Hanoi attractors and the Sierpiński Gasket

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Abstract The famous game *Towers of Hanoi* is related with a family of so–called Hanoi–graphs. We regard these non self–similar graphs as geometrical objects and obtain a sequence of fractals HG_α converging to the Sierpiński gasket which is one of the best studied fractals. It is shown that this convergence holds not only with respect to the Hausdorff distance, but that also Hausdorff dimension does converge. Moreover, it is shown that each of the approximating sets has non-trivial Hausdorff measure.

Keywords: Fractals, Hausdorff dimension, Iterated function system, attractor, Hausdorff measure, Hanoi graph, Sierpiński gasket.

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Uta Freiberg was born in Greifswald (former East Germany) in 1971. She studied mathematics and economics at the Humboldt University of Berlin where she got her diploma in Stochastic Analysis in 1997 under the supervision of Uwe Küchler. After that she moved to Jena where she got her PhD in Fractal Geometry and Measure Geometric Analysis under supervision of Martina Zähle. She spent longer research periods in Rome (La Sapienza) and Canberra (ANU). Since October 2009, she is leader of the junior research group "'Fractal Geometry and Stochastics"' at the University of Siegen.

1 Introduction

For the whole discussion, let $\mathcal{A} = \{1, 2, 3\}$ be the alphabet consisting of three symbols $1, 2$ and 3. Further, denote $\mathbb{R}^2$ the Euclidean plane and define $p_1, p_2, p_3 \in \mathbb{R}^2$ by

$$p_1 := (0, 0), \qquad p_2 := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad p_3 := (1, 0).$$

Hence, p_1, p_2 and p_3 are the vertices of a unilateral triangle of side length one.

Our work centers on one of the most famous fractals, the Sierpiński gasket, that we denote by SG. This is the unique non-empty compact subset of $\mathbb{R}^2$ such that

$$SG = \bigcup_{i=1}^{3} S_i(SG),$$

where

$$\begin{aligned} S_i \colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x \; &\mapsto \frac{x + p_i}{2}, \qquad i = 1, 2, 3. \end{aligned}$$

This fractal is a *p.c.f. self-similar set* (see Kigami, J. (1993) for details) and it is usually approximated by an increasing sequence of finite sets defined by

$$V_0 := \{p_1, p_2, p_3\},$$

and for every $n \geq 1$

$$V_n := \bigcup_{w \in \mathcal{A}^n} S_w(V_0), \tag{1}$$

where $S_w \colon \mathbb{R}^2 \to \mathbb{R}^2$, $S_w(x) := S_{w_1} \circ \cdots \circ S_{w_n}(x)$ for $w = w_1 \dots w_n \in \mathcal{A}^n$. The set

$$V_* := \bigcup_{n \geq 0} V_n$$

is dense in SG with respect to the Euclidean metric. For further details we refer to Kigami, J. (1993).

Definition 1.1: *The n-th approximating graph Γ_n of SG is the graph with vertex set $V(\Gamma_n)$ and edge set $E(\Gamma_n)$ defined by*

$$\begin{cases} V(\Gamma_n) := V_n \\ E(\Gamma_n) := \{\{x, y\} : \; x \neq y \text{ and } \exists\, w \in \mathcal{A}^n \text{ such that } x, y \in S_w(V_0)\}, \end{cases}$$

where V_n is the set defined in (1).

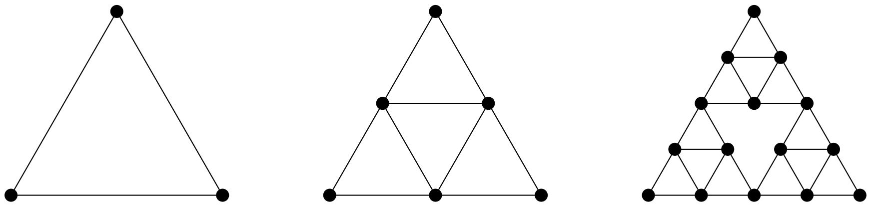


Figure 1 Approximating graphs Γ_0, Γ_1 and Γ_2.

Hanoi graphs have their origin as mathematical representation of the so called *Tower of Hanoi game* (*TH game* for short), intensely studied since its invention due to the French mathematician Édouard Lucas in 1883.

This game consists of *three* (or in general $p \geq 3$) *vertical pegs*, named $1, 2$ and 3, and n discs numbered 1 through n according to rising diameters, $n \in \mathbb{N}$. At the beginning, these discs are situated on the first peg so that the largest disc lies at the bottom and the smallest at the top, building a tower. The goal of the game is to construct the tower again on one of the other pegs (see Figure 2).

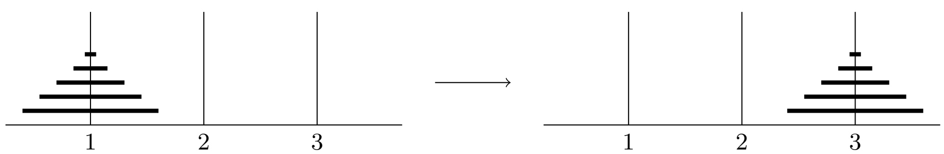


Figure 2 Starting state with the tower on the first peg and expected final state with the tower on the third peg.

