The regularizing properties of the trapezoidal method for weakly singular Volterra integral equations of the first kind

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Abstract

The repeated trapezoidal method was considered by P. Eggermont for the numerical solution of weakly singular Volterra integral equations of the first kind with exactly given right-hand sides ([7]). In the present paper we consider the regularizing properties of this method for perturbed right-hand sides. Finally, numerical results are presented.

1 Introduction

In this paper we consider linear weakly singular Volterra integral equations of the following form,

$$(Au)(x) \;=\; \tfrac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} k(x,y) u(y)\, dy \;=\; f(x) \quad \text{for } 0 \le x \le L, \tag{1.1}$$

with $0 < \alpha < 1$ and $L > 0$, and with a sufficiently smooth kernel function $k : [0,L] \times [0,L] \to \mathbb{R}$, and Γ denotes Euler's gamma function. Moreover, the function $f : [0,L] \to \mathbb{R}$ is supposed to be approximately given, and a function $u : [0,L] \to \mathbb{R}$ satisfying equation (1.1) is to be determined.

For applications see e.g. Durbin [6] and Lerche/Zeitler [15], where crossing probabilities for Brownian motions and the inversion of the two-dimensional Radon transform are considered, respectively. In the sequel we suppose that the kernel function does not vanish on the diagonal $0 \le x = y \le L$, and without loss of generality we may assume that

$$k(x,x) \;=\; 1 \quad \text{for } 0 \le x \le L \tag{1.2}$$

holds.

There exists many methods for the approximate solution of equation (1.1) if the right-hand side f is exactly given, see e.g., Brunner/van der Houwen [4] and Hackbusch [10]. One of these methods is the repeated trapezoidal rule which is considered in detail, e.g., in Eggermont [7]. In the present paper we consider its regularizing properties for perturbed right-hand sides in equation (1.1). Finally, some numerical illustrations are presented.

2 The numerical scheme

2.1 Preparations

As a first step we consider in (1.1) the special situation $k \equiv 1$, with the corresponding integral operator being the classical Abel integral operator

$$(\mathcal{V}_\alpha u)(x) \;=\; \tfrac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} u(y)\, dy \quad \text{for } 0 \le x \le L, \tag{2.1}$$

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where $u : [0, L] \to \mathbb{R}$ is supposed to be a continuous function. One of the basic properties of the Abel integral operator is as follows,

$$(\mathcal{V}_\alpha y^q)(x) \;=\; \tfrac{\Gamma(q+1)}{\Gamma(q+1+\alpha)}\, x^{q+\alpha} \quad \text{for } x \ge 0 \qquad (q \ge 0) \tag{2.2}$$

where y^q is short notation for the mapping $y \mapsto y^q$. This and other properties of the Abel integral operator can be found e.g. in Gorenflo/Vessella [9] or Hackbusch [10]. For the numerical approximation of the integral operator (2.1) we introduce grid points

$$x_n \;=\; nh, \qquad n = 0, 1, \ldots, N \quad \text{with } h \;=\; \frac{L}{N}, \tag{2.3}$$

where N is a positive integer. Consider the space of linear splines with respect to that grid,

$$\mathcal{S}_N = \big\{\, s_N : [0, L] \to \mathbb{R} \mid s_N \text{ continuous on } [0, L], \text{ linear on } [x_{j-1}, x_j], j = 1, \ldots, N \,\big\}. \tag{2.4}$$

The following lemma is used as a preparation for the numerical scheme to be considered.

Lemma 2.1 (cf. [7]). *For each $s_N \in \mathcal{S}_N$ we have*

$$(\mathcal{V}_\alpha s_N)(x_n) \;=\; h^\alpha \Big\{ \tilde{a}_n s_N(0) + \sum_{j=1}^{n} a_{n-j} s_N(x_j) \Big\} \quad \textit{for } n \;=\; 1, 2, \ldots, N. \tag{2.5}$$

Here, the coefficients $a_0, a_1, \ldots$ and $\tilde{a}_1, \tilde{a}_2, \ldots$ are given by

$$a_n \;=\; \frac{1}{\Gamma(\alpha+2)} \big[(n+1)^{\alpha+1} - 2n^{\alpha+1} + (n-1)^{\alpha+1} \big],\ n = 1, 2, \ldots, \quad a_0 = \frac{1}{\Gamma(\alpha+2)}, \tag{2.6}$$

$$\tilde{a}_n \;=\; \frac{1}{\Gamma(\alpha+2)} \big[(n-1)^{\alpha+1} - n^{\alpha+1} + (\alpha+1) n^\alpha \big] \quad \textit{for } n = 1, 2, \ldots\,. \tag{2.7}$$

PROOF. The proof is elementary and is presented for convenience of the reader. We write $u_j = s_N(x_j)$ and obtain the following:

$$(\mathcal{V}_\alpha s_N)(x_n) \;=\; \frac{1}{\Gamma(\alpha)} \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} (x_n - y)^{\alpha-1} \big[u_{j-1} \frac{x_j - y}{h} + u_j \frac{y - x_{j-1}}{h} \big]\, dy. \tag{2.8}$$

We proceed with the terms considered in the integral in (2.8):

$$\int_{x_{j-1}}^{x_j} (x_n - y)^{\alpha-1} (x_j - y)\, dy$$
$$= \; -\frac{1}{\alpha(\alpha+1)} \big[(n-j+1)^{\alpha+1} - (n-j)^{\alpha+1} \big] h^{\alpha+1} \;+\; \frac{1}{\alpha} (n-j+1)^\alpha h^{\alpha+1}, \tag{2.9}$$

and

$$\int_{x_{j-1}}^{x_j} (x_n - y)^{\alpha-1} (y - x_{j-1})\, dy$$
$$= \; \frac{1}{\alpha(\alpha+1)} \big[(n-j+1)^{\alpha+1} - (n-j)^{\alpha+1} \big] h^{\alpha+1} - \frac{1}{\alpha} (n-j)^\alpha h^{\alpha+1}. \tag{2.10}$$

Summing up the terms in (2.9) and (2.10) finally gives the two representations (2.6) and (2.7). This completes the proof. $\square$

2.2 The numerical scheme

In the sequel we suppose that the values of the right-hand side of equation (1.1) are only approximately given with

$$|f_n^\delta - f(x_n)| \ \le \ \delta \quad \text{for } n \ = \ 1, 2, \ldots, N, \tag{2.11}$$

where $\delta > 0$ is a known noise level. In this situation representation (2.5) leads to the following scheme for the numerical solution of equation (1.1):

$$h^\alpha \sum_{j=1}^{n} a_{n-j}\, k(x_n, x_j)\, u_j^\delta \ = \ f_n^\delta - h^\alpha \tilde{a}_n k(x_n, 0) u_0^\delta, \qquad n = 1, 2, \ldots, N. \tag{2.12}$$

For each n the representation in (2.12) follows from the representation (2.5), considered for $L = x_n = nh$ and with the function u replaced by the function $y \mapsto k(x_n, y) u(y)$ for $0 \le y \le nh$.

