SELFCOINCIDENCES IN HIGHER CODIMENSIONS

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ABSTRACT. When can a map between manifolds be deformed away from itself? We describe a (normal bordism) obstruction which is often computable and in general much stronger than the classical primary obstruction in cohomology. In particular, it answers our question completely in a large dimension range.

As an illustration we give explicit criteria in three sample settings: projections from Stiefel manifolds to Grassmannians, sphere bundle projections and maps defined on spheres. In the first example a theorem of Becker and Schultz concerning the framed bordism class of a compact Lie group plays a central role; our approach yields also a very short geometric proof (included as an appendix) of this result.

I. Introduction

Throughout this paper $M$ and $N$ denote smooth connected manifolds without boundary, of dimensions $m$ and $n$, resp., $M$ being compact. We say a map $f : M \to N$ is loose (or $f \vert_0 f$ in the notation of Dold and Gonçalves [DG]) if $f$ is homotopic to some map $f'$ which has no coincidences with $f$, i.e. $f(x) \neq f'(x)$ for all $x \in M$.

Problem: Give strong and computable criteria (expressed in a language of algebraic topology) for $f$ to be loose.

In this paper we present some results and examples which seem to indicate that normal bordism theory offers an appropriate language. Indeed, a careful analysis of the coincidence behaviour (of a suitable approximation) of $(f, f) : M \to N \times N$ yields a triple $(C, g, \overline{g})$ where

(i) $C$ is a smooth $(m - n)$-dimensional manifold (the coincidence locus);
(ii) $g : C \to M$ is a continuous map (the inclusion); and
(iii) $\overline{g}$ is a vector bundle isomorphism which describes the stable normal bundle of $C$ in terms of the pullback $g^*(\varphi)$ of the virtual coefficient bundle $\varphi = f^*(TN) - TM$ over $M$.

This leads to a well-defined looseness obstruction

$$\omega(f) := [C, g, \overline{g}] \in \Omega_{m-n}(M; f^*(TN) - TM)$$

in the normal bordism group which consists of the bordism classes of triples as above.

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Selfcoincidence theorem. Assume \( m < 2n - 2 \). Then \( f \) is loose if and only if \( \omega(f) = 0 \).

This is our central result. In \( \S \) 2 below we give the proof which is based on the singularity theory for vector bundle morphisms (see [Ko 1]). As a by-product we show also that if the map \( f \) can be homotoped away from itself, then this can be achieved by an arbitrarily small deformation. Furthermore we obtain a formula expressing \( \omega(f) \) in terms of the Euler number \( \chi(N) \) of \( N \) and of the (normal bordism) degree of \( f \). Often this makes explicit calculations possible.

The natural Hurewicz homomorphism maps our invariant \( \omega(f) \) to the Poincaré dual of the classical primary obstruction \( o_n(f, f) \) in the (co-)homology of \( M \) (in general with twisted coefficients; compare [GJW], theorem 3.3). This transition forgets the vector bundle isomorphism \( g \) nearly completely, keeping track only of the orientation information it carries. If \( m = n \), this is no loss. However, in higher codimensions \( m - n > 0 \) the knowledge of \( g \) is usually crucial.

Example. Consider the canonical projections

\[
p : V_{r,k} \rightarrow G_{r,k} \quad \text{and} \quad \tilde{p} : V_{r,k} \rightarrow \tilde{G}_{r,k}
\]

from the Stiefel manifold of orthonormal \( k \)-frames in \( \mathbb{R}^r \) to the Grassmannian of (unoriented or oriented, resp.) \( k \)-planes through the origin in \( \mathbb{R}^r \).

Then \( \omega(p) = \omega(\tilde{p}) \) lies in the framed bordism group \( \Omega^{fr}_{d(k)}(V_{r,k}) \) where \( d(k) = \frac{1}{2}k(k - 1) \). In nearly all interesting cases the map \( g \) from the coincidence locus \( C \) into \( V_{r,k} \) factors – up to homotopy – through a lower dimensional manifold so that the primary obstruction vanishes. Frequently \( g \) is even nulhomotopic.

