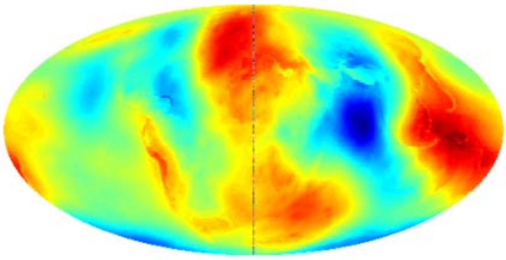

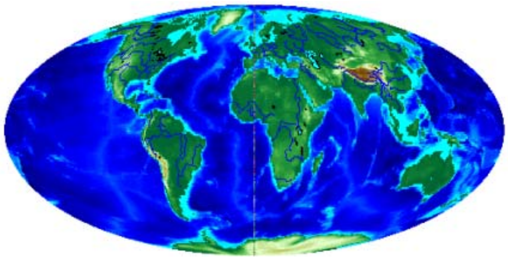

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Regularisation of the Helmholtz Decomposition and its Application to Geomagnetic Field Modelling

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Regularisation of the Helmholtz Decomposition and its Application to Geomagnetic Field Modelling

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Abstract

In this paper, we present a regularised Green's function for $-\nabla^2$ in $\mathbb{R}^3$ and a regularised scalar and vector potential of a vector field on a bounded domain in $\mathbb{R}^3$. We prove that the regularised Green's function for $-\nabla^2$ converges to the actual Green's function. We also show that the regularised parts of the Helmholtz decomposition converge to the actual decomposition. This convergence is given in the pointwise and the distributional sense as well as with respect to the L^1-norm. Moreover, applications to the modelling of static electromagnetic fields are discussed. Finally, numerical results for a synthetic example are presented.

Key Words: Green's function, Helmholtz decomposition, weak convergence, Maxwell's equations, regularisation.

AMS(2000) Classification: 41A30, 41A35, 41A63, 44A35, 65N80, 86-08

1 Introduction

It is well-known that the Helmholtz theorem decomposes an arbitrary vector function into two parts: One is an irrotational component which can be expressed by the gradient of a scalar function and the other one is a rotational part which can be expressed by the curl of a vector function. As a particular decomposition form of a vector function, the theorem has very important applications in electromagnetics because its decomposition terms have been closely related to the scalar potential and the vector potential of a vector field. As we know, the Helmholtz theorem is based on the Green's functions of $-\nabla^2$ in the space $\mathbb{R}^3$ and we also know that the Green's functions have singular points. To avoid the singularity, we introduce a regularised version of the Green's function for $-\nabla^2$ in the space $\mathbb{R}^3$ and we prove that the irrotational as well as the rotational part of the vector function can be approximated by using our regularised Green's function for $-\nabla^2$ in the space $\mathbb{R}^3$.

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2 Preliminaries

The function G_0 defined by

$$G_0(x, y) := \frac{1}{4\pi|x - y|}$$

for all $x, y \in \mathbb{R}^3$ with $x \neq y$ is a Green's function for the operator $-\nabla^2$ in the space $\mathbb{R}^3$ (see [7]). For $\delta \in (0, 1)$, we may define the regularised Green's function with respect to $-\nabla^2$ for all $x, y \in \mathbb{R}^3$ by

$$G_0^\delta(x, y) := \frac{1}{4\pi\left(|x - y|^2 + \delta\right)^{\frac{1}{2}}}.$$

The graph of the regularised Green's function for the operator $-\nabla^2$ for different values of the parameter δ can be seen in Figure 1.

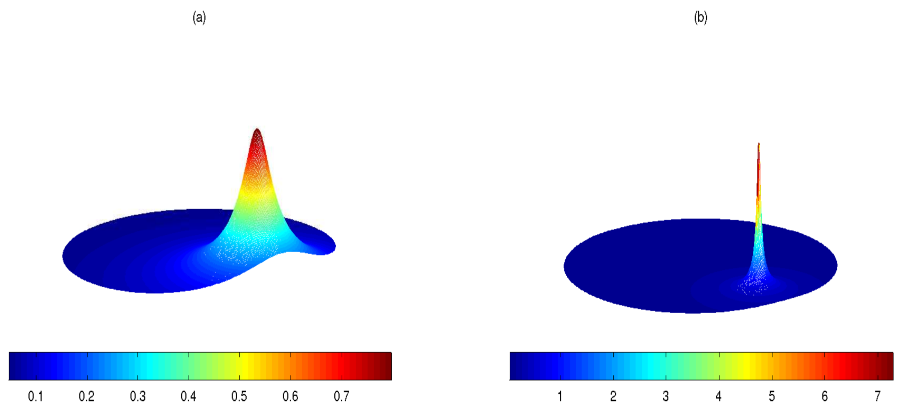

Figure 1: (a,b) are the graphs of the regularised Green's function for the operator $-\nabla^2$ (restricted on the 3-D unit ball) at the fixed point $x = (0, 0, -0.7)$, where the function is plotted in the $y_1 = 0$ plane. In (a) we chose the parameter $\delta = 0.01$ and in (b) $\delta = 0.0001$

Lemma 2.1 *Let G_0 and G_0^δ be as defined above then the following equalities hold for arbitrary but fixed x in $\mathbb{R}^3$.*

(i)

$$\lim_{\delta\to 0+} G_0^\delta(x,y) = G_0(x,y) \quad \textit{for all } y \in \mathbb{R}^3 \textit{ with } y \neq x.$$

(ii)

$$\lim_{\delta\to 0+} \nabla_x G_0^\delta(x,y) = \nabla_x G_0(x,y) \textit{ for all } y \in \mathbb{R}^3 \textit{ with } y \neq x.$$

The proof of this lemma is easy and only involves elementary analytical operations.

