Dirichlet forms on non self-similar sets: Hanoi attractors and the Sierpiński gasket

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Lda. Patricia Alonso Ruiz

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Abstract

In this thesis we study a class of non self-similar fractals $\{K_\alpha : \alpha \in (0, 1/3)\}$, the so–called Hanoi attractors of parameter α. We investigate the geometric and analytic relationships between the Hanoi attractors and the Sierpiński gasket, which is one of the most studied self-similar fractals.

The first part of the thesis treats the problem from a geometric point of view: For each $\alpha \in (0, 1/3)$ we construct the Hanoi attractor K_α and prove that the sequence $(K_\alpha)_\alpha$ converges to the Sierpiński gasket in the Hausdorff metric as α tends to zero. Moreover, we prove convergence of the Hausdorff dimension as α tends to zero.

The second part of the thesis deals with the construction of an analysis on Hanoi attractors. To this end, we introduce an appropriate resistance form on K_α, choose a suitable Radon measure and obtain a local and regular Dirichlet form that acts on the associated L^2-space. This form defines a Laplacian on K_α, whose spectral properties we then investigate.

The study of the asymptotic behaviour of the eigenvalue counting function of this Laplacian allows us to calculate the spectral dimension of K_α, which turns out to coincide with the one of the Sierpiński gasket for all $\alpha \in (0, 1/3)$.

Zusammenfassung

Diese Arbeit behandelt eine Klasse nicht selbstähnlicher Fraktale $\{K_\alpha : \alpha \in (0, 1/3)\}$, die sogenannten *Hanoi–Attraktoren* zum Parameter α. Die analytischen und geometrischen Zusammenhänge zwischen den Hanoi–Attraktoren und dem Sierpiński–Dreieck, einem der bekanntesten selbstähnlichen Fraktale, werden untersucht.

Der erste Teil der Arbeit betrachtet das Problem von einem geometrischen Standpunkt: Für jedes $\alpha \in (0, 1/3)$ konstruieren wir den Hanoi–Attraktor K_α und beweisen, dass die Folge $(K_\alpha)_\alpha$ für $\alpha \to 0$ in der Hausdorff–Metrik gegen das Sierpiński–Dreieck konvergiert. Darüberhinaus beweisen wir auch die Konvergenz der Hausdorff Dimension für $\alpha \to 0$.

Der zweite Teil der Dissertation befasst sich mit der Konstruktion einer Analysis auf Hanoi–Attraktoren. Zu diesem Zweck konstruieren wir eine *resistance form* auf K_α und definieren ein geeignetes Radon–Maß. Dadurch erhalten wir eine lokale und reguläre Dirichletform auf dem zugehörigen L^2–Raum. Diese Form definiert einen Laplace–Operator, dessen spektrale Eigenschaften wir untersuchen.

Die Untersuchung des asymptotischen Verhaltens der Eigenwertzählfunktion des Laplace–Operators dient dazu, die spektrale Dimension von K_α bestimmen zu können. Für alle $\alpha \in (0, 1/3)$ stimmt sie mit der spektralen Dimension des Sierpiński–Dreiecks überein.

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Chapter 1

Introduction

What is a fractal? From the Latin adjective *fractus*, which means irregular or broken, the word *fractal* was coined by Mandelbrot in [35] in order to describe (primarily natural) objects whose shape could not be identified with anyone known from classical geometry. A fractal is certainly not a strictly mathematical concept, since it has not been well defined yet. Indeed, its definition is not unique and it will probably remain so, because there will always be a mathematician able to define a new "rare" object that does not fit any of the already existent definitions. A list of the properties of a "typical" fractal can be found in [11].

Nevertheless, *fractals are everywhere*, so says the title of the book by M. Barnsley [6]. Already in the 1970s, Mandelbrot described in [35] how fractal structures are present from earth reliefs to lungs or even the universe. Later on, scenarios for numerous physical phenomena like sponges, clouds or blood vessels have been modelled at best by fractals. As the name itself means, one of the principal properties of a fractal is its great irregularity, that is reflected in its lack of smoothness. This prevents the description of physical phenomena by means of classical partial differential equations like the heat-, wave-, or Schrödinger equation. In search of an analogue emerges what has been called *fractal analysis*, an area of analysis which deals with classical analytic questions where the underlying space is a fractal.