Theorem. Assume \( r \geq 2k \geq 2 \). Then: \( p \) and \( \tilde{p} \) are loose if and only if

\[
0 = 2\chi(G_{r,k}) \cdot [SO(k)] \in \pi^{S}_d(k).
\]

This condition holds e.g. if \( k \) is even or \( k = 7 \) or \( 9 \) or \( \chi(G_{r,k}) \equiv 0(12) \).

Here a fascinating problem enters our discussion: to determine the order of a Lie group, when equipped with a left invariant framing and interpreted – via the Pontryagin-Thom isomorphism – as an element in the stable homotopy group of spheres \( \pi^S_\ast \cong \Omega^W_\ast \). Deep contributions were made e.g. by Atiyah and Smith [AS], Becker and Schultz [BS], Knapp [Kn], and Ossa [O], to name but a few (consult the summary of results and the references in [O]). In particular, it is known that the invariantly framed special orthogonal group \( SO(k) \) is nulbordant for \( 4 \leq k \leq 9, k \neq 5 \) (cf. table 1 in [O]) and that \( 24[SO(k)] = 0 \) and \( 2[SO(2\ell)] = 0 \) for all \( k \) and \( \ell \) (cf. [O], p. 315, and [BS], 4.7; for a short proof of this last claim see also our appendix).

On the other hand, the Euler number \( \chi(G_{r,k}) \) is easily calculated: it vanishes if \( k \neq r \equiv 0(2) \) and equals \( \left( \frac{r/2}{k/2} \right) \) otherwise (compare [MS], 6.3 and 6.4).

Corollary 1. Assume \( r > k = 2 \). Then \( p \) and \( \tilde{p} \) are loose.

Corollary 2. Assume \( r \geq k = 3 \). Then \( p \) (or, equivalently, \( \tilde{p} \)) is loose if and only if \( r \) is even or \( r \equiv 1(12) \).

This follows from the fact that \( [SO(3)] \in \pi^S_3 = \mathbb{Z}_{24} \) has order 12 (cf. [AS]).
Corollary 3. Assume $r \geq k = 5$, $r \neq 7$. Then $p$ (or, equivalently, $\tilde{p}$) is loose if and only if $r \neq 5(6)$.

This follows since $[SO(5)]$ has order 3 in $\pi^S_{10} \cong \mathbb{Z}_6$ (cf. [O]).

The details of this example will be discussed in § 3.

Next consider the case when a map $f : M \to N$ allows a section $s : N \to M$ (i.e. $f \circ s = \text{id}_N$). Then clearly $f$ is loose if and only if $\text{id}_N$ is loose or, equivalently, $\chi(N) = 0$ whenever $N$ is closed. In § 4 we refine this simple observation in case $f$ is the projection of a suitable sphere bundle $S(\xi)$. Here the relative importance of the $\mathcal{g}$- and $\mathcal{G}$-data (fiber inclusion and “twisted framing”) in $\omega(f)$ can be studied explicitly via Gysin sequences. We obtain divisibility conditions for $\chi(N)$ in terms of the Euler class of $\xi$.

As a last illustration we discuss the case $M = S^m$ in § 5. Our looseness obstruction determines (and is determined by) a group homomorphism

$$\omega : \pi_m(N; y_0) \to \Omega^f_{m-n},$$

Thus when $m < 2n - 2$ a map $f : S^m \to N$ is loose precisely if its homotopy class $[f]$ lies in the kernel of this homomorphism. In the case $N = S^n$ this holds if $2[f] = 0$ (when $n$ is even) and for all $f$ (when $n$ is odd).

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§ 1. The coincidence invariant and the degree

Consider two maps $f_1, f_2 : M \to N$.

If the resulting map $(f_1, f_2) : M \to N \times N$ is smooth and transverse to the diagonal

$$\Delta := \{(y, y) \in N \times N \mid y \in N\}$$

then the coincidence locus

$$(1.1) \quad C := \{x \in M \mid f_1(x) = f_2(x)\} = (f_1, f_2)^{-1}(\Delta)$$

is a closed $(m - n)$-dimensional manifold canonically equipped with the following two data:

a continuous map

$$(1.2) \quad g : C \to M \quad \text{(namely the inclusion)}$$

and

a stable tangent bundle isomorphism

$$(1.3) \quad \overline{g} : TC \oplus g^*(f_1^*(TN)) \cong g^*(TM)$$

(since the normal bundle $\nu(\Delta, N \times N)$ of $\Delta$ in $N \times N$ is canonically isomorphic to the pullback of the tangent bundle $TN$ under the first projection $p_1$).

