Lavrentiev's Method for Linear Volterra Integral Equations of the First Kind, with Applications to the Non-Destructive Testing of Optical-Fibre Preforms

Robert Plato

Fachbereich Mathematik, Technische Universität Berlin,
D - 10623 Berlin, Germany

1 Introduction

1.1 Linear Volterra Integral Equations of the First Kind

In the non-destructive testing of optical-fibre preforms, the problem of determining the axial stress components from measurements of the phase retardation of laser lights sent through the object reduces to a generalized Abel integral equation of the first kind, for more details we refer to Section 4. The methods presented in this paper can be applied to solve those problems efficiently, and we begin more generally with the consideration of a linear Volterra integral equation of the first kind,

$$(Au)(t) := \int_0^t k(t,s)u(s)\ ds \ = \ f_*(t), \qquad t \in [0,a]. \tag{1.1}$$

Later on we shall present specific equations of type (1.1) like the mentioned generalized Abel integral equations and integral equations with a completely monotone convolution kernel, but for the moment we suppose that $k : [0,a]\times[0,a] \to \mathbb{F}$ in (1.1) denotes an arbitrary kernel, and $f_* : [0,a] \to \mathbb{F}$ is an approximately given function, where either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. We moreover suppose that $A \in \mathcal{L}(\mathcal{H})$ and $f_* \in \mathcal{R}(A)$, where $\mathcal{H}$ is a given Hilbert space, and

$$\mathcal{L}(\mathcal{H}) \ = \ \{\ T : \mathcal{H} \to \mathcal{H} \mid \ T \text{ is bounded and linear } \},$$

and finally,

$$\mathcal{R}(A) \ = \ \{\ Au \mid \ u \in \mathcal{H}\ \} \ \subset \ \mathcal{H}$$

denotes the range of A. If the kernel k in (1.1) is non-degenerated, then $\mathcal{R}(A)$ is non-closed in $\mathcal{H}$, and thus equation (1.1) is ill-posed. This means that if merely an approximation $f \in \mathcal{H}$ for f_* is available,

$$f \in \mathcal{H}, \qquad f_* \in \mathcal{R}(A), \qquad f \approx f_*,$$

then the minimum norm solution of $Au = f$ (if it exists) may have an arbitrarily large distance to the minimum norm solution of (1.1). Thus some careful regularization is needed, and to this end, in this paper we shall consider Lavrentiev's m-times iterated method, see the following subsection for its introduction.

1.2 Lavrentiev's m-Times Iterated Method

In Hilbert spaces, Volterra integral equations of the first kind (1.1) in general may be regularized by methods like Tikhonov regularization, Landweber iteration or the classical conjugate gradient method (the latter applied to the normal equations $A^\star Au = A^\star f_*$, where $A^\star$ denotes the adjoint operator of A). However, these approaches do not profit from the triangular form of (1.1), or even more worse, the triangular structure is destroyed by any of these methods. Triangularity of the Volterra integral equation (1.1) here means that a discretization of (1.1), e.g. by a projection method, typically leads to a left triangular system or at least to an almost triangular system of equations.

In order to introduce a regularization method that in fact benefits from the triangular structure, we assume that the Volterra operator A in (1.1) is *accretive* with respect to $\mathcal{H}$, i.e.,

$$\mathbf{Re}\,\langle Au, u\rangle \;\geq\; 0, \qquad u \in \mathcal{H}.$$

Here, $\mathbf{Re}\, z$ denotes the real part of $z \in \mathbb{C}$, and $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{F}$ denotes the inner product in $\mathcal{H}$. Then the mentioned (almost) triangular system of equations associated with the Volterra integral equation (1.1) typically has small positive entries near the diagonal, and thus is seems to be natural and efficient to stabilize this system of equations by adding a small positive constant term on the diagonal. In the infinite-dimensional setting this corresponds to Lavrentiev's classical method, and in this paper we shall consider more generally Lavrentiev's m-times iterated method (with fixed integer m). It generates for a regularization parameter $\gamma > 0$ an $u_\gamma \in \mathcal{H}$ by

$$\begin{aligned} (A + \gamma I)v_n \;&=\; \gamma v_{n-1} \;+\; f, \qquad n = 1, 2, ..., m, \\ u_\gamma \;&:=\; v_m \end{aligned}$$

with $v_0 = 0$, and I denotes the identity operator in $\mathcal{H}$. For $m = 1$ one gets Lavrentiev's classical method while for $m > 1$, $m - 1$ stabilized residual corrections are employed. Note, however, that m is fixed so that Lavrentiev's m-times iterated method is a parametric method and not an iterative method. A good choice of the regularization parameter $\gamma > 0$ for Lavrentiev's m-times iterated method is important, and to this end in this paper some discrepancy principles are presented.

1.3 Outline of the Paper

The outline of the paper is as follows: In Section 2 we consider generalized Abel operators as well as integral operators with a completely monotone convolution kernel as examples for Volterra integral operators that are accretive, and in Section 3 some discrepancy principles as specific parameter choices for Lavrentiev's m-times method are considered and the associated convergence results are stated. In Section 4 we present some details about the non-destructive testing of optical-fibre preforms, and numerical illustrations are presented. Finally, in Section 5 we shall consider the case where the underlying space is a Banach space $\mathcal{X}$ and provide some results for Lavrentiev's m-times method for that case; moreover, also a subsection on two stationary iteration methods is included.

2 Specific Linear Accretive Volterra Integral Operators

2.1 Introduction

In this section we present some specific Volterra operators that are accretive in the following sense.