The procedure for determining these approximations is as follows:

(a) First determine a starting value $u_0^\delta \approx u(0)$ of sufficient accuracy, i.e.,

$$u_0^\delta - u(0) \ = \ \mathcal{O}(h^2 + \delta / h^\alpha) \quad \text{as } (h, \delta) \to 0. \tag{2.13}$$

A simple extrapolation scheme is considered at the end of this section.

(b) Then successively determine approximations $u_n^\delta \approx u(x_n)$ for $n = 1, 2, \ldots, N$ by using scheme (2.12).

Remark 2.2. The considered scheme coincides, for exact given right-hand sides, with a spline-collocation method. In the special case $k \equiv 1$ and $\alpha = 1/2$ it also can be obtained by a repeated one-point Gauss rule followed by piecewise linear interpolation, cf. Branca [3] or Hackbusch [10]. $\triangle$

3 The integration error

3.1 Preparations

We now consider the quadrature error corresponding to method (2.12). The basic assumptions are as follows.

Assumption 3.1. (a) The kernel function k in the integral operator (1.1) has continuous partial derivatives up to the order 3 on $[0, L] \times [0, L]$, and there exists a solution u of the integral equation (1.1) which is twice continuously differentiable on the interval $[0, L]$, and the second derivative u'' is assumed to be Lipschitz continuous.

(b) $k(x, x) = 1$ holds for each $0 \ \le \ x \ \le \ L$,

(c) the grid points x_n are given by (2.3),

(d) the values of the right-hand side of equation (1.1) are approximately given at the grid points, cf. (2.11).

At the end of Section 4, conditions on the right-hand side f of the considered weakly singular Volterra integral equation of the first kind are given which guarantee the existence of a solution u satisfying (a) in Assumption 3.1.

We start our numerical analysis of the regularizing properties of the product trapezoidal method method for the numerical integration of the weakly singular Volterra integral operator considered in (1.1). As a preparation we consider the Abel operator (2.1), this is, $k \equiv 1$. We approximate a given function u on each subinterval by a linear polynomial interpolating u at both ends of the considered subinterval:

$$\begin{aligned}
(\mathcal{V}_\alpha u)(x_n) &= \frac{1}{\Gamma(\alpha)} \int_0^{x_n} (x_n - y)^{\alpha-1} u(y)\, dy \\
&= \frac{1}{\Gamma(\alpha)} \int_0^{x_n} (x_n - y)^{\alpha-1} s_N(y)\, dy + \frac{1}{\Gamma(\alpha)} \int_0^{x_n} (x_n - y)^{\alpha-1} (u(y) - s_N(y))\, dy \\
&= h^\alpha \Big\{ \tilde{a}_n u(0) + \sum_{j=1}^n a_{n-j} u(x_j) \Big\} - E_{N,n} \quad \text{for } n = 1, 2, \ldots, N, \qquad (3.1)
\end{aligned}$$

where

$$E_{N,n} = \frac{1}{\Gamma(\alpha)} \int_0^{x_n} (x_n - y)^{\alpha-1} \big[s_N(y) - u(y) \big]\, dy \quad \text{for } n = 1, 2, \ldots, N, \qquad (3.2)$$

denotes the integrated interpolation error. It is well known, cf. e.g., [17], that for each index $j \in \{1, 2, \ldots, n\}$ and each $x_{j-1} \le y \le x_j$ the error representation

$$s_N(y) - u(y) = \tfrac{1}{2} (y - x_{j-1})(x_j - y)\, u''(\xi) \qquad (3.3)$$

holds, with some intermediate value $x_{j-1} \le \xi = \xi(y) \le x_j$. From that we obtain the following representations for the integrated interpolation errors $E_{N,n}$ introduced in (3.2):

Lemma 3.2 (cf. [7]). *We have*

$$E_{N,n} = h^{\alpha+2} \sum_{j=1}^n c_{n-j} u''(x_j - \theta_{n,j} h) \quad \textit{for } n = 1, 2, \ldots, N \qquad (3.4)$$

with certain coefficients $0 \le \theta_{n,j} \le 1$. The coefficients c_n have the following asymptotic behavior:

$$c_n = \frac{1}{12\Gamma(\alpha)} n^{\alpha-1} + \mathcal{O}(n^{\alpha-2}) \quad \textit{as } n \to \infty. \qquad (3.5)$$

PROOF. From the representations (3.2) and (3.3) it follows that for $j = 1, 2, \ldots, n$ we have

$$\begin{aligned}
&\int_{x_{j-1}}^{x_j} (x_n - y)^{\alpha-1} \big[s_N(y) - u(y) \big]\, dy \\
&\quad = \tfrac{1}{2} \Big[\int_{x_{j-1}}^{x_j} (x_n - y)^{\alpha-1} (y - x_{j-1})(x_j - y)\, dy \Big] u''(x_j - \theta_{n,j} h) \\
&\quad = \frac{h^{\alpha+2}}{2} \Big[\int_0^1 t(1-t)(n-j+1-t)^{\alpha-1}\, dt \Big] u''(x_j - \theta_{n,j} h),
\end{aligned}$$

where the mean value theorem for integrals and substitution has been employed. Integration by parts yields the following representations:

$$\begin{aligned}
&\int_0^1 t(1-t)(n+1-t)^{\alpha-1}\, dt \\
&\quad = \frac{1}{\alpha(\alpha+1)} \Big\{ (n+1)^{\alpha+1} + n^{\alpha+1} - \frac{2}{\alpha+2} \big((n+1)^{\alpha+2} - n^{\alpha+2} \big) \Big\} \text{ for } n = 1, 2, \ldots .
\end{aligned}$$

The statement (3.5) on the asymptotic behavior of the coefficients $c_0, c_1, \ldots$ now follows from the Taylor expansions

$$\begin{aligned}
(n+1)^{\alpha+2} &= n^{\alpha+2} + (\alpha+2) n^{\alpha+1} + \frac{(\alpha+1)(\alpha+2)}{2} n^\alpha + \frac{\alpha(\alpha+1)(\alpha+2)}{6} n^{\alpha-1} + \mathcal{O}(n^{\alpha-2}), \\
(n \pm 1)^{\alpha+1} &= n^{\alpha+1} \pm (\alpha+1) n^\alpha + \frac{\alpha(\alpha+1)}{2} n^{\alpha-1} + \mathcal{O}(n^{\alpha-2}) \qquad (3.6)
\end{aligned}$$

as $n \to \infty$. This completes the proof of the lemma. □

As a further preparation we note that the coefficients $a_0, a_1, \ldots$ and $\tilde{a}_0, \tilde{a}_1, \ldots$ (cf. (2.6) and (2.7)) have the following asymptotic behavior (cf. (3.6)):

$$a_n = \frac{1}{\Gamma(\alpha)} n^{\alpha-1} + \mathcal{O}(n^{\alpha-2}) \quad \text{as } n \to \infty, \tag{3.7}$$

$$\tilde{a}_n = \mathcal{O}(n^{\alpha-1}) \quad \text{as } n \to \infty. \tag{3.8}$$

As an immediate consequence we obtain the following corollary.

