{K}_n(A, r_0)$.
$(d) \Rightarrow (a)$: The assumption implies $A^j r_0 \in M, \quad j = 0, 1, ...,$ and therefore $\dim \operatorname{span}(r_0, ..., A^n r_0) \le n$, and this implies (a). □

The next proposition is helpful to study the situations under which the algorithm breaks down.

Proposition 5.2.3 *Assume that for $n \ge 1$,*

$$\mathcal{K}_{n-1}(A, r_0) \;\ne\; \mathcal{K}_n(A, r_0), \qquad \mathcal{N}(A) \cap \mathcal{K}_n(A, r_0) = \{0\}. \tag{5.8}$$

Then there is a unique u_n with (5.1), (5.2), and then

$$r_n \;=\; Au_n - f \;\in\; \mathcal{K}_{n+1}(A, r_0). \tag{5.9}$$

Moreover:

(a) If $Ar_n \ne 0$ then $r_n \notin \mathcal{K}_n(A, r_0)$ and the method of conjugate residuals can continue,

$$\mathcal{K}_{n+1}(A, r_0) \;=\; \mathcal{K}_n(A, r_0) \oplus \operatorname{span}(r_n), \qquad \mathcal{N}(A) \cap \mathcal{K}_{n+1}(A, r_0) \;=\; \{0\}.$$

(b) Consider now the case $Ar_n = 0$. Then the cr-method stops, and $u_n = u_0 - A^\dagger(Au_0 - f)$ (that is the best one can expect). Moreover, for that break-down case we distinguish two different situations:

- *If $r_n = 0$ then $Au_n = f$, and*

$$\mathcal{K}_n(A, r_0) \;=\; \mathcal{K}_{n+1}(A, r_0).$$

- *If $r_n \ne 0$ (and $Ar_n = 0$) then*

$$\mathcal{N}(A) \cap \mathcal{K}_{n+1}(A, r_0) \;\ne\; \{0\}, \qquad \mathcal{K}_n(A, r_0) \;=\; \mathcal{K}_{n+1}(A, r_0).$$

Proof. First, uniqueness of u_n follows from the second equality in (5.8), and (5.9) follows from

$$\begin{aligned} r_n \;&=\; Au_n - f \;\in\; r_0 - A(\mathcal{K}_n(A, r_0)) \\ &\subset\; r_0 - \mathcal{K}_{n+1}(A, r_0) \;=\; \mathcal{K}_{n+1}(A, r_0). \end{aligned} \tag{5.10}$$

In order to show that the assertion in (a) is satisfied we observe first that (5.3) implies

$$A^{1/2} r_n \in \Big(A^{1/2}\big(\mathcal{K}_n(A, r_0)\big)\Big)^{\perp}. \tag{5.11}$$

(5.11) yields $A^{1/2} r_n \notin A^{1/2}\big(\mathcal{K}_n(A, r_0)\big)$, and then $r_n \notin \mathcal{K}_n(A, r_0)$. Again (5.11) implies $\mathcal{N}(A^{1/2}) \cap \mathcal{K}_{n+1}(A, r_0) = \{0\}$ and thus $\mathcal{N}(A) \cap \mathcal{K}_{n+1}(A, r_0) = \{0\}$. The assertion for the first case in (b) follows with Lemma 5.2.2, since (5.10) implies $r_0 \in A(\mathcal{K}_n(A, r_0))$ and hence $r_0, Ar_0, ..., A^n r_0$ are linearly dependent. Finally, the second case in (b) follows again with (5.9): $0 \ne r_n \in \mathcal{K}_{n+1}(A, r_0), \;\; Ar_n = 0$. □

We can put Lemma 5.1.2 and Proposition 5.2.3 together and look at it from a different point of view:

Corollary 5.2.4 *Termination occurs if and only if r_0 is a linear combination of (a finite number of) eigenvectors of A. If that case happens, then r_0 can be written uniquely as a sum*

$$r_0 \;= \sum_{k=1}^{m} v_m$$

of eigenvectors corresponding to distinct eigenvalues. If none of the v_k's belong to the (possible) eigenvalue 0 of A, then the method breaks down after m steps; if otherwise $Av_k = 0$ for some k, then it breaks down after $m - 1$ steps.

5.2.2 Minimizing $J(u) = \|Au - f\|$ over subspaces with conjugate directions

Let n_* be any integer, and let $d_0,\ d_1, ...d_{n_*}$ be non-degenerated A^2-conjugate vectors, i.e.,

$$\begin{aligned} \langle Ad_j, Ad_k\rangle &= 0, \qquad j \neq k, \\ Ad_k &\neq 0, \qquad k = 0, ..., n_*, \end{aligned} \tag{5.12}$$

and let

$$\mathcal{D}_n := \text{span}(d_0, ...d_{n-1}), \qquad n = 0, 1, ..., n_* + 1,$$

and let $u_0 \in X$. Then there are unique $u_n,\ n = 1, ..., n_* + 1$, with

$$\begin{aligned} u_n &\in u_0 + \mathcal{D}_n, \\ \|Au_n - f\| &= \min_{u \in u_0 + \mathcal{D}_n} \|Au - f\|, \end{aligned}$$

and due to (5.12) and $Au_{n+1} - f \in A(\mathcal{D}_n)^\perp$, the computation of those u_n is very easy: we have for $n = 0, 1, ..., n_*$,

$$u_{n+1} = u_0 + \sum_{j=0}^{n} \alpha_j d_j, \tag{5.13}$$

$$\alpha_j := -\frac{\langle Ar_0, d_j\rangle}{\|Ad_j\|^2}, \qquad j = 0, 1, ..., n. \tag{5.14}$$

(5.13) and (5.14) imply that

$$u_{n+1} = u_n + \alpha_n d_n, \qquad n = 0, 1, ..., n_*, \tag{5.15}$$

i.e., the minimization of the defect over a sequence of subspaces $u_0 + \mathcal{D}_j,\ j = 0, 1, ...$, can be conceived as iteration process. Note that the directions $d_{n+1}, ..., d_{n_*}$ are not needed to compute u_{n+1}, and this allows to construct the search directions in the course of iteration.

We show that the coefficents in (5.14) are optimal with respect to the minimization of the functional $t \mapsto \|A(u_n + td_n) - f\|$. To this end, we observe that (5.13) yields $r_n = r_0 + \sum_{j=0}^{n-1} \alpha_j Ad_j$, and then (5.12) implies

$$\langle Ar_n, d_n\rangle = \langle Ar_0, d_n\rangle,$$

therefore the coefficients in (5.14) take the form

$$\alpha_n = -\frac{\langle Ar_n, d_n\rangle}{\|Ad_n\|^2}, \qquad n = 0, 1, ..., n_*, \tag{5.16}$$

i.e., the coefficients are optimal as claimed.

5.2.3 How to create conjugate directions in Krylov subspaces

Assume that the method of conjugate residuals does not terminate before step n, and that it has created nontrivial A^2-conjugate directions $d_0 = -r_0,\ d_1, ...,\ d_{n-1}$ and approximations $u_1,\ u_2, ..., u_n$ as described in Subsection 5.2.2, and that

$$\text{span}(d_0, ..., d_{j-1}) = \text{span}(r_0, ..., r_{j-1}) = \mathcal{K}_j(A, r_0), \qquad j = 1, ..., n,$$

If $Ar_n \neq 0$, then Schmidt orthogonalization of $d_0,\ d_1, ...,\ d_{n-1},\ -r_n$, with respect to $\langle\cdot,\cdot\rangle_{A^2}$, yield ($d_0,\ d_1, ..., d_{n-1}$ and) d_n. Note that (5.3) and $A(\mathcal{K}_{n-2}(A, r_0)) \subset \mathcal{K}_{n-1}(A, r_0)$ gives

$$\langle Ar_n, Ad_j\rangle = 0, \qquad j \leq n-2,$$

hence

$$\begin{aligned} d_n &= -r_n + \sum_{j=0}^{n-1} \frac{\langle Ar_n, Ad_j\rangle}{\|Ad_j\|^2} d_j \\ &= -r_n + \beta_{n-1} d_{n-1}, \end{aligned} \tag{5.17}$$

with

$$\beta_{n-1} := \frac{\langle Ar_n, Ad_{n-1}\rangle}{\|Ad_{n-1}\|^2}, \tag{5.18}$$

and we then have

$$\mathrm{span}(d_0, ..., d_n) \ = \ \mathrm{span}(r_0, ..., r_n) \ = \ \mathcal{K}_{n+1}(A, r_0). \tag{5.19}$$

The first equality in (5.19) follows immediately from the construction, and (5.9) gives $\mathrm{span}(r_0, ..., r_n) \subset \mathcal{K}_{n+1}(A, r_0)$.

Application of A to (5.17) leads to

$$-\langle Ar_n, d_n\rangle \ = \ \langle Ar_n, r_n\rangle, \tag{5.20}$$

and this yields the desired form (see (5.4)) $\alpha_n \ = \ \frac{\langle Ar_n, r_n\rangle}{\|Ad_n\|^2}$.

Finally, (5.7) gives (with formulas (5.16) and (5.18) for α_{n-1} and β_{n-1}, respectively, and with (5.20)),

$$\langle Ar_n, r_n\rangle \ = \ \langle Ar_n, r_{n-1}\rangle \ + \ \alpha_{n-1}\langle Ar_n, Ad_{n-1}\rangle \ = \ 0 + \beta_{n-1}\langle Ar_{n-1}, r_{n-1}\rangle,$$

hence

$$\beta_{n-1} \ = \ \frac{\langle Ar_n, r_n\rangle}{\langle Ar_{n-1}, r_{n-1}\rangle},$$

and we obtain the version described by Algorithms 5.2.1.