3 Convergence of the Regularised Scalar and Vector Potential

Theorem 3.1 *[12] (Helmholtz Theorem in a Bounded Domain) Let S be an arbitrary closed curved surface that is smooth or piecewise smooth and let V be the volume surrounded by the surface S, where $\overline{V} = V \cup S$ represents a closure of the domain V. Moreover, let f be a vector function which is defined in the domain $\overline{V}$. If f satisfies the following conditions:*
(i) f is continuously differentiable in V,
(ii) f is almost everywhere continuous and bounded on the boundary S,
then f can be decomposed into the sum of an irrotational field and a solenoidal field in the following sense:

$$f = -\nabla\rho + \nabla\times\Lambda \tag{1}$$

with

$$\rho(x) = \int_V \frac{\nabla_y \cdot f(y)}{4\pi|x-y|}dy - \int_S \frac{f(y)\cdot\hat{n}(y)}{4\pi|x-y|}dy \tag{2}$$

$$\Lambda(x) = \int_V \frac{\nabla_y \times f(y)}{4\pi|x-y|}dy + \int_S \frac{f(y)\times\hat{n}(y)}{4\pi|x-y|}dy, \tag{3}$$

where $\hat{n}$ is the exterior unit normal vector of the closed surface S.

Theorem 3.2 *Let f be a vector-valued function which is defined in the domain $\overline{V}$ such that f is continuous on $\overline{V}$ and has continuous first order derivatives in V. Then $f = -\nabla\rho + \nabla\times\Lambda$ such that*

(i)

$$\rho(x) = \lim_{\delta\to 0+} \rho^\delta(x)$$

with

$$\rho^\delta(x) = \int_V \nabla_x G_0^\delta(x, y) \cdot f(y)\ dy$$

and

(ii)

$$\Lambda(x) = \lim_{\delta \to 0+} \Lambda^\delta(x)$$

with

$$\Lambda^\delta(x) = \int_V \nabla_x G_0^\delta(x, y) \times f(y)\ dy,$$

for all $x \in \overline{V}$, *where* G_0^δ *is the regularised Green's function for* $-\nabla^2$ *on* $\mathbb{R}^3$.

Proof. Due to Theorem 3.1, we have $f = -\nabla\rho + \nabla \times \Lambda$ with

$$\rho(x) = \int_V G_0(x, y)\nabla_y \cdot f(y)\ dy - \int_S G_0(x, y) f(y) \cdot \hat{n}(y)\ dy \tag{4}$$

and

$$\Lambda(x) = \int_V G_0(x, y)\nabla_y \times f(y)\ dy + \int_S G_0(x, y) \left(f(y) \times \hat{n}(y)\right)\ dy, \tag{5}$$

where $\hat{n}(y)$ is the outward unit surface normal at the surface point y. From the vector calculus (see [15]), we have

$$G_0(\nabla \cdot f) = \nabla \cdot (G_0 f) - (\nabla G_0) \cdot f \tag{6}$$

and

$$G_0(\nabla \times f) = \nabla \times (G_0 f) - (\nabla G_0) \times f. \tag{7}$$

Using Equation (6), we have

$$\begin{aligned}\rho(x) \quad &= \quad \int_V \nabla_y \cdot (G_0(x, y) f(y))\, dy - \int_V (\nabla_y G_0(x, y)) \cdot f(y)\ dy \\ &\quad\; - \int_S G_0(x, y) f(y) \cdot \hat{n}(y)\ dy.\end{aligned}$$

Now applying Gauß's Divergence Theorem (see [15]) on the first integral of the equation given above and using the fact $\nabla_x G_0(x, y) = -\nabla_y G_0(x, y)$ we have

$$\begin{aligned}\rho(x) \quad &= \quad \int_S G_0(x, y) f(y) \cdot \hat{n}(y)\ dy - \int_V (\nabla_y G_0(x, y)) \cdot f(y)\ dy \\ &\quad\; - \int_S G_0(x, y) f(y) \cdot \hat{n}(y)\ dy \\ &= \quad -\int_V (\nabla_y G_0(x, y)) \cdot f(y)\ dy \\ &= \quad \int_V (\nabla_x G_0(x, y)) \cdot f(y)\ dy. \qquad (8)\end{aligned}$$

Similarly, using Equation (7) we get

$$\begin{aligned}\Lambda(x) &= \int_V \nabla_y \times (G_0(x,y)f(y))\,dy - \int_V \nabla_y G_0(x,y) \times f(y)\ dy \\ &\quad + \int_S G_0(x,y)\,(f(y) \times \hat{n}(y))\ dy.\end{aligned}$$

From Gauß's Divergence Theorem we can deduce that

$$\int_V \nabla \times f\ dv = \int_S -(f \times \hat{n})\ ds \tag{9}$$

(see [1], p. 15). Now using Equation (9) and the equality $\nabla_x G_0(x,y) = -\nabla_y G_0(x,y)$ we have

$$\begin{aligned}\Lambda(x) &= \int_S -(G_0(x,y)f(y) \times \hat{n}(y))\,dy - \int_V \nabla_y G_0(x,y) \times f(y)\ dy \\ &\quad + \int_S G_0(x,y)\,(f(y) \times \hat{n}(y))\ dy \\ &= \int_V \nabla_x G_0(x,y) \times f(y)\ dy = \int_V \nabla_x \times (G_0(x,y)f(y))dy.\end{aligned} \tag{10}$$

To prove (i) and (ii), we have to show that

$$\lim_{\delta \to 0+} \int_V \nabla_x G_0^\delta(x,y) \cdot f(y)\ dy = \int_V \nabla_x \cdot (G_0(x,y)f(y))\ dy$$

and

$$\lim_{\delta \to 0+} \int_V \nabla_x G_0^\delta(x,y) \times f(y)\ dy = \int_V \nabla_x \times (G_0(x,y)f(y))\ dy.$$

Using Lemma 2.1 and the continuity of f on V we can say that for arbitrary but fixed x

$$\lim_{\delta \to 0+} \Big(\nabla_x G_0^\delta(x,y) \cdot f(y)\Big) = \nabla_x G_0(x,y) \cdot f(y) \quad \text{on } V \text{ with } x \neq y$$

and

$$\Big|\nabla_x G_0^\delta(x,y) \cdot f(y)\Big| \leq g_x(y) \text{ for all } \delta > 0, \quad y \in V, \text{with } x \neq y, \tag{11}$$

where g_x is defined as

$$g_x(y) := \begin{cases} |\nabla_x G_0(x,y) \cdot f(y)| & \text{for} \quad y \neq x \\ 0 & \text{for} \quad y = x. \end{cases}$$