Throughout the construction one must follow two basic rules:

- Each time one and only one disc has to be moved.
- No larger disc lies on a smaller one.

This means that only one disc lying at the top of a peg will be moved each time. The states of the Tower of Hanoi for n discs and p pegs can be represented by words $w = w_1 \dots w_n \in \{1, 2, \dots p\}^n$, where w_i indicates the peg on which the disc i is stacked in state w. A move is a pair of states (w, w'), where w' results from w by a legal transfer of a disc (see Figure 3).

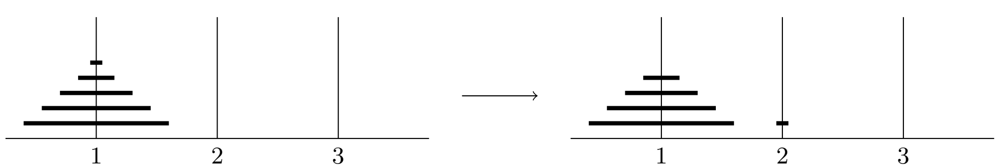


Figure 3 Legal move $(11111, 21111)$ of the TH game with $p = 3$, $n = 5$.

A mathematical representation of this game is given by the so called *Hanoi graphs*, H_p^n, p being the number of pegs and n being the number of discs.
Of our interest are Hanoi graphs with $p = 3$. So the states will be represented by words $w \in \{1, 2, 3\}^n = \mathcal{A}^n$. From now on, we will drop the index p and write H^n only.

For any $n \geq 1$, the *Hanoi graph* H^n is defined by

$$\begin{cases} V(H^n) := \mathcal{A}^n \\ E(H^n) := \{\{w, w'\} :\ (w, w') \text{ is a legal move in the TH game}\}. \end{cases}$$

The different moves done during the game build a path in the corresponding Hanoi graph. An example is given in Figures 4 and 5.

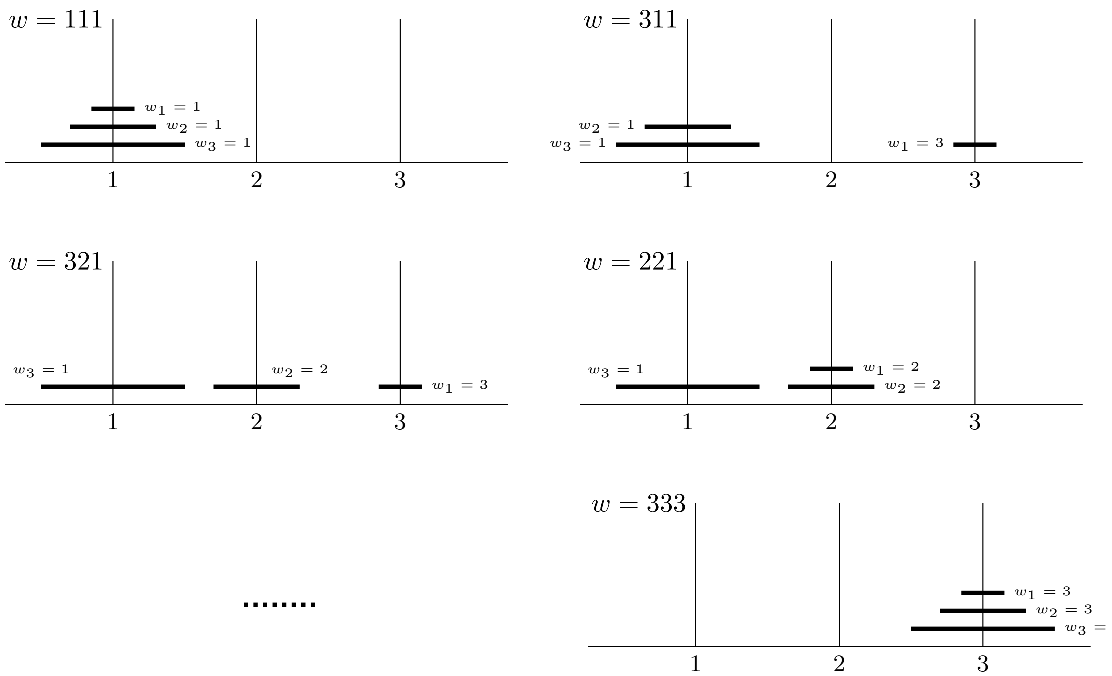


Figure 4 Moves in the TH game with $p = 3$ and $n = 3$.

