SELFCOINCIDENCES IN HIGHER CODIMENSIONS

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Abstract. When can a map between manifolds be deformed away from itself? We describe a (normal bordism) obstruction which is often computable and in general much stronger than the classical primary obstruction in cohomology. In particular, it answers our question completely in a large dimension range.

As an illustration we give explicit criteria in three sample settings: projections from Stiefel manifolds to Grassmannians, sphere bundle projections and maps defined on spheres. In the first example a theorem of Becker and Schultz concerning the framed bordism class of a compact Lie group plays a central role; our approach yields also a very short geometric proof (included as an appendix) of this result.

I. Introduction

Throughout this paper M and N denote smooth connected manifolds without boundary, of dimensions m and n, resp., M being compact. We say a map $f : M \longrightarrow N$ is *loose* (or $f| \wr f$ in the notation of Dold and Gonçalves [DG]) if f is homotopic to some map f' which has no coincidences with f, i.e. $f(x) \neq f'(x)$ for all $x \in M$.

Problem: *Give strong and computable criteria* (*expressed in a language of algebraic topology*) *for* f *to be loose.*

In this paper we present some results and examples which seem to indicate that normal bordism theory offers an appropriate language. Indeed, a careful analysis of the coincidence behaviour (of a suitable approximation) of $(f, f) : M \longrightarrow N \times N$ yields a triple $(C, g, \overline{g})$ where

(i) C is a smooth $(m-n)$–dimensional manifold (the coincidence locus);
(ii) $g : C \longrightarrow M$ is a continuous map (the inclusion); and
(iii) $\overline{g}$ is a vector bundle isomorphism which describes the stable normal bundle of C in terms of the pullback $g^*(\varphi)$ of the virtual coefficient bundle $\varphi = f^*(TN) - TM$ over M.

This leads to a well-defined looseness obstruction

$$\omega(f) := [C, g, \overline{g}] \in \Omega_{m-n}(M; f^*(TN) - TM)$$

in the normal bordism group which consists of the bordism classes of triples as above.

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Selfcoincidence theorem. *Assume $m < 2n-2$. Then f is loose if and only if $\omega(f) = 0$.*

This is our central result. In § 2 below we give the proof which is based on the singularity theory for vector bundle morphisms (see [Ko 1]). As a by-product we show also that if the map f can be homotoped away from itself, then this can be achieved by an arbitrarily small deformation. Furthermore we obtain a formula expressing $\omega(f)$ in terms of the Euler number $\chi(N)$ of N and of the (normal bordism) degree of f. Often this makes explicit calculations possible.

The natural Hurewicz homomorphism maps our invariant $\omega(f)$ to the Poincaré dual of the classical primary obstruction $o_n(f,f)$ in the (co-)homology of M (in general with twisted coefficients; compare [GJW], theorem 3.3). This transition forgets the vector bundle isomorphism $\overline{g}$ nearly completely, keeping track only of the orientation information it carries. If $m = n$, this is no loss. However, in higher codimensions $m - n > 0$ the knowledge of $\overline{g}$ is usually crucial.

Example. Consider the canonical projections

$$p : V_{r,k} \longrightarrow G_{r,k} \quad \text{and} \quad \widetilde{p} : V_{r,k} \longrightarrow \widetilde{G}_{r,k}$$

from the Stiefel manifold of orthonormal k–frames in $\mathbb{R}^r$ to the Grassmannian of (unoriented or oriented, resp.) k–planes through the origin in $\mathbb{R}^r$.

Then $\omega(p) = \omega(\widetilde{p})$ lies in the framed bordism group $\Omega^{fr}_{d(k)}(V_{r,k})$ where $d(k) = \frac{1}{2}k(k-1)$. In nearly all interesting cases the map g from the coincidence locus C into $V_{r,k}$ factors – up to homotopy – through a lower dimensional manifold so that the primary obstruction vanishes. Frequently g is even nulhomotopic.

Theorem. *Assume $r \geq 2k \geq 2$. Then: p and $\widetilde{p}$ are loose if and only if*

$$0 \ = \ 2\chi(G_{r,k}) \cdot [SO(k)] \ \in \ \pi^S_{d(k)} \ .$$

This condition holds e.g. if k is even or $k = 7$ or 9 or $\chi(G_{r,k}) \equiv 0(12)$.