5.3 Fundamental properties of the method of conjugate residuals

Let us assume that the iteration process, described by Algorithm 5.2.1, does not terminate before step $n_* \geq 1$, i.e., $Ar_{n_*-1} \neq 0$ and produce iterates $u_1, \ u_2, ..., u_{n_*}$. Then, for any $0 \leq n \leq n_*$ there is a (unique) polynomial (depending on $A, \ u_0$ and f)

$$q_n \in \Pi_{n-1} \tag{5.21}$$

such that

$$u_n \ = \ u_0 - q_n(A)r_0. \tag{5.22}$$

Here, $\Pi_{-1} := \{0\}$, and

$$\Pi_{n-1} \ = \ \{ \ q : q \text{ is a polynomial of degree} \leq n-1 \ \}.$$

Furthermore, for the defect $r_n = Au_n - f$ we have

$$r_n \ = \ p_n(A)r_0, \tag{5.23}$$

with residual polynomials

$$p_n(t) \ = \ 1 - tq_n(t). \tag{5.24}$$

We now state one of the most important properties of the cr-method.

Minimum property. For $0 \leq n \leq n_*$,

$$\|r_n\| \ = \ \|\big(I - Aq_n(A)\big)r_0\| \ \leq \ \|\big(I - Aq(A)\big)r_0\| \qquad \text{for all} \ \ q \in \Pi_{n-1},$$

or equivalently,

$$\|r_n\| \ \leq \ \|s(A)r_0\| \qquad \text{for any } s \in \Pi_n, \ \ s(0) = 1. \tag{5.25}$$

In the following two subsections we recall some well-kown properties of the polynomials q_n and p_n.

5.3.1 Some properties of the residual polynomials p_n

We denote by $\{E_t\}_{t\in\mathbb{R}}$ the resolution of the identity (with respect to A).

Residual polynomials are conjugate. For $0 \le n, m \le n_*$, $n \ne m$, we have

$$\begin{aligned} 0 &= \langle Ar_n, r_m\rangle = \langle Ap_n(A)r_0,\ p_m(A)r_0\rangle \\ &= \int_0^a tp_n(t)p_m(t)\ d\|E_t r_0\|^2, \end{aligned} \tag{5.26}$$

i.e., $p_0, ..., p_{n_*}$ is a Sturm sequence and hence for $n = 1, ..., n_*$, the zeros $(t_{j,n})_{j=1,...,n}$ of p_n, also called Ritz-values of A with respect to $\mathcal{K}_n(A, r_0)$, are simple and have an intertwining property, more explicitly, if the zeros are ordered,

$$0 < t_{1,n} < t_{2,n} < ... < t_{n,n} \le \|A\|, \qquad n = 1, ..., n_*, \tag{5.27}$$

then

$$t_{k,n} < t_{k,n-1} < t_{k+1,n}, \qquad k = 1, ..., n-1, \qquad n = 2, ..., n_*, \tag{5.28}$$

is satisfied. Furthermore, due to $p_n(0) = 1$,

$$p_n(t) = \prod_{k=1}^{n}\Big(1 - \frac{t}{t_{k,n}}\Big), \tag{5.29}$$

which implies

$$p_n(t) \in [0, 1] \qquad \text{for all } \ 0 \le t \le t_{1,n}. \tag{5.30}$$

5.3.2 Some properties of the polynomials $q_n(t)$

(5.24) yields

$$q_n(t) = \frac{1 - p_n(t)}{t}, \qquad t > 0, \tag{5.31}$$

and other properties of q_n are listed in the following lemma.

Lemma 5.3.1 *For $1 \le n \le n_*$,*

$$q_n(0) = -p_n'(0) = \sum_{k=1}^{n} t_{k,n}^{-1}, \tag{5.32}$$

$$q_n(0) \le q_{n+1}(0), \tag{5.33}$$

$$q_n(0) \le t_{1,n}^{-1} + q_{n-1}(0), \tag{5.34}$$

$$q_n(0) = \sup_{t\in[0,t_{1,n}]} q_n(t). \tag{5.35}$$

Proof. (5.32) follows from (5.29), and the intertwining property (5.28) yield

$$\begin{aligned} q_n(0) &= \sum_{k=1}^{n} t_{k,n}^{-1} \le \sum_{k=1}^{n} t_{k,n+1}^{-1} \\ &\le \sum_{k=1}^{n+1} t_{k,n+1}^{-1} = q_{n+1}(0), \end{aligned}$$

this is (5.33). (5.28) yield also (for $n \ge 2$; the case $n = 1$ in (5.34) is trivial)

$$q_n(0) = t_{1,n}^{-1} + \sum_{k=2}^{n} t_{k,n}^{-1} \le t_{1,n}^{-1} + \sum_{k=1}^{n-1} t_{k,n-1}^{-1} = t_{1,n}^{-1} + q_{n-1}(0),$$

this is (5.34). Finally, in order to show (5.35) we observe that p_n is convex on $[0, t_{1,n}]$, since

$$\begin{aligned} p_n'(t) &= -\sum_{k=1}^{n} \frac{1}{t_{k,n}} \prod_{k=1}^{n}\Big(1 - \frac{t}{t_{k,n}}\Big), \\ p_n''(t) &= \sum_{k} \frac{1}{t_{k,n}} \sum_{l\ne k} \frac{1}{t_{l,n}} \prod_{j\ne k,l}\Big(1 - \frac{t}{t_{j,n}}\Big). \end{aligned}$$

Now, (5.35) follows from (5.31). □

Lemma 5.3.2 *Let $\alpha > 0$, and let n be some integer. Let q_n and p_n be as in (5.21), (5.24), and let p_n have (increasingly ordered) positive roots $\{t_{k,n}\}_{1\le k\le n}$. Then*

$$\Phi(t) := t^\alpha p_n(t) \ \le \ (\alpha q_n(0)^{-1})^\alpha, \qquad 0 \le t \le t_{1,n}. \tag{5.36}$$

Proof. From the definition of Φ we get immediately $\Phi(0) = \Phi(t_{1,n}) = 0$ and $\Phi(t) > 0$. Moreover,

$$p_n'(t) \ = \ -p_n(t)\sum_{k=1}^n \frac{1}{t_{k,n}-t}.$$

Now let $0 < \overline{t} < t_{1,n}$ with

$$\Phi(\overline{t}) \ = \ \sup_{0\le t\le t_{1,n}} \Phi(t).$$

Hence $0 = \Phi'(\overline{t})$ and then

$$\begin{aligned} \alpha\overline{t}^{\alpha-1}p_n(\overline{t}) \ &= \ \overline{t}^\alpha p_n(\overline{t})\sum_{k=1}^n \frac{1}{t_{k,n}-\overline{t}} \\ &\ge \ \overline{t}^\alpha p_n(\overline{t})q_n(0), \end{aligned}$$

therefore $\overline{t} \le \alpha q_n(0)^{-1}$, and thus

$$\sup_{0\le t\le t_{1,n}} \Phi(t) \ = \ \Phi(\overline{t}) \ = \ \overline{t}^\alpha p_n(\overline{t}) \ \le \ \overline{t}^\alpha \ \le \ (\alpha q_n(0)^{-1})^\alpha,$$

and we have shown the assertion (5.36). □

5.4 The discrepancy principle for the method of conjugate residuals, and the main results

We now assume that some noise level for the approximations of the exact right-hand side $Au = f_*$ is given: let $f^\delta = f \in X$ and $\delta > 0$ with

$$\|f_* - f^\delta\| \ \le \ \delta, \qquad \delta > 0.$$

This requires a further change of notation, e.g., iterates are denoted by u_n^δ instead of u_n, and the defects are denoted by r_n^δ instead of r_n. For convenience of the reader the dependence of other quantities like steplengths and polynomials on δ will not be indicated explicitly. We assume that the initial guess $u_0 \in X$ is independent of δ, i.e.,

$$u_0 \ = \ u_0^\delta.$$

For the cr-method we use the discrepancy principle as stopping criterium:

Stopping Rule 5.4.1 *Let $b > 1$. Stop the cr-method if for the first time*

$$\|r_n^\delta\| \ \le \ b\delta.$$

n_δ denotes the stopping index.

Note that this in fact leads to a finite breakdown, since

$$\inf_{n\ge 0} \|Au_n^\delta - f^\delta\| \ \le \ \delta.$$

This follows from (5.25), take there e.g. the polynomials that appear in the Richardson iteration. Note that the 'termination case' $Ar_n^\delta = 0$ implies $Au_n^\delta = Pf^\delta$, and thus $\|Au_n^\delta - f^\delta\| \le \delta$ and therefore $n_\delta \le n$.

In main Theorem 5.4.3 a measure for the efficiency of Stopping Rule 5.4.1 is given in terms of the function $F_\alpha^{\sigma(A)}(n)$. This function is introduced in the following definition, and bounds for it are stated in Theorem 5.4.4.

Definition 5.4.2 *For any bounded set $M \subset [0,\infty)$ let*

$$\Pi_n^M \;:=\; \Big\{ s_n \in \Pi_n : \;\; s_n(0) = 1, \;\; \sup_{\lambda \in M} |s_n(\lambda)| \;\le\; 1 \Big\}, \tag{5.37}$$

$$F_\alpha^M(n) \;:=\; \Big(\inf_{s_n \in \Pi_n^M} \sup_{\lambda \in M} |s_n(\lambda)| \lambda^\alpha \Big)^{1/\alpha}. \tag{5.38}$$

For reasons of notational convenience we set $F_\alpha^M(-1) = \infty$. Obviously, $F_\alpha^M(n+1) \le F_\alpha^M(n)$ for any $n \ge 0$, and $F_\alpha^M(0) = \sup_{\lambda \in M} \lambda$.

Theorem 5.4.3 *1. If the solution u_* of $Au = f_*$ with minimal distance to u_0 is smooth, i.e., if*

$$u_0 - u_* \;\in\; \mathcal{R}(A^\alpha), \qquad \varrho := \|u_0 - u_*\|_\alpha,$$

holds for some $\alpha > 0$, then

$$\|u_{n_\delta}^\delta - u_*\| \;\le\; c_{\alpha,b} (\varrho \delta^\alpha)^{1/(\alpha+1)}, \tag{5.39}$$

$$\Big(F_{\alpha+1}^{\sigma(A)}(n_\delta - 1)\Big)^{-1} \;\le\; e_{\alpha,b} \big(\varrho \delta^{-1}\big)^{1/(\alpha+1)}, \tag{5.40}$$

where $c_{\alpha,b}$ denotes some constant, and $e_{\alpha,b} := (b-1)^{-1/(\alpha+1)}$.