Now first we confirm that g_x is integrable on V. For this purpose, we consider the integral

$$\begin{aligned}
\int_V |\nabla_x G_0(x,y)\cdot f(y)|\ dy &= \int_V \left|\frac{-(x-y)}{4\pi|x-y|^3}\cdot f(y)\right| dy \\
&= \int_V \frac{\left|-\frac{x-y}{|x-y|}\cdot f(y)\right|}{4\pi|x-y|^2} dy \\
&\leq \int_V \frac{\left|-\frac{x-y}{|x-y|}\right| |f(y)|}{4\pi|x-y|^2} dy \\
&= \int_V \frac{|f(y)|}{4\pi|x-y|^2} dy.
\end{aligned}$$

Since the function f is continuous on the compact domain $\overline{V}$, therefore it is bounded on $\overline{V}$. Let

$$M = \sup_{x\in\overline{V}} |f(x)|,$$

then we get

$$\int_V |\nabla_x G_0(x,y)\cdot f(y)|\ dy \leq \frac{M}{4\pi}\int_V \frac{1}{|x-y|^2} dy.$$

From the proof of a theorem (see [13], p. 158-159) we can see that

$$\int_V \frac{1}{|x-y|^2} dy \leq \frac{4\pi^{\frac{3}{2}} R}{\Gamma(\frac{3}{2})}$$

is bounded, where R is the radius of the ball containing $\overline{V}$. Therefore, we can conclude

$$\int_V |\nabla_x G_0(x,y)\cdot f(y)|\ dy \leq \frac{M}{4\pi}\int_V \frac{1}{|x-y|^2} dy \leq M\frac{\pi^{\frac{1}{2}} R}{\Gamma(\frac{3}{2})} < \infty. \tag{12}$$

This gives that $|\nabla_x G_0(x,y)\cdot f(y)|$ is integrable with respect to y on V. Since $g_x(y) = |\nabla_x G_0(x,y)\cdot f(y)|$ almost everywhere on V as a function of y, therefore g_x is also integrable on V. Now by using Lebesgue's Dominated Convergence Theorem (see [6, 8]) we have

$$\begin{aligned}
\lim_{\delta\to 0+}\int_V \nabla_x G_0^\delta(x,y)\cdot f(y)\ dy &= \int_V \lim_{\delta\to 0} \nabla_x G_0^\delta(x,y)\cdot f(y)\ dy \\
&= \int_V \nabla_x G_0(x,y)\cdot f(y)\ dy = \rho(x).
\end{aligned}$$

For part (ii), let us consider $\left\{\nabla_x \times (G_0^\delta(x,y) f(y))\right\}_\delta = \left\{\nabla_x G_0^\delta(x,y) \times f(y)\right\}_\delta$ which is a family of vectors with real components and by [1], p. 102, (A.1.7)

$$\lim_{\delta \to 0+} \nabla_x G_0^\delta(x,y) \times f(y) = \nabla_x G_0(x,y) \times f(y)$$

almost everywhere on V, for arbitrary but fixed x in $\overline{V}$. By using again equation (10), we get

$$\Lambda(x) = \int_V \nabla_x G_0(x,y) \times f(y)\; dy$$

for all $x \in \overline{V}$. Now, we show that the function Λ is bounded on $\overline{V}$, that is the components $\Lambda_1,\ \Lambda_2$ and Λ_3 of Λ are bounded. We denote the components of f by F_1, F_2 and F_3 and get

$$\begin{aligned} |\Lambda_1(x)| &= \left| \int_V (\nabla_x G_0(x,y) \times f(y))_1\, dy \right| \\ &\leq \int_V |(\nabla_x G_0(x,y) \times f(y))_1|\, dy \\ &\leq \int_V \left| \frac{(x_2 - y_2)F_3(y) - (x_3 - y_3)F_2(y)}{4\pi |x-y|^3} \right| dy, \end{aligned}$$

see also the proof of Lemma A.1.1. in [1]. This gives

$$\begin{aligned} |\Lambda_1(x)| &\leq \int_V \frac{|(x_2 - y_2)F_3(y)| + |(x_3 - y_3)F_2(y)|}{4\pi|x-y|^3} dy \\ &\leq \int_V \frac{\left|\frac{x_2 - y_2}{|x-y|}\right| |f(y)| + \left|\frac{x_3 - y_3}{|x-y|}\right| |f(y)|}{4\pi|x-y|^2} dy \\ &\leq \int_V \frac{2\,|f(y)|}{4\pi|x-y|^2} dy \\ &\leq \frac{M}{2\pi} \int_V \frac{1}{|x-y|^2} dy = \frac{2M\pi^{\frac{1}{2}} R}{\Gamma(\frac{3}{2})} < \infty. \end{aligned} \tag{13}$$

Similarly, we can show that Λ_2 and Λ_3 are also bounded on $\overline{V}$. We can also conclude that

$$|(\nabla_x G_0(x,y) \times f(y))_1|\,,\ |(\nabla_x G_0(x,y) \times f(y))_2|$$

and $|(\nabla_x G_0(x,y) \times f(y))_3|$ are integrable with respect to y. The real components of

$$\nabla_x G_0^\delta(x,y) \times f(y)$$

satisfy the following relations

$$\begin{aligned}\left|\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)_1\right| &= \left|\frac{(x_2-y_2)F_3(y)-(x_3-y_3)F_2(y)}{4\pi\left(|x-y|^2+\delta\right)^{\frac{3}{2}}}\right| \\ &\leq \left|\frac{(x_2-y_2)F_3(y)-(x_3-y_3)F_2(y)}{4\pi|x-y|^3}\right| \\ &= \left|\left(\nabla_x G_0(x,y)\times f(y)\right)_1\right| \qquad (14)\end{aligned}$$