Here a fascinating problem enters our discussion: to determine the order of a Lie group, when equipped with a left invariant framing and interpreted – via the Pontryagin-Thom isomorphism – as an element in the stable homotopy group of spheres $\pi^S_* \cong \Omega^{fr}_*$. Deep contributions were made e.g. by Atiyah and Smith [AS], Becker and Schultz [BS], Knapp [Kn], and Ossa [O], to name but a few (consult the summary of results and the references in [O]). In particular, it is known that the invariantly framed special orthogonal group $SO(k)$ is nulbordant for $4 \leq k \leq 9$, $k \neq 5$ (cf. table 1 in [O]) and that $24[SO(k)] = 0$ and $2[SO(2\ell)] = 0$ for all k and ℓ (cf. [O], p. 315, and [BS], 4.7; for a short proof of this last claim see also our appendix).

On the other hand, the Euler number $\chi(G_{r,k})$ is easily calculated: it vanishes if $k \not\equiv r \equiv 0(2)$ and equals $\binom{[r/2]}{[k/2]}$ otherwise (compare [MS], 6.3 and 6.4).

Corollary 1. *Assume $r > k = 2$. Then p and $\widetilde{p}$ are loose.*

Corollary 2. *Assume $r \geq k = 3$. Then p (or, equivalently, $\widetilde{p}$) is loose if and only if r is even or $r \equiv 1(12)$.*

This follows from the fact that $[SO(3)] \in \pi^S_3 \cong \mathbb{Z}_{24}$ has order 12 (cf. [AS]).

Corollary 3. *Assume* $r \geq k = 5$, $r \neq 7$. *Then* p *(or, equivalently,* $\widetilde{p}$ *) is loose if and only if* $r \not\equiv 5(6)$.

This follows since $[SO(5)]$ has order 3 in $\pi_{10}^S \cong \mathbb{Z}_6$ (cf. [O]).
The details of this example will be discussed in § 3.

Next consider the case when a map $f : M \longrightarrow N$ allows a section $s : N \longrightarrow M$ (i.e. $f \circ s = \mathrm{id}_N$). Then clearly f is loose if and only if id_N is – or, equivalently, $\chi(N) = 0$ whenever N is closed. In § 4 we refine this simple observation in case f is the projection of a suitable sphere bundle $S(\xi)$. Here the relative importance of the g- and $\overline{g}$-data (fibre inclusion and "twisted framing") in $\omega(f)$ can be studied explicitly via Gysin sequences. We obtain divisibility conditions for $\chi(N)$ in terms of the Euler class of ξ.

As a last illustration we discuss the case $M = S^m$ in § 5. Our looseness obstruction determines (and is determined by) a group homomorphism

$$\omega \ : \ \pi_m(N; y_0) \ \longrightarrow \ \Omega^{fr}_{m-n} \ .$$

Thus when $m < 2n - 2$ a map $f : S^m \longrightarrow N$ is loose precisely if its homotopy class $[f]$ lies in the kernel of this homomorphism. In the case $N = S^n$ this holds if $2[f] = 0$ (when n is even) and for all f (when n is odd).

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§ 1. The coincidence invariant and the degree

Consider two maps $f_1, f_2 : M \longrightarrow N$.

If the resulting map $(f_1, f_2) : M \longrightarrow N \times N$ is smooth and transverse to the diagonal

$$\Delta \ := \{(y, y) \ \in \ N \times N \mid y \in N\}$$

then the coincidence locus

$$C \ := \{x \in M \mid f_1(x) = f_2(x)\} \ = \ (f_1, f_2)^{-1}(\Delta) \tag{1.1}$$

is a closed $(m - n)$–dimensional manifold canonically equipped with the following two data:

a continuous map

$$g : C \longrightarrow M \qquad \text{(namely the inclusion)} \quad ; \text{ and} \tag{1.2}$$

a stable tangent bundle isomorphism

$$\overline{g} \ : \ TC \oplus g^*(f_1^*(TN)) \ \cong \ g^*(TM) \tag{1.3}$$

(since the normal bundle $\nu(\Delta, N \times N)$ of Δ in $N \times N$ is canonically isomorphic to the pullback of the tangent bundle TN under the first projection p_1).