Definition 1. Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and let $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{F}$ be the associated inner product. An operator $A \in \mathcal{L}(\mathcal{H})$ is called accretive, if

$$\mathbf{Re}\,\langle Au, u \rangle \ \geq \ 0, \qquad u \in \mathcal{H}. \tag{2.1}$$

Accretiveness usually is introduced for unbounded operators, cf. Tanabe [24], but for the applications we have in mind it is sufficient to consider bounded operators. Note that (2.1) is valid if and only if (a) the resolvent set $\rho(-A)$ corresponding to $-A$ contains $(0, \infty)$, and (b) the following estimate is valid,

$$\|(A + \gamma I)^{-1}\| \ \leq \ 1/\gamma, \qquad \gamma > 0,$$

where $\|\cdot\|$ denotes the associated operator norm.

2.2 Abel Integral Operators

In the sequel we consider generalized Abel integral operators (cf. Gorenflo & Vessella [9] for an introduction) with respect to specific Hilbert spaces. To this end, throughout this subsection let $\beta > 0$ and $a > 0$ be arbitrary but fixed finite numbers, if not further specified. We then denote by $L^2([0,a],\ s^{\beta-1}ds)$ the Hilbert space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ containing all real- or complex-valued, measurable functions u on $[0,a]$, such that $|u|^2$ is integrable with respect to the measure $s^{\beta-1}ds$, and the associated inner product is

$$\langle u, v \rangle \ = \ \int_0^a u(s)\overline{v(s)}\ s^{\beta-1}ds, \qquad u,\ v \in L^2([0,a],\ s^{\beta-1}ds).$$

For the case $\beta = 1$, this space will be simplified denoted by $L^2[0,a]$.

Abel Integral Operators: Part I For $0 < \alpha < 1$, a generalized Abel integral operator $A = A_{\alpha,\beta,a}$ is given by

$$(Au)(t) \ = \ \int_0^t \frac{s^{\beta-1}u(s)}{(t^\beta - s^\beta)^{1-\alpha}}\ ds, \qquad t \in [0,a]. \tag{2.2}$$

The operator A defined by (2.2) is accretive with respect to $\mathcal{H} = L^2([0,a],\ s^{\beta-1}ds)$; for the details see [17].

Note that the operator A in (2.2) is moderately ill-posed. In fact, if $\beta = 1$ and $a = 1$, then for the singular values $\sigma_n(A)$, $n \geq 1$, of A one has

$$\sigma_n(A) \asymp n^{-\alpha} \qquad \text{as} \quad n \to \infty,$$

cf. Dostanić [7]. Here, $a_n \asymp b_n$ for positive numbers a_n, b_n, $n \geq 1$, means that there are constants $0 < \kappa_1$, κ_2 with $\kappa_1 a_n \leq b_n \leq \kappa_2 a_n$ for $n \geq 1$.

Abel Integral Operators: Part II For $0 < \alpha < 1$, another generalized Abel integral operator $A = A_{\alpha,\beta,a}$ is given by

$$(Au)(t) \; = \; \int_t^a \frac{s^{\beta-1} u(s)}{(s^\beta - t^\beta)^{1-\alpha}} \, ds, \qquad t \in [0,a]. \tag{2.3}$$

The lower bound in the integral in (2.3) depends on t, thus A defined by (2.3) has not the form (1.1). This operator A given by (2.3) nevertheless is accretive with respect to $\mathcal{H} = L^2([0,a],\ s^{\beta-1} ds)$, since A is the adjoint of the operator given by (2.2). We return to this generalized Abel integral operator in Section 4 where the non-destructive testing of optical-fibre preforms is considered.

Other Abel type integral equations with generalized kernels arise in X-ray tomography, see e.g., Cormack [5], Natterer [12].

2.3 Volterra Integral Operators with Convolution Kernels

We next consider the Volterra integral operator

$$(Au)(t) \; = \; \int_0^t \kappa(t-s) u(s) \, ds, \qquad t \in [0,a], \tag{2.4}$$

with a Lebesgue-integrable convolution kernel $\kappa : [0,\infty) \to \mathbb{R}$. Here, A as in (2.4) is accretive with respect to $\mathcal{H} = L^2[0,a]$ if κ is completely monotone, i.e.,

$$\mathbf{Re}\, (\mathbf{L}\kappa)(z) \; \geq \; 0, \qquad \mathbf{Re}\, z > 0,$$

where $(\mathbf{L}\kappa)(z) := \int_0^\infty e^{-zs} \kappa(s) \, ds$, $\mathbf{Re}\, z > 0$, denotes the Laplace transform of κ. For more details see Nohel & Shea [14] or Gripenberg, Londen & Staffans [10], Theorem 16.2.4.

3 Parameter Choices for Lavrentiev's m-Times Iterated Method

For an arbitrary accretive $A \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ denotes a given Hilbert space, we consider the equation

$$Au = f_*, \tag{3.1}$$

with $\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}$ in general, i.e., equation (3.1) is then ill-posed. Here $\overline{\mathcal{R}(A)}$ denotes the closure of $\mathcal{R}(A)$.

As in the introduction we admit the right-hand side in (3.1) to be disturbed, and in the sequel we additionally suppose that an estimate for the noise level is known, i.e.,

$$f^\delta \in \mathcal{H}, \qquad f_* \in \mathcal{R}(A), \qquad \|f_* - f^\delta\| \le \delta, \tag{3.2}$$

where $\delta > 0$ is a known error bound, and $\|\cdot\| : \mathcal{H} \to \mathbb{R}$ denotes the underlying norm.

In the sequel we consider (for fixed integer m) Lavrentiev's m-times iterated method which for $\gamma > 0$ generates an $u_\gamma^\delta \in \mathcal{H}$ by

$$\left.\begin{array}{rl} (A+\gamma I)v_n &= \gamma v_{n-1} \; + \; f^\delta, \qquad n = 1,2,\dots,m \\ u_\gamma^\delta &:= v_m \end{array}\right\} \tag{3.3}$$

where $v_0 = 0$, and for notational convenience we set

$$u_\infty^\delta \;=\; 0.$$

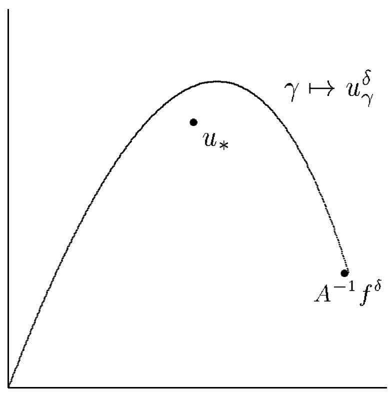

Fig. 1. Semiconvergence of Lavrentiev's m-times iterated method.