Corollary 3.3 (cf. [7]). *We have*

$$E_{N,n} = \frac{h^{\alpha+2}}{12} \sum_{j=1}^{n} a_{n-j} u''(x_j) + \mathcal{O}(h^{\alpha+2}) \quad \textit{for } n = 1, 2, \ldots, N. \tag{3.9}$$

PROOF. This follows immediately from the representations (3.4), (3.5) and (3.7):

$$\begin{aligned} c_{n-j} u''(x_j - \theta_{n,j} h) &= c_{n-j} u''(x_j) + \mathcal{O}((n-j)^{\alpha-1} h) \\ &= \tfrac{1}{12} a_{n-j} u''(x_j) + \mathcal{O}((n-j)^{\alpha-2}) + \mathcal{O}((n-j)^{\alpha-1} h) \quad \text{for } j \le n-1. \end{aligned}$$

□

4 Error analysis of the trapezoidal method for first kind Volterra integral equations with perturbations

We now present the main result on the convergence order of the approximations obtained by the scheme (2.12). As a first step we observe that, under the conditions in Assumption 3.1, we have

$$\begin{aligned} & h^\alpha \sum_{j=1}^{n} a_{n-j}\, k(x_n, x_j)\, u(x_j) \\ &= f(x_n) + \frac{h^{\alpha+2}}{12} \sum_{j=1}^{n} a_{n-j} \varphi(x_n, x_j) - h^\alpha \tilde{a}_n k(x_n, 0) u(0) + \mathcal{O}(h^{\alpha+2}) \quad \text{for } n = 1, 2, \ldots, N \end{aligned} \tag{4.1}$$

uniformly with respect to n, with the function $\varphi(x, z) = \frac{d^2}{dy^2}\{\, k(x, y) u(y) \,\}_{|y=z}$ for $0 \le z \le x \le L$. For each n the representation in (4.1) follows from the representation (3.1) and (3.9), considered for $L = x_n = nh$, and with the function u replaced by the function $y \mapsto k(x_n, y) u(y)$ for $0 \le y \le nh$.

As a further preparation, in the sequel we identify sequences $(b_n)_{n \ge 0}$ of complex numbers with their (formal) power series $b(\xi) = \sum_{n=0}^{\infty} b_n \xi^n$. Pointwise multiplication of two power series

$$\Big(\sum_{k=0}^{\infty} b_k \xi^k \Big) \cdot \Big(\sum_{j=0}^{\infty} c_j \xi^j \Big) = \sum_{n=0}^{\infty} d_n \xi^n, \quad \text{with } d_n := \sum_{k=0}^{n} b_k c_{n-k} \quad \text{for } n = 0, 1, \ldots$$

makes the set of power series into a complex commutative algebra with unit element $1 + 0 \cdot \xi + 0 \cdot \xi^2 + \ldots$. For any power series $b(\xi) = \sum_{n=0}^{\infty} b_n \xi^n$ with $b_0 \neq 0$ there exists a power series which inverts the power series b with respect to pointwise multiplication and is denoted by $1/b(\xi)$ or by $[b(\xi)]^{-1}$. For a thorough introduction to formal power series see, e.g., Henrici [12].

In the sequel we consider the inverse

$$[a(\xi)]^{-1} = \sum_{n=0}^{\infty} a_n^{(-1)} \xi^n \tag{4.2}$$

of the generating function $a(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$. These coefficients satisfy

$$a_n^{(-1)} \;=\; \mathcal{O}(n^{-\alpha-1}) \quad \text{as } \; n \to \infty \tag{4.3}$$

which is shown in [7]. Another proof of (4.3) which uses Banach algebra theory and may be of independent interest is presented in Section 6 of the present paper.

The representations (4.1)–(4.3) provide basic tools for the convergence analysis.

Theorem 4.1. *Let the conditions of Assumption 3.1 be satisfied, and let $u_0^\delta \in \mathbb{R}$ be a starting value with $u_0^\delta - u(0) = \mathcal{O}(h^2 + \delta/h^\alpha)$ as $(h, \delta) \to 0$. Then the approximations $u_1^\delta, u_2^\delta, \ldots, u_N^\delta$ determined by (2.12) can be estimated as follows,*

$$\max_{n=1,2,\ldots,N} |u_n^\delta - u(x_n)| \;=\; \mathcal{O}(h^2 + \delta/h^\alpha) \quad \textit{as} \;\; (h, \delta) \to 0.$$

PROOF. The approximation property (2.13) of the starting value, the representations (4.1) and (2.12) and the estimate $\tilde{a}_n = \mathcal{O}(1)$ as $n \to \infty$ (cf. (3.8)) imply the following,

$$h^\alpha \sum_{j=1}^{n} a_{n-j}\, k(x_n, x_j)\, e_j^\delta \;=\; -\frac{h^{\alpha+2}}{12} \sum_{j=1}^{n} a_{n-j}\, \varphi(x_n, x_j) + \mathcal{O}(h^{\alpha+2} + \delta) \quad \text{for } \; n = 1, 2, \ldots, N \tag{4.4}$$

as $(h, \delta) \to 0$ uniformly with respect to n, where

$$e_n^\delta = u_n^\delta - u(x_n), \quad n = 1, 2, \ldots, N.$$

We next consider a matrix-vector formulation of (4.4). As a preparation we consider the matrix $A_h \in \mathbb{R}^{N \times N}$ given by

$$A_h \;=\; \begin{pmatrix} a_0 k_{1,1} & 0 & \cdots & \cdots & 0 \\ a_1 k_{2,1} & a_0 k_{2,2} & \ddots & & 0 \\ \vdots & a_1 k_{3,2} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ a_{N-1} k_{N,1} & \cdots & \cdots & a_1 k_{N,N-1} & a_0 k_{N,N} \end{pmatrix}$$

with the notation

$$k_{n,j} \;=\; k(x_n, x_j) \quad \text{for } \; 1 \le j \le n \le N.$$

Additionally we consider the matrix $B_h \in \mathbb{R}^{N\times N}$ given by

$$B_h \;=\; \begin{pmatrix} a_0\varphi(x_1,x_1) & 0 & \cdots & \cdots & 0 \\ a_1\varphi(x_2,x_1) & a_0\varphi(x_2,x_2) & \ddots & & 0 \\ \vdots & a_1\varphi(x_3,x_2) & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ a_{N-1}\varphi(x_N,x_1) & \cdots & \cdots & a_1\varphi(x_N,x_{N-1}) & a_0\varphi(x_N,x_N) \end{pmatrix},$$

and the vectors

$$\Delta_h^\delta \;=\; (e_1^\delta, e_2^\delta, \ldots, e_N^\delta)^\top, \qquad G_h \;=\; (1,1,\ldots,1)^\top \in \mathbb{R}^N.$$