If f_1 and f_2 are arbitrary continuous maps, apply the preceding construction to a smooth map (f_1', f_2') which approximates (f_1, f_2) and is transverse to Δ. Then a (sufficiently small) homotopy from f_1 to f_1' determines an isomorphism $f_1^*(TN) \cong f_1'^*(TN)$ which is canonical up to regular homotopy. In any case we obtain a well-defined normal bordism class

$$\omega(f_1, f_2) \ := \ [C, g, \overline{g}] \ \in \ \Omega_{m-n}(M; f_1^*(TN) - TM) \tag{1.4}$$

which depends only on f_1 and on the homotopy class of f_2.

Proposition 1.5. *If there exist maps* $f'_i : M \longrightarrow N$ *which are homotopic to* f_i, $i = 1, 2$, *and such that* $f'_1(x) \not\equiv f'_2(x)$ *for all* $x \in M$, *then* $\omega(f_1, f_2) = 0$.

Proof. The homotopy $f_1 \sim f'_1$ yields a nulbordism for $\omega(f_1, f'_2) = \omega(f_1, f_2)$.

■

Our approach also leads us to define the (normal bordism) degree of any map $f : M \longrightarrow N$ by

(1.6) $$\deg(f) \ := \ \omega(f, \text{constant map}) \ .$$

It is represented by the inverse image F of a regular value of (a smooth approximation of) f, together with the inclusion map and the obvious stable description of the tangent bundle TF.

§ 2. Selfcoincidences

Given any continuous map $f : M \longrightarrow N$, we apply the previous discussion to the special case $f_1 = f_2 = f$. We obtain the two invariants

(2.1) $$\omega(f) := \omega(f, f), \quad \deg(f) \ \in \ \Omega_{m-n}(M; f^*(TN) - TM)$$

(cf. 1.4 and 1.6), both lying in the same normal bordism group.

Any generic section s of the vector bundle $f^*(TN)$ over M gives rise to a map (which is homotopic to f) from M to a tubular neighbourhood $U \cong \nu(\Delta, N \times N) \cong p_1^*(TN)$ of the diagonal Δ in $N \times N$ (compare 1.3). The resulting coincidence locus, together with its normal bordism data, equals the zero set of s (interpreted as a vector bundle homomorphism from the trivial line bundle $\underline{\mathbb{R}}$ to $f^*(TN)$), together with its singularity data (cf. [Ko 1]). This locus consists of $f^{-1}\{y_1, \dots\}$ if s is the pullback of a generic section of TN with zeroes $\{y_1, \dots\}$, which are regular values of (a smooth approximation of) f. In particular, if N admits a nowhere zero vector field v – e.g. when N is open – then the map f is loose (since it can be "pushed slightly along v" to get rid of all selfcoincidences). We conclude:

Theorem 2.2. *Let* $f : M^m \longrightarrow N^n$ *be a continuous map between smooth closed connected manifolds.*

Then the selfcoincidence invariant $\omega(f)$ *(cf.* 2.1*) is equal to the singularity invariant* $\omega(\underline{\mathbb{R}}, f^*(TN))$ *(cf.* [Ko 1], § 2) *and hence also to* $\chi(N) \cdot \deg(f)$ *(cf.* 1.6*; here* $\chi(N)$ *denotes the Euler number of* N*).*

Moreover, each of the following conditions implies the next one:

(i) $f^*(TN)$ *allows a nowhere zero section over* M*;*
(ii) f *can be* approximated *by a map which has no coincidences with* f *;*
(iii) f *is loose ;*
(iv) *there exist maps* $f', f'' : M \longrightarrow N$ *which have no coincidences and which are both homotopic to* f *; and*
(v) $\omega(f) = 0$.

If $m < 2n - 2$, *all these conditions are equivalent.*

Indeed, in this dimension range $\omega(\underline{\mathbb{R}}, f^*(TN))$ is the only obstruction to the existence of a monomorphism $\underline{\mathbb{R}} \hookrightarrow f^*(TN)$ (see theorem 3.7 in [Ko 1]).