For simplicity we will choose another labelling for the vertices of these graphs. This labelling was introduced in Klavžar, S. and Milutinović, U. (1997) with the definition of the so called *Sierpiński graphs* $\mathcal{S}_3^n$, $n \geq 0$. These graphs are defined recursively by

$$V(\mathcal{S}_3^0) := w_0, \qquad E(\mathcal{S}_3^0) := \emptyset,$$

where w_0 denotes the empty word, and

$$\begin{cases} V(\mathcal{S}_3^{n+1}) := \mathcal{A}^{n+1} \\ E(\mathcal{S}_3^{n+1}) := \{\{iw, iw'\},\ i \in \mathcal{A},\ \{w, w'\} \in E(\mathcal{S}_3^n)\} \\ \qquad\qquad\qquad \cup \left\{\{ij \ldots j, ji \ldots i\} \in \binom{\mathcal{A}^{n+1}}{2}\right\} \end{cases}$$

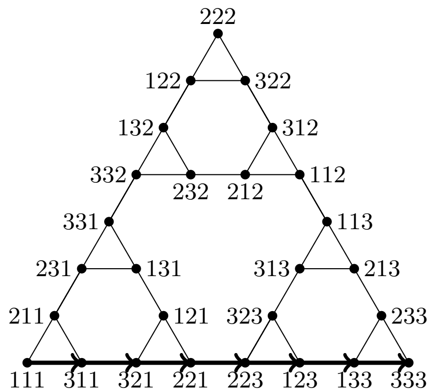


Figure 5 The marked edges build the path corresponding to the moves done in Figure 4.

Since Sierpiński graphs are isomorphic to Hanoi graphs (see Lemma 2 in Hinz, A.M. and Schief, A. (1990)), we can use equivalently this alternative labelling (see Figure 6).

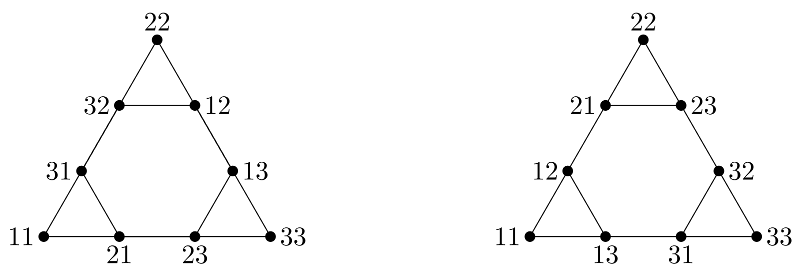


Figure 6 Representation of the graphs H_3^2 and $\mathcal{S}_3^2$.

From now on, we regard Hanoi graphs not only as topological objects, but also as geometrical ones. In particular, we will equip the edges with a length. The purpose of the rest of the paper is to investigate geometric properties of these sets.

We define the parameter $\alpha > 0$ to be the length of the only edges that belong to $E(H^n)$ for every $n \geq 2$. This length corresponds to the following distance

$$\alpha := |x_{ijj\ldots j} - x_{jii\ldots i}|\,, \tag{2}$$

for any $i, j \in \mathcal{A}$, $i \neq j$. Here, $x_{ijj\ldots j}$ represents the point that geometrically corresponds to the vertex $ijj\ldots j \in \mathcal{A}^n$. New similar edges appearing over the levels will have a length which is a multiple of α.

It is easy to understand (see Figure 7) that, if we let α tend to zero, any Hanoi graph H^n will end up to the $(n-1)-$th approximating graph of SG, Γ_{n-1}.

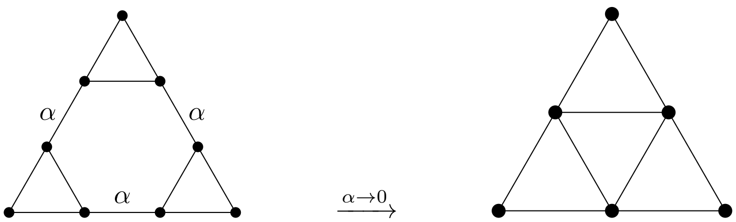


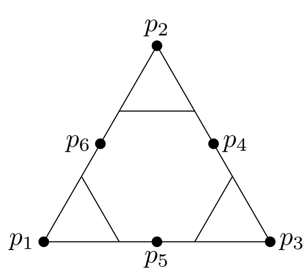
Figure 7 H^2 will become Γ_1 if $\alpha \to 0$.

This fact brings up some interesting questions. Firstly, how one can describe properly this geometric convergence. Secondly, if this convergence works for other quantities than Hausdorff dimension. Thirdly, if and how analytic convergence holds. The first two questions are answered in the present paper. Hereby some proofs concerning the second question are sketched only. For details we refer to the authors paper Alonso-Ruiz, P. and Freiberg, U.R (2012). The last question will be investigated in the forthcoming paper Alonso-Ruiz, P. and Freiberg, U.R (2012).