Using these notations, the linear system (4.4) becomes

$$h^\alpha A_h \Delta_h^\delta \;=\; -\frac{h^{\alpha+2}}{12} B_h G_h + F_h^\delta, \quad \text{with } F_h^\delta \in \mathbb{R}^N,\ \|F_h^\delta\|_\infty \;=\; \mathcal{O}(h^{\alpha+2}+\delta) \quad \text{as } (h,\delta)\to 0, \tag{4.5}$$

where $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{R}^N$. For a further treatment of the identity (4.5) we consider now the matrix $D_h \in \mathbb{R}^{N\times N}$ given by

$$D_h \;=\; \begin{pmatrix} a_0^{(-1)} & 0 & \cdots & \cdots & 0 \\ a_1^{(-1)} & a_0^{(-1)} & 0 & & 0 \\ a_2^{(-1)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_{N-1}^{(-1)} & \cdots & \cdots & a_1^{(-1)} & a_0^{(-1)} \end{pmatrix}. \tag{4.6}$$

We apply the matrix $h^{-\alpha}D_h$ to both sides of (4.5) and show in the sequel that

$$\|D_h A_h \Delta_h^\delta\|_\infty \;=\; \mathcal{O}(h^2 + \delta/h^\alpha) \quad \text{as } (h,\delta)\to 0 \tag{4.7}$$

holds. For this purpose we first observe that

$$\|D_h\|_\infty \;=\; \mathcal{O}(1) \quad \text{as } h\to 0, \qquad \|G_h\|_\infty \;=\; 1, \tag{4.8}$$

holds, where in the first term $\|\cdot\|_\infty$ denotes the matrix norm induced by the maximum vector norm on $\mathbb{R}^N$. The first estimate in (4.8) follows from the decay of the coefficients of the inverse of the generating function a, cf. estimate (4.3). The lower triangular matrix $D_h B_h$ can be written as follows,

$$D_h B_h \;=\; M_h + hC_h \quad \text{with} \quad \begin{aligned} & M_h = \mathrm{diag}(\varphi(x_1,x_1), \varphi(x_2,x_2), \ldots, \varphi(x_N,x_N)), \\ & C_h \;=\; (c_{h,n,j}) \in \mathbb{R}^{N\times N} \quad \text{strictly lower triangular}, \\ & \max_{1\le j<n\le N} |c_{h,n,j}| \;=\; \mathcal{O}(1) \quad \text{as } h\to 0, \end{aligned}$$

cf. the proof of Lemma 4.2 in Eggermont [7] for more details. This in particular means

$$\| D_h B_h \|_\infty \; = \; \mathcal{O}(1) \quad \text{as} \;\; h \to 0. \tag{4.9}$$

From (4.5), (4.8) and (4.9) we then obtain estimate (4.7). For a further treatment of estimate (4.7) we note that the lower triangular matrix $D_h A_h$ can be written as follows,

$$D_h A_h \; = \; I + h K_h \quad \text{with} \qquad K_h \; = \; (k_{h,n,j}) \in \mathbb{R}^{N\times N} \quad \text{strictly lower triangular,}$$
$$\max_{1 \le j < n \le N} | k_{h,n,j} | \; = \; \mathcal{O}(1) \quad \text{as} \;\; h \to 0.$$

This follows similarly as the representation of $D_h B_h$ considered above, cf. again Lemma 4.2 in [7] for more details. This representation and the discrete version of Gronwall's inequality now yields

$$\| (D_h A_h)^{-1} \|_\infty \; = \; \mathcal{O}(1) \quad \text{as} \;\; h \to 0. \tag{4.10}$$

The statement of the theorem now follows from estimates (4.7) and (4.10). $\Box$

In the sequel we consider (for $L > 0$ fixed) step sizes $h = L/N$ which depend on the noise level δ as follows,

$$h \sim \delta^{1/(\alpha+2)} \quad \text{as} \;\; \delta \to 0, \tag{4.11}$$

i. e., there exist real constants $c_2 \ge c_1 > 0$ such that $c_1 h \le \delta^{1/(\alpha+2)} \le c_2 h$ holds for $\delta \to 0$. As an immediate consequence of Theorem 4.1 we obtain the following main result of this paper.

Corollary 4.2. *Let Assumption 3.1 be satisfied, and let $h = h(\delta)$ be step sizes satisfying (4.11). Let $u_0^\delta \in \mathbb{R}$ be a starting value with $u_0^\delta - u(0) = \mathcal{O}(\delta^{2/(\alpha+2)})$ as $\delta \to 0$. Then the error for the approximations given by (2.12) can be estimated as follows:*

$$\max_{n=1,2,\ldots,N} | u_n^\delta - u(x_n) | \; = \; \mathcal{O}(\delta^{2/(\alpha+2)}) \quad \textit{as} \;\; \delta \to 0.$$

We conclude this section with some remarks.

Remark 4.3. (a) The error estimate presented in Corollary 4.2 is order optimal within the class of functions $\mathcal{R}(A^p) = \{ \, u = A^p z, z : [0,L] \to \mathbb{R} \text{ continuous} \, \}$ with $p = 2/\alpha$. For a consideration of fractional powers of Abel integral operators see, e.g., [16].

(b) The smoothness conditions on the solution u considered in Assumption 3.1 are satisfied (and additionally, the existence of the solution u can be guaranteed then), if the exact right-hand side f can be written in the form $f(x) = x^\alpha g(x)$ with a function $g \in C^4[0,L]$ and if in addition the kernel $k(x,y)$ has for $0 \le y \le x \le L$ continuous partial derivatives up to the order 5, cf. Atkinson [2] for the details.