2. In the general case $u_0 - u_* \in \mathcal{N}(A)^\perp$ we have convergence,

$$\|u_{n_\delta}^\delta - u_*\| \;\to\; 0 \qquad \text{as } \;\delta \to 0, \tag{5.41}$$

$$F_2^{\sigma(A)}(n_\delta - 1)^{-1} \delta \;\to\; 0 \qquad \text{as } \;\delta \to 0, \tag{5.42}$$

The proof of Theorem 5.4.3 is given in the next section. In the next theorem, bounds for $F_\alpha^\sigma(A)(n)$ (that depend explicitly on $\sigma(A)$) are given, and these bounds provide us informations about the efficiency of the method of conjugate residuals. The first estimate is for the general case; note that $\sigma(A) \subset [0, \|A\|]$. If $A \in \mathcal{L}(X)$ is a compact operator with eigensystem $\{\lambda_k, v_k\}$, where $\lambda_1 \ge \lambda_2 \ge ... > 0$, with $\lambda_k \to 0$ as $k \to \infty$, then the estimates for the stopping index can be improved as can be seen in (5.44) and (5.45). In other terms, the efficiency of the method of conjugate residuals reacts to particular properties of the underlying operator: The faster the decay of the λ_j's are, the faster is the decay of $F_\alpha^{\{\lambda_j\}}(n)$ as n increases. Note, however, that the faster the eigenvalues λ_k decay, the smaller becomes $\mathcal{R}(A^\alpha)$ (for fixed α.)

Theorem 5.4.4 *(1) For intervals $M = [0,a]$ one has*

$$F_\alpha^{[0,a]}(n) \;\le\; d_\alpha n^{-2}, \qquad n \ge 1, \tag{5.43}$$

with $d_\alpha = a\lceil 2\alpha \rceil^2$.

(2) Now assume that $\lambda_1 \ge \lambda_2 \ge ... > 0$, with $\lambda_k \to 0$ as $k \to \infty$.

(a) Assume that for some $\tau > 0$,

$$\lambda_j \;=\; \mathcal{O}(j^{-\tau}) \qquad \text{as } \; j \to \infty.$$

Then,

$$F_\alpha^{\{\lambda_j\}}(n) \;\le\; d_\alpha n^{-(2+\tau)}, \qquad n \ge 1, \tag{5.44}$$

with $d_\alpha = \lceil 2\alpha \rceil^2 2^{2+\tau} \sup_j \{\lambda_j j^\tau\}$.

(b) Assume now that for some $0 < \theta < 1$,

$$\lambda_j \;=\; \mathcal{O}(\theta^j) \qquad \text{as } \; j \to \infty.$$

Then

$$F_\alpha^{\{\lambda_j\}}(n) \;\le\; d_\alpha \kappa^{-n}. \tag{5.45}$$

with $\kappa_\alpha \;=\; \theta^\alpha, \;\; d_\alpha \;=\; \theta \sup_j \{\lambda_j \theta^{-j}\}^\alpha$.

Section 5.6 is devoted to the proof of Theorem 5.4.4

Any reasonable method is expected to supply approximations that fulfill order optimal estimates (5.39) and (5.41). However, e.g. for stationary iterative methods like the Richardson iteration, an estimate similar to (5.40) and (5.42) can be obtained for n_δ only, see Section 3. Since the cr-method as well as the stationary methods require one matrix-vector multiplication in each iteration step (in the finite-dimensional case), the results (5.40) and (5.42) for the stopping index in conjunction with Theorem 5.4.4 describe the advantage of the cr-method.

From the results for noisy data one can recover convergence results in the exact data case.

Theorem 5.4.5 *Denote by u_n^* the iterates corresponding to the exact data case (where $r_0 = Au_0 - f_*$), and let u_* be the solution of $Au = f_*$ which has minimal distance to u_0.*

(a) Assume that

$$u_0 - u_* \in \mathcal{R}(A^\alpha), \qquad \varrho := \|u_0 - u_*\|_\alpha,$$

for some $\alpha > 0$. Then

$$\|u_n^* - u_*\| \;\le\; \gamma_\alpha \varrho F_{\alpha+1}^{\sigma(A)}(n), \tag{5.46}$$

with some constant $\gamma_\alpha > 0$.

(b) In the general case $u_0 - u_* \in \mathcal{N}(A)^\perp$,

$$\|u_n^* - u_*\| \to 0 \qquad \text{as} \;\; n \to \infty.$$

The proof of Thoerem 5.4.5 is given, in a more general frame, in Section 5.7. We now have stated the main results for the method of conjugate residuals and the discrepancy principle, and the next sections are devoted to the proofs of these results.

5.5 The proof of the rates (5.38) and (5.39) for the approximations and the stopping indices, respectively

5.5.1 The proof of the rates (5.38) for the approximations

We first show that (5.39) holds. For convenience of the reader, in the following diagramm the connections between the different lemmas and corollaries are shown.

$$\left.\begin{array}{r} \text{Lemma 5.5.1} \Longrightarrow \text{Corollary 5.5.2} \\ \left.\begin{array}{r}\text{Lemma 5.5.3}\\ \text{Lemma 5.5.4}\end{array}\right\} \Longrightarrow \text{Corollary 5.5.5} \end{array}\right\} \Longrightarrow (5.39).$$

Corollary 5.5.2 contains an estimate for the error $\|u_n^\delta - u_*\|$, and Corollary 5.5.5 provides an estimate for $\|r_{n-1}^\delta\|$. These estimates enable us to prove rates for the approximations (5.39).

The first lemma is formulated in a general form.

Lemma 5.5.1 *Assume that $\|Au - f\| \le \delta$ for $u,\ f \in X$, and assume that $u \in \mathcal{R}(A^\alpha)$, $\|u\|_\alpha =: \varrho$. Let $q : [0, a] \to \mathrm{I\!R}$ be a continuous function (with $a := \|A\|$), and let $\Delta = \|\big(I - Aq(A)\big)f\|$. Then for all $0 < \tau \le a$,*

$$\|u - q(A)f\| \;\le\; \varrho \sup_{0\le t\le\tau} |1 - tq(t)|t^\alpha \;+\; \tau^{-1}(\Delta + \delta) \;+\; \delta \sup_{0\le t\le\tau} |q(t)|.$$

Proof. We have

$$\|u - q(A)f\| \;\le\; \|(I - E_\tau)\big(u - q(A)f\big)\| \;+\; \|E_\tau\big(u - q(A)f\big)\|, \tag{5.47}$$

and we shall estimate both terms of the right-hand side. First,

$$\begin{aligned}
\|(I-E_\tau)\big(u-q(A)f\big)\|^2 &= \int_\tau^a d\|E_t\Big((I-E_\tau)(u-q(A)f)\Big)\|^2 \\
&\le \tau^{-2}\int_\tau^a t^2\, d\|E_t\Big((I-E_\tau)(u-q(A)f)\Big)\|^2 \\
&= \tau^{-2}\int_0^a t^2\, d\|E_t\Big((I-E_\tau)(u-q(A)f)\Big)\|^2 \\
&= \tau^{-2}\|(I-E_\tau)A\big(u-q(A)f\big)\|^2 \\
&\le \tau^{-2}\|A\big(u-q(A)f\big)\|^2 \\
&\le \tau^{-2}\big(\|Au-f\|+\Delta\big)^2 \\
&\le \tau^{-2}(\delta+\Delta)^2.
\end{aligned}$$

We shall estimate the second term in the right-hand side of (5.47). Since $u = A^\alpha z,\ \|z\| = \varrho$, for some $z \in X$, it follows

$$u - q(A)f \;=\; \big(I-q(A)A\big)A^\alpha z \;+\; q(A)(Au-f)$$

and

$$\|E_\tau\big(u-q(A)f\big)\| \;\le\; \|\big(I-q(A)A\big)A^\alpha E_\tau z\| \;+\; \|q(A)E_\tau(Au-f)\|. \tag{5.48}$$

We estimate the first term of the right-hand side in (5.48):

$$\begin{aligned}
\|\big(I-q(A)A\big)A^\alpha E_\tau z\|^2 &= \int_0^a |1-tq(t)|^2 t^{2\alpha}\, d\|E_t(E_\tau z)\|^2 \\
&= \int_0^\tau |1-tq(t)|^2 t^{2\alpha}\, d\|E_t z\|^2 \\
&\le \varrho^2 \sup_{0\le t\le\tau} |1-tq(t)|^2 t^{2\alpha}.
\end{aligned}$$

The second term in the right-hand side (5.48) can be estimated similarly, and we obtain

$$\|E_\tau\big(u-q(A)f\big)\| \;\le\; \varrho \sup_{0\le t\le\tau}|1-tq(t)|t^\alpha \;+\; \delta \sup_{0\le t\le\tau}|q(t)|. \quad \square$$

We return to the method of conjugate residuals. Lemma 5.5.1 implies the following corollary.