For ill-posed equations (3.1), a possible shape of the trajectory $\gamma \mapsto u_\gamma^\delta$ is
described in Figure 1. Here $\delta > 0$ small is fixed, and for notational
it is assumed that $f^\delta \in \mathcal{R}(A)$ and that A has a trivial nullspace,
denotes the solution of (3.1). In Figure 1, the point $u_\infty^\delta \in \mathcal{H}$ on the
orresponds to the origin, and u_0^δ corresponds to $A^{-1}f^\delta$.

behavior of $\gamma \mapsto u_\gamma^\delta$ as described in Figure 1, also known as semicon-
es it necessary to choose the regularization parameter γ appropri-
his end in the following subsections we present certain discrepancy

3.1 Discrepancy Principles

In the sequel we present discrepancy principles as rules for choosing $\gamma_\delta > 0$ in order to get good approximations $u^\delta_{\gamma_\delta} \in \mathcal{H}$ for some solution $u_* \in \mathcal{H}$ of (3.1). To this end, let $\Delta^\delta_\gamma \in \mathcal{H}$ denote the defect, i.e.,

$$\Delta^\delta_\gamma := Au^\delta_\gamma - f^\delta. \tag{3.4}$$

In fact, if $A \in \mathcal{L}(\mathcal{H})$ is accretive, then for fixed $\delta > 0$ the norm of the defect $\|\Delta^\delta_\gamma\|$ is continuous and nondecreasing in γ, and $\lim_{\gamma\to 0}\|\Delta^\delta_\gamma\| \le \delta$; see Figure 2 for the illustration of a typical situation. Thus the following two versions of the discrepancy principle can be implemented numerically.

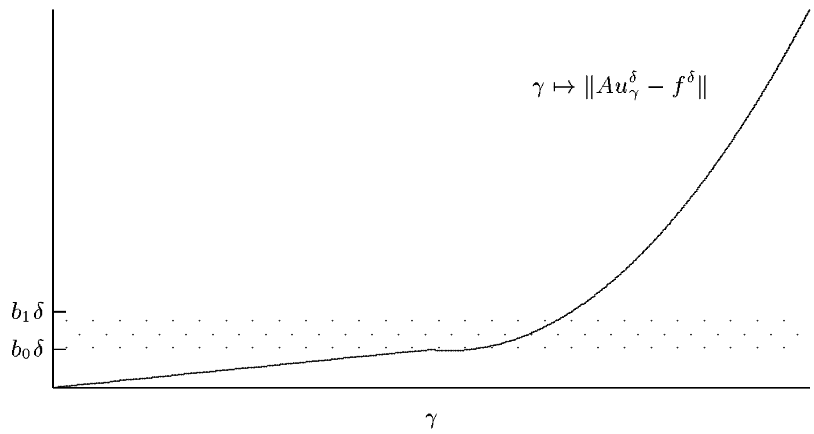

Fig. 2. Illustration for a typical behavior of the functional $\gamma \mapsto \|Au^\delta_\gamma - f^\delta\|$.

Discrepancy Principle 1. Fix positive constants $b_0,\ b_1$ with $b_1 \ge b_0 > 1$.
(a) If $\|\Delta^\delta_\infty\| \le b_1\delta$ then choose $\gamma_\delta = \infty$.
(b) If $\|\Delta^\delta_\infty\| > b_1\delta$ then choose $\gamma_\delta > 0$ such that

$$b_0\delta \;\le\; \|\Delta^\delta_{\gamma_\delta}\| \;\le\; b_1\delta.$$

Discrepancy Principle 2. Fix a real $b > 1$. Moreover, fix $\theta > 0$ and $\tau > 0$, and set $\gamma(k) = \theta/k^\tau$. Terminate computation of $u^\delta_{\gamma(k)} \in \mathcal{H},\ k = 0, 1, 2, \ldots$, if for the first time

$$\|\Delta^\delta_{\gamma(k)}\| \;\le\; b\delta,$$

and let $\gamma_\delta := \gamma(k_\delta)$, where k_δ denotes the stopping index.

The following result can be derived from the results in [16]. (3.5) shows that each discrepancy principle for Lavrentiev's m-times iterated method (3.3) with $m \geq 2$ defines a regularization method, and (3.8) provides, under additional smoothness assumptions, order-optimal convergence rates. The asymptotic behavior (3.6) and the estimate (3.9) show that the parameter $\gamma_\delta > 0$ cannot be arbitrarily small, respectively.

Theorem 2. *Let $\mathcal{H}$ be a Hilbert space, let $A \in \mathcal{L}(\mathcal{H})$ be accretive, and suppose that (3.2) is valid. Let $\{u_\gamma^\delta\} \subset \mathcal{H}$ be as in (3.3), with $m \geq 2$. Fix one of the two described Discrepancy Principles 1 or 2, and let the parameter $\gamma_\delta > 0$ be chosen according to it.*
1. If $u_* \in \overline{\mathcal{R}(A)}$ solves (3.1) then

$$\|u_{\gamma_\delta}^\delta - u_*\| \to 0 \qquad as \quad \delta \to 0, \tag{3.5}$$