If $f_1$ and $f_2$ are arbitrary continuous maps, apply the preceding construction to a smooth map $(f'_1, f'_2)$ which approximates $(f_1, f_2)$ and is transverse to $\Delta$. Then a (sufficiently small) homotopy from $f_1$ to $f'_1$ determines an isomorphism $f^*_1(TN) \cong f'_1^*(TN)$ which is canonical up to regular homotopy. In any case we obtain a well-defined normal bordism class

$$(1.4) \quad \omega(f_1, f_2) := [C, g, \overline{g}] \in \Omega_{m-n}(M; f_1^*(TN) - TM)$$

which depends only on $f_1$ and on the homotopy class of $f_2$. 


Proposition 1.5. If there exist maps \( f'_i : M \to N \) which are homotopic to \( f_i \), \( i = 1, 2 \), and such that \( f'_1(x) \neq f'_2(x) \) for all \( x \in M \), then \( \omega(f_1, f_2) = 0 \).

Proof. The homotopy \( f_1 \sim f'_1 \) yields a nulbordism for \( \omega(f_1, f'_2) = \omega(f_1, f_2) \).

Our approach also leads us to define the (normal bordism) degree of any map \( f : M \to N \) by

\[
\deg(f) := \omega(f, \text{constant map}) .
\]

It is represented by the inverse image \( F \) of a regular value of (a smooth approximation of) \( f \), together with the inclusion map and the obvious stable description of the tangent bundle \( TF \).

§ 2. Selfcoincidences

Given any continuous map \( f : M \to N \), we apply the previous discussion to the special case \( f_1 = f_2 = f \). We obtain the two invariants

\[
\omega(f) := \omega(f, f), \quad \deg(f) \in \Omega_{m-n}(M; f^*(TN) - TM)
\]

(cf. 1.4 and 1.6), both lying in the same normal bordism group.

Any generic section \( s \) of the vector bundle \( f^*(TN) \) over \( M \) gives rise to a map (which is homotopic to \( f \)) from \( M \) to a tubular neighbourhood \( U \cong \nu(\Delta, N \times N) \cong p_1^*(TN) \) of the diagonal \( \Delta \) in \( N \times N \) (compare 1.3). The resulting coincidence locus, together with its normal bordism data, equals the zero set of \( s \) (interpreted as a vector bundle homomorphism from the trivial line bundle \( \mathbb{R} \) to \( f^*(TN) \)), together with its singularity data (cf. [Ko 1]). This locus consists of \( f^{-1}\{y_1, \ldots\} \) if \( s \) is the pullback of a generic section of \( TN \) with zeroes \( \{y_1, \ldots\} \), which are regular values of (a smooth approximation of) \( f \). In particular, if \( N \) admits a nowhere zero vector field \( v \) – e.g. when \( N \) is open – then the map \( f \) is loose (since it can be “pushed slightly along \( v \)” to get rid of all selfcoincidences). We conclude:

Theorem 2.2. Let \( f : M^m \to N^n \) be a continuous map between smooth closed connected manifolds.

Then the selfcoincidence invariant \( \omega(f) \) (cf. 2.1) is equal to the singularity invariant \( \omega(\mathbb{R}, f^*(TN)) \) (cf. [Ko 1], § 2) and hence also to \( \chi(N) \cdot \deg(f) \) (cf. 1.6; here \( \chi(N) \) denotes the Euler number of \( N \)).

Moreover, each of the following conditions implies the next one:

(i) \( f^*(TN) \) allows a nowhere zero section over \( M \);
(ii) \( f \) can be approximated by a map which has no coincidences with \( f \);
(iii) \( f \) is loose;
(iv) there exist maps \( f', f'' : M \to N \) which have no coincidences and which are both homotopic to \( f \); and
(v) \( \omega(f) = 0 \).