Special case 2.3 (codimension zero). Assume $m = n \geq 0$. Then f is loose if and only if $\omega(f) = 0$. Here the relevant normal bordism group $\Omega_0(M; f^*(TN) - TM)$ is isomorphic to $\mathbb{Z}$ if $w_1(M) = f^*(w_1(N))$ and to $\mathbb{Z}_2$ otherwise. $\omega(f)$ counts the isolated zeroes of a generic section in $f^*(TN)$. We can concentrate these zeroes in a ball in M and (after isotoping some of them – if needed – around loops where $w_1(M) \neq f^*(w_1(N))$, in order to change their signs) cancel all of them if $\omega(f) = 0$.

For a very special case in higher codimensions compare [DG], 1.15.

Remark 2.4. In order to understand and compute normal bordism obstructions, it is often helpful to use the natural forgetful homomorphisms

$$\Omega_i(M;\varphi) \xrightarrow{\text{forg}} \overline{\Omega}_i(M;\varphi) \xrightarrow{\mu} H_i(M;\widetilde{\mathbb{Z}}_\varphi) \quad .$$

Here forg retains only the orientation information contained in the $\overline{g}$–components of a normal bordism class, and μ denotes the Hurewicz homomorphism to homology with coefficients which are twisted like the orientation line bundle of φ. The detailed analysis of forg, given in § 9 of [Ko 1], yields computing techniques which often permit to calculate obstructions in low dimensional normal bordism groups.

§ 3. Principal bundles

As a first example consider the projection $p : M \longrightarrow N$ of a smooth principal G–bundle (cf. [S], 8.1) over the closed manifold N, with G a compact Lie group. A fixed choice of an orientation of G at its unit element equips G with a left invariant framing (which we will drop from the notation); it also yields a (stable) trivialization of the tangent bundle along the fibres of p and hence of the coefficient bundle $p^*(TN) - TM$. Thus by theorem 2.2 our selfcoincidence invariant takes the form

$$\omega(p) = \chi(N) \cdot [G, g = \text{fibre inclusion}] \in \Omega^{fr}_{m-n}(M) \ . \tag{3.1}$$

If we concentrate on the normal bundle information – which, in a way, represents the highest order component of this obstruction – and neglect its g–part, we obtain the weaker invariant

$$\omega'(p) \ := \ \text{const}_*(\omega(p)) \ = \ \chi(N)[G] \ \in \ \Omega^{fr}_{m-n} \tag{3.2}$$

which must also vanish whenever p is loose. In other words, the Euler number $\chi(N)$ must be a multiple of the order of $[G]$ in $\Omega^{fr}_* \cong \pi^S_*$.

For $i \leq 6$ the stable stem π^S_i is generated by the class $[G]$ of some compact connected Lie group (e.g. $\pi^S_1 = \mathbb{Z}_2 \cdot [S^1]$ and $\pi^S_3 = \mathbb{Z}_{24} \cdot [SU(2)]$). However, this is not typical, and only the divisors of 72 (if not of 24) can be the order of such a class (see [O], theorem 1.1; note also Ossa's table 1).

As an illustration let us work out the details for the projections p and $\widetilde{p}$ discussed in the example of the introduction. We may assume $r > k \geq 1$.

Let us first dispose of two elementary cases.

Case 1: $k = 1$. $p : S^{r-1} \longrightarrow \mathbb{R}P^{r-1}$ and $\widetilde{p} = \mathrm{id}_{S^{r-1}}$ are loose if and only if r is even.

This follows from 2.3 and 2.2.

Case 2: $k = r-1$ *or* $k \equiv r-1 \not\equiv 0(2)$*: both* p *and* $\widetilde{p}$ *are loose.*

Here $p^*(TG_{r,k}) \cong p^*(\mathrm{Hom}(\gamma,\gamma^\perp)) \cong \oplus^k p^*(\gamma^\perp)$ (cf. [MS], p. 70) has a nowhere zero section, be it for orientation reasons or since $G_{r,k}$ is odd–dimensional.