$$\delta/\gamma_\delta \to 0 \qquad as \quad \delta \to 0. \tag{3.6}$$

2. If moreover for some real $0 < \nu \leq m-1$ and $z \in \mathcal{H}$,

$$u_* \;=\; A^\nu z, \qquad \varrho := \|z\|, \tag{3.7}$$

then with some constants $d_\nu,\ e_\nu > 0$ we have the estimates

$$\|u_{\gamma_\delta}^\delta - u_*\| \;\leq\; d_\nu(\varrho\delta^\nu)^{1/(\nu+1)}, \tag{3.8}$$

$$\gamma_\delta \;\geq\; e_\nu(\varrho^{-1}\delta)^{1/(\nu+1)}. \tag{3.9}$$

d_ν and e_ν depend also on $b_0,\ b_1$ and on b introduced in the Discrepancy Principles 1 and 2, respectively. Moreover, for $0 < \nu < 1$ fractional powers $A^\nu \in \mathcal{L}(\mathcal{H})$ of accretive operators $A \in \mathcal{L}(\mathcal{H})$ can be defined e.g. by formula (6.16) in Chapter 2 of Pazy [15]; for arbitrary $\nu > 0$, fractional powers $A^\nu \in \mathcal{L}(\mathcal{H})$ then are given recursively by $A^\nu := A^{\nu - \lfloor\nu\rfloor} A^{\lfloor\nu\rfloor}$, where $\lfloor\nu\rfloor$ denotes the greatest integer $\leq \nu$.

The proof of Theorem 2 depends basically on the estimate

$$\|u_\gamma^\delta - u_*\| \;\leq\; \|\big[\gamma(A+\gamma I)^{-1}\big]^m u_*\| \;+\; m\delta/\gamma, \tag{3.10}$$

and (3.10) follows immediately from the representation

$$u_* - u_\gamma^\delta \;=\; \big[\gamma(A+\gamma I)^{-1}\big]^m u_* \;+\; \gamma^{-1}\sum_{j=1}^m \big[\gamma(A+\gamma I)^{-1}\big]^j (Au_* - f^\delta).$$

For later notational convenience we use for $\gamma = \infty$ the notation $\gamma(A+\gamma I)^{-1} := I$, and then estimate (3.10) is valid also for $\gamma = \infty$.

Remarks. 1. Theorem 2 generalizes similar results for symmetric, positive semi-definite operators $A \in \mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ denotes a Hilbert space, cf. Vainikko [25].

2. For Abel integral operators A as in (2.2) with $0 < \alpha < 1$, $\beta = 1$ and $a > 0$, and for the underlying space $\mathcal{H} = L^2[0,a]$ we shall give an illustration of the smoothness condition (3.7) and the corresponding convergence rates (3.8). To this end for integer $k \geq 1$ we denote by $W^{k,2}[0,a]$ the Sobolev space of all functions $u : [0,a] \to \mathbb{C}$ such that u and its distributional derivates $u^{(j)}$ of order $j \leq k$ all belong to $L^2[0,a]$. If

$$u_* \in W^{k,2}[0,a], \qquad u_*(0) = u_*'(0) = \ldots = u_*^{(k-1)}(0) = 0,$$

for some integer k with $1 \leq k \leq (m-1)\alpha$, then for $\nu = k/\alpha$ one has $u_* \in \mathcal{R}(A^\nu)$, and due to (3.8) we then can expect the following speed of convergence,

$$\|u_{\gamma_\delta}^\delta - u_*\| \;=\; \mathcal{O}\big(\delta^{k/(k+\alpha)}\big) \qquad \text{as} \;\; \delta \to 0.$$

The statement (3.5) means by definition that for $m \geq 2$, Lavrentiev's m-times iterated method associated with any of the mentioned discrepancy principles yields a regularization method, respectively. For Lavrentiev's classical method, however, we have the following negative result (the proof is given in [18]):

Proposition 3. *Let $0 \neq A \in \mathcal{L}(\mathcal{H})$ be accretive, and suppose that $0 \in \sigma_{ap}(A)$. Then Discrepancy Principle 1 for Lavrentiev's classical method, this is (3.3) with $m = 1$, yields not a regularization method.*

Here, $\sigma_{ap}(A)$ denotes the approximate point spectrum of A.

Remark. It can be shown similarly that the Discrepancy Principle 2 for Lavrentiev's method also fails. Moreover, Proposition 3 shows that in the case $\{0\} \neq \mathcal{N}(A) \neq \mathcal{H}$, the Discrepancy Principle 1 for Lavrentiev's classical method fails even in the well-posed case $\mathcal{R}(A) \;=\; \overline{\mathcal{R}(A)}$.

3.2 Pseudo-Optimal Parameter Choice

It would be desirable to find parameters $\gamma_\delta > 0$ for Lavrentiev's m-times iterated method (3.3) such that an estimate of the following type is fulfilled,

$$\|u_{\gamma_\delta}^\delta - u_*\| \;\leq\; K \inf_{\gamma > 0} \|u_\gamma^\delta - u_*\|,$$

with some constant K not depending on δ, u_* and f^δ. In the sequel we shall consider a parameter choice that provides a similar estimate at least for the right-hand side in the basic estimate (3.10). To this end we introduce the following notation, which is similar to that used by Leonov, see e.g. [11].

Definition 4. Let $\mathcal{H}$ be a Hilbert space, and let $A \in \mathcal{L}(\mathcal{H})$ be accretive. A parameter choice strategy for Lavrentiev's m-times iterated method (3.3) is called <u>pseudo-optimal</u>, if it provides, for any $\delta > 0$ and for any u_*, $f^\delta \in \mathcal{H}$ with $\|Au_* - f^\delta\| \leq \delta$, a parameter $\gamma_\delta > 0$ such that

$$\|\big[\gamma_\delta(A+\gamma_\delta I)^{-1}\big]^m u_*\| + \delta/\gamma_\delta \;\leq\; K \inf_{\gamma>0} \Big(\|\big[\gamma(A+\gamma I)^{-1}\big]^m u_*\| + \delta/\gamma\Big), \quad (3.11)$$

with some constant K not depending on δ, u_* and f^δ.