Corollary 5.5.2 *For the cr-method with perturbed data we have for all* $0 \le n \le n_*$ *and for all* $\tau \in (0, t_{1,n}]$ *(if* $n \ge 1$*; if* $n = 0$ *then* $\tau > 0$ *can be taken arbitrarily)*

$$\|u_n^\delta - u_*\| \;\le\; \varrho\tau^\alpha \;+\; \tau^{-1}(\delta + \|r_n^\delta\|) \;+\; q_n(0)\delta \qquad \textit{for all}\ \ \tau \in (0, t_{1,n}]. \tag{5.49}$$

(if $n \ge 1$*; if* $n = 0$ *then* $\tau > 0$ *can be taken arbitrarily in (5.49)).*

Proof. We have $u_n^\delta = u_0 - q_n(A)(Au_0 - f^\delta)$ and therefore $u_n^\delta - u_* = p_n(A)(u_0 - u_*) - q_n(A)(Au_0 - f^\delta)$, hence we apply Lemma 5.5.1 with $u = u_0 - u_*$ and $f = Au_0 - f^\delta$. The assertion then follows from the fact that for $0 \le t \le \tau$ we have $0 \le p_n(t) \le 1$ (see (5.30)) and $0 \le q_n(t) \le q_n(0)$ (see (5.35)). $\square$

(5.49) provides a first estimate for the approximation error $\|u_n^\delta - u_*\|$. Note that due to the stopping rule, $\|r_{n_\delta}^\delta\| \le b\delta$, and (5.39) is shown if we can prove

$$q_{n_\delta}(0) = \mathcal{O}((\varrho\delta^{-1})^{1/(\alpha+1)}) \;\; \text{(uniformly in } \delta > 0,\ \varrho > 0).$$

In order to prove this result, we need the following two lemmata which provide estimates for $\|r_n^\delta\|$. These two estimates are then combined in Corollary 5.8.

Lemma 5.5.3 *For the cr-method with perturbed data we have for* $1 \le n \le n_*$

$$\|r_n^\delta\| \;\le\; \varrho(2\alpha+2)^{\alpha+1} q_n(0)^{-(\alpha+1)} \;+\; \delta.$$

Proof. We have to estimate

$$\|r_n^\delta\|^2 \;=\; \int_{0-0}^{a} p_n^2(t)\; d\|E_t r_0^\delta\|^2. \tag{5.50}$$

For that we define

$$s(t) := p_n(t)(1-\frac{t}{t_{1,n}})^{-1}.$$

Then $s \in \Pi_{n-1}$, the orthogonality property (5.26) leads to

$$\int_{t_{1,n}}^{a} tp_n^2(t)(\frac{t}{t_{1,n}}-1)^{-1}\; d\|E_t r_0^\delta\|^2 \;=\; \int_0^{t_{1,n}} tp_n(t)s(t)\; d\|E_t r_0^\delta\|^2.$$

From this and $t(\frac{t}{t_{1,n}}-1)^{-1} \geq t_{1,n}$ for $t_{1,n} < t$ we obtain

$$\begin{aligned}\int_{t_{1,n}}^{a} p_n^2(t)\; d\|E_t r_0^\delta\|^2 \;&\leq\; t_{1,n}^{-1}\int_{t_{1,n}}^{a} tp_n^2(t)(\frac{t}{t_{1,n}}-1)^{-1}\; d\|E_t r_0^\delta\|^2 \\ &=\; \int_0^{t_{1,n}} \frac{t}{t_{1,n}-t}p_n^2(t)\; d\|E_t r_0^\delta\|^2. \end{aligned}\tag{5.51}$$

(5.50), (5.51) and (5.30) yield

$$\begin{aligned}\|r_n^\delta\|^2 \;&\leq\; \int_{0-0}^{t_{1,n}} p_n^2(t)(1+\frac{t}{t_{1,n}-t})\; d\|E_t r_0^\delta\|^2 \\ &=\; \int_{0-0}^{t_{1,n}} p_n^2(t)(1-\frac{t}{t_{1,n}})^{-1}\; d\|E_t r_0^\delta\|^2 \\ &\leq\; \int_{0-0}^{t_{1,n}} p_n(t)\; d\|E_t r_0^\delta\|^2 \\ &=\; \|\big(\chi_{[0,t_{1,n}]}p_n\big)^{1/2}(A)E_{t_{1,n}} r_0^\delta\|^2,\end{aligned}$$

where $\chi_{[0,t_{1,n}]}$ denotes the characteristic function corresponding to $[0,t_{1,n}]$. With $u_0-u_* = A^\alpha z,\ \|z\| = \varrho$, one has

$$r_0^\delta \;=\; Au_0 - f^\delta \;=\; A^{\alpha+1}z \;+\; Au_* - f^\delta, \tag{5.52}$$

and then

$$\|r_n^\delta\| \;\leq\; \varrho \sup_{0\leq t\leq t_{1,n}} p_n(t)^{1/2}t^{\alpha+1} \;+\; \delta \sup_{\leq t\leq t_{1,n}} p_n(t)^{1/2} \tag{5.53}$$

$$\leq\; \varrho\Big((2\alpha+2)q_n(0)^{-1}\Big)^{\alpha+1} \;+\; \delta, \tag{5.54}$$

where the same technique as in the proof of Lemma 5.14 applies to get (5.53), and estimate (5.54) follows with Lemma 5.3.2 and the fact that $0 \leq p_n(t) \leq 1$ for all $0 \leq t \leq t_{1,n}$. □

Lemma 5.5.4 *We again consider the cr-method with perturbed data. Let* $1 \leq n \leq n_*,\ \beta > 2$ *and* $2 < \mu \leq 2(\beta-1)$. *If* $\beta q_{n-1}(0) \leq q_n(0)$, *then with* $\gamma := (1-\beta^{-1})/\mu$,

$$\frac{\mu-2}{\mu-1}\|r_{n-1}^\delta\| \;\leq\; \varrho\big(\gamma q_n(0)\big)^{-(\alpha+1)} \;+\; \delta.$$

Proof. Let

$$s(t) := p_n(t)(1-\frac{t}{t_{1,n}})^{-1}.$$

Then $s \in \Pi_{n-1}$ und $s(0) = 1$, the minimum property (5.25) leads to $\Big($using $\|r_n^\delta\| \leq \|r_{n-1}^\delta\|$ and the fact that for $t \geq \mu t_{1,n}$ we have $(1-\frac{t}{t_{1,n}})^{-2} \leq (\mu-1)^{-2}\Big)$

$$\|r_{n-1}^\delta\|^2 \;\;\leq\;\; \|s(A)r_0^\delta\|^2$$

$$
\begin{aligned}
&= \int_{0-0}^{\mu t_{1,n}} s^2(t)\; d\|E_t r_0^\delta\|^2 \;+\; \int_{\mu t_{1,n}}^{a} s^2(t)\; d\|E_t r_0^\delta\|^2 \\
&\le \int_{0-0}^{\mu t_{1,n}} s^2(t)\; d\|E_t r_0^\delta\|^2 \;+\; (\mu-1)^{-2}\int_{\mu t_{1,n}}^{a} p_n^2(t)\; d\|E_t r_0^\delta\|^2 \\
&\le \|s(A)E_{\mu t_{1,n}} r_0^\delta\|^2 \;+\; (\mu-1)^{-2}\|r_n^\delta\|^2, \\
&\le \|s(A)E_{\mu t_{1,n}} r_0^\delta\|^2 \;+\; (\mu-1)^{-2}\|r_{n-1}^\delta\|^2,
\end{aligned}
$$

hence

$$\frac{\mu-2}{\mu-1}\|r_{n-1}^\delta\| \;\le\; \|s(A)E_{\mu t_{1,n}} r_0^\delta\|. \tag{5.55}$$

In order to estimate the right-hand side in (5.55) we now prove the estimates

$$\mu t_{1,n} \;\le\; 2t_{2,n}, \qquad \text{if } n \ge 2, \tag{5.56}$$

$$t_{1,n} \;\le\; (1-\beta^{-1})^{-1} q_n(0)^{-1}, \qquad \text{if } n \ge 1. \tag{5.57}$$

To this end we recall that $q_n(0) \le t_{1,n}^{-1} + q_{n-1}(0)$ (this is (5.34)), and the assumption $\beta q_{n-1}(0) \le q_n(0)$ already implies (5.57) as well as

$$(\beta-1)t_{1,n-1}^{-1} \;\le\; (\beta-1)q_{n-1}(0) \;\le\; t_{1,n}^{-1}.$$

The last estimate then yields $\mu t_{1,n} \le \frac{\mu}{\beta-1} t_{1,n-1} \le 2t_{2,n}$ because of $t_{1,n-1} \le t_{2,n}$ (see (5.28)), hence also (5.56) holds.

As a final preparation for estimating the right-hand side of (5.55) we observe that $s(t) = \prod_{k=2}^{n}\left(1-\frac{t}{t_{k,n}}\right)$ implies $|s(t)| \le 1$ for all $t \le 2t_{2,n}$ (this makes sense only in the case $n \ge 2$; if $n = 1$, then, however, $s(t) = 1$ for all $t \in \mathbb{R}$).

We now can estimate the right-hand side of (5.55) and obtain

$$
\begin{aligned}
\frac{\mu-2}{\mu-1}\|r_{n-1}^\delta\| &\le \sup_{0\le t\le \mu t_{1,n}} |s(t)|t^{\alpha+1}\varrho \;+\; \sup_{0\le t\le \mu t_{1,n}} |s(t)|\delta \\
&\le \varrho(\mu t_{1,n})^{\alpha+1} \;+\; \delta \\
&\le \varrho\Big(\mu(1-\beta^{-1})^{-1}q_n(0)^{-1}\Big)^{\alpha+1} \;+\; \delta,
\end{aligned} \tag{5.58}
$$

and this is the assertion. □

Lemma 5.5.3 and Lemma 5.5.4 imply the following result.

Corollary 5.5.5 *(a) For all $0 < \theta < 1$ there is a $d_{\theta,\alpha} > 0$, such that for all $1 \le n \le n_*$*

$$\theta\|r_{n-1}^\delta\| \;\le\; d_{\theta,\alpha}\varrho q_n(0)^{-(\alpha+1)} \;+\; \delta.$$

(b) If $n_\delta \ne 0$, then for all $0 < \theta < 1/b$,

$$q_{n_\delta}(0) \;\le\; e_{\alpha,b}(\varrho\delta^{-1})^{1/(\alpha+1)}, \tag{5.59}$$

with $e_{\alpha,b} = \left(\frac{2d_{\theta,\alpha}}{b-1}\right)^{1/(\alpha+1)}$, where $\theta := (b+1)/(2b)$.