Next recall that in the general setting $\dim(G_{r,k}) = k(r-k)$; the fibre dimension is given by

$$d(k) \ := \ \dim(S)O(k) \ = \ \frac{1}{2}k(k-1) \ . \tag{3.3}$$

We have $p^*(TG_{r,k}) \cong \widetilde{p}^*(T\widetilde{G}_{r,k})$ and hence $\omega(p) = \omega(\widetilde{p})$. According to theorem 2.2 this is the only looseness obstruction if

$$r \ \geq \ \frac{3}{2}k \ - \ \frac{1}{2} \ + \ \frac{3}{k} \ . \tag{3.4}$$

Clearly the fibre of p (or $\widetilde{p}$) over the point $(\mathbb{R}^k \subset \mathbb{R}^r)$ in the Grassmannian is $V_{k,k} = O(k)$ (or $SO(k)$, resp.). Also, up to homotopy the fibre inclusion g factors through $V_{k,\ell}$ where $\ell := \max\{2k-r, 0\} < k$; this is seen by rotating the vectors $v_{\ell+1}, \ldots, v_k$ of a k–frame in $\mathbb{R}^k$ into the standard basis vectors $e_{k+1}, \ldots, e_{2k-\ell}$ in $\mathbb{R}^r$. Except in situations which are already settled by the cases 1 and 2 above we see that the dimension of the intermediate manifold $V_{k,\ell}$ is strictly less (and often considerably so) than the fibre dimension $d(k)$ (cf. 3.3) so that the cohomological primary obstruction detects nothing.

In particular, if $r \geq 2k$ then g is nulhomotopic and therefore all the information contained in the (complete!) non-selfcoincidence obstruction is already given by

$$\omega'(\widetilde{p}) \ = \ 2 \cdot \chi(G_{r,k})[SO(k)] \ \in \ \Omega^{fr}_{d(k)}$$

(cf. 3.2 and 3.3). The theorem of the introduction and its corollaries follow. (If $(r,k) = (3,2), (4,3), (6,5)$ or $(8,5)$ refer to case 2 above; if $(r,k) = (9,5)$ the bordism class $a = [SO(5) \subset V_{9,5}] \in \Omega^{fr}_{10}(V_{9,5})$ lies in the image of $\Omega^{fr}_{10}(S^4) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2$ and hence $\omega(f) = 12a = 0$.)

§ 4. Sphere bundles

Let ξ be a $(k+1)$–dimensional real vector bundle over a closed manifold N^n. We want to study the coincidence question for the projection of the corresponding sphere bundle

$$p: \quad M := S(\xi) \ \longrightarrow \ N \ .$$

Decomposing the tangent bundle of M into a "horizontal" and a "vertical" part, we obtain the canonical isomorphism

$$TM \ \oplus \ \underline{\mathbb{R}} \ \cong \ p^*(TN) \ \oplus \ p^*(\xi) \ .$$

Thus the following commuting diagram of Gysin sequences (cf. [Sa], 5.3 or [Ko 1], 9.20) turns out to be relevant.

$$
\begin{tikzcd}
 & & & \omega(p) \arrow[d, phantom, sloped, "\in"] & \\
\cdots \arrow[r, "p_*"] & \Omega_{k+1}(N;-\xi) \arrow[r, "\pitchfork"] & \Omega_0^{fr}(N) \arrow[r, "\partial"] \arrow[d, equal] & \Omega_k(M;-p^*(\xi)) \arrow[r, "p_*"] & \cdots \\
 & & \mathbb{Z} \arrow[d, equal] & & \\
 & 0 \arrow[r] & \Omega_0^{fr}(\{y_0\}) \arrow[r] & \Omega_k^{fr}(S(\xi_{y_0})) \arrow[uu, "\mathrm{incl}_*"] & .
\end{tikzcd}
\tag{4.1}
$$

Here the transverse intersection homomorphism $\pitchfork$ can also be defined by applying $\mu \circ \mathrm{forg}$ (cf. 2.4) and then evaluating the (possibly twisted) Euler class $e(\xi)$. $\partial(1)$ is given by the inclusion of a typical fibre $S(\xi_{y_0})$, $y_o \in N$, with boundary framing induced from the compact unit ball in ξ_{y_0}; in other words, $\partial(1) = \deg(p)$ (cf. 1.6). Thus $\omega(p) = \partial(\chi(N))$ (cf. theorem 2.2) vanishes if and only if

$$
\chi(N) \ \in \ e(\xi)\,(\mu \circ \mathrm{forg}(\Omega_{k+1}(N;-\xi))) \quad . \tag{4.2}
$$

We also have the successively weaker *necessary* conditions that $\chi(N)$ lies in the subgroups $e(\xi)(\mu(\overline{\Omega}_{k+1}(N;-\xi)))$ and $e(\xi)(H_{k+1}(N;\widetilde{Z}_\xi))$ (compare 2.4).