We next introduce a modified discrepancy principle for Lavrentiev's m-times iterated method which is pseudo-optimal, cf. Theorem 5.

Discrepancy Principle 3. Fix real numbers $b_1 \geq b_0 > 1$, and let

$$B_\gamma := \gamma(A + \gamma I)^{-1}, \qquad \gamma > 0.$$

If $\|\Delta_\infty^\delta\| \leq b_1\delta$, then choose $\gamma_\delta = \infty$. Otherwise choose $\gamma_\delta > 0$ such that

$$b_0\delta \ \leq \ \|B_{\gamma_\delta}\Delta_{\gamma_\delta}^\delta\| \ \leq \ b_1\delta.$$

Here, $\Delta_\gamma^\delta \in \mathcal{H}$ again denotes the defect associated with Lavrentiev's m-times iterated method, cf. (3.4). Note that $B_\gamma\Delta_\gamma^\delta \in \mathcal{H}$ is the defect associated with the $(m+1)$-times iterated method of Lavrentiev, while the approximations $u_\gamma^\delta \in \mathcal{H}$ are generated by Lavrentiev's m-times iterated method.

The following theorem is proved in [19], and it extends analog results for symmetric operators in Hilbert spaces, cf. Raus [20]; similar results for normal equations can be found in Raus [21] and Engl & Gfrerer [8].

Theorem 5. *Let $\mathcal{H}$ be a Hilbert space, and let $A \in \mathcal{L}(\mathcal{H})$ be accretive. Then Discrepancy Principle 3 for Lavrentiev's m-times iterated method (with $m \geq 1$) is pseudo-optimal.*

4 Non-Destructive Testing of Optical-Fibre Preforms and Numerical Tests

4.1 Non-Destructive Testing of Optical-Fibre Preforms

In the sequel we consider the non-destructive testing of optical-fibre preforms, cf. Anderssen & Calligaro [1], or Calligaro, Payne, Anderssen & Ellen [4].

The properties of optical-fibre preforms can be studied in terms of the intrinsic stress components $\sigma_r(r)$, $\sigma_\theta(r)$ and $\sigma_z(r)$, $r \in [0, R]$, in the cylindrical coordinate directions r, θ and z, respectively. Here the independent variable r denotes the distance to the axis, and R denotes the radius of the preform.

In order to determine those intrinsic stress components, laser lights are sent through the optical-fibre preform in the direction normal to the axial direction, and the phase retardation $\psi(x)$, $x \in [0, R]$, of the laser beam then is measured. The required intrinsic stress components then in fact can be recovered from the phase retardation ψ. For example, the retardation ψ and the axial stress component σ_z are related via an Abel integral equation of type (2.3),

$$-\frac{4\pi C}{\lambda}\int_x^R \frac{r\sigma_z(r)}{\sqrt{r^2 - x^2}}\,dr \ = \ \psi(x), \qquad x \in [0, R], \tag{4.1}$$

where C denotes the photoelastic constant, and λ denotes the wavelength of the laser light. Moreover, the radial stress σ_r and the retardation ψ are related vice versa,

$$\sigma_r(r) \;=\; \frac{\lambda}{2C\pi^2} \cdot \frac{1}{r^2} \int_r^R \frac{x\psi(x)}{\sqrt{x^2 - r^2}}\, dx, \qquad 0 < r \le R.$$

Finally, the tangential stress σ_θ then is easily obtained, $\sigma_\theta = \sigma_z - \sigma_r$.

4.2 Numerical Experiments

In the sequel we solve numerically the classical Abel integral equation

$$(Au)(t) := \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} u(s)\, ds \;=\; f_*(t), \qquad t \in [0,1], \tag{4.2}$$

which (up to a scalar multiple) can be obtained from (4.1) by the substitution $f_*(t) = \psi(R\sqrt{1-t}),\ u(s) = \sigma_z(R\sqrt{1-s})$ for $t,\ s \in [0,1]$. In our numerical tests we use the right-hand side

$$f_*(t) \;=\; \frac{3\sqrt{\pi}}{8} t^2, \qquad t \in [0,1],$$

and then the solution of (4.2) is given by $\big($cf. [9], Chapter 1.1$\big)$

$$u_*(s) \;=\; s^{3/2}, \qquad s \in [0,1].$$

We choose perturbed right-hand sides $f^\delta = f_* + \delta \cdot v$, where $v \in \mathcal{H} := L^2[0,1]$ has uniformly distributed random values so that $\|v\| \le 1$, and where

$$\delta \;=\; \|f_*\| \cdot \%/100,$$

with % noise $\in \big\{\ 0.11,\ 0.33,\ 1.00,\ 3.00,\ 10.00\ \big\}$ in our implementations.

We next present the results of our experiments with Lavrentiev's m-times iterated method for

$$m = 5,$$

and as parameter choice strategy the Discrepancy Principle 2 is applied with $b = 1.5,\ \theta = 1$ and $\tau = 2$. One can show, cf. [17] and Chapter 1.1 in Gorenflo & Vessella [9], that

$$u_* \in \mathcal{R}(A^\nu), \qquad 0 < \nu < 4, \tag{4.3}$$

$$u_* \notin \mathcal{R}(A^4), \tag{4.4}$$

and due to (4.4) we cannot derive from Theorem 2 that the entries in the third column stay bounded as % of noise decreases. On the other hand, however, due to (4.3) it is no surprise that these entries in our experiments in fact stay bounded.

Lavrentiev's 5-times iterated method				
% noise	$\|\|u_{\gamma_\delta}^\delta - u_*\|\|$	$\|\|u_{\gamma_\delta}^\delta - u_*\|\|/\delta^{4/5}$	γ_δ	$\sharp$ flops
10.00	0.0839	1.40	1.0	0.3e+06
3.00	0.0918	4.00	0.25	0.5e+06
1.00	0.0292	3.07	0.25	0.5e+06
0.33	0.0084	2.15	0.25	0.5e+06
0.11	0.0029	1.79	0.25	0.5e+06

In Figure 3, the approximations $u_{\gamma_\delta}^\delta$ are shown for two noise levels.