(c) In the situation of part (b) of this remark there holds $\Gamma(\alpha+1)g(0) = u(0)$, see e.g., Theorem 1.3.11 in Brunner / van der Houwen [4]. Thus, a possible strategy for determining a starting value u_0^δ satisfying (2.13) is to consider the interpolating polynomial P^δ of degree not larger than 1 which satisfies $P^\delta(x_n) = f_n^\delta / x_n^\alpha$ for $n = 1, 2$. The choice $u_0^\delta = \alpha P^\delta(0) = u(0) + \mathcal{O}(h^2 + \delta/h^\alpha)$ (cf. Lemma 4.4 below, applied with $\varepsilon = \delta/h^\alpha, \psi(x) = g(x)$ and $\psi_n^\varepsilon = f_n^\delta / x_n^\alpha$) then gives a starting value of sufficient accuracy.

(d) For other special regularization methods for the approximate solution of Volterra integral equations of the first kind with perturbed right-hand sides and with possibly weakly singular kernels, see e.g., Bughgeim [5], Gorenflo / Vessella [9], Lamm [14], and [18], and the references therein.

We conclude this section with an elementary lemma on extrapolation with perturbed data which completes the considerations in part (b) of the Remark 4.3. In the sequel, Π_1 denotes the space of polynomials with degree ≤ 1.

Lemma 4.4. *Let $\psi : [0, 2h] \to \mathbb{R}$ be a twice continuously differentiable function, with $h > 0$. Let ψ_1^ε and ψ_2^ε be real numbers with $|\psi_n^\varepsilon - \psi(nh)| \le \varepsilon$ for $n = 1, 2$, with some $\varepsilon > 0$. Let the polynomial $P^\varepsilon \in \Pi_1$ satisfy $P^\varepsilon(nh) = \psi_n^\varepsilon$ for $n = 1, 2$. Then there holds*

$$|P^\varepsilon(0) - \psi(0)| \;\le\; \max_{0 \le x \le 2h} |\psi''(x)| h^2 + 3\varepsilon.$$

PROOF. We consider the polynomial $P \in \Pi_1$ with $P(nh) = \psi(nh)$ for $n = 1, 2$. The standard error representation for polynomial interpolation (see e.g., [17]) gives $|P(0) - \psi(0)| \le \max_{0 \le x \le 2h} |\psi''(x)| h^2$. In addition, an expansion of $P^\varepsilon(0) - P(0)$ in terms of Lagrange basis polynomials $L_n \in \Pi_1, n = 1, 2$, with respect to the two grid points $h, 2h$ gives $|P^\varepsilon(0) - P(0)| \le c\varepsilon$ with $c = |L_1(0)| + |L_2(0)| = 3$. This completes the proof. □

5 Numerical experiments

As an illustration of the main result considered in Corollary 4.2, we next present the results of some numerical experiments. First we consider the following linear weakly singular Volterra integral equation of the first kind,

$$\frac{1}{\sqrt{\pi}} \int_0^x (x-y)^{-1/2} e^{-(x-y)} u(y)\, dy \;=\; \underbrace{e^{-x}(x^4 + x^6)}_{=:\, f(x)} \quad \text{for } 0 \le x \le 1, \tag{5.1}$$

with exact solution (cf. (2.2))

$$u(y) \;=\; e^{-y}\big(\tfrac{4!}{\Gamma(4.5)} y^{3.5} + \tfrac{6!}{\Gamma(6.5)} y^{5.5}\big) \quad \text{for } 0 \le y \le 1,$$

so that the conditions in (a) and (b) of Assumption 3.1 are satisfied. Here are some additional remarks on the numerical tests:

- numerical experiments with step sizes $h = 1/2^q$ for $q = 5, 6, \ldots, 11$ are employed, respectively;
- for each considered step size h, the noise level $\delta = h^{2.5}$ is considered;
- in the numerical experiments, the perturbations are of the form $f_n^\delta = f(x_n) + \Delta_n$ with uniformly distributed random values Δ_n with $|\Delta_n| \le \delta$;
- in each experiment, the starting value u_0^δ is determined by the strategy described in part (c) of Remark 4.3.

Experiments are employed using the program system OCTAVE (http://www.octave.org). The results are shown in Table 1. There, $\|f\|_\infty$ denotes the maximum norm of the function f.

N	δ	$100 \cdot \delta / \|f\|_\infty$	$\max_n \lvert u_n^\delta - u(x_n) \rvert$	$\max_n \lvert u_n^\delta - u(x_n) \rvert \,/ \delta^{4/5}$
32	$1.7 \cdot 10^{-4}$	$2.35 \cdot 10^{-2}$	$2.95 \cdot 10^{-3}$	3.02
64	$3.1 \cdot 10^{-5}$	$4.15 \cdot 10^{-3}$	$1.04 \cdot 10^{-3}$	4.26
128	$5.4 \cdot 10^{-6}$	$7.33 \cdot 10^{-4}$	$2.03 \cdot 10^{-4}$	3.32
256	$9.5 \cdot 10^{-7}$	$1.30 \cdot 10^{-4}$	$5.79 \cdot 10^{-5}$	3.79
512	$1.7 \cdot 10^{-7}$	$2.29 \cdot 10^{-5}$	$1.72 \cdot 10^{-5}$	4.50
1024	$3.0 \cdot 10^{-8}$	$4.05 \cdot 10^{-6}$	$4.24 \cdot 10^{-6}$	4.45
2048	$5.3 \cdot 10^{-9}$	$7.16 \cdot 10^{-7}$	$1.09 \cdot 10^{-6}$	4.58

Table 1: Numerical results for equation (5.1)

Next we present some numerical results for another right-hand side (and the same operator as above):

$$\frac{1}{\sqrt{\pi}} \int_0^x (x-y)^{-1/2} e^{-(x-y)} u(y)\, dy \;=\; \underbrace{e^{-x}(\sqrt{x} + x^4)}_{=:\, f(x)} \quad \text{for } 0 \le x \le 1, \tag{5.2}$$

with exact solution (cf. (2.2))

$$u(y) \;=\; e^{-y}\big(\tfrac{\sqrt{\pi}}{2} + \tfrac{4!}{\Gamma(4.5)} y^{3.5}\big) \quad \text{for } 0 \,\le\, y \,\le\, 1,$$

so that the conditions in (a) and (b) of Assumption 3.1 are again satisfied, but this time we have $u(0) \neq 0$. Step sizes, noise levels and starting value are chosen similar to the example considered above. The results are shown in Table 2.