2 The Hanoi attractor

Let $0 < \alpha < 1/3$ be the parameter defined in (2). Moreover, recall the points p_1, p_2, p_3, defined at the beginning of the introduction and set

$$p_4 := \frac{1}{2}(p_2 + p_3), \quad p_5 := \frac{1}{2}(p_1 + p_3), \quad p_6 := \frac{1}{2}(p_1 + p_2).$$



Further, define the maps $G_{\alpha,i} \colon \mathbb{R}^2 \to \mathbb{R}^2,\ i = 1, \ldots 6$ by

$$G_{\alpha,i}(x) := \begin{cases} \frac{1-\alpha}{2}(x - p_i) + p_i \text{ for } i = 1, 2, 3, \\ A_i\alpha(x - p_i) + p_i \text{ for } i = 4, 5, 6, \end{cases} \tag{3}$$

where

$$A_4 = \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix} \quad A_5 = \begin{pmatrix} 1\,0 \\ 0\,0 \end{pmatrix} \quad A_6 = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$

Note that $G_{\alpha,1}, G_{\alpha,2},\ G_{\alpha,3}$ are contractive similitudes of ratio $\frac{1-\alpha}{2}$, while $G_{\alpha,4}, G_{\alpha,5},\ G_{\alpha,6}$ are contractive (singular) mappings of ratio α. Nevertheless, since all the ratios are less than one, $\{G_{\alpha,i}\}_{i=1}^{6}$ is a family of contractions. Thus we know (see Hutchinson, John E. (1981)) that there exists a unique non-empty compact set, HG_α, such that

$$HG_\alpha = \bigcup_{i=1}^{6} G_{\alpha,i}(HG_\alpha).$$

This is what we call the *Hanoi attractor* of parameter α. The reason for this attractor to be called *Hanoi* is simply that it almost looks like the drawing of the graph of the TH game if we would have three pegs and "infinitely many discs".

Observe that HG_α is not *self-similar* (see Section 9.2 in Falconer, K. (2003) for definition), because $G_{\alpha,4}, G_{\alpha,5}, G_{\alpha,6}$ are not similitudes. This will cause difficulties when doing calculations (see Section 4).

If we now consider the sequence of Hanoi attractors $(HG_\alpha)_{0<\alpha<1/3}$, we ask ourselves about its geometric properties when approaching SG.

3 Convergence in the Hausdorff metric

Let $(\mathscr{H}(\mathbb{R}^2), h)$ be the complete metric space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff metric h.
Recall that this metric $h\colon \mathscr{H}(\mathbb{R}^2) \times \mathscr{H}(\mathbb{R}^2) \to [0, +\infty)$ is defined by

$$h(A,B) := \inf\{\epsilon > 0 : \text{ such that } A \subseteq B_\epsilon \text{ and } B \subseteq A_\epsilon\}, \; A, B \in \mathscr{H}(\mathbb{R}^2),$$

where

$$A_\epsilon := \left\{x \in \mathbb{R}^2 : \; d(x,A) \le \epsilon\right\}$$

and

$$d(x,A) := \min\{|x-y| : \; y \in A\}.$$

Theorem 3.1: *Let SG be the Sierpiński gasket and HG_α the Hanoi attractor of parameter α, $0 < \alpha < 1/3$. Then it holds that*

$$h(HG_\alpha, SG) \xrightarrow{\alpha \to 0} 0.$$

Before proving this, we need some previous work.

For each $n \ge 1$ and each $0 < \alpha < 1/3$, define the sets $W_{\alpha,n}$ by

$$W_{\alpha,n} := \bigcup_{w \in \mathcal{A}^n} G_{\alpha,w}(W_{\alpha,0}),$$

where $W_{\alpha,0} := \{p_1, p_2, p_3\}$ and

$$G_{\alpha,w}(x) := G_{\alpha,w_1} \circ G_{\alpha,w_2} \circ \cdots G_{\alpha,w_n}(x), \qquad w = w_1, \dots w_n \in \mathcal{A}^n.$$

Note that we still work with the alphabet $\mathcal{A} = \{1,2,3\}$.
Further, define

$$W_{\alpha,*} := \bigcup_{n \ge 0} W_{\alpha,n}.$$

Lemma 3.2: *It holds that*

$$h(V_*, W_{\alpha,*}) \to 0, \qquad \text{as } \alpha \downarrow 0.$$

Proof: From the definition of Hausdorff metric we know that

$$h(V_*, W_{\alpha,*}) = \inf\left\{\epsilon > 0 : \ V_* \subseteq (W_{\alpha,*})_\epsilon \text{ and } W_{\alpha,*} \subseteq (V_*)_\epsilon\right\}.$$

Fix $\alpha \in (0, 1/3)$. We will show that

$$h(V_*, W_{\alpha,*}) \leq \alpha.$$

We firstly show that $V_* \subseteq (W_{\alpha,*})_\alpha$. We prove by complete induction over m that

$$V_m \subseteq (W_{\alpha,*})_\alpha \quad \text{ for all } m \geq 0.$$

Case $m = 0$: If $x \in V_0$ there is nothing to prove since $V_0 = W_{\alpha,0}$.
Let us prove the first non-trivial case $m = 1$: Consider $x \in V_1$. Then, $x \in V_0$ (trivial case, see above) or $x \in V_1 \setminus V_0$. In the latter case, there exist $i, j \in \mathcal{A}$, $i \neq j$ such that

$$x = S_i(p_j). \tag{4}$$

For the same $i, j \in \mathcal{A}$ as in (4), consider the point $y := G_{\alpha,i}(p_j)$. Its distance to x is:

$$\begin{aligned}
|x - y| = |S_i(p_j) - G_{\alpha,i}(p_j)| &= \left|\frac{1}{2}p_j + \frac{1}{2}p_i - \frac{1-\alpha}{2}p_j - \frac{1+\alpha}{2}p_i\right| \\
&= \left|\frac{\alpha}{2}p_j - \frac{\alpha}{2}p_i\right| = \frac{\alpha}{2}\left|p_j - p_i\right| = \frac{\alpha}{2} \leq \alpha.
\end{aligned}$$

Thus $V_1 \in (W_{\alpha,*})_\alpha$.