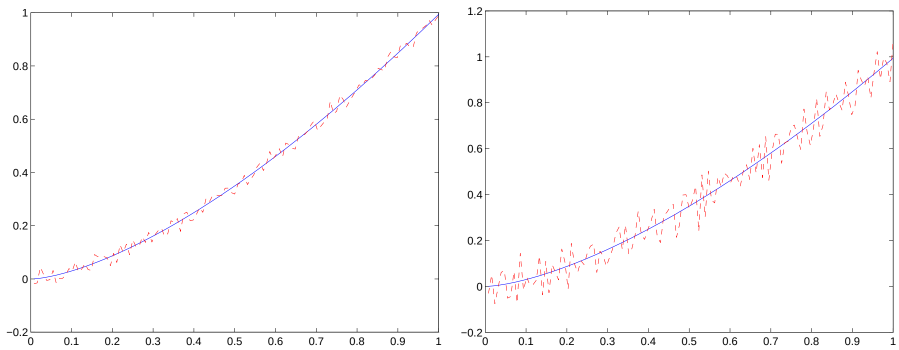

Fig. 3. Reconstruction for 1.0% perturbation (left) and 3.0% perturbation (right) of the right-hand side f_*; solid and dashed lines correspond to u_* and $u_{\gamma_\delta}^\delta$, respectively.

In our implementations, equation (4.2) has been discretized by a Bubnov-Galerkin method with piecewise constant trial functions of length $h = 1/N$, $N = 128$, and Lavrentiev's 5-times iterated method then in fact is applied to the corresponding finite system of equations. All computations are performed in MATLAB on an IBM RISC/6000.

5 Several Extensions

The most important and natural norm is the maximum norm, and therefore it is desirable to provide a theory for the numerical solution of integral equations of the first kind that allows error estimates with respect to this norm. Hence in this subsection we shall drop the assumption that the underlying space is a Hilbert space, and in the sequel $\mathcal{X}$ denotes a general complex Banach space, if not further specified. For ill-posed problems in the Banach space $\mathcal{X}$ we will present briefly some results for Lavrentiev's m-times iterated method. Finally we shall consider two stationary iterative schemes for ill-posed problems in $\mathcal{X}$.

5.1 Lavrentiev's m-Times Iterated Method in Banach Spaces

Introduction In Banach spaces $\mathcal{X}$, the results about Lavrentiev's m-times iterated method presented in this paper can be generalized for those operators $A \in \mathcal{L}(\mathcal{X})$ that are weakly sectorial in the following sense.

Definition 6. Let $\mathcal{X}$ be a complex Banach space. An operator $A \in \mathcal{L}(\mathcal{X})$ is weakly sectorial, if $(0, \infty) \subset \rho(-A)$ and if

$$\|(A+\gamma I)^{-1}\| \leq M_0/\gamma, \qquad \gamma > 0, \tag{5.1}$$

for some $M_0 \geq 1$.

This notation is introduced in [17], and the notation is justified by the fact that a weakly sectorial operator A fulfils a resolvent condition over a small sector in the complex plane. Here and in the sequel symbols like $\mathcal{L}(\mathcal{X})$ or the operator norm in (5.1) have the same meaning as in Sections 1 and 2 (for Hilbert spaces).

Example 1. For $0 < \alpha < 1$, $\beta > 0$ and $a > 0$, the Abel integral operators (2.2) and (2.3) are weakly sectorial with respect to the spaces $\mathcal{X} = C[0, a]$ and $\mathcal{X} = L^p\big([0, a],\ s^{\beta-1} ds\big)$, $1 \leq p \leq \infty$, respectively, with $M_0 = 2$ in each case. For a reasoning we refer to [17].

Here, $C[0, a]$ denotes the complex space of complex-valued continuous functions on the finite interval $[0, a]$, supplied with the maximum norm $\|\cdot\|_\infty$. Moreover, for $1 \leq p < \infty$, $L^p\big([0, a],\ s^{\beta-1} ds\big)$ denotes the complex space of complex-valued, measurable functions u on $[0, a]$, such that $|u|^p$ is integrable with respect to the measure $s^{\beta-1} ds$, and this space is supplied with the norm

$$\|u\| := \Big(\int_0^a |u(s)|^p \ s^{\beta-1} ds \Big)^{1/p}, \qquad u \in L^p\big([0, a],\ s^{\beta-1} ds\big).$$

Similarly, $L^\infty\big([0, a],\ s^{\beta-1} ds\big)$ denotes the space of complex-valued, measurable functions u on $[0, a]$ which are essentially bounded with respect to the measure $s^{\beta-1} ds$, and then $\|u\|_\infty$ denotes the essential supremum of $|u|$ with respect to the measure $s^{\beta-1} ds$.

Note that the space $L^2\big([0, a],\ s^{\beta-1} ds\big)$ in Example 1 is already considered in Section 2, and there it is already observed that estimate (5.1) for the mentioned Abel integral operators is valid with $M_0 = 1$.

Discrepancy Principle Let $\mathcal{X}$ be a Banach space, let $A \in \mathcal{L}(\mathcal{X})$ be weakly sectorial, and let $f^\delta \in \mathcal{X}$, $f_* \in \mathcal{R}(A)$ such that $\|f_* - f^\delta\| \leq \delta$ is satisfied. Then the statements in Theorems 2 and 5 remain valid, if in the Discrepancy Principles 1-3, respectively, the conditions "$b_1 > b_0 > 1$" and "$b > 1$" are replaced by "$b_1 > b_0 > M_0^m$", "$b > M_0^m$", and "$b_1 > b_0 > M_0^{m+1}$", respectively. The proofs are given in [16] and in [19], respectively.