If \( m < 2n - 2 \), all these conditions are equivalent.

Indeed, in this dimension range \( \omega(\mathbb{R}, f^*(TN)) \) is the only obstruction to the existence of a monomorphism \( \mathbb{R} \hookrightarrow f^*(TN) \) (see theorem 3.7 in [Ko 1]).
Special case 2.3 (codimension zero). Assume \( m = n \geq 0 \). Then \( f \) is loose if and only if \( \omega(f) = 0 \). Here the relevant normal bordism group \( \Omega_0(M; f^*(TN) - TM) \) is isomorphic to \( \mathbb{Z} \) if \( w_1(M) = f^*(w_1(N)) \) and to \( \mathbb{Z}_2 \) otherwise. \( \omega(f) \) counts the isolated zeroes of a generic section in \( f^*(TN) \). We can concentrate these zeroes in a ball in \( M \) and (after isotoping some of them – if needed – around loops where \( w_1(M) \neq f^*(w_1(N)) \), in order to change their signs) cancel all of them if \( \omega(f) = 0 \).

For a very special case in higher codimensions compare [DG], 1.15.

Remark 2.4. In order to understand and compute normal bordism obstructions, it is often helpful to use the natural forgetful homomorphisms

\[
\Omega_i(M; \varphi) \xrightarrow{\text{forg}} \tilde{\Omega}_i(M; \varphi) \xrightarrow{\mu} H_i(M; \tilde{\mathbb{Z}}) .
\]

Here \( \text{forg} \) retains only the orientation information contained in the \( \tilde{\varphi} \)-components of a normal bordism class, and \( \mu \) denotes the Hurewicz homomorphism to homology with coefficients which are twisted like the orientation line bundle of \( \varphi \). The detailed analysis of \( \text{forg} \), given in § 9 of [Ko 1], yields computing techniques which often permit to calculate obstructions in low dimensional normal bordism groups.

§ 3. Principal bundles

As a first example consider the projection \( p : M \longrightarrow N \) of a smooth principal \( G \)-bundle (cf. [S], 8.1) over the closed manifold \( N \), with \( G \) a compact Lie group. A fixed choice of an orientation of \( G \) at its unit element equips \( G \) with a left invariant framing (which we will drop from the notation); it also yields a (stable) trivialization of the tangent bundle along the fibres of \( p \) and hence of the coefficient bundle \( p^*(TN) - TM \). Thus by theorem 2.2 our selfcoincidence invariant takes the form

\[
\omega(p) = \chi(N) \cdot [G, g = \text{ fibre inclusion}] \in \Omega^{fr}_{m-n}(M) .
\]

If we concentrate on the normal bundle information – which, in a way, represents the highest order component of this obstruction – and neglect its \( g \)-part, we obtain the weaker invariant

\[
\omega'(p) := \text{ const}_*(\omega(p)) = \chi(N)[G] \in \Omega^{fr}_{m-n}
\]

which must also vanish whenever \( p \) is loose. In other words, the Euler number \( \chi(N) \) must be a multiple of the order of \( [G] \) in \( \Omega^{fr}_* \approx \pi^S_* \).

For \( i \leq 6 \) the stable stem \( \pi^S_* \) is generated by the class \( [G] \) of some compact connected Lie group (e.g. \( \pi^S_1 = \mathbb{Z}_2 \cdot [S^1] \) and \( \pi^S_3 = \mathbb{Z}_{24} \cdot [SU(2)] \)). However, this is not typical, and only the divisors of 72 (if not of 24) can be the order of such a class (see [O], theorem 1.1; note also Ossa’s table 1).

As an illustration let us work out the details for the projections \( p \) and \( \overline{p} \) discussed in the example of the introduction. We may assume \( r > k \geq 1 \).

Let us first dispose of two elementary cases.

Case 1: \( k = 1 \). \( p : S^{r-1} \longrightarrow \mathbb{R}P^{r-1} \) and \( \overline{p} = \text{id}_{S^{r-1}} \) are loose if and only if \( r \) is even.
This follows from 2.3 and 2.2.