$$\begin{aligned}\left|\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)_2\right| &= \left|\frac{(x_3-y_3)F_1(y)-(x_1-y_1)F_3(y)}{4\pi\left(|x-y|^2+\delta\right)^{\frac{3}{2}}}\right| \\ &\leq \left|\frac{(x_3-y_3)F_1(y)-(x_1-y_1)F_3(y)}{4\pi|x-y|^3}\right| \\ &= \left|\left(\nabla_x G_0(x,y)\times f(y)\right)_2\right| \qquad (15)\end{aligned}$$

$$\begin{aligned}\left|\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)_3\right| &= \left|\frac{(x_1-y_1)F_2(y)-(x_2-y_2)F_1(y)}{4\pi\left(|x-y|^2+\delta\right)^{\frac{3}{2}}}\right| \\ &\leq \left|\frac{(x_1-y_1)F_2(y)-(x_2-y_2)F_1(y)}{4\pi|x-y|^3}\right| \\ &= \left|\left(\nabla_x G_0(x,y)\times f(y)\right)_3\right| \qquad (16)\end{aligned}$$

for $y \neq x$. Let for i =1,2,3

$$h_{x,i}(y) := \begin{cases} \left|\left(\nabla_x G_0(x,y)\times f(y)\right)_i\right| & \text{for} \quad y\neq x \\ 0 & \text{for} \quad y = x. \end{cases}$$

These $h_{x,i}$ are integrable because $h_{x,i}$ is equal to $\left|\left(\nabla_x G_0(x,y)\times f(y)\right)_i\right|$ almost everywhere and $\left|\left(\nabla_x G_0(x,y)\times f(y)\right)_i\right|$ is integrable. Furthermore,

$$\left|\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)_i\right| \leq h_{x,i}(y) \qquad (17)$$

for all $\delta > 0$ and for all $y \in V$ with $y \neq x$.
Hence, the real families

$$\left(\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)_i\right) \text{ for } i = 1,2,3$$

satisfy all the requirements of Lebesgue's Dominated Convergence Theorem, so we have for $i = 1,2,3$

$$\begin{aligned}\lim_{\delta\to 0+}\int_V \left(\nabla_x G_0^\delta(x,y)\times f(y)\right)_i dy &= \int_V \lim_{\delta\to 0}\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)_i dy \\ &= \int_V \left(\nabla_x G_0(x,y)\times f(y)\right)_i dy.\end{aligned}$$

Finally, we have

$$\begin{aligned}\lim_{\delta\to 0+}\int_V \nabla_x G_0^\delta(x,y)\times f(y)dy &= \int_V \lim_{\delta\to 0}\nabla_x G_0^\delta(x,y)\times f(y)dy\\ &= \int_V \nabla_x G_0(x,y)\times f(y)dy = \Lambda(x)\end{aligned}$$

which proves the result. ∎

Corollary 3.3 *Let f be a vector-valued function which is defined in the domain $\overline{V}$ such that f is continuous on $\overline{V}$ and has continuous first order derivatives in V. Then $f = -\nabla\rho + \nabla\times\Lambda$ with*

(i)

$$\rho = \lim_{\delta\to 0+}\rho^\delta \ \ in \ \ \mathrm{L}^1(V),$$

where

$$\rho^\delta(x) := \int_V \nabla_x G_0^\delta(x,y)\cdot f(y)\ dy$$

for all $x\in\overline{V}$ and

(ii)

$$\Lambda = \lim_{\delta\to 0+}\Lambda^\delta \ \ in \ \ \mathrm{L}^1(V),$$

where

$$\Lambda^\delta(x) := \int_V \nabla_x G_0^\delta(x,y)\times f(y)\ dy$$

for all $x\in\overline{V}$, where G_0^δ is the regularised Green's function for $-\nabla^2$ on $\mathbb{R}^3$.

Proof. In Theorem 3.1, we can say that ρ is also a continuous function of x. In Theorem 3.2, we proved that $\rho^\delta(x)\to\rho(x)$ as $\delta\to 0+$ for all $x\in\overline{V}$, where

$$\begin{aligned}|\rho^\delta(x)| &= \left|\int_V \nabla_x G_0^\delta(x,y)\cdot f(y)dy\right|\\ &\le \int_V \left|\nabla_x G_0^\delta(x,y)\cdot f(y)\right| dy\\ &\le \int_V |\nabla_x G_0(x,y)|\cdot|f(y)|\,dy =: H(x). \end{aligned}\tag{18}$$

Now, we show that H is integrable on V as follows

$$\begin{aligned}\int_V H(x)dx &= \int_V\int_V |\nabla_x G_0(x,y)|\cdot|f(y)|\,dydx\\ &\le \|f\|_{\mathrm{C}(V)}\int_V\int_V |\nabla_x G_0(x,y)|\,dydx,\end{aligned}$$

because f is continuous. We can easily see that

$$|\nabla_x G_0(x,y)| \leq \frac{1}{4\pi|x-y|^2}.$$

Therefore, using the inequality given above we have

$$\int_V H(x)dx \quad \leq \quad \int_V \int_V \frac{1}{4\pi|x-y|^2} dydx.$$

Due to Inequality (7) on page 159 in [13], we get

$$\begin{aligned} \int_V H(x)dx &\leq \int_V \frac{\pi^{\frac{1}{2}} R}{\Gamma(\frac{3}{2})} dx \\ &= \frac{4\pi^{\frac{3}{2}} R^4}{3\Gamma(\frac{3}{2})} < \infty. \end{aligned}$$

From the discussion given above, we can conclude that $\left\{\rho^\delta\right\}_{\delta>0}$ is a family of uniformly integrable functions and $\rho^\delta(x) \to \rho(x)$ as $\delta \to 0+$ almost everywhere on V. Therefore, using Vitali's convergence theorem (see [14], p. 133), we can say $\rho^\delta \to \rho$ as $\delta \to 0+$ in $\mathrm{L}^1(V)$.