Now assume that $V_m \subseteq (W_{\alpha,*})_\alpha$ for some m.
Let $x \in V_{m+1} \setminus V_m$. Then, there exist a point $\overline{x} \in V_m$ and a symbol $k \in \mathcal{A}$ such that

$$x = S_k(\overline{x}). \tag{5}$$

Since $\overline{x} \in V_m$, by hypothesis of induction we know that there exists $\overline{y} \in W_{\alpha,*}$ such that

$$|\overline{x} - \overline{y}| \leq \alpha. \tag{6}$$

For the same $k \in \mathcal{A}$ as in (5), consider the point $y := G_{\alpha,k}(\overline{y}) \in W_{\alpha,*}$. For the point y it holds that

$$\begin{aligned}
|x - y| = |S_k(\overline{x}) - G_{\alpha,k}(\overline{y})| &= \left|\frac{1}{2}\overline{x} + \frac{1}{2}p_k - \frac{1-\alpha}{2}\overline{y} - \frac{1+\alpha}{2}p_k\right| \\
&= \frac{1}{2}\left|\overline{x} - \overline{y} + \alpha\overline{y} - \alpha p_k\right| \leq \frac{1}{2}\left|\overline{x} - \overline{y}\right| + \frac{\alpha}{2}\left|\overline{y} - p_k\right| \\
&\overset{(6)}{\leq} \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.
\end{aligned}$$

Thus we have proven, that for every $m \geq 0$ the following holds: For any $x \in V_m$ there exists an $y \in W_{\alpha,*}$ such that $|x - y| \leq \alpha$, i.e

$$V_m \subseteq (W_{\alpha,*})_\alpha \quad \text{for all } m \geq 0.$$

Since $V_* = \bigcup_{m \geq 0} V_m$, we get

$$V_* \subseteq (W_{\alpha,*})_\alpha.$$

This holds for all $\alpha \in (0, 1/3)$.

It would remain to prove the inclusion $W_{\alpha,*} \subseteq (V_*)_\alpha$ for all $0 < \alpha < 1/3$, but the proof of this is analogous to the latter one by simply changing the roles of V_* and $W_{\alpha,*}$. Thus,

$$W_{\alpha,m} \subseteq (V_*)_\alpha \quad \text{for all } m \geq 0,$$

and therefore

$$h(V_*, W_{\alpha,*}) \leq \alpha \xrightarrow{\alpha \to 0} 0,$$

as we wanted to prove. □

Note that the bound $h(V_*, W_{\alpha,*}) \leq \alpha$ is quite rough, one easily could obtain sharper estimates. However, the bound α is sufficient for our purposes.

We now introduce a new set, F_α, which is the unique non-empty compact set such that

$$F_\alpha = \bigcup_{i=1}^{3} G_{\alpha,i}(F_\alpha), \tag{7}$$

where $G_{\alpha,1}, G_{\alpha,2}$ and $G_{\alpha,3}$ are the three similitudes of ratio $\frac{1-\alpha}{2}$ defined in (3). Note that $W_{\alpha,*}$ is dense in F_α with respect to the Euclidean metric in $\mathbb{R}^2$.

Lemma 3.3: *Let $0 < \alpha < 1/3$ and consider the sets HG_α and F_α. Then,*

$$F_\alpha \subseteq HG_\alpha.$$

Proof: Define the map

$$\begin{aligned} T\colon \mathscr{H}(\mathbb{R}^2) &\longrightarrow \mathscr{H}(\mathbb{R}^2) \\ B \quad &\mapsto \bigcup_{i=1}^{3} G_{\alpha,i}(B), \qquad B \in \mathscr{H}(\mathbb{R}^2), \end{aligned}$$

whose unique fixed point is F_α. From Theorem 9.1 in Falconer, K. (2003) we know that for any starting set $B_0 \in \mathscr{H}(\mathbb{R}^2)$ such that $G_{\alpha,i}(B_0) \subseteq B_0$ for all $i = 1, 2, 3$, the sequence $(B_n)_{n=1}^{\infty}$ defined recursively by

$$B_1 := T(B_0), \qquad B_n := T(B_{n-1}) \qquad \text{for every } n > 1,$$

converges to F_α in the Hausdorff distance h and it holds that

$$F_\alpha = \bigcap_{n=1}^{\infty} B_n.$$

On the other hand, define

$$\begin{aligned} \widetilde{T}\colon \mathscr{H}(\mathbb{R}^2) &\longrightarrow \mathscr{H}(\mathbb{R}^2) \\ B \qquad &\mapsto \bigcup_{i=1}^{6} G_{\alpha,i}(B), \qquad B \in \mathscr{H}(\mathbb{R}^2). \end{aligned}$$