N	δ	$100 \cdot \delta / \lVert f \rVert_\infty$	$\max_n \lvert u_n^\delta - u(x_n) \rvert$	$\max_n \lvert u_n^\delta - u(x_n) \rvert \,/ \delta^{4/5}$
32	$1.7 \cdot 10^{-4}$	$2.35 \cdot 10^{-2}$	$1.94 \cdot 10^{-3}$	1.98
64	$3.1 \cdot 10^{-5}$	$4.15 \cdot 10^{-3}$	$6.14 \cdot 10^{-4}$	2.51
128	$5.4 \cdot 10^{-6}$	$7.33 \cdot 10^{-4}$	$1.75 \cdot 10^{-4}$	2.86
256	$9.5 \cdot 10^{-7}$	$1.30 \cdot 10^{-4}$	$4.00 \cdot 10^{-5}$	2.62
512	$1.7 \cdot 10^{-7}$	$2.29 \cdot 10^{-5}$	$9.64 \cdot 10^{-6}$	2.53
1024	$3.0 \cdot 10^{-8}$	$4.05 \cdot 10^{-6}$	$2.54 \cdot 10^{-6}$	2.66
2048	$5.3 \cdot 10^{-9}$	$7.16 \cdot 10^{-7}$	$6.72 \cdot 10^{-7}$	2.82

Table 2: Numerical results for equation (5.2)

Note that the relative errors in the right-hand side presented in the third column (of both tables in fact) are rather small, respectively.

6 Estimates for the inverse of the generating function

We now present a proof of (4.3) for the coefficients of the inverse of the considered generating power series $\sum_{n=0}^{\infty} a_n \xi^n$ which differs from that given in [7]. The proof presented here uses Banach algebra theory and may be of independent interest.

6.1 Some results for power series

In the sequel we consider classes of power series $b(\xi) = \sum_{n=0}^{\infty} b_n \xi^n$ with $(b_n)_{n \ge 0} \subset \mathbb{C}$ and convergence radius of at least 1, e.g., power series with absolutely summable coefficients,

$$\lVert b \rVert_1 = \sum_{n=0}^{\infty} \lvert b_n \rvert < \infty.$$

In the latter case the series $\sum_{n=0}^{\infty} b_n \xi^n$ converges absolutely for each $\xi \in \mathbb{C}$ with $\lvert \xi \rvert \le 1$, and it is continuous on the closed unit disc in the complex plane. The following lemma turns out to be useful.

Lemma 6.1. *For each q > 1 the space of power series $\sum_{n=0}^{\infty} b_n \xi^n$ satisfying $b_n = \mathcal{O}(n^{-q})$ as $n \to \infty$ and endowed with pointwise multiplication defines a complex algebra.*

PROOF. We only show that the considered space is closed with respect to pointwise multiplication. For two power series $\sum_{n=0}^{\infty} b_n \xi^n$ and $\sum_{n=0}^{\infty} c_n \xi^n$ satisfying $b_n = \mathcal{O}(n^{-q})$ and $c_n = \mathcal{O}(n^{-q})$ as $n \to \infty$ we have:

$$\begin{aligned}
\lvert \sum_{k=0}^{n} b_k c_{n-k} \rvert = \lvert \sum_{k \le n/2} b_k c_{n-k} \rvert + \lvert \sum_{k > n/2} b_k c_{n-k} \rvert \\
&\le 2^q \big(\sum_{k \le n/2} \lvert b_k \rvert \cdot \lvert c_{n-k} \rvert (n-k)^q + \sum_{k > n/2} \lvert b_k \rvert \cdot k^q \lvert c_{n-k} \rvert \big) n^{-q} \\
&\le 2^q (\lVert b \rVert_1 \mathcal{O}(1) + \mathcal{O}(1) \lVert c \rVert_1) n^{-q} \;=\; \mathcal{O}(n^{-q}) \quad \text{as } n \to \infty.
\end{aligned}$$

This completes the proof. □

For similar considerations and further analysis of similar spaces see Rogozin ([19], [20]).

6.2 The power series $\sum_{n=0}^{\infty}(n+1)^{\alpha+1}\xi^n$

Our analysis continues with a special representation of the power series $\sum_{n=0}^{\infty}(n+1)^{\alpha+1}\xi^n$. For this, binomial expansions will be useful:

$$(1-\xi)^\beta = \sum_{n=0}^{\infty}(-1)^n\binom{\beta}{n}\xi^n \quad \text{for } \xi\in\mathbb{C},\ |\xi|<1 \qquad (\beta\in\mathbb{R}), \tag{6.1}$$

$$(-1)^n\binom{\beta}{n} \;=\; \sum_{s=0}^{m-1} d_{\beta,s}n^{-\beta-1-s} + \mathcal{O}(n^{-\beta-1-m}) \quad \text{as } n\to\infty, \tag{6.2}$$

with certain real coefficients $d_{\beta,s}$ for $s=0,1,\ldots,m-1$, $m=0,1,\ldots$, where $d_{\beta,0}=1/\Gamma(-\beta), \beta\neq 0,1,\ldots$, cf. e.g., Abramowitz/Stegun [1]. We need the following result.

Lemma 6.2. *We have*

$$\frac{1}{\Gamma(\alpha+2)}\sum_{n=0}^{\infty}(n+1)^{\alpha+1}\xi^n \;=\; (1-\xi)^{-\alpha-2}r(\xi) \quad \textit{for } \xi\in\mathbb{C},\ |\xi|<1, \tag{6.3}$$

$$\textit{with } r(\xi)=\sum_{n=0}^{\infty}r_n\xi^n, \quad r(1)=1, \quad r_n=\mathcal{O}(n^{-\alpha-3}) \ \textit{as } n\to\infty. \tag{6.4}$$

PROOF. We first observe that for each $m\geq 0$ there exist real coefficients $c_0,c_1,\ldots,c_m$ with

$$\frac{1}{\Gamma(\alpha+2)}\sum_{n=0}^{\infty}(n+1)^{\alpha+1}\xi^n \;=\; \sum_{j=0}^{m}c_j(1-\xi)^{-\alpha-2+j}+s(\xi) \quad \text{for } \xi\in\mathbb{C},\ |\xi|<1, \tag{6.5}$$

with $s(\xi)=\sum_{n=0}^{\infty}s_n\xi^n$, where $s_n=\mathcal{O}(n^{\alpha-m})$ as $n\to\infty$, and we have $c_0=1$. This follows by comparing the coefficients in the Taylor expansion $(n+1)^{\alpha+1}=\sum_{t=0}^{m}e_tn^{\alpha+1-t}+\mathcal{O}(n^{\alpha-m})$ with the coefficients in the expansions considered in (6.1) and (6.2).

A reformulation of (6.5) gives, with $m=5$,

$$\frac{1}{\Gamma(\alpha+2)}\sum_{n=0}^{\infty}(n+1)^{\alpha+1}\xi^n = (1-\xi)^{-\alpha-2}\Big(\sum_{j=0}^{5}c_j(1-\xi)^j+(1-\xi)^{\alpha+2}s(\xi)\Big) \quad \text{for } \xi\in\mathbb{C},\ |\xi|<1,$$

$$\text{with } s(\xi)=\sum_{n=0}^{\infty}s_n\xi^n, \quad s_n=\mathcal{O}(n^{\alpha-5}) \quad \text{as } n\to\infty.$$

Lemma 6.1 together with (6.1), (6.2) applied with $\beta=\alpha+2, m=0$ then gives the statement of the lemma. □

As a consequence of Lemma 6.2 we obtain the following representation.