For part (ii), let $\Lambda_1^\delta, \Lambda_2^\delta$ and Λ_3^δ be the real components of the vector-valued function Λ^δ and Λ_1, Λ_2 and Λ_3 be the components of Λ. From Equation (14), we get

$$\begin{aligned} \left|\Lambda_1^\delta(x)\right| &= \left|\int_V \left(\nabla_x G_0^\delta(x,y) \times f(y)\right)_1 dy\right| \\ &\leq \int_V \left|\left(\nabla_x G_0^\delta(x,y) \times f(y)\right)_1\right| dy \\ &\leq \int_V \left|(\nabla_x G_0(x,y) \times f(y))_1\right| dy =: H_1(x) \end{aligned} \tag{19}$$

Now, we show that H_1 is integrable on V as follows

$$\int_V H_1(x)dx \quad = \quad \int_V \int_V \left|(\nabla_x G_0(x,y) \times f(y))_1\right| dydx.$$

From Equation (13), we have

$$\int_V \left|(\nabla_x G_0(x,y) \times f(y))_1\right| dy \leq \frac{2\|f\|_{\mathrm{C}(\overline{V})}\pi^{\frac{1}{2}} R}{\Gamma(\frac{3}{2})},$$

therefore, we have

$$\begin{aligned}\int_V H_1(x)dx &\leq \frac{2\|f\|_{\mathrm{C}(\overline{V})}\pi^{\frac{1}{2}}R}{\Gamma(\frac{3}{2})}\int_V dx \\ &= \frac{4\|f\|_{\mathrm{C}(\overline{V})}\pi^{\frac{3}{2}}R^4}{3\Gamma(\frac{3}{2})} < \infty.\end{aligned}$$

This gives that Λ_1^δ is a family of uniformly integrable functions in V. We have proved in Theorem 3.2 that $\Lambda_1^\delta(x) \to \Lambda_1(x)$ as $\delta \to 0+$, for all x in V, therefore using Vitali's theorem (see [14], p. 133), we get $\Lambda_1^\delta \to \Lambda_1$ as $\delta \to 0+$ in $\mathrm{L}^1(\mathrm{V})$. Similarly, we can prove that $\Lambda_2^\delta \to \Lambda_2$ as $\delta \to 0+$ in $\mathrm{L}^1(\mathrm{V})$ and $\Lambda_3^\delta \to \Lambda_3$ as $\delta \to 0+$ in $\mathrm{L}^1(\mathrm{V})$. Hence, combining these components we can say that $\Lambda^\delta \to \Lambda$ as $\delta \to 0+$ in $\mathrm{L}^1(\mathrm{V})$. ■

Corollary 3.4 *Let f be a vector-valued function which is defined in the domain $\overline{V}$ such that f is continuous on $\overline{V}$ and has continuous first order derivatives in V. Then $f = -\nabla\rho + \nabla \times \Lambda$ with*

(i)

$$\rho = \lim_{\delta \to 0+} \rho^\delta \;\; in \;\; \mathscr{D}'(V)$$

with

$$\rho^\delta(x) = \int_V \nabla_x G_0^\delta(x,y) \cdot f(y)\; dy$$

(ii)

$$\Lambda = \lim_{\delta \to 0+} \Lambda^\delta \;\; in \;\; \mathscr{D}'(V),$$

with

$$\Lambda^\delta(x) = \int_V \nabla_x G_0^\delta(x,y) \times f(y)\; dy$$

for all $x \in V$, where G_0^δ is the regularised Green's function for $-\nabla^2$ on $\mathbb{R}^3$ and $\mathscr{D}'(V)$ denotes the space of all distributions corresponding to the domain V.

Proof. From Theorem 3.2 and the proof of Corollary 3.3, we can see that ρ and the components of Λ satisfy all the requirements of Theorem 1.5.1 of [11]. Therefore, by applying Theorem 1.5.1 of [11], we get the required result. ■

Remark 3.5 *When we say a sequence of functions (f_k) converges to a function f in $\mathscr{D}'(V)$ it means that the distribution T_{f_k} of f_k defined by*

$$T_{f_k}(\phi) = \int_V f_k(x)\phi(x)dx, \quad \phi \in \mathscr{D}(V),$$

converges to the distribution T_f of f defined in a similar way (see [2]).

4 Applications

4.1 The Electromagnetic Field

In this section, we show how the electric potential ρ of an electric field E and the magnetic potential Λ of a magnetic field B can be approximated using Theorem 3.2 and Maxwell's equations in a static case.
Due to Theorem 3.2, a vector field f defined on $\overline{V}$ can be decomposed as

$$f = -\nabla\rho + \nabla \times \Lambda \tag{20}$$

with

$$\rho(x) = \lim_{\delta\to 0+} \rho^\delta(x) = \lim_{\delta\to 0+} \int_V \nabla_x G_0^\delta(x,y) \cdot f(y)\ dy \tag{21}$$

and

$$\Lambda(x) = \lim_{\delta\to 0+} \Lambda^\delta(x) = \lim_{\delta\to 0+} \int_V \nabla_x G_0^\delta(x,y) \times f(y)\ dy. \tag{22}$$

If we use the fact that $\nabla_x G_0^\delta(x,y) = -\nabla_y G_0^\delta(x,y)$ in Equation (21), we get

$$\begin{aligned} \rho^\delta(x) &= -\int_V \nabla_y G_0^\delta(x,y) \cdot f(y)\ dy \\ &= \int_S G_0^\delta(x,y) f(y) \cdot \hat{n}(y)\ dy - \int_V (\nabla_y G_0^\delta(x,y)) \cdot f(y)\ dy \\ &\quad - \int_S G_0^\delta(x,y) f(y) \cdot \hat{n}(y)\ dy. \end{aligned} \tag{23}$$

Now applying Gauß's Divergence Theorem on the first term of Equation (23), we get

$$\begin{aligned} \rho^\delta(x) &= \int_V \nabla_y \cdot \Big(G_0^\delta(x,y) f(y) \Big)\ dy - \int_V (\nabla_y G_0^\delta(x,y)) \cdot f(y)\ dy \\ &\quad - \int_S G_0^\delta(x,y) f(y) \cdot \hat{n}(y)\ dy. \end{aligned}$$

Now combining the first two terms of the equation given above and using Equation (6) we obtain

$$\rho^\delta(x) = \int_V G_0^\delta(x,y) \nabla_y \cdot f(y)\ dy - \int_S G_0^\delta(x,y) f(y) \cdot \hat{n}(y)\ dy. \tag{24}$$