Proof. (a) Take $\mu > 2$ such that $\theta = \frac{\mu-2}{\mu-1}$ is satisfied. Moreover let $\beta > 2$ such that $\mu = 2(\beta-1)$. Applying Lemma 5.5.3 for the case '$\beta q_{n-1}(0) > q_n(0)$' and Lemma 5.5.4 for the case '$\beta q_{n-1}(0) \le q_n(0)$' yields the assertion (a). (5.59) then follows from (a). □

Corollaries 5.5.2 and 5.56 immediately yield the first part of our main Theorem 5.4.3.

Proof of (5.39), this is, the convergence rates for the approximations. Corollary 5.5.2 yields

$$\|u_{n_\delta}^\delta - u_*\| \;\le\; \varrho\tau^\alpha \;+\; (b+1)\tau^{-1}\delta \;+\; q_{n_\delta}(0)\delta \qquad \text{for all } 0 < \tau \le t_{1,n_\delta} \tag{5.60}$$

(in the case $n_\delta \neq 0$; if $n_\delta = 0$ then $\tau > 0$ can be taken arbitrary in (5.60)). (5.59) then shows that $q_{n_\delta}(0)$ is small enough, and we now indicate how to choose τ. If $n_\delta = 0$, then we may take the optimal $\tau = \big((b+1)\delta/(\varrho\alpha)\big)^{1/(\alpha+1)}$. If $n_\delta \geq 1$ then take

$$\tau \;=\; c_{\alpha,b}^{-1}(\delta/\varrho)^{1/(\alpha+1)};$$

this is possible since (5.59) then yields

$$\tau \;\leq\; q_{n_\delta}(0)^{-1} \;\leq\; t_{1,n_\delta}. \qquad \Box$$

5.5.2 The proof of the rates (5.39) for the stopping indices

We conclude this subsection with the proof of (5.40): For any $0 \leq n \leq n_*$ and any $s_n \in \Pi_n^{\sigma(A)}$ we obtain from the minimum property,

$$\begin{aligned} \|r_n^\delta\| &\leq \|s_n(A) r_0^\delta\| \\ &\leq \|s_n(A) A^{\alpha+1}\|\varrho \;+\; \|s_n(A)\|\delta \\ &\leq \sup_{\lambda\in\sigma(A)} |s_n(\lambda)|\lambda^{\alpha+1}\varrho \;+\; \delta, \end{aligned}$$

and then also

$$\|r_n^\delta\| \;\leq\; \varrho\Big(F_{\alpha+1}^{\sigma(A)}(n)\Big)^{\alpha+1} \;+\; \delta,$$

this leads immediately to the assertion (5.40). $\Box$

5.6 Proof of Theorem 5.4.4, i.e, the bounds for $F_\alpha^{\sigma(A)}(n)$.

We use the Tschebyscheff polynomials of the first kind,

$$T_n(t) \;=\; \cos(n \arccos t), \qquad t \in [-1,1], \qquad n \geq 0,$$

to introduce the following polynomials

Definition 5.6.1 *For $n \geq 0$ let*

$$\mathcal{P}_n(t) := \frac{(-1)^n}{2n+1}\,\frac{T_{2n+1}(\sqrt{t})}{\sqrt{t}}, \qquad 0 < t \leq 1.$$

Within the class of polynomials $p \in \Pi_n$ with $p(0) = 1$, these polynomials $\mathcal{P}_n$ have nice properties like modulus less then 1 and a decay behavior with respect to the weight $\sqrt{t}$.

Proposition 5.6.2 *For any $n \geq 0$, $\mathcal{P}_n$ defines a polynomial, and*

$$\mathcal{P}_n \in \Pi_n, \qquad \mathcal{P}_n(0) = 1, \tag{5.61}$$

$$\sup_{0\leq t\leq 1} |\mathcal{P}_n(t)| \;=\; 1, \tag{5.62}$$

$$\sup_{0\leq t\leq 1} |\mathcal{P}_n(t)|\sqrt{t} \;=\; \frac{1}{2n+1}. \tag{5.63}$$

Proof. If

$$T_{2n+1}(t) \;=\; \sum_{k=0}^{n} a_k t^{2k+1} \quad \text{for} \;\; t \in [-1,1],$$

then for $0 \leq t \leq 1$,

$$T_{2n+1}(\sqrt{t})/\sqrt{t} \;=\; \sum_{k=0}^{n} a_k t^k \;\in \Pi_n,$$

which gives the first part of (5.61). Moreover

$$\lim_{t\to 0} T_{2n+1}(t)/t \;=\; \lim_{t\to 0} T'_{2n+1}(t) \;=\; (-1)^n(2n+1),$$

and this completes the proof of (5.61).

We show that (5.62) holds. To this end, let $0 \le t \le 1$ and $0 \le \theta \le \pi/2$ with

$$\sqrt{t} \;=\; \sin\theta \;=\; \cos(\theta - \pi/2).$$

Then

$$\begin{aligned} T_{2n+1}(\sqrt{t}) &= \cos\Big((2n+1)(\theta-\pi/2)\Big) \\ &= (-1)^n \cos\Big((2n+1)\theta - \pi/2\Big) \\ &= (-1)^n \sin\Big((2n+1)\theta\Big). \end{aligned}$$

Since

$$|\sin k\omega| \;\le\; k\sin\omega \quad \text{for} \quad 0 \le \omega \le \pi/2 \quad \text{and integer } k \ge 0,$$

(follows by induction over k), we get

$$|T_{2n+1}(\sqrt{t})| \;\le\; (2n+1)\sin\theta \;=\; (2n+1)\sqrt{t},$$

and this yields (5.62).

We finally get (5.63), since

$$\mathcal{P}_n(t)\sqrt{t} \;=\; \frac{(-1)^n}{2n+1} T_{2n+1}(\sqrt{t}). \qquad \square$$

Proposition 5.6.2 implies that $\mathcal{P}_n \in \Pi_n^{[0,1]}$ and $F_\alpha^{[0,1]}(n) \le 1/(2n+1)$ (with an equality here, in fact, which can be shown with an alternation type theorem). These polynomials $\mathcal{P}_n$ enable us to define polynomials having modulus less than 1 and fast decay with respect to different weight functions (on intervals $[0,a]$):

Proposition 5.6.3 *For any $\alpha > 0$, $a > 0$ and $n \ge 0$ let l the biggest integer less or equal to $n/\lceil 2\alpha \rceil$,*

$$l := \lfloor n/\lceil 2\alpha\rceil \rfloor.$$

Then for

$$\overline{s}_{n,a}(t) := \mathcal{P}_l^{\lceil 2\alpha\rceil}(t/a), \qquad t \in [0,a],$$

(with $\mathcal{P}_n$ as in Definition 5.6.1) one has

$$\begin{aligned} \overline{s}_{n,a} &\in \Pi_n^{[0,a]}, \\ \sup_{0\le t\le a} |\overline{s}_{n,a}(t)|t^\alpha &\le a^\alpha \lceil 2\alpha\rceil^{2\alpha} \cdot n^{-2\alpha}. \end{aligned} \tag{5.64}$$

Note that (5.43) in Theorem 5.4.4, this is $F_\alpha^{[0,a]}(n) = \mathcal{O}(n^{-2})$ as $n \to \infty$, is an immediate consequence of Proposition 5.6.3.

Proof of Proposition 5.6.3. We consider first the case $a = 1$ and use the notation

$$\overline{s}_n := \overline{s}_{n,1}. \tag{5.65}$$

Then $\overline{s}_n \in \Pi_n$, since $l \cdot \lceil 2\alpha \rceil \le n$, and obviously $\overline{s}_n(0) = 1$ and $|\overline{s}_n(t)| \le 1$ for $0 \le t \le 1$. To prove the result on the speed of convergence, consider first the case $n \le \lceil 2\alpha \rceil$. Here for $0 \le t \le 1$ we have

$$|\overline{s}_n(t)|t^\alpha \;\le\; 1 \;\le\; \lceil 2\alpha\rceil^{2\alpha} \cdot n^{-2\alpha}.$$

Now let $n > \lceil 2\alpha \rceil$. Then for $0 \le t \le 1$

$$\begin{aligned}
|\overline{s}_n(t)|t^\alpha &= \Big(|\mathcal{P}_l^{\lceil 2\alpha\rceil/2\alpha}(t)|\sqrt{t}\Big)^{2\alpha} \le \Big(|\mathcal{P}_l(t)|\sqrt{t}\Big)^{2\alpha} \\
&\le (2l+1)^{-2\alpha} \\
&\le \Big(2(\frac{n}{\lceil 2\alpha\rceil} - 1) + 1\Big)^{-2\alpha} \qquad \Big(\text{since } l \ge \frac{n}{\lceil 2\alpha\rceil} - 1\Big) \\
&= \Big(\frac{2}{\lceil 2\alpha\rceil} - n^{-1}\Big)^{-2\alpha} \cdot n^{-2\alpha} \\
&\le \Big(\frac{2}{\lceil 2\alpha\rceil} - \frac{1}{\lceil 2\alpha\rceil}\Big)^{-2\alpha} \cdot n^{-2\alpha} \qquad \Big(\text{since } n > \lceil 2\alpha\rceil\Big) \\
&= \lceil 2\alpha\rceil^{2\alpha} n^{-2\alpha},
\end{aligned}$$

and this completes the proof for $a = 1$. We now consider the general case for a in (5.38). Obviously, $\overline{s}_{n,a}(t) = \overline{s}_n(t/a)$, with $\overline{s}_n$ as in (5.65), and thus $\overline{s}_{n,a} \in \Pi_n^{[0,a]}$, and

$$|\overline{s}_{n,a}(t)|t^\alpha \;=\; a^\alpha |\overline{s}_n(t/a)|(t/a)^\alpha \;\le\; a^\alpha \lceil 2\alpha\rceil^{2\alpha} n^{-2\alpha} \qquad \text{for } \; t \in [0,a],$$

and this completes the proof of Proposition 5.6.3. □

As mentioned in the introduction to this section, the estimates for the stopping index can be improved for compact operators, where one has a discrete spectrum $\sigma(A) = \{\lambda_j\}$, or in other terms, we have better estimates for $F_\alpha^{\sigma(A)}$ then. The proof of these improved estimates is based on the following lemma.