Example 4.3. Let ξ be an oriented real plane bundle. Then according to [Ko 1], 9.3

$$
\mu \circ \mathrm{forg}(\Omega_2(N;-\xi)) \ = \ \ker(w_2(\xi) : H_2(N;\mathbb{Z}) \ \longrightarrow \ \mathbb{Z}_2) \ .
$$

Thus $\omega(p) = 0$ if and only if $\chi(N) \in e(\xi)(H_2(N;\mathbb{Z}))$ *and* $\chi(N)$ is even. For all $n \geq 1$ this is also the precise condition for p to be loose (if $n = 2$ it implies – via a cohomology Gysin sequence – that $e(p^*(TN)) = 0$; therefore $p^*(TN)$ allows a nowhere vanishing section over the 2-skeleton and hence over all of M, since $\pi_2(S^1) = 0$).

As an illustration let us consider the case when ξ is the r–th tensor power of the canonical complex line bundle over $\mathbb{C}P(q)$, $q > 1$. Then p is loose if and only if $q + 1 \in r\mathbb{Z} = e(\xi)(H_1(\mathbb{C}P(q);\mathbb{Z}))$ *and* q is odd. This last condition is captured by normal bordism, but not by the weaker conditions (expressed in terms of oriented bordism or homology) mentioned above (cf. 4.2; compare also theorem 2.2 in [DG]).

§ **5. Homotopy groups**

Our last example deals with maps which are not fibre projections in general. Choose a local orientation of N at a base point $y_0 \in N$. Then our looseness obstruction determines a group homomorphism

$$
\omega \ : \ \pi_m(N;y_0) \ \longrightarrow \ \Omega^{fr}_{m-n}
$$

as follows. If $n = 1$, then $\omega \equiv 0$. So assume $n \geq 2$ and let x_0 and $*$ denote the base point of S^m and its antipode. Given $[f] \in \pi_m(N;y_0)$, the inclusions $\{x_0\} \subset S^m - \{*\} \subset S^m$ determine canonical isomorphisms (use transversality!)

$$
\Omega^{fr}_{m-n} \xrightarrow{\cong} \Omega_{m-n}(S^m - \{*\}; f^*(TN)) \xrightarrow{\cong} \Omega_{m-n}(S^m; f^*(TN))
$$

which we apply to the obstruction $\omega(f)$. Clearly, we just obtain a multiple of a similarly defined degree homomorphism (which in the case $N = S^n$ is the stable Freudenthal suspension). The relevant multiplying factor is the Euler number of N (whether N is closed or not).

Appendix

Our approach yields also a short proof of the following result which is very useful for calculations as in § 3.

Theorem of Becker and Schultz (cf. [BS], 4.5). *Let B be a compact connected Lie group and $G \subset B$ a proper closed subgroup. Then*

$$\chi(B/G) \cdot [G] = 0 \qquad in \quad \Omega_*^{fr} \quad .$$

Proof. The left hand term is the (weak) selfcoincidence invariant $\omega'(p)$ of the projection $p : B \longrightarrow B/G$ (cf. 3.2). But right multiplication with a path in B from the unit to some element $b_0 \notin G$, when composed with p, yields a deformation from p to a map p' which has no coincidences with p. Thus p is loose and $\omega'(p) = 0$.

More directly: the left hand term is represented by the zero set of the pullback (under p) of a generic section of $T(B/G)$. But clearly $p^*(T(B/G))$ allows a (left invariant) section with empty zero set, and the two zero sets are framed bordant.

Corollary. $2 \cdot [SO(k)] = 0$ *for all even* $k \geq 2$.

Indeed, $SO(k+1)/SO(k) \cong S^k$ has Euler number 2.

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