In a similar way if we use the fact $\nabla_x G_0^\delta(x,y) = -\nabla_y G_0^\delta(x,y)$ in Equation (22), we get

$$\begin{aligned}
\Lambda^\delta(x) &= \int_V -\nabla_y G_0^\delta(x,y) \times f(y)\ dy \\
&= -\int_V \nabla_y \times \Big(G_0^\delta(x,y) f(y)\Big)\ dy + \int_V G_0^\delta(x,y) \nabla_y \times f(y)\ dy \\
&= \int_S G_0^\delta(x,y) f(y) \times \hat{n}(y)\ dy + \int_V G_0^\delta(x,y) \nabla_y \times f(y)\ dy. \qquad (25)
\end{aligned}$$

We combine Equations (24) and (25) and their ability to provide approximate solutions with Maxwell's equations in a vacuum; again classically we view the charge density and the current density as further vector fields, which are continuously differentiable. Below are the Maxwell's equations.

$$\begin{aligned}
\nabla \times E &= -\partial_t B \qquad (26) \\
\nabla \cdot E &= \varrho/\varepsilon_0 \qquad (27) \\
\nabla \times B &= \mu_0(\mathbf{J} + \varepsilon_0 \partial_t E) \qquad (28) \\
\nabla \cdot B &= 0 \qquad (29)
\end{aligned}$$

where E is the electric field, B is the magnetic field, ϱ is the charge density in the volume V, $\mathbf{J}$ is the current density, μ_0 is the vacuum permeability and ε_0 is the vacuum capacitivity. Many phenomena of geomagnetism take place on a long time scale and we might imagine the static approximation would be a good one. Therefore, the Maxwell equations in the static case have the form

$$\begin{aligned}
\nabla \times E &= 0 \qquad (30) \\
\nabla \cdot E &= \varrho/\varepsilon_0 \qquad (31) \\
\nabla \times B &= \mu_0 \mathbf{J} \qquad (32) \\
\nabla \cdot B &= 0. \qquad (33)
\end{aligned}$$

From the Maxwell equation (30), the curl of the electric field vanishes. Hence, the electric field may be written simply as the gradient of some scalar, ρ, which of course is the electric potential: $E = -\nabla \rho$. From the Maxwell equation (31) it follows that $-\nabla^2 \rho = \varrho/\varepsilon_0$. Further, if we take

$$\begin{aligned}
\sigma &= -E \cdot n \qquad (34) \\
\Gamma &= B \times n \qquad (35)
\end{aligned}$$

with, for example, σ taken as a surface charge density and Γ taken as a surface current density then by combining all this together with (24), we obtain

$$\rho(x) = \lim_{\delta\to 0+} \rho^{\delta}(x) = \lim_{\delta\to 0+} \left(\frac{1}{\varepsilon_0} \int_V G_0^{\delta}(x,y)\varrho(y)\ dy \right. \tag{36}$$

$$\left. + \int_S G_0^{\delta}(x,y)\sigma(y)\ dy \right). \tag{37}$$

For the magnetic field, it follows from the Maxwell equation (33), due to the Mie decomposition (see [3, 4, 5]), that the magnetic field may be written as the curl of some vector, Λ i.e. $B = \nabla \times \Lambda$. The vector field Λ is called the magnetic vector potential. Then using equation (25), Maxwell equation (32) and equation (35), we get

$$\Lambda(x) = \lim_{\delta\to 0+} \Lambda^{\delta}(x) = \lim_{\delta\to 0+} \left(\mu_0 \int_V G_0^{\delta}(x,y)J(y)\ dy + \int_S G_0^{\delta}(x,y)\Gamma(y)\ dy \right). \tag{38}$$

4.2 Reconstruction of the Decomposition Parts of a Vector-valued Function

In this section, we present some numerical tests in which we reconstruct the decompositions ρ and Λ of a synthetic vectorial function f by using the kernels given in Theorem 3.2 for different parameters $\delta = 0.01$, $\delta = 0.001$, $\delta = 0.0001$, $\delta = 0.00001$ and $\delta = 0.000001$. We have also calculated the rooted mean square error for these values of δ which are given in Table 1. For all values of the parameter δ we used $200 \times 256 \times 256$ grid points for the integration on $B_1(0) = V$.
To compute our convolution on $B_1(0)$, we use the standard separation of an integral on $B_1(0)$

$$\int_{B_1(0)} F(x)dx = \int_0^1 r^2 \int_{\partial B_1(0)} F(r\eta)d\omega(\eta)dr. \tag{39}$$

For the integral over the sphere, the most commonly used sampling measures have support on either an equiangular grid or a Gaussian grid. We use an equiangular grid and the quadrature theorem given in [9] and for the line integral we use the composite Simpson's rule. Our vectorial synthetic function and its analytically calculated decompositions ρ and Λ using Theorem 3.1 are given by

$$f(y) = f(r\eta) = r^3 \eta Y_{3,2}(\eta), \tag{40}$$

Table 1: Rooted mean square error in the reconstruction of the decompositions ρ and Λ of the synthetic vectorial function f for different values of the parameter δ.

δ	Rooted mean square error in ρ	Rooted mean square error in $\|\Lambda\|$
10^{-2}	0.0023	0.0015
10^{-3}	$5.8856 \cdot 10^{-04}$	$2.7607 \cdot 10^{-04}$
10^{-4}	$1.3743 \cdot 10^{-04}$	$4.1590 \cdot 10^{-05}$
10^{-5}	$4.4364 \cdot 10^{-05}$	$1.7110 \cdot 10^{-04}$
10^{-6}	$4.2001 \cdot 10^{-04}$	$2.7607 \cdot 10^{-04}$

where $r = |y|$ and $\eta = \frac{y}{|y|}$ for $y \in \overline{B_1(0)}$ and

$$\rho(x) \qquad = \rho(s\xi) = \left(\frac{-5s^4}{8} + \frac{4s^3}{7} \right) Y_{3,2}\left(\xi\right), \tag{41}$$

$$\Lambda(x) \qquad = \Lambda(s\xi) = \left(\frac{s^4}{8} - \frac{s^3}{7} \right) L^{\star} Y_{3,2}(\xi), \tag{42}$$

where $s = |x|$ and $\xi = \frac{x}{|x|}$ for $x \in \overline{B_1(0)}$. $Y_{3,2}$ is a spherical harmonic of degree 3 and order 2 and $L^{\star}$ represents the surface curl gradient (see [10]). Analysing the graphs in Figures 3 to 8 and the errors in Table 1, we can say that we get a very good approximation of the scalar potential ρ and the vector potential Λ of the vectorial function f. However, we observe that the optimal values of the parameter δ for the scalar potential ρ and for the vector potential Λ are not the same. We also observe that when the parameter δ becomes close to the singularity the rooted mean square error may increase, this indicates why the regularisation of the Green's function was necessary.