Lemma 5.6.4 *Let $\{\lambda_j\}$ be a sequence of reals with $\lambda_1 \ge \lambda_2 \ge ... > 0$ and $\lambda_j \to 0$ as $j \to \infty$. Then for any $\alpha > 0$, $n \ge 1$, and $1 \le k \le n$,*

$$F_\alpha^{\{\lambda_j\}}(n) \;\le\; \begin{cases} \lceil 2\alpha\rceil^2 \lambda_{k+1} \cdot (n-k)^{-2}, & \text{if } k < n, \\ \lambda_{n+1} \quad , & \text{if } k = n, \end{cases}$$

(and one then is free to choose k in order to estimate $F_\alpha^{\{\lambda_j\}}(n)$ best possible).

Proof. For n and k fixed let

$$v_k(t) \;=\; \prod_{j=1}^k \Big(1 - \frac{\lambda}{\lambda_j}\Big).$$

Obviously $v \in \Pi_k$, $v(0) = 1$ and $v(\lambda_j) = 0$, $j \le k$, and consider then

$$s(t) := \overline{s}_{n-k,\lambda_{k+1}}(t) \cdot v_k(t),$$

where $\overline{s}_{n-k,\lambda_{k+1}}$ is taken from Proposition 5.6.3. Then obviously $s \in \Pi_n^{\{\lambda_j\}}$, and moreover

$$\begin{aligned}
\sup_j |s(\lambda_j)|\lambda_j^\alpha &= \sup_{j \ge k+1} |\overline{s}_{n-k,\lambda_{k+1}}(\lambda_j)|\lambda_j^\alpha \\
&\le \sup_{0 \le t \le \lambda_{k+1}} |\overline{s}_{n-k,\lambda_{k+1}}(t)|t^\alpha \qquad (5.66) \\
&\le \lceil 2\alpha\rceil^{2\alpha} \lambda_{k+1}^\alpha \cdot (n-k)^{-2\alpha}, \qquad (5.67)
\end{aligned}$$

where the last estimate follows with (5.64), for $a = \lambda_{k+1}$, and it is valid for the case $k < n$; the assertion for $k = n$ follows trivially from (5.66). □

Proof of (5.44) (5.45) in Theorem 5.4.4, this are the bounds for $F_\alpha^{\{\lambda_j\}}(n)$.
(a) Let $k := \lfloor n/2 \rfloor$. Then $n - k \ge n/2$, hence

$$(n-k)^{-2} \;\le\; 2^2 n^{-2}.$$

Secondly, $k + 1 \ge n/2$ and therefore

$$(k+1)^{-\tau} \;\le\; 2^\tau n^{-\tau},$$

and this and Lemma 5.6.4 yields

$$\begin{aligned} F_\alpha^{\{\lambda_j\}}(n) &\le \lceil 2\alpha \rceil^2 \lambda_{k+1} \cdot (n-k)^{-2} \\ &\le \sup_j \{\lambda_j j^\tau\} \lceil 2\alpha \rceil^2 (k+1)^{-\tau} \cdot (n-k)^{-2} \\ &\le \sup_j \{\lambda_j j^\tau\} \lceil 2\alpha \rceil^2 2^{(2+\tau)} \cdot n^{-(2+\tau)}. \end{aligned}$$

(b) follows from Lemma 5.6.4, with $k = n$. □

5.7 The convergence of the discrepancy principle for the method of conjugate residuals

It is the purpose of this section to show that the cr-method for ill-posed problems is a regularization method in the sense of Definition 3.1.1, if iteration is terminated according to the discrepancy principle, and to prove an asymptotic behavior for the stopping index (in dependence of the noise level). Moreover, convergence results for exact data are provided.

These results are presented in a more general form For that, let X be a real or complex Banach space, and let $A \in \mathcal{L}(X)$ be weakly sectorial and $f_* \in \mathcal{R}(A)$. To solve $Au = f_*$ approximately, we consider some iterative method which for (fixed underlying space X and operator A) and arbitrary start vector $u_0 = X$ and any $f \in X$ generates a sequence $u_n,\ n = 1, 2, \ldots$, thus u_n can be written in the form

$$u_n \;=\; T_n(u_0, f), \qquad n = 1, 2, \ldots,$$

for some (non-linear) operator T_n which is not further specified. Note that the method of conjugate residuals fit into this framework, if we set for formal reasons $u_m = u_n$ for all $m \ge n$, if $Au_n - f \in \mathcal{N}(A)$.

We now again assume that some approximation $f^\delta \in X$ to the right-hand side of $Au = f_*$ is given, and some noise level $\delta > 0$ is available,

$$\|f_* - f^\delta\| \;\le\; \delta, \qquad \delta > 0,$$

and denote by

$$u_n^\delta \;=\; T_n(u_0, f^\delta) \tag{5.68}$$

the corresponding iterates. Note that u_n^δ does not depend on the noise level δ; our notation is chosen in order to simplify notations. We introduce the defects

$$r_n^\delta := Au_n^\delta - f^\delta, \qquad n = 0, 1, 2, \ldots\,,$$

and the first assumption on the method is that there is a constant $\gamma_0 > 0$ such that for any

$$b > \gamma_0$$

the iteration terminates after a finite number of steps if the following stopping rule is applied (and if $Au_0 - f_* \in A(\overline{\mathcal{R}(A)})$).

Stopping Rule 5.7.1 *(Discrepancy principle) Stop iteration (5.68), if for the first time*

$$\|r_n^\delta\| \;\le\; b\delta.$$

$n_\delta := n$ *denotes the stopping index.*

This defines a method generating an $u_{n_\delta}^\delta$ (which is supposed to be an approximation to a solution u_* of $Au = f_*$) and a stopping index n_δ (depending also on f^δ which is not further indicated).

The further *assumptions* on that method are those stated in Theorem 5.7.2. First it is supposed that for some *fixed* smoothness degree $\alpha = \alpha_0 > 0$, this method, incorporated by (5.68) and Stopping Rule 5.7.1, allows estimates (5.71) and (5.72) of $\|u_{n_\delta}^\delta - u_*\|$ and n_δ for all those $u_*,\ f^\delta,\ u_0 \in X$ which fulfill (5.69) and (5.70). Assumption (5.72) is motivated by (5.40) for the cr-method.

The first *statement* of the theorem then is that similar estimates for the the approximation and the stopping index also hold for other degrees of smoothness for the initial error, and the statement of (1), (b) is that this procedure defines a regularization method in the sense of Definition 3.1.1, and that moreover some asymptotic behavior for the stopping index holds. In (2), the exact data case is considered.

Theorem 5.7.2 *Assume that $A \in \mathcal{L}(X)$ is weakly sectorial, and let*

$$\phi : [-1, \infty) \to [0, \infty]$$

be some given function. We assume that for some

$$0 < \alpha = \alpha_0$$

the following hold: For all $u_*,\ f^\delta,\ u_0 \in X$ such that

$$\begin{aligned} \|Au_* - f^\delta\| &\le \delta, && (5.69)\\ u_0 - u_* &\in \mathcal{R}(A^\alpha), \qquad \varrho := \|u_0 - u_*\|_\alpha, && (5.70) \end{aligned}$$

iteration (5.68) and Stopping Rule 5.7.1 supplies (for arbitrary $b > \gamma_0$) an approximation $u^\delta_{n_\delta}$ and a stopping index n_δ with

$$\begin{aligned} \|u^\delta_{n_\delta} - u_*\| &\le c_{\alpha,b}(\varrho\delta^\alpha)^{1/(\alpha+1)}, && (5.71)\\ \phi(n_\delta - 1)^{-1} &\le e_{\alpha,b}\big(\varrho\delta^{-1}\big)^{1/(\alpha+1)}, && (5.72) \end{aligned}$$

where $c_{\alpha,b}$ and $e_{\alpha,b}$ denote some constants.

Then the following assertions (1) and (2) are valid:
(1) Then we have moreover for iteration (5.68) and its termination rule:
(a) For any $b > \gamma_0$ and for any

$$0 < \alpha \le \alpha_0,$$

and for all $u_*,\ f^\delta,\ u_0 \in X$ with (5.69), (5.70), we have the estimates (5.71) and (5.72).
(b) (No smoothness assumption) For any $b > \gamma_0$, and for all $u_*,\ f^\delta,\ u_0 \in X$ with (5.69) and $u_0 - u_* \in \overline{\mathcal{R}(A)}$, we have for the approximations and stopping indices

$$\begin{aligned} \|u^\delta_{n_\delta} - u_*\| &\to 0 \qquad \text{as } \delta \to 0, && (5.73)\\ \phi(n_\delta - 1)^{-1}\delta &\to 0 \qquad \text{as } \delta \to 0. && (5.74) \end{aligned}$$

(2) (Convergence in case of exact data:) Let $Au_0 - f_* \in A(\overline{\mathcal{R}(A)})$, and set $u^*_n := T_n(u_0, f_*)$ and $r^*_n := Au^*_n - f_*,\ n = 0, 1, ...,$ and let $u_* := u_0 - A^\dagger(A_0 - f_*)$. We additionally assume for any n: if $Ar^*_n = 0$ then $u^*_m = u^*_n$ for $m \ge n$, and if otherwise $Ar^*_n \ne 0$ then $\|r^*_{n+1}\| < \|r^*_n\|$. Then:

(a) If

$$u_0 - u_* \in \mathcal{R}(A^\alpha), \qquad \varrho := \|u_0 - u_*\|_\alpha,$$

for some $0 < \alpha \le \alpha_0$, then

$$\|u^*_n - u_*\| \le a_\alpha \varrho \phi(n)^\alpha, \qquad n = 1, 2, ...,$$

with some constant $a_\alpha > 0$.