Recall that HG_α is the fixed point of $\widetilde{T}$ in $\mathscr{H}(\mathbb{R}^2)$.
Analogously, it holds that for any starting set $C_0 \in \mathscr{H}(\mathbb{R}^2)$ such that $G_{\alpha,i}(C_0) \subseteq C_0$ for all $i = 1, \dots, 6$, the sequence $(C_n)_{n=1}^{\infty}$ defined recursively by

$$C_1 := \widetilde{T}(C_0), \qquad C_n := \widetilde{T}(C_{n-1}) \qquad \text{for every } n > 1,$$

converges to HG_α in the Hausdorff distance h and it holds that

$$HG_\alpha = \bigcap_{n=1}^{\infty} C_n.$$

Denote by Δ the triangle with vertices p_1, p_2, p_3. Set $B_0 = C_0 = \Delta$. Then,

$$F_\alpha = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} T^n(\Delta) \subseteq \bigcap_{n=1}^{\infty} \widetilde{T}^n(\Delta) = HG_\alpha,$$

as required. $\square$

Lemma 3.4: *It holds that*

$$h(HG_\alpha, F_\alpha) \xrightarrow{\alpha \to 0} 0.$$

Proof: Note that, since $F_\alpha \subseteq HG_\alpha$ for all $0 < \alpha < 1/3$, it follows directly that

$$F_\alpha \subseteq (HG_\alpha)_\epsilon \qquad \text{for all } \epsilon > 0.$$

In order to obtain the other direction, we prove that

$$HG_\alpha \subseteq (F_\alpha)_{\frac{\alpha}{2}} \qquad \text{for every } 0 < \alpha < 1/3. \tag{8}$$

Let $x \in HG_\alpha$. If $x \in F_\alpha$, then we are done. So let us assume, $x \in HG_\alpha \setminus F_\alpha$. Then, there exists $n \geq 1$ and a word $w = w_1 w_2, \dots w_n \in \{1, \dots 6\}$ with at least one letter in $\{4,5,6\}$ (note, there is such a letter since $x \notin F_\alpha$) such that

$$x = G_{\alpha,w}(HG_\alpha).$$

Consider w_k, $k \leq n$ the first letter of w such that $w_k \in \{4,5,6\}$ and define $\overline{w} := w_1 w_2 \dots w_{k-1} \in \mathcal{A}^{k-1}$. Further, observe that

$$x \in G_{\alpha,\overline{w}w_k}(HG_\alpha).$$

Therefore, there exists a point $z \in G_{\alpha,w_k}(HG_\alpha)$ such that $x = G_{\alpha,\overline{w}}(z)$. By construction (see e.g. Figure 3) we can find a point $y \in \{G_{\alpha,i}(p_j), G_{\alpha,j}(p_i)\}$ for $i, j \in \mathcal{A}$, $i + j + w_k = 9$ such that

$$|z - y| \leq \frac{\alpha}{2}.$$

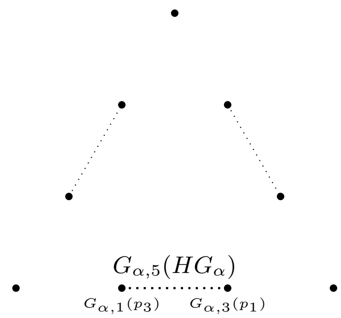


Figure 8 For any $x \in G_{\alpha,5}(HG_\alpha)$ it holds that $|x - G_{\alpha,1}(p_3)| \leq \frac{\alpha}{2}$ or $|x - G_{\alpha,3}(p_1)| \leq \frac{\alpha}{2}$.

Define $\overline{y} := G_{\alpha,\overline{w}}(y) \in F_\alpha$. Since $G_{\alpha,1}, G_{\alpha,2}$ and $G_{\alpha,3}$ are similitudes of ratio $\frac{1-\alpha}{2}$, $G_{\alpha,\overline{w}}$ is a similitude of ratio $\left(\frac{1-\alpha}{2}\right)^{k-1}$ and therefore

$$|G_{\alpha,\overline{w}}(z) - G_{\alpha,\overline{w}}(y)| = \left(\frac{1-\alpha}{2}\right)^{k-1} |z - y| \,.$$

Thus,

$$|x - \overline{y}| = |G_{\alpha,\overline{w}}(z) - G_{\alpha,\overline{w}}(y)| = \left(\frac{1-\alpha}{2}\right)^{k-1} |z - y| \quad \leq \left(\frac{1-\alpha}{2}\right)^{k-1} \cdot \frac{\alpha}{2} \leq \frac{\alpha}{2},$$

and (8) is proved. Therefore

$$h(HG_\alpha, F_\alpha) \leq \frac{\alpha}{2} \xrightarrow{\alpha \to 0} 0$$

as we wanted to prove. $\square$

Now we are ready to prove Theorem 3.1.