Case 2: \( k = r - 1 \) or \( k \equiv r - 1 \not\equiv 0(2) \): both \( p \) and \( \overline{p} \) are loose.

Here \( p^*(TG_{r,k}) \cong p^*(\text{Hom}(\gamma, \gamma^\perp)) \cong \oplus k^*(\gamma^\perp) \) (cf. [MS], p. 70) has a nowhere zero section, be it for orientation reasons or since \( G_{r,k} \) is odd-dimensional.

Next recall that in the general setting \( \dim(G_{r,k}) = k(r-k) \); the fibre dimension is given by

\[
(3.3) \quad d(k) := \dim(S)O(k) = \frac{1}{2}k(k-1).
\]

We have \( p^*(TG_{r,k}) \cong \overline{p}^*(T\overline{G}_{r,k}) \) and hence \( \omega(p) = \omega(\overline{p}) \). According to theorem 2.2 this is the only looseness obstruction if

\[
(3.4) \quad r \geq \frac{3}{2}k - \frac{1}{2} + \frac{3}{k}.
\]

Clearly the fibre of \( p \) (or \( \overline{p} \)) over the point \((\mathbb{R}^k \subset \mathbb{R}^r)\) in the Grassmannian is \( V_{k,k} = O(k) \) (or \( SO(k) \), resp.). Also, up to homotopy the fibre inclusion \( g \) factors through \( V_{k,\ell} \) where \( \ell := \max\{2k-r,0\} < k \); this is seen by rotating the vectors \( v_{\ell+1}, \ldots, v_k \) of a \( k \)-frame in \( \mathbb{R}^k \) into the standard basis vectors \( e_{k+1}, \ldots, e_{2k-\ell} \) in \( \mathbb{R}^r \). Except in situations which are already settled by the cases 1 and 2 above we see that the dimension of the intermediate manifold \( V_{k,\ell} \) is strictly less (and often considerably so) than the fibre dimension \( d(k) \) (cf. 3.3) so that the cohomological primary obstruction detects nothing.

In particular, if \( r \geq 2k \) then \( g \) is nullhomotopic and therefore all the information contained in the (complete!) non-selfcoincidence obstruction is already given by

\[
\omega'(\overline{p}) = 2 \cdot \chi(G_{r,k})[SO(k)] \in \Omega^{fr}_{d(k)}
\]

(cf. 3.2 and 3.3). The theorem of the introduction and its corollaries follow. (If \((r,k) = (3,2), (4,3), (6,5) \) or \((8,5) \) refer to case 2 above; if \((r,k) = (9,5) \) the bordism class \( a = [SO(5) \subset V_{9,5}] \in \Omega_{10}^{fr}(V_{9,5}) \) lies in the image of \( \Omega_{10}^{fr}(S^4) \cong \mathbb{Z}_6 \oplus \mathbb{Z}_2 \) and hence \( \omega(f) = 12a = 0 \).)

§ 4. Sphere bundles

Let \( \xi \) be a \((k+1)\)-dimensional real vector bundle over a closed manifold \( N^n \). We want to study the coincidence question for the projection of the corresponding sphere bundle

\[
p : \quad M := S(\xi) \to N.
\]

Decomposing the tangent bundle of \( M \) into a “horizontal” and a “vertical” part, we obtain the canonical isomorphism

\[
TM \oplus \mathbb{R} \cong p^*(TN) \oplus p^*(\xi).
\]

Thus the following commuting diagram of Gysin sequences (cf. [Sa], 5.3 or [Ko 1], 9.20) turns out to be relevant.
Here the transverse intersection homomorphism $\cap$ can also be defined by applying $\mu \circ \text{forg}$ (cf. 2.4) and then evaluating the (possibly twisted) Euler class $e(\xi)$. $\partial(1)$ is given by the inclusion of a typical fibre $S(\xi_{y_0})$, $y_0 \in N$, with boundary framing induced from the compact unit ball in $\xi_{y_0}$; in other words, $\partial(1) = \text{deg}(p)$ (cf. 1.6). Thus $\omega(p) = \partial(\chi(N))$ (cf. theorem 2.2) vanishes if and only if
\begin{equation}
\chi(N) \in e(\xi)(\mu(\Omega_{k+1}(N; -\xi))) \quad (4.2)
\end{equation}

We also have the successively weaker necessary conditions that $\chi(N)$ lies in the subgroups $e(\xi)(\mu(\Omega_{k+1}(N; -\xi)))$ and $e(\xi)(H_{k+1}(N; \tilde{Z}_\xi))$ (compare 2.4).