Some Related Results It is possible to admit also unbounded operators in Definition 6, and in fact certain a priori parameter choices for Lavrentiev's m-times iterated method for linear weakly sectorial unbounded operators A in Banach spaces are provided in Schock & Phóng [22]. The regularizing properties of Lavrentiev's classical method and other specific regularization methods for Volterra integral equations of the first kind with smooth kernels are e.g. considered in Denisov [6] and in Srazhidinov [23], including error estimates with respect to the maximum norm.

5.2 Iterative Regularization in Banach Spaces

Introduction This subsection is devoted to the iterative solution of ill-posed problems in a Banach space $\mathcal{X}$. More specifically, we consider the discrepancy principle as a stopping rule for the Richardson iteration and an implicit iteration method, respectively, and we present associated order-optimal error estimates for those ill-posed equations $Au = f_*$ where $A \in \mathcal{L}(\mathcal{X})$ is strictly sectorial in the sense of the following Definition 7. To this end we introduce the sector $\Sigma_\theta \subset \mathbb{C}$,

$$\Sigma_\theta := \{ \ \lambda = re^{i\varphi} : \ r > 0, \ |\varphi| \ \le \ \theta \ \}, \qquad \theta \in [0, \pi].$$

Definition 7. Let $\mathcal{X}$ be a complex Banach space. An operator $A \in \mathcal{L}(\mathcal{X})$ is <u>strictly sectorial</u>, if there is an $0 < \varepsilon \le \pi/2$ such that $\Sigma_{\pi/2+\varepsilon} \subset \rho(-A)$ and

$$\|(A + \lambda I)^{-1}\| \ \le \ M/|\lambda|, \qquad \lambda \in \Sigma_{\pi/2+\varepsilon},$$

for some $M \ge 1$.

This notation is introduced in [17]. If $A \in \mathcal{L}(\mathcal{X})$ is strictly sectorial, then $-A$ in fact is the infinitesimal generator of a semigroup $T(t) = e^{-tA} \in \mathcal{L}(\mathcal{X}), \ t \ge 0$, that can be extended on a sector Σ_ε (for a small $\varepsilon > 0$) to an analytical, uniformly bounded semigroup (cf. Tanabe [24], Theorem 3.3.1).

Example 2. Let $0 < \alpha < 1, \ \beta > 0$ and $a > 0$. In the spaces $\mathcal{X} = C[0, a]$ and $\mathcal{X} = L^p([0, a], \ s^{\beta-1} ds), \ 1 \le p \le \infty$, the Abel integral operators (2.2) and (2.3) are strictly sectorial, respectively, see again [17] for a reasoning.

In the sequel we suppose that $A \in \mathcal{L}(\mathcal{X})$ is an arbitrary but fixed strictly sectorial operator, where $\mathcal{X}$ denotes some Banach space. Moreover, let again $f^\delta \in \mathcal{X}, \ f_* \in \mathcal{R}(A)$ such that $\|f_* - f^\delta\| \le \delta$ is fulfilled.

Two Stationary Iteration Methods First we consider the *Richardson iteration* which for initial vector $u_0^\delta = 0$ and $\mu > 0$ small enough generates iteratively the sequence

$$u_{n+1}^\delta \ = \ u_n^\delta \ - \ \mu(Au_n^\delta - f^\delta), \qquad n = 0, 1, 2, \ldots .$$

The assumption that $A \in \mathcal{L}(\mathcal{X})$ is strictly sectorial implies that $I - \mu A$ is power bounded for $\mu > 0$ small enough (this follows from Nevanlinna [13], Theorem 4.5.4), and thus the stopping rule discussed below is applicable.

For fixed $\mu > 0$ we next consider the *implicit iteration method*

$$(I + \mu A)u_{n+1}^\delta \;=\; u_n^\delta \;+\; \mu f^\delta, \qquad n = 0, 1, 2, \dots ,$$

for initial vector $u_0^\delta = 0$. Since $A \in \mathcal{L}(\mathcal{X})$ is strictly sectorial, $(I + \mu A)^{-1}$ is power bounded (this can be derived by standard results in semigroup theory, see e.g., Pazy [15], Theorem 1.7.7), and thus the stopping rule discussed below is applicable.

Discrepancy Principle We fix one of the considered iteration methods and denote by

$$\Delta_n^\delta := Au_n^\delta - f^\delta, \qquad n = 0, 1, 2, \dots,$$

the associated defect.

Discrepancy principle 4. For the Richardson iteration let $b > \sup_{n \geq 0} ||(I - \mu A)^n||$, and for the implicit method let $b > \sup_{n \geq 0} ||(I + \mu A)^{-n}||$. If $||\Delta_0^\delta|| \leq b\delta$ then set $n_\delta = 0$. Otherwise stop the iteration after $n_\delta \geq 1$ iteration steps, if

$$||\Delta_{n_\delta}^\delta|| \;\leq\; b\delta \;<\; ||\Delta_{n_\delta - 1}^\delta||.$$

For a strictly sectorial $A \in \mathcal{L}(\mathcal{X})$ and the Discrepancy Principle 4 as a stopping rule for the Richardson iteration and the implicit scheme, respectively, the statements 1. and 2. in Theorem 2 are valid, if the condition "$0 < \nu \leq m - 1$" is replaced by the weaker condition "$0 < \nu < \infty$", and if moreover γ_δ is replaced (a) by n_δ in (3.5), (3.8); and (b) by n_δ^{-1} in (3.6), (3.9). For the proofs see [16].