Proof: Since h is a metric, we know from the triangular inequality that

$$h(HG_\alpha, SG) \leq h(HG_\alpha, F_\alpha) + h(F_\alpha, SG). \tag{9}$$

On the one hand, by Lemma 3.4 we know that

$$h(HG_\alpha, F_\alpha) \xrightarrow{\alpha \to 0} 0.$$

On the other hand, $W_{\alpha,*}$ and V_* are dense in F_α and SG respectively, therefore we have

$$h(F_\alpha, SG) \leq \underbrace{h(F_\alpha, W_{\alpha,*})}_{=0} + h(W_{\alpha,*}, V_*) + \underbrace{h(V_*, SG)}_{=0} = h(W_{\alpha,*}, V_*).$$

From Lemma 3.2 we know that $h(W_{\alpha,*}, V_*)$ tends to zero if α tends to zero. Thus from (9) we get

$$h(HG_\alpha, SG) \leq h(HG_\alpha, F_\alpha) + h(W_{\alpha,*}, V_*) \xrightarrow{\alpha \to 0} 0,$$

as we wanted to prove. $\square$

4 Geometric convergence

4.1 Hausdorff dimension

We are interested in the geometric properties of the sets HG_α, $\alpha \in (0, 1/3)$, and in particular we look for their Hausdorff dimensions. From the result of the last section, one could conjecture that the Hausdorff dimension of HG_α converges to the Hausdorff dimension of SG. In the present section we show that this is true.

First of all, let us recall some definitions, that can be found for example in Falconer, K. (2003).

Definition 4.1: *Let (X, d) be a complete metric space and $(\mathscr{H}(X), h)$ its associated Hausdorff space.*

(1) Let $U \subseteq X$, $U \neq \emptyset$. The diameter of U is defined as

$$|U| := \sup \{d(x, y) : x, y \in U\}.$$

(2) Let $K \in \mathscr{H}(X)$ and $\{U_i\}_{i \in I}$ a countable (or finite) collection of sets of diameter at most δ covering K, i.e

$$K \subseteq \bigcup_{i \in I} U_i, \qquad \textit{and} \qquad 0 \leq |U_i| \leq \delta.$$

We say that $\{U_i\}_{i \in I}$ is a $\delta-$covering of K.

(3) Let $s \geq 0$. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(K) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\}_{i \in I} \textit{ is a } \delta - \textit{covering of } K \right\}.$$

(4) The limit

$$\mathcal{H}^s(K) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(K)$$

is called the s-dimensional Hausdorff measure of K. This limit exists as an element of $[0, +\infty]$ because the sequence $(\mathcal{H}^s_\delta(K))_{\delta > 0}$ is monotone and non-decreasing for $\delta \downarrow 0$.

(5) If we consider $\mathcal{H}^s(K)$ as a function over s, there is a critical value of s such that the function jumps down from ∞ to 0. This critical value is called the Hausdorff dimension of K, $\dim_H(K)$. So it holds

$$\dim_H(K) = \inf \{s \geq 0 : \mathcal{H}^s(K) = 0\} = \sup \{s \geq 0 : \mathcal{H}^s(K) = \infty\}.$$

Observe that for $s = \dim_H(K)$, $\mathcal{H}^s(K)$ may be zero, infinite, or some positive finite number. The latter case makes possible to get a deep analysis of the object.

In our particular case, we work in the metric space $\mathbb{R}^2$ equipped with the Euclidean norm. Recall, that a contractive *c-similitude* is a map $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$S(x) = cUx + x_0, \quad x \in \mathbb{R}^2,$$

where U is a unitary matrix, $x_0 \in \mathbb{R}^2$ and $0 < c < 1$.

A finite family of similitudes $\{S_i\}_{i=1}^N$ is said to satisfy the *open set condition* (OSC) if there exists a non-empty bounded open set $V \subseteq \mathbb{R}^2$ such that

$$\bigcup_{i=1}^N S_i(V) \subseteq V$$

and $S_i(V) \cap S_j(V) = \emptyset$ for $i \neq j$. Note that $\{\mathbb{R}^2; S_1, \ldots, S_N\}$ is also called *Iterated Function System* (IFS for short).

Theorem 4.2: *Provided that the family $\{S_i\}_{i=1}^N$ with ratios $c_1, \ldots c_N$ satisfies (OSC) and K is the unique non-empty compact set such that*

$$K = \bigcup_{i=1}^N S_i(K),$$

then $\dim_H K = s$, where $s > 0$ is the solution of the equation given by

$$\sum_{i=1}^N c_i^s = 1. \tag{10}$$

Moreover, for this value of s, it holds that $0 < \mathcal{H}^s(K) < \infty$.

Proof: See Hutchinson, John E. (1981) as the original source, or Theorem 9.3 in Falconer, K. (2003). $\square$

Our next goal is to determine the Hausdorff dimension of HG_α, but there is a problem: Since HG_α is not self-similar, we can not apply the formula in (10). So we will have to solve the problem using the original definition of Hausdorff dimension, which involves $\delta-$coverings.
Here we state our result.