Corollary 6.3. *For the coefficients considered in (2.6) we have, with the power series r from (6.3), (6.4),*

$$\sum_{n=0}^{\infty}a_n\xi^n \;=\; (1-\xi)^{-\alpha}r(\xi) \quad \textit{for } \xi\in\mathbb{C},\ |\xi|<1. \tag{6.6}$$

PROOF. The power series $\sum_{n=0}^{\infty}(n+1)^{\alpha+1}\xi^n$ and the power series $a(\xi)=\sum_{n=0}^{\infty}a_n\xi^n$ with coefficients as in (2.6) are related as follows,

$$\sum_{n=0}^{\infty}a_n\xi^n \;=\; \frac{1}{\Gamma(\alpha+2)}(1-\xi)^2\sum_{n=0}^{\infty}(n+1)^{\alpha+1}\xi^n,$$

which follows from elementary computations. The representation (6.3) now implies the statement of the corollary. □

Inverting (6.6) now immediately gives the power series representation

$$\sum_{n=0}^{\infty} a_n^{(-1)} \xi^n \;=\; (1-\xi)^\alpha [r(\xi)]^{-1} \tag{6.7}$$

where $a_n^{(-1)}$ denote the coefficients of the inverse of the power series $a(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$, cf. (4.2).

In the sequel we examine the asymptotic behavior of the coefficients in the power series

$$[r(\xi)]^{-1} \;=\; \sum_{n=0}^{\infty} r_n^{(-1)} \xi^n. \tag{6.8}$$

Lemma 6.4. *We have* $r_n^{(-1)} = \mathcal{O}(n^{-\alpha-3+\varepsilon})$ *as* $n \to \infty$ *for* $\varepsilon > 0$ *arbitrarily small.*

PROOF. (a) We consider, for $q > 1$ fixed, the following space of power series,

$$c_{0,q} := \Big\{ \sum_{n=0}^{\infty} b_n \xi^n \mid (b_n)_{n \ge 0} \subset \mathbb{C},\ b_n n^q \to 0 \text{ for } n \to \infty \Big\}$$

with norm $\|b\|_{\infty,q} = \sup_{n\ge 0} |b_n|(n+1)^q$ for $b \in c_{0,q}$. It is easy to show that the space $c_{0,q}$, endowed with pointwise multiplication and after a renorming

$$\|b\|'_{\infty,q} := \sup_{a \in c_{0,q}, \|a\|_{\infty,q} \le 1} \|a \cdot b\|_{\infty,q} \quad \text{for } b \in c_{0,q},$$

is a complex commutative Banach algebra with unit element.

(b) For each complex homomorphism $h : c_{0,q} \to \mathbb{C}$ not vanishing identically we have $h = \delta_\lambda$ for some $\lambda = \lambda(h) \in \mathbb{C}$ with $|\lambda| \le 1$, where δ_λ means evaluation of a power series $\in c_{0,q}$ at λ. This correspondence between a homomorphism h and the complex number λ is as follows: $h(\xi) = \lambda$, where ξ is short notation for the power series $0 + \xi + 0 \cdot \xi^2 + \ldots$. Details are left to the reader; we only note that the estimate $|\lambda| \le 1$ follows from $\|\xi^n\|_{\infty,q}^{1/n} \to 1$ as $n \to \infty$, and polynomials are dense in $c_{0,q}$. Now it follows from Banach algebra theory that a power series $\sum_{n=0}^{\infty} b_n \xi^n \in c_{0,q}$ has an inverse in $c_{0,q}$ with respect to pointwise multiplication if and only if $\sum_{n=0}^{\infty} b_n \xi^n \ne 0$ for each $\xi \in \mathbb{C}$ with $|\xi| \le 1$. See, e.g., Rudin [21] for a general treatment of Banach algebras, and see also Rogozin ([19], [20]).

(c) It follows from (6.4) that the power series r considered in (6.3) satisfies $r \in c_{0,\alpha+3-\varepsilon}$ for each $\varepsilon > 0$ small enough, and let us assume that $r(\xi) \ne 0$ for $\xi \in \mathbb{C}, |\xi| \le 1$. From (6.7) and parts (a) and (b) of this lemma we then obtain that $[r(\xi)]^{-1} \in c_{0,\alpha+3-\varepsilon}$ and thus in particular $r_n^{(-1)} = \mathcal{O}(n^{-\alpha-3+\varepsilon})$ as $n \to \infty$.

(d) In view of parts (a)–(c) of this proof it remains to show that $r(\xi) \ne 0$ holds for $\xi \in \mathbb{C}, |\xi| \le 1$. For this purpose we consider a reformulation of (6.6),

$$r(\xi) \;=\; (1-\xi)^\alpha \sum_{n=0}^{\infty} a_n \xi^n \quad \text{for } \xi \in \mathbb{C},\ |\xi| < 1.$$

We have, for some $\tau > 0$,

$$\Big| \sum_{n=0}^{\infty} a_n \xi^n \Big| \ge \tau \quad \text{for } \xi \in \mathbb{C},\ |\xi| < 1, \tag{6.9}$$

a proof of (6.9) is presented in the next section. Since $r(1) \ne 0$ and r is continuous on $\{ \xi \in \mathbb{C} \mid |\xi| \le 1 \}$, (6.9) then implies $r(\xi) \ne 0$ for $\xi \in \mathbb{C}, |\xi| \le 1$ as desired, and thus the statement of the lemma is proved. □

From the representation (6.1), (6.2) with $\beta = \alpha$ it follows that the coefficients in the expansion $(1-\xi)^\alpha = \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} \xi^n$ satisfy $(-1)^n \binom{\alpha}{n} = \mathcal{O}(n^{-\alpha-1})$ as $n \to \infty$. This together with Lemma 6.4 (which means in particular $r_n^{(-1)} = \mathcal{O}(n^{-\alpha-1})$) and Lemma 6.1 finally results in the desired estimate (4.3) for the coefficients of the power series $[a(\xi)]^{-1}$.

6.3 The proof of the lower bound (6.9)

To complete our proof of (4.3) we need to show that (6.9) holds. We start with a useful lemma.