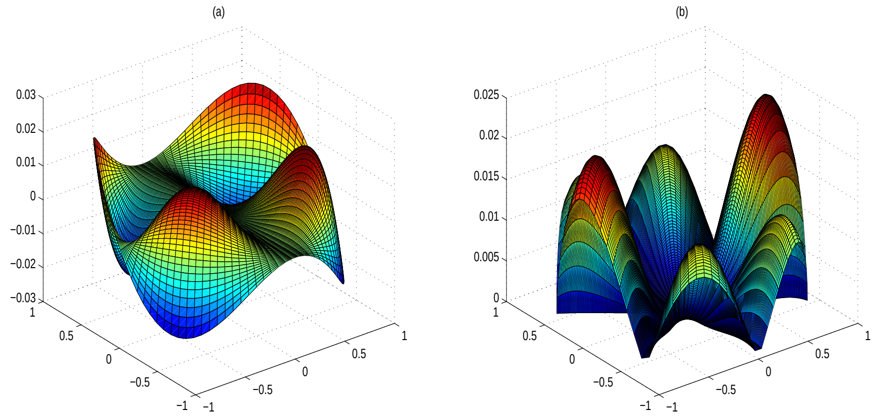

Figure 2: (a) is the graph of the analytically calculated scalar potential ρ of the function f defined in Equation (40) plotted in the $y_1 = 0$ plane and (b) is the graph of the absolute values of the analytically calculated vector potential Λ of the function f defined in Equation (40) plotted in the $y_1 = 0$ plane.

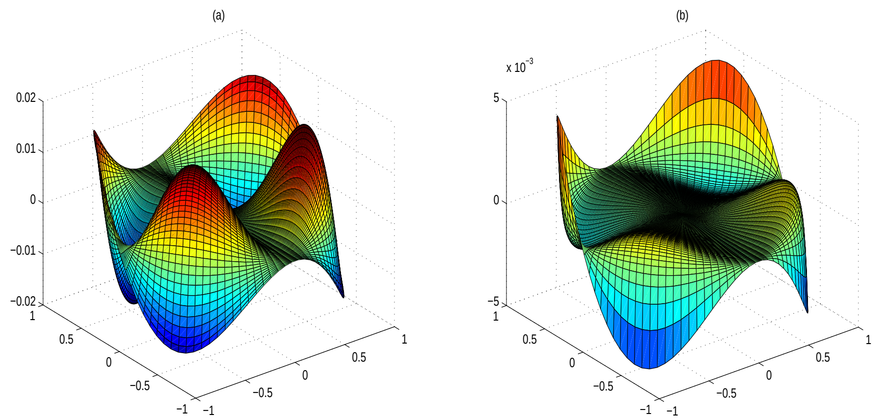

Figure 3: (a) is the graph of the reconstructed scalar potential ρ using the kernel of Theorem 3.2 (i) with the parameter $\delta = 0.001$ and (b) is the graph of the difference of the analytically calculated scalar potential ρ and the numerically calculated scalar potential ρ, using the kernel of Theorem 3.2 (i), with the parameter $\delta = 0.001$. In both cases, the function is plotted in the $y_1 = 0$ plane.

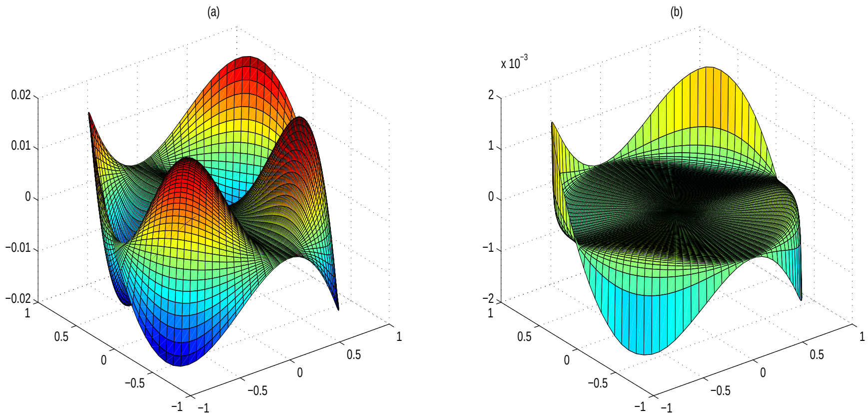

Figure 4: (a) is the graph of the reconstructed scalar potential ρ using the kernel of Theorem 3.2 (i) with the parameter $\delta = 0.0001$ and (b) is the graph of the difference of the analytically calculated scalar potential ρ and the numerically calculated scalar potential ρ, using the kernel of Theorem 3.2 (i), with the parameter $\delta = 0.0001$. In both cases, the function is plotted in the $y_1 = 0$ plane.