Fractal analysis has two different approaches, called *extrinsic* and *intrinsic*. The first one embeds the fractal set K in a suitable Euclidean space and applies the analytic theory of this space to its restriction to K (see [22, 41] for references). The present work will deal with the intrinsic approach, where the construction of the analysis is based on the fractal itself. Here, the fractal set K is typically *post critically finite* (it can be disconnected by removing finitely many points) and *self-similar* (made of "copies" of itself). These properties allow K to be defined as the renormalised limit of a sequence of finite sets V_n, as $n \to \infty$. Diverse methods have been developed to construct this analysis: on one hand, probability theory uses Brownian motion to reach the definition of an operator on the fractal as the infinitesimal generator of a stochastic process that arises as the limit of renormalized random walks on the finite approximations V_n (see [5, 15] for original sources). On the other hand, Jun Kigami set up in [24] a calculus by means of *Laplacians* and *Dirichlet forms*, that are defined on K as the renormalized limit of sequences of finite Laplacians and Dirichlet forms on V_n. These coincide with the known *graph Laplacian* and *graph energy* of certain graphs whose vertex set is V_n. Later on, he presented in [26] the theory of *resistance forms*, extending his earlier research to much more general settings.

This work is devoted to the detailed construction of an analysis on a class of fractals that we call *Hanoi attractors*, whose principal properties are the lack of self-similarity and its relationship to one of the most studied post critically finite self-similar fractals, the *Sierpiński gasket* (see Figure 1.1). In fact, Hanoi attractors are closely related to a particular case of so-called *deformed Sierpiński gaskets*, defined in [36].

As we already pointed out, we will perform this construction in the spirit of Kigami's approach, although we will modify his discrete approximation of the Dirichlet form on these fractals in order to get an easier and more natural form. The relationship of Hanoi attractors to the Sierpiński gasket will provide us with the possibility of comparing our results with the known ones for this specific fractal.

The origin of this investigation came from the following observation:

Let $\mathbb{R}^2$ denote the Euclidean plane, $\mathcal{A} := \{1,2,3\}$ the alphabet consisting of three symbols $1, 2$ and 3, and define the points $p_1, p_2, p_3 \in \mathbb{R}^2$ by

$$p_1 := (0,0), \qquad p_2 := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad p_3 := (1,0).$$

For each $i \in \mathcal{A}$, define the mappings

$$\begin{aligned} S_i\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \frac{x+p_i}{2}, \end{aligned} \tag{1.0.1}$$

where p_i is the fixed point of S_i for each $i \in \mathcal{A}$. Notice that p_1, p_2, p_3 are the vertices of an equilateral triangle of side length one.

The Sierpiński gasket, that we denote by K, is defined in [26, p.3] as the unique non-empty compact subset of $\mathbb{R}^2$ such that

$$K = \bigcup_{i=1}^{3} S_i(K). \tag{1.0.2}$$

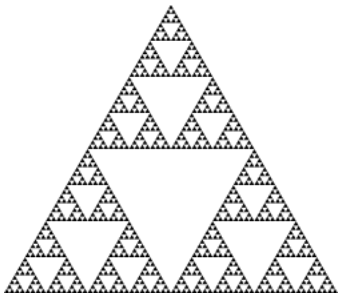


Figure 1.1: The Sierpiński gasket K.

We will see in Chapter 3 that this fractal is a post critically finite self-similar set (p.c.f. for short) and that this property allows us to approximate it by an increasing sequence of finite sets defined for each $n \in \mathbb{N}_0$ by

$$V_n := \bigcup_{w \in \mathcal{A}^n} S_w(\{p_1, p_2, p_3\}), \tag{1.0.3}$$

where for each $w = w_1 \dots w_n \in \mathcal{A}^n$ and $x \in \mathbb{R}^2$, $S_w(x) = S_{w_1} \circ \cdots \circ S_{w_n}(x)$ and $S_\emptyset := \mathrm{Id}_{\mathbb{R}^2}$ for the empty word $\emptyset$.