Theorem 4.3: *Let $0 < \alpha < 1/3$ and HG_α be the corresponding Hanoi attractor. Then it holds that*

$$\dim_H(HG_\alpha) = \frac{\ln 3}{\ln 2 - \ln(1-\alpha)}.$$

Proof: Fix $\alpha \in (0, 1/3)$ and denote $d := \frac{\ln 3}{\ln 2 - \ln(1-\alpha)}$.

(1) $d \leq \dim_H(HG_\alpha)$.
Let F_α be the set defined in (7). It satisfies the open set condition and therefore we can easily compute its Hausdorff dimension. This is the unique number $s > 0$ such that

$$\sum_{i=1}^3 \left(\frac{1-\alpha}{2}\right)^s = 1,$$

thus,

$$s = \frac{\ln 3}{\ln 2 - \ln(1-\alpha)} = d.$$

From the monotonicity of Hausdorff dimension (see (Falconer, K., 2003, p. 32)), we get that

$$d = \dim_H(F_\alpha) \le \dim_H(HG_\alpha),$$

because we have that $F_\alpha \subseteq HG_\alpha$ by Lemma 3.3. This proves Step (1).

(2) For the proof of the inequality $\dim_H(HG_\alpha) \le d$ we refer to Alonso-Ruiz, P. and Freiberg, U.R (2012). We only note that it requires the original definition of Hausdorff dimension based on $\delta-$coverings. Since this is quite technical, it will be skipped here. $\square$

The most important consequence, and the reason for our great interest in this theorem is the following observation.

Corollary 4.1:

$$\dim_H(HG_\alpha) \xrightarrow{\alpha \to 0} \dim_H(SG) = \frac{\ln 3}{\ln 2}.$$

4.2 Hausdorff measure

We already mentioned that the d-dimensional Hausdorff measure of a set with Hausdorff dimension d, can be zero or even infinity. Although we may not use this fact until further work (see Alonso-Ruiz, P. and Freiberg, U.R (2012)), it will become important to determine, if the d-dimensional Hausdorff measure of HG_α is positive and finite or not. Hereby $d = d(\alpha)$ is the Hausdorff dimension of HG_α, given in Theorem 4.3. The following result answers this question.

Theorem 4.4: *Let $0 < \alpha < 1/3$ and $d = d(\alpha) = \dim_H HG_\alpha$. Then it holds that*

$$0 < \mathcal{H}^d(HG_\alpha) < \infty.$$

Before proving this, we need some previous work. Recall that F_α is defined by

$$F_\alpha = \bigcup_{i=1}^{3} G_{\alpha,i}(F_\alpha).$$

If we decompose HG_α into the (disjoint) union of F_α and $HG_\alpha \setminus F_\alpha$, we obtain obviously that

$$\mathcal{H}^d(HG_\alpha) = \mathcal{H}^d(F_\alpha) + \mathcal{H}^d(HG_\alpha \setminus F_\alpha).$$

From Theorem 4.2 we know that $0 < \mathcal{H}^d(F_\alpha) < \infty$.

Hence, we have already proven one of the inequalities stated in Theorem 4.4, since

$$\mathcal{H}^d(HG_\alpha) \ge \mathcal{H}^d(F_\alpha) > 0.$$

However, we still cannot assure that the measure remains finite. The question if it holds that $\mathcal{H}^d(HG_\alpha \setminus F_\alpha) < \infty$ is not trivial. One could think, since this set is the countable union of segments of finite 1-dimensional Lebesgue measure (and hence finite 1-dimensional Hausdorff measure), it should have "almost" finite 1-dimensional Hausdorff measure and therefore $d-$dimensional Hausdorff measure zero for any $d > 1$. But this set has infinite 1-dimensional Hausdorff measure, as the following calculation shows.

For any $m \in \mathbb{N}$ it holds that

$$\begin{aligned}\lambda^1(HG_\alpha \setminus F_\alpha) &= 3\alpha + 3^2\alpha\frac{1-\alpha}{2} + \cdots + 3^m\alpha\left(\frac{1-\alpha}{2}\right)^{m-1} \\ &= 3\alpha\sum_{k=0}^{m-1}\left(\frac{3(1-\alpha)}{2}\right)^k \xrightarrow{m\to\infty} \infty\end{aligned}$$

which implies $\mathcal{H}^1(HG_\alpha) = \infty$.

So the proof of Theorem 4.4 consists of proving that $\mathcal{H}^d(HG_\alpha \setminus F_\alpha) < \infty$. For this we have to work again with δ-coverings. Since the proof is rather technical, we refer to Alonso-Ruiz, P. and Freiberg, U.R (2012) for details.

These results make us conjecture that the $d(\alpha)$-dimensional Hausdorff measures of HG_α, $\alpha \in (0, 1/3)$ may converge to the $\frac{ln3}{ln2}$-dimensional Hausdorff measure of SG.

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