**Example 4.3.** Let $\xi$ be an oriented real plane bundle. Then according to [Ko 1], 9.3

\[ \mu \circ \text{forg}(\Omega_2(N; -\xi)) = \ker(w_2(\xi) : H_2(N; \mathbb{Z}) \to \mathbb{Z}_2). \]

Thus $\omega(p) = 0$ if and only if $\chi(N) \in e(\xi)(H_2(N; \mathbb{Z}))$ and $\chi(N)$ is even. For all $n \geq 1$ this is also the precise condition for $p$ to be loose (if $n = 2$ it implies – via a cohomology Gysin sequence – that $e(p^*(TN)) = 0$; therefore $p^*(TN)$ allows a nowhere vanishing section over the 2-skeleton and hence over all of $M$, since $\pi_2(S^1) = 0$).

As an illustration let us consider the case when $\xi$ is the $r$-th tensor power of the canonical complex line bundle over $\mathbb{C}P(q)$, $q > 1$. Then $p$ is loose if and only if $q + 1 \in r\mathbb{Z} = e(\xi)(H_1(\mathbb{C}P(q); \mathbb{Z}))$ and $q$ is odd. This last condition is captured by normal bordism, but not by the weaker conditions (expressed in terms of oriented bordism or homology) mentioned above (cf. 4.2; compare also theorem 2.2 in [DG]).

§ 5. Homotopy groups

Our last example deals with maps which are not fibre projections in general. Choose a local orientation of $N$ at a base point $y_0 \in N$. Then our looseness obstruction determines a group homomorphism

\[ \omega : \pi_m(N; y_0) \to \Omega_m^{fr} \]

as follows. If $n = 1$, then $\omega \equiv 0$. So assume $n \geq 2$ and let $x_0$ and $\ast$ denote the base point of $S^m$ and its antipode. Given $[f] \in \pi_m(N; y_0)$, the inclusions $\{x_0\} \subset S^m - \{\ast\} \subset S^m$ determine canonical isomorphisms (use transversality!)

\[ \Omega_{m-n}^{fr} \xrightarrow{\cong} \Omega_{m-n}(S^m - \{\ast\}; f^*(TN)) \xrightarrow{\cong} \Omega_{m-n}(S^m; f^*(TN)) \]
which we apply to the obstruction $\omega(f)$. Clearly, we just obtain a multiple of a similarly defined degree homomorphism (which in the case $N = S^n$ is the stable Freudenthal suspension). The relevant multiplying factor is the Euler number of $N$ (whether $N$ is closed or not).

Appendix

Our approach yields also a short proof of the following result which is very useful for calculations as in § 3.

Theorem of Becker and Schultz (cf. [BS], 4.5). Let $B$ be a compact connected Lie group and $G \subset B$ a proper closed subgroup. Then

$$\chi(B/G) \cdot [G] = 0 \quad \text{in} \quad \Omega^{fr}_s.$$

Proof. The left hand term is the (weak) selfcoincidence invariant $\omega'(p)$ of the projection $p : B \to B/G$ (cf. 3.2). But right multiplication with a path in $B$ from the unit to some element $b_0 \notin G$, when composed with $p$, yields a deformation from $p$ to a map $p'$ which has no coincidences with $p$. Thus $p$ is loose and $\omega'(p) = 0$.

More directly: the left hand term is represented by the zero set of the pullback (under $p$) of a generic section of $T(B/G)$. But clearly $p^*(T(B/G))$ allows a (left invariant) section with empty zero set, and the two zero sets are framed bordant.

Corollary. $2 \cdot [SO(k)] = 0$ for all even $k \geq 2$.

Indeed, $SO(k + 1)/SO(k) \cong S^k$ has Euler number 2.

References


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