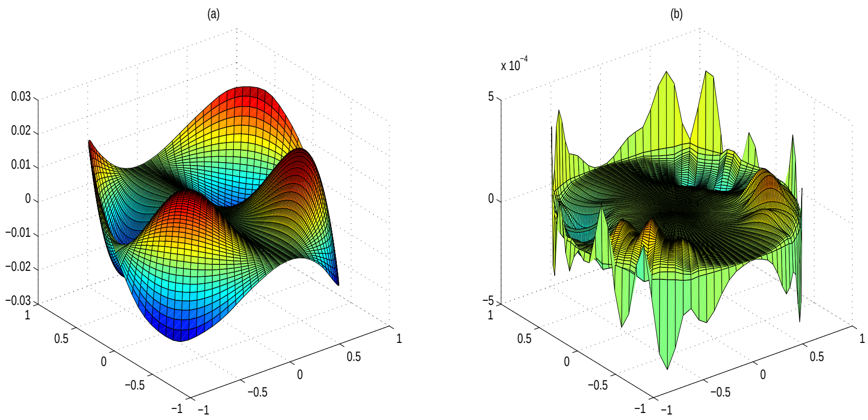

Figure 5: (a) is the graph of the reconstructed scalar potential ρ using the kernel of Theorem 3.2 (i) with the parameter $\delta = 0.00001$ and (b) is the graph of the difference of the analytically calculated scalar potential ρ and the numerically calculated scalar potential ρ, using the kernel of Theorem 3.2 (i), with the parameter $\delta = 0.00001$. In both cases, the function is plotted in the $y_1 = 0$ plane.

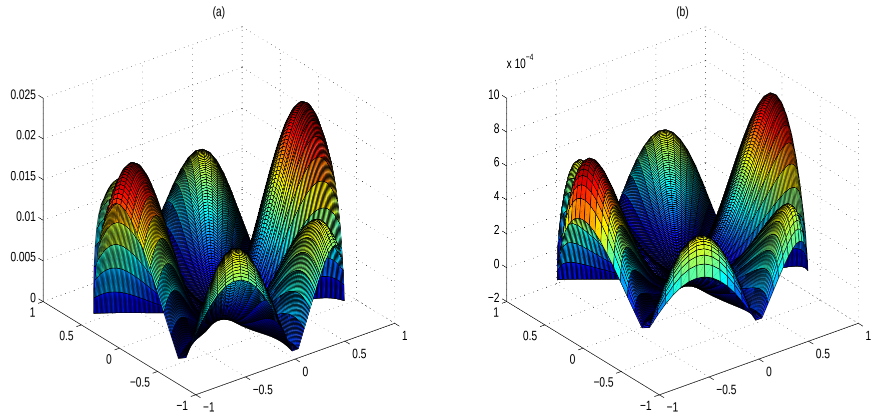

Figure 6: (a) is the graph of the absolute values of the reconstructed vector potential Λ using the kernel of Theorem 3.2 (ii) with the parameter $\delta = 0.001$ and (b) is the graph of the difference of the absolute values of the analytically calculated vector potential Λ and the absolute values of the numerically calculated vector potential Λ, using the kernel of Theorem 3.2 (ii), with the parameter $\delta = 0.001$. In both cases, the function is plotted in the $y_1 = 0$ plane.

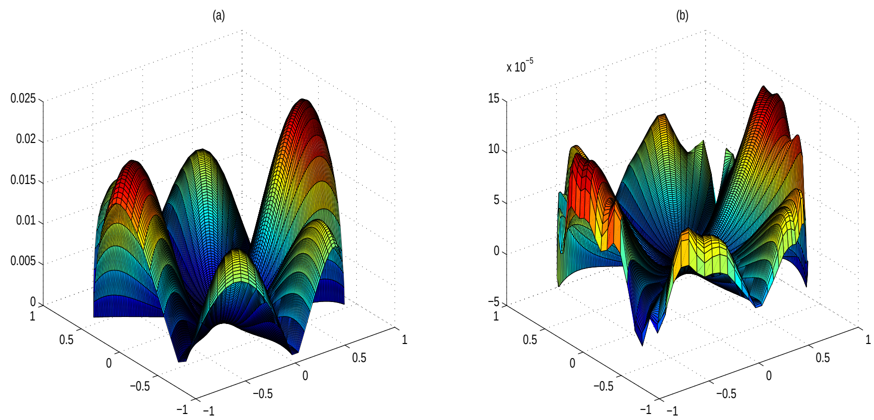

Figure 7: (a) is the graph of the absolute values of the reconstructed vector potential Λ using the kernel of Theorem 3.2 (ii) with the parameter $\delta = 0.0001$ and (b) is the graph of the difference of the absolute values of the analytically calculated vector potential Λ and the absolute values of the numerically calculated vector potential Λ, using the kernel of Theorem 3.2 (ii), with the parameter $\delta = 0.0001$. In both cases, the function is plotted in the $y_1 = 0$ plane.

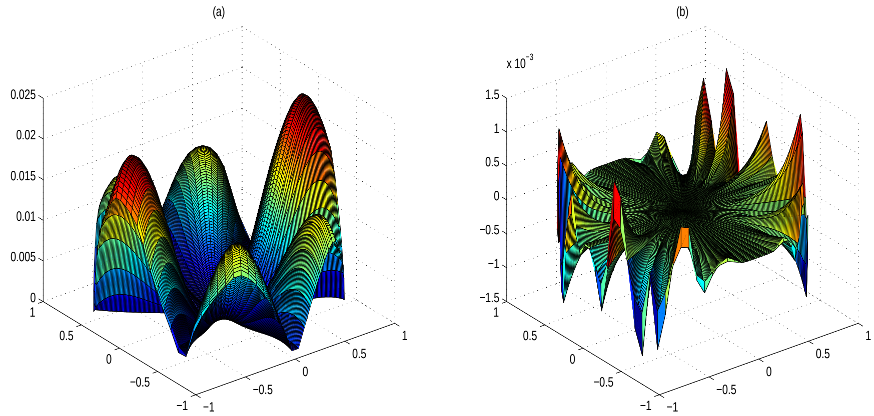

Figure 8: (a) is the graph of the absolute values of the reconstructed vector potential Λ using the kernel of Theorem 3.2 (ii) with the parameter $\delta = 0.00001$ and (b) is the graph of the difference of the absolute values of the analytically calculated vector potential Λ and the absolute values of the numerically calculated vector potential Λ, using the kernel of Theorem 3.2 (ii), with the parameter $\delta = 0.00001$. In both cases, the function is plotted in the $y_1 = 0$ plane.

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