(b) In the general case $u_0 - u_* \in \overline{\mathcal{R}(A)}$,

$$\|u^*_n - u_*\| \to 0 \qquad \text{as } n \to \infty.$$

Remark. For the linear methods considered in Chapters 2 and 3, first convergence results for precisely data are derived, and they are applied to obtain results in the case of noisy data. Therefore here part (2) of Theorem 5.7.2 does not have a reasonable application. For the proof of Theorem 5.4.3 on the discrepancy principle for the method of conjugate residuals, however, no convergence result for precise data are used, and here it is convenient to derive these convergence results from Theorem 5.4.3.

Proof of Theorem 5.7.2. We start with part (1) of the theorem and assume that for any $b > \gamma_0$, estimates (5.71) and (5.72) are available for some

$$0 < \alpha = \alpha_1, \tag{5.75}$$

and for all $u_*,\ f^\delta,\ u_0 \in X$ with (5.69), (5.70). We show the following:
(i) For any $b > \gamma_0$, and for any $\alpha > 0$ with $\alpha_1 - 1 \le \alpha \le \alpha_1$, and for all $u_*,\ f^\delta,\ u_0 \in X$ with (5.69), (5.70) we have (5.71), (5.72).
(ii) if moreover $\alpha_1 < 1$ in (5.75) then assertion (b) in part (1) holds.
(From this we can deduce that (1) in Theorem 5.7.2 holds in its general form.)

We prove (i) and (ii) simultaneously, and for that we now *fix* $b,\ u_*,\ u_0,\ f^\delta$ such that the conditions in (i) or (ii) are fulfilled, respectively. This means in particular, that $u_0 - u_* \in \mathcal{R}(A^\alpha)$ in case (i), and $u_0 - u_* \in \overline{\mathcal{R}(A)}$ in case (ii), respectively.

In order to involve the assumption we generate an u^δ with

$$u_0 - u^\delta \in \mathcal{R}(A^{\alpha_1}), \tag{5.76}$$

with several further requirements on it, such as proximity to u_* in the A-norm $\|\cdot\|_A = \|A(\cdot)\|$ as well as in the underlying norm itself, and $\|u_0 - u^\delta\|_{\alpha_1}$ has to be small enough. Let

$$b_1 \ = \ \big(\frac{b}{\gamma_0} - 1\big)/2.$$

(Then $b_1 = b_1(b) > 0$.) The first specific requirement on u^δ is

$$\|A(u^\delta - u_*)\| \ \le \ b_1\delta. \tag{5.77}$$

(5.69) then implies

$$\|Au^\delta - f^\delta\| \ \le \ (b_1+1)\delta =: \delta_1.$$

The choice

$$b_2 := b/(b_1+1)$$

then ensures $b_2\delta_1 = b\delta$ so that iteration terminates with n_δ, either the stopping rule is applied with $b,\ \delta$, or with $b_2,\ \delta_1$, respectively. Our choice of b_1 yields $b_2 > \gamma_0$, with the notation

$$\varrho_\delta := \|u_0 - u^\delta\|_{\alpha_1}$$

we obtain (from (5.71), (5.72) for $\alpha = \alpha_1$, and with u_* replaced by u^δ),

$$\begin{aligned} \|u^\delta_{n_\delta} - u^\delta\| \ &\le \ c_{\alpha_1,b_2}(\varrho_\delta\delta_1^{\alpha_1})^{1/(\alpha_1+1)} \\ &\le \ \Big(c_{\alpha_1,b_2}(b_1+1)^{\alpha_1/(\alpha_1+1)}\Big)\big(\varrho_\delta\delta^{\alpha_1}\big)^{1/(\alpha_1+1)}, \qquad (5.78) \\ \phi(n_\delta - 1)^{-1} \ &\le \ e_{\alpha_1,b_2}\big(\varrho_\delta\delta_1^{-1}\big)^{1/(\alpha_1+1)} \\ &= \ \Big(e_{\alpha_1,b_2}(b_1+1)^{-1/(\alpha_1+1)}\Big)\big(\varrho_\delta\delta^{-1}\big)^{1/(\alpha_1+1)}. \qquad (5.79) \end{aligned}$$

From (5.78) we easily obtain

$$\begin{aligned} \|u^\delta_{n_\delta} - u_*\| \ &\le \ \|u^\delta - u_*\| \ + \ \|u^\delta_{n_\delta} - u^\delta\| \\ &\le \ \|u^\delta - u_*\| \ + \ \Big(c_{\alpha_1,b_2}(b_1+1)^{\alpha_1/(\alpha_1+1)}\Big)(\varrho_\delta\delta^{\alpha_1})^{1/(\alpha_1+1)}, \end{aligned} \tag{5.80}$$

and now further estimates of the right-hand sides in (5.79), (5.80) are necessary. For that we impose requirements on $\|u^\delta - u_*\|$ and ϱ_δ. In case (i) we require that

$$\begin{aligned} \|u^\delta - u_*\| \ &\le \ C_1\big(\varrho\delta^\alpha\big)^{1/(\alpha+1)}, \qquad (5.81) \\ \varrho_\delta^{1/(\alpha_1+1)} \ &\le \ C_2\varrho^{1/(\alpha+1)}\delta^{\frac{\alpha}{\alpha+1} - \frac{\alpha_1}{\alpha_1+1}}, \qquad (5.82) \end{aligned}$$

with constants C_1 and C_2 depending on α and b. This together with (5.79), (5.80) yields

$$\begin{aligned} \|u_{n_\delta}^\delta - u_*\| &\le \Big(C_1 + C_2 c_{\alpha_1,b_2}(b_1+1)^{\alpha_1/(\alpha_1+1)}\Big)(\varrho\delta^\alpha)^{1/(\alpha+1)}, \\ \phi(n_\delta - 1)^{-1} &\le \Big(e_{\alpha_1,b_2}(b_1+1)^{-1/(\alpha_1+1)}C_2\Big)\big(\varrho\delta^{-1}\big)^{1/(\alpha+1)}, \end{aligned}$$

that is the assertion for case (i). In case (ii) the requirements are

$$\|u^\delta - u_*\| \to 0 \qquad \text{as } \delta \to 0, \tag{5.83}$$
$$\varrho_\delta \delta^{\alpha_1} \to 0 \qquad \text{as } \delta \to 0. \tag{5.84}$$

Then, again the estimates (5.79) and (5.80) imply (5.73) and (5.74) which completes the assertion stated at the beginning if this proof.

What remains is to generate u^δ that fulfills (5.77) and moreover (5.81), (5.82) in case (i), and (5.83), (5.84) in case (ii), respectively. To this end we use the (iterated) method of Lavrentiev which for fixed integer $m \ge \alpha_0 + 1$ and $u_0 = 0$ generates for any parameter $t \ge 0$ and $y \in X$ an

$$G_t y \in X,$$

with $G_t = t\sum_{j=1}^m (I + tA)^{-j}$, see (2.6), (2.7). Now take any $k > M_0^m$ (with M_0 as in (1.2)), and let $u \in \overline{\mathcal{R}(A)}$, and $y^\epsilon \in X$ with $\|Au - y^\epsilon\| \le \epsilon$. Theorem 3.2.3 on the discrepancy principle for the parameter dependent methods implies that there is a parameter

$$t_\epsilon \ge 0$$

such the following conditions are satisfied:

$$\|(AG_{t_\epsilon} - I)y^\epsilon\| \le k\epsilon, \tag{5.85}$$
$$\|u - G_{t_\epsilon} y^\epsilon\| \to 0 \qquad \text{as } \epsilon \to 0, \tag{5.86}$$
$$t_\epsilon \epsilon \to 0 \qquad \text{as } \epsilon \to 0. \tag{5.87}$$

If furthermore

$$u \in \mathcal{R}(A^\alpha), \qquad \rho := \|u\|_\alpha,$$

for some $\alpha \le m - 1$, then we can replace (5.86) and (5.87) by the estimates

$$\|u - G_{t_\epsilon} y^\epsilon\| \le C_3(\rho\epsilon^\alpha)^{1/(\alpha+1)}, \tag{5.88}$$
$$t_\epsilon \le C_4\big(\rho\epsilon^{-1}\big)^{1/(\alpha+1)}, \tag{5.89}$$

respectively. Here C_3 and C_4 denote constants which depend on α and k.

We apply this result to $u = u_0 - u_*$, $\epsilon = (b_1/k)\delta$ and $y^\epsilon = A(u_0 - u_*)$. (Note that $\epsilon = \epsilon(\delta)$, and that $y^\epsilon = Au$ so that y^ϵ in fact does not depend on ϵ.) Let

$$u^\delta := u_0 - G_{t_\epsilon} A(u_0 - u_*).$$

Thus, (5.85) implies (5.77), and moreover (5.88) implies (5.81) in case (i), and (5.86) implies (5.83) in case (ii). Therefore (5.82) in case (i) and (5.84) in case (ii) remain to prove. We first consider case (i) and hence assume that (5.70) holds. Then

$$u_0 - u^\delta \;=\; G_{t_\epsilon} A(u_0 - u_*) \;=\; A^{\alpha_1} z^\delta,$$

where

$$z^\delta \;=\; A^{\alpha+1-\alpha_1} G_{t_\epsilon} z,$$

with $z \in X$ such that $u_0 - u_* = A^\alpha z$. The interpolation inequality (1.15) yields with some $C_5 > 0$

$$\|z^\delta\| \;\le\; \|A^{\alpha+1-\alpha_1} G_{t_\epsilon}\| \cdot \|z\| \;\le\; C_5\|z\| t_\epsilon^{\alpha_1 - \alpha}. \tag{5.90}$$

and here it is used that $\|AG_t\|$ and $\|G_t\|t^{-1}$ are uniformly bounded for $t > 0$, see (2.2), (2.3) for that. Going over to the infimum in (5.90) yields

$$\varrho_\delta \;=\; \|u_0 - u^\delta\|_{\alpha_1} \;\le\; C_5 \varrho t_\epsilon^{\alpha_1 - \alpha}. \tag{5.91}$$

From this and (5.89) we obtain (with another constant C_6 depending on α and k)

$$\begin{aligned}\varrho_\delta &\le \varrho C_6 (\varrho\delta^{-1})^{\frac{\alpha_1-\alpha}{\alpha+1}} = C_6 \varrho^{1+\frac{\alpha_1-\alpha}{\alpha+1}} \delta^{\frac{\alpha-\alpha_1}{\alpha+1}} \\ &= C_6 \varrho^{\frac{\alpha_1+1}{\alpha+1}} \delta^{\frac{\alpha-\alpha_1}{\alpha+1}},\end{aligned}$$

which yield the desired estimate (5.82). Finally, in case (ii) we have $\alpha_1 < 1$, and then (5.91) with $\alpha = 0$ and (5.87) imply (5.84), and this completes the proof of part (1) of the theorem.