Lemma 6.5 (cf. [7]). *The coefficients $a_0, a_1, \ldots$ in (2.6) are positive, respectively, with $\sum_{n=0}^{\infty} a_n = \infty$. Moreover they satisfy*

$$\frac{a_{n+1}}{a_n} > \frac{a_n}{a_{n-1}} \quad \textit{for } n = 2, 3, \ldots, \qquad \frac{a_2}{a_1} > \frac{a_1}{2/\Gamma(\alpha+2)}. \tag{6.10}$$

PROOF. From the asymptotic behavior $a_n = n^{\alpha-1}/\Gamma(\alpha) + \mathcal{O}(n^{\alpha-2})$ as $n \to \infty$, cf. (3.7), it follows $\sum_{n=0}^{\infty} a_n = \infty$. The second estimate in (6.10) is obtained by direct computation, and now we will consider the first estimate in (6.10). Using the notation

$$f(x) = (x+1)^{\alpha+1} - x^{\alpha+1} \quad \text{for } x \geq 0$$

we obtain the following,

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{f(n+1) - f(n)}{f(n) - f(n-1)} \overset{(*)}{=} \frac{f'(t_n)}{f'(t_n - 1)} = \frac{(t_n+1)^{\alpha} - t_n^{\alpha}}{t_n^{\alpha} - (t_n - 1)^{\alpha}} \\ &= \frac{(1 + 1/t_n)^{\alpha} - 1}{1 - (1 - 1/t_n)^{\alpha}} \quad =: \quad h(1/t_n) \quad \text{for } n = 1, 2, \ldots, \end{aligned}$$

with some real number $n < t_n < n+1$. The identity $(*)$ follows from a generalized mean value theorem. From the identities

$$\begin{aligned} (1 - (1-s)^{\alpha})^2 h'(s) &= \alpha(1+s)^{\alpha-1}(1-s)^{\alpha-1} g(s), \\ &\text{with } g(s) = (1-s)^{1-\alpha} + (1+s)^{1-\alpha} - 2 \quad \text{for } 0 < s < 1, \end{aligned}$$

and from the inequality $g(s) < 0$ for $0 < s < 1$ it follows that $h(s)$ is monotonically decreasing for $0 < s < 1$ which yields the first of the two estimates in (6.10). Finally, we have $a_1 > 0$, and from the inequalities (6.10) it follows by induction then that the coefficients $a_2, a_3, \ldots$ are positive, respectively. This completes the proof. □

From Lemma 6.5 it follows that the conditions of the following lemma are satisfied for the specific choice $p_0 = 2$ and $p_n = \Gamma(\alpha+2) a_n$ for $n = 1, 2, \ldots$.

Lemma 6.6 (cf. Kaluza [13], see also Szegö [22], Hardy [11]). *Let $p_0, p_1, \ldots$ be real numbers satisfying*

$$p_n > 0 \quad \textit{for } n = 0, 1, \ldots, \qquad \frac{p_{n+1}}{p_n} > \frac{p_n}{p_{n-1}} \quad \textit{for } n = 1, 2, \ldots . \tag{6.11}$$

Then the inverse $[p(\xi)]^{-1}$ of the power series $p(\xi) = \sum_{n=1}^{\infty} p_n \xi^n$ can be written as follows,

$$[p(\xi)]^{-1} = c_0 - \sum_{n=1}^{\infty} c_n \xi^n, \tag{6.12}$$

with coefficients $c_0, c_1, \ldots$ satisfying $c_n > 0$ for $n = 0, 1, \ldots$. If moreover $\sum_{n=0}^{\infty} p_n = \infty$ holds, then we have $\sum_{n=1}^{\infty} c_n = c_0$.

The following lemma is closely related to results in Erdös, Feller and Pollard [8].

Lemma 6.7. *Let $c_1, c_2, \ldots$ be a sequence of real numbers satisfying $c_n > 0$ for $n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} c_n = 1/2$. Then the power series $q(\xi) = 1/2 - \sum_{n=1}^{\infty} c_n \xi^n$ satisfies $|q(\xi)| < 1$ for each complex number ξ with $|\xi| \leq 1$.*

PROOF. For complex numbers ξ with $|\xi| < 1$ we have

$$|q(\xi)| \le \tfrac{1}{2} + \sum_{n=1}^{\infty} \underbrace{c_n}_{>0} \underbrace{|\xi|^n}_{<1} \;<\; \tfrac{1}{2} + \sum_{n=1}^{\infty} c_n \;=\; 1.$$

We now consider a complex number ξ with $|\xi| = 1$ and assume contradictory that $|q(\xi)| \ge 1$ holds. From that we obtain

$$1 \le |q(\xi)| \le |\tfrac{1}{2} - c_1\xi| + \sum_{n=2}^{\infty} c_n \;=\; |\tfrac{1}{2} - c_1\xi| + \tfrac{1}{2} - c_1 \tag{6.13}$$

which shows $(1/2 + c_1)^2 \;\le\; |1/2 - c_1\xi|^2 = 1/4 - c_1 \mathrm{Re}\,\xi + c_1^2$ so that necessarily $\xi = -1$ holds. With a similar reasoning as in (6.13) we then get

$$1 \;\le\; |q(\xi)| \le |\tfrac{1}{2} + c_1 - c_2| + \sum_{n=3}^{\infty} c_n \;=\; |\tfrac{1}{2} + c_1 - c_2| + \tfrac{1}{2} - c_1 - c_2$$

which results in the contradiction $1/2 + c_1 + c_2 \le |1/2 + c_1 - c_2|$. This completes the proof of the lemma.□

We are now in a position to present a proof of the lower bound (6.9). In fact, from Lemma 6.5 it follows that the coefficients of the power series $p(\xi) = 1 + \Gamma(\alpha+2)a(\xi)$ with a_n as in (2.6) satisfy the conditions of Lemma 6.6. This implies that the coefficients of the power series

$$\frac{1}{1 + \Gamma(\alpha+2)a(\xi)} \;=\; c_0 - \sum_{n=1}^{\infty} c_n \xi^n$$

satisfy $c_n > 0$ for $n = 0, 1, \ldots$ as well as $\sum_{n=1}^{\infty} c_n = c_0 = 1/2$. Lemma 6.7 and continuity then implies that for some $\tau > 0$ we have $|1 + \Gamma(\alpha+2)a(\xi)| \ge 1 + \tau$ for $\xi \in \mathbb{C}, |\xi| < 1$ and thus

$$\Gamma(\alpha+2)|a(\xi)| \ge |1 + \Gamma(\alpha+2)a(\xi)| - 1 \ge \tau \quad \text{for } \xi \in \mathbb{C}, |\xi| < 1.$$

This finally gives the desired estimate (6.9) needed in the proof of Lemma 6.4.

7 Conclusions

In the present paper we have considered the repeated trapezoidal rule for the regularization of weakly singular Volterra integral equations of the first kind with perturbed given right-hand sides. The applied techniques are closely related to those used in Eggermont [7]. The results presented here (which include some numerical experiments) have useful applications for the stable solution of inverse problems. In addition we have given a new proof of the stability estimate for the inverse of the generating sequence, cf. (4.3), which may be of independent interest.

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