Pseudo-Optimality For a strictly sectorial $A \in \mathcal{L}(\mathcal{X})$, the Discrepancy Principle 4 as a stopping rule for the Richardson iteration and the implicit scheme, respectively, is even pseudo-optimal in a sense similar to that of Definition 4: for the Richardson iteration we have

$$||(I - \mu A)^{n_\delta} u_*|| \;+\; n_\delta \delta \;\leq\; K \inf_{n \geq 0} \Big(||(I - \mu A)^n u_*|| \;+\; n\delta \Big), \tag{5.2}$$

with some constant K not depending on $\delta > 0$, $u_* \in \mathcal{X}$, $f^\delta \in \mathcal{X}$ with $||Au_* - f^\delta|| \leq \delta$. Moreover, an estimate similar to (5.2) also holds for the implicit method; in fact, $I - \mu A$ in (5.2) has to be replaced by $(I + \mu A)^{-1}$ then. The proofs are given in [19].

Concluding Remarks If the strictly sectorial $A \in \mathcal{L}(\mathcal{X})$ is a Volterra integral operator, then Lavrentiev's m-times iterated method is superior to the presented iterative methods due to the reasons mentioned in Section 1. If A is a Fredholm operator and not a Volterra operator, however, then in fact the mentioned iterative methods are more efficient than Lavrentiev's m-times iterated method, since the computation of a parameter $\gamma_\delta > 0$ in the discrepancy principles usually requires a large computational effort then.

Some Related Results Early results on the iterative regularization of linear ill-posed problems in Banach spaces can be found in papers by Bakushinskiĭ, see e.g., [2], [3].

Acknowledgements. The author thanks the referee for helpful comments on a first version of this paper.

References

1. **R.S. Anderssen and R.B. Calligaro**. Non-destructive testing of optical-fibre preforms. *Austral. Math. Soc. (Ser. B)*, 23:127–135, **1981**.
2. **A.B. Bakushinskiĭ**. The problem of constructing linear regularizing algorithms in Banach spaces. *U.S.S.R. Comput. Math. Math. Phys.*, 13(1):261–270, **1973**.
3. **A.B. Bakushinskiĭ**. Regularization algorithms in Banach spaces, based on the generalized discrepancy principle (in Russian). In A.C. Alekseev, editor, *Incorrect Problems in Mathematical Physics and Analysis*, pages 18–21, Novosibirsk, **1984**. Nauka.
4. **R.B. Calligaro, D.N. Payne, R.S. Anderssen, and B.A. Ellen**. Determination of stress profiles in optical-fibre preforms. *Electronics Letters*, 18(11):474–475, **1982**.
5. **A.M. Cormack**. Representation of a function by its line integrals, with some radiological applications. *J. Appl. Phys.*, 34(9):2722–2727, **1963**.
6. **A.M. Denisov**. The approximate solution of Volterra equation of the first kind associated with an inverse problem for the heat equation. *Moscow Univ. Comput. Math. Cybern.*, 15(3):57–60, **1980**.
7. **M.R. Dostanić**. Asymptotic behavior of the singular values of fractional integral operators. *J. Math. Anal. Appl.*, 175:380–391, **1993**.
8. **H.W. Engl and H. Gfrerer**. A posteriori parameter choice for general regularization methods for solving linear ill-posed problems. *Appl. Numer. Math.*, 4:395–417, **1988**.
9. **R. Gorenflo and S. Vessella**. *Abel Integral Equations.* Springer, New York, 1st edition, **1991**.
10. **G. Gripenberg, S.-O. Londen, and O. Staffans**. *Volterra and Integral Functional Equations.* Cambridge University Press, Cambridge, 1st edition, **1990**.
11. **A.S. Leonov**. On the accuracy of Tikhonov regularizing algorithms and quasioptimal selection of a regularization parameter. *Soviet Math. Dokl.*, 44(3):711–716, **1991**.
12. **F. Natterer**. Recent developments in X-ray tomography. In E.T. Todd, editor, *Tomography, impedance imaging, and integral geometry*, pages 177–198, Providence, Rhode Island, **1994**. AMS, Lect. Appl. Math. 30.
13. **O. Nevanlinna**. *Convergence of Iterations for Linear Equations.* Birkhäuser, Basel, 1st edition, **1993**.
14. **J.A. Nohel and D.F. Shea**. Frequency domain methods for Volterra equations. *Adv. Math.*, 22:278–304, **1976**.
15. **T. Pazy**. *Semigroups and Applications to Partial Differential Operators.* Springer, New York, 1st, reprint edition, **1983**.
16. **R. Plato**. The discrepancy principle for iterative and parametric methods to solve linear ill-posed equations. *Numer. Math.*, 75(1):99–120, **1996**.

17. **R. Plato**. On resolvent estimates for Abel integral operators and the regularization of associated first kind integral equations. *Submitted for publication.*
18. **R. Plato**. *Iterative and parametric methods for linear ill-posed equations.* Habilitationsschrift, Fachbereich Mathematik, TU Berlin, **1995**.
19. **R. Plato and U. Hämarik**. On the pseudo-optimality of parameter choices and stopping rules for regularization methods in Banach spaces. *Numer. Funct. Anal. Optim.*, 17(2):181–195, **1996**.
20. **T. Raus**. Residue principle for ill-posed problems (in Russian). *Acta et comment. Univers. Tartuensis*, 672:16–26, **1984**.
21. **T. Raus**. Residue principle for ill-posed problems with nonselfadjoint operators (in Russian). *Acta et comment. Univers. Tartuensis*, 715:12–20, **1985**.
22. **E. Schock and Vũ Quôc Phóng**. Regularization of ill-posed problems involving unbounded operators in Banach spaces. *Hokkaido Math. J.*, 20:559–569, **1991**.
23. **A. Srazhidinov**. Regularization of Volterra integral equations of the first kind. *Differential Equations*, 26(3):390–398, **1990**.
24. **H. Tanabe**. *Equations of Evolution.* Pitman, London, 1st edition, **1979**.
25. **G.M. Vainikko**. The discrepancy principle for a class of regularization methods. *U.S.S.R. Comput. Math. Math. Phys.*, 22(3):1–19, **1982**.