(2) In order to prove the results for exact data, we shall use the results of part (1) of this theorem: take any $b > \gamma_0$, and fix n. If $Ar_n^* = 0$, then by assumption on the method $u_m^* = u_*$ for $m \ge n$, and in the sequel we consider the case $\|r_{m+1}^*\| < \|r_m^*\|$ for $0 \le m \le n$, and thus $\delta := \|Au_n - f_*\|/b > 0$. Stopping Rule 5.7.1 with $f^\delta = f^*$ then yields $n_\delta = n$.

We consider the case (2), (a) first, and then for any $\epsilon > 0$ with $\|r_{n+1}^*\| \le b(\delta - \epsilon) < \|r_n^*\|$ we obtain from (1), (a), with δ replaced by $\delta - \epsilon$, and with $f^\delta = f_*$,

$$\phi(n)^{-1} \;\le\; e_{\alpha,b}\big(\varrho(\delta-\epsilon)^{-1}\big)^{1/(\alpha+1)},$$

and letting $\epsilon \to 0$ yields $\phi(n)^{-1} \;\le\; e_{\alpha,b}(\varrho\delta^{-1})^{1/(\alpha+1)}$. Again with (1), (a) we get

$$\begin{aligned}\|u_n^* - u_*\| &\le c_{\alpha,b}(\varrho\delta^\alpha)^{1/(\alpha+1)} \\ &\le c_{\alpha,b} e_{\alpha,b}^\alpha \varrho \phi(n)^\alpha.\end{aligned}$$

Finally, case (b) in (2) follows immediately from (1), (b), with the same approach as in the proof of (2), (a). □

Remark. 1. Theorem 5.4.5 is an immediate consequence of Theorem 5.7.2.
2. Let us assume for simplicity that $u_0 = 0$. We can then write the method, composed by iteration (5.68) and its termination rule, in the form

$$\mathcal{P} : X \times \mathbb{R}_+ \to X \times \mathbb{R}_+ \cup \{0\}, \quad (f^\delta,\ \delta) \mapsto (u_{n_\delta}^\delta,\ n_\delta),$$

where $\mathbb{R}_+ = \{\ \delta > 0\ \}$, and this formulation can be the set-up for modifications of Theorem 5.7.2 for other than iterative methods and/or stopping rules. In order to be most instructive, however, we restrict the considerations to iterations.

5.8 Numerical Illustrations

To illustrate the method of conjugate residuals and the discrepancy principle as its stopping criterium we consider the problem of harmonic continuation. To this end, let

$$D = \big\{x \in \mathbb{R}^2 :\ x_1^2 + x_2^2 < 1\big\}$$

be the unit disk in the plane, and let $v : \overline{D} \to \mathbb{R}$ be continuous on $\overline{D}$ and harmonic on D, i.e.,

$$\Delta v(x) = 0, \qquad x \in D.$$

We assume that v is known on the boundary of a concentric disk of radius $0 < r < 1$, i.e.,

$$f(\varphi) \;=\; v(r,\varphi), \qquad 0 \le \varphi \le 2\pi,$$

is assumed to be known. From these informations we want to determine v on the boundary of D, i.e,

$$u(\varphi) \;=\; v(1,\varphi), \qquad 0 \le \varphi \le 2\pi,$$

is the function we are looking for. The relation between u and f can be stated in terms of the following operator A,

$$\begin{aligned} Au(t) &:= \int_0^{2\pi} k(t,s)u(s)\,ds \;=\; f(t), \qquad 0 \le t \le 2\pi, \\ k(t,s) &:= \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(t-s)}. \end{aligned} \tag{5.92}$$

and (5.92) is an integral equation of the first kind. We now consider the equation $Au = f$ in the L^2-setting; as underlying space we take the real space

$$X = L^2[0, 2\pi]$$

with corresponding inner product

$$\langle u, v\rangle \;=\; \int_0^{2\pi} u(t)v(t)\,dt.$$

$A : X \to X$ then is compact, selfadjoint and positive semidefinite, i.e., $A = A^* \ge 0$. Moreover, dim $\mathcal{R}(A) = \infty$. One can use the residue theorem to show that the eigenvalues of A are

$$\lambda_j = r^j, \qquad j = 0, 1, ...,$$

with eigenfunction 1 corresponding to λ_0, and eigenfunctions

$$\sin(js), \quad \cos(js), \qquad 0 \le s \le 2\pi,$$

corresponding to $\lambda_j,\ j = 1, 2, ...$ In our numerical example we consider the following functions.

Example 5.8.1 *Let*

$$f_*(t) \;=\; e^{r\cos t}\cos(r\sin t), \qquad 0 \le t \le 2\pi.$$

Then

$$u_*(s) \;=\; e^{\cos s}\cos(\sin s), \qquad 0 \le s \le 2\pi$$

solves $Au = f_*$, where A is as in (5.92). Note, that for any $\alpha > 0$ and

$$z(t) := e^{r^{-\alpha}\cos t}\cos(r^{-\alpha}\sin t) \;=\; \mathrm{Re}\,\Big(\sum_{k=0}^{\infty}\frac{(r^{-\alpha}e^{it})^k}{k!}\Big), \qquad 0 \le t \le 2\pi,$$

one has $A^\alpha z = u_*$, thus in particular

$$u_* \in \mathcal{R}(A^\alpha), \qquad \|u_*\|_\alpha \;\le\; \sqrt{2\pi}\exp(r^{-\alpha}).$$

We test the method of conjugate residuals and the stopping rule for

$$r = 0.5.$$

In our numerical experiments, equation (5.92) is discretized with Ritz's method, and as ansatz space we take

$$X_h = \big\{\ \psi \in C([0,2\pi]) :\ \psi_{|[ih,\ (i+1)h]} \text{ linear},\ 0 \le i \le N-1\ \big\},$$

with $N = 128,\quad h = 2\pi/N$. We always start with $u_0 = 0$, and the results of two experiments are illustrated in Figures 5.1 and 5.2.

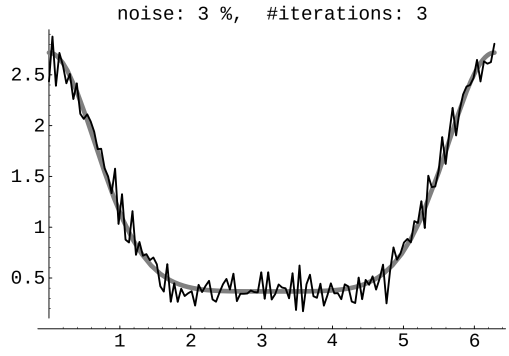


Abbildung 5.1: Grey: solution u_*; Black: approximation $u_{n_\delta}^\delta$

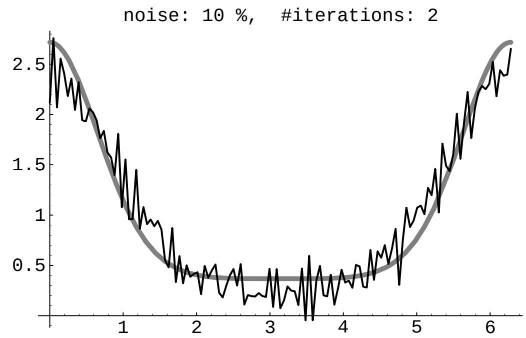


Abbildung 5.2: Grey: solution u_*; Black: approximation $u_{n_\delta}^\delta$

Bibliographical notes and remarks

We start with a short history about conjugate gradient type methods for linear ill-posed problems. The first convergence rates for the classical conjugate gradient method (of Hestenes and Stiefel) for solving normal equations $A^*Au = A^*f$ were provided by Kammerer and Nashed [32]. A survey on general convergence results for conjugate gradient type methods can be found in Nemirovskiĭ and Polyak [51]. Speed of convergence for the conjugate gradient method for normal equations can be found in Brakhage [9] and Louis [42].

We now consider perturbed data and the discrepancy principle, and here best possible convergence rates for the classical conjugate gradient method for solving normal equations $A^*Au = A^*f$ and the cr-method for symmetric, positive semidefinite equations $Au = f$ were obtained by Nemirovskiĭ [50], and it is proven in Plato [54] that these methods are regularization methods in the sense of Theorem 3.1.1.

In Hanke [24] it can be found that the classical conjugate gradient method (of Hestenes and Stiefel) for solving symmetric positive semidefinite equations $Au = f$ defines a regularization method if stopped appropriately (not by the discrepancy principle, however).

Section 5.4 Here the main results are listed, and in the sequel sources are designated.

Section 5.5 The complete section is due to Nemirovskiĭ [50].

Section 5.6 Propositions 5.6.2 and 5.6.3 and Lemma 5.6.4 are (in a slightly different form) due to Nemirovskiĭ and Polyak [51]. The bounds (5.43) (in conjunction with (5.40)) and (5.46)) has been proved by Nemirovskiĭ [50], and for (5.44), (5.45) (in conjunction with (5.40) and (5.46)) see Hanke [24].

Section 5.7 The general ideas of this section are developed in [54] and are extended in [56].

Section 5.8 The problem of harmonic continuation is considered in Kress [37], Problem 15.3, and our functions u_* and f_* are considered at the same source, see e.g., Example 11.6 there. For interior and exterior Dirichlet and Neumann problems for the Laplace equation see also Mikhlin [47], Chapter 13.

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