The sets V_n can be considered as the vertex sets of the following graphs (see Figure 1.2).

Definition 1.0.1. The *n-th approximating graph* of K is the graph Γ^n with vertex set V_n and edge set given by

$$E(\Gamma^n) = \{\{x, y\} \mid x \neq y \text{ and } \exists\, w \in \mathcal{A}^n \text{ such that } x, y \in S_w(V_0)\}.$$

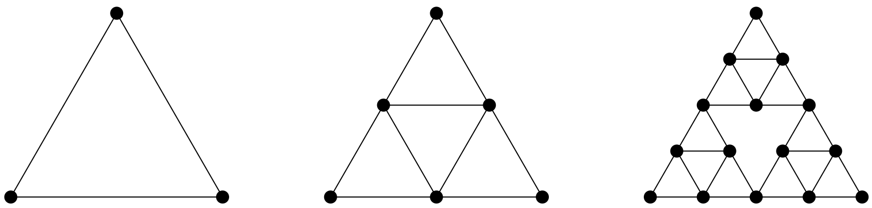


Figure 1.2: Approximating graphs Γ^0, Γ^1 and Γ^2.

In contrast to these graphs, but at the same time closely related to them, we find the so-called *Hanoi graphs*, that have their origin as mathematical representation of the *Tower of Hanoi game* (*TH game* for short), invented by the French mathematician Édouard Lucas in 1883 (see [33]) and intensely studied since then. We refer to [18, 19, 30] for a selection of results concerning these graphs.

This game consists of *three* (or in general $p \geq 3$) *vertical pegs*, named $1, 2$ and 3, and n discs numbered 1 through n according to rising diameters, $n \in \mathbb{N}$. At the beginning, these discs are situated on the first peg so that the largest disc lies at the bottom and the smallest at the top, building a tower. The goal of the game is to reconstruct the tower on one of the other pegs as Figure 1.3 shows.

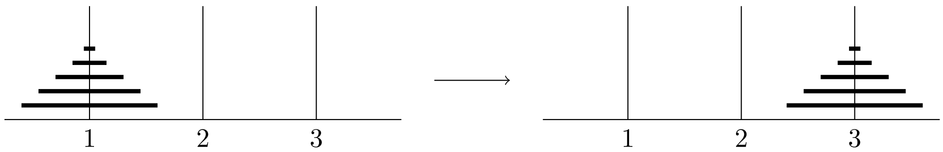


Figure 1.3: Starting state with the tower on the first peg and required final state with the tower on the third peg.

Throughout the construction one must follow two basic rules:

- Each time one and only one disc has to be moved.
- No larger disc lies on a smaller one.

This means that only one disc lying at the top of a peg will be moved each time. The states of the Tower of Hanoi for n discs and 3 pegs can be represented by words $w = w_1 \dots w_n \in \{1, 2, 3\}^n = \mathcal{A}^n$, where w_i indicates the peg on which the disc i is stacked in state w. A move is a pair of states (w, w'), where w' results from w by a legal transfer of a disc (see Figure 1.4).

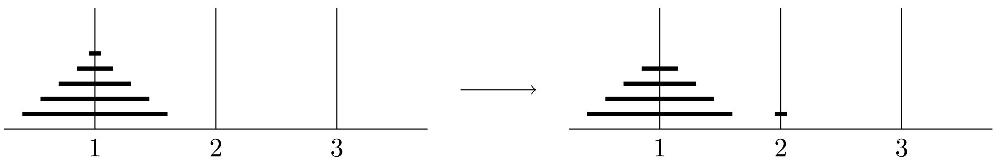


Figure 1.4: Legal move $(11111, 21111)$ of the TH game with $p = 3$, $n = 5$.

This game can be modelled by the Hanoi graph H^n, which is defined for any $n \in \mathbb{N}_0$ as the graph with vertex set $\mathcal{A}^n$ and edge set given by

$$E(H^n) = \big\{\{w, w'\} \mid (w, w') \text{ is a legal move in the TH game}\big\}.$$

The different moves done during the game build a trail in the corresponding Hanoi graph. An example is given in Figures 1.5 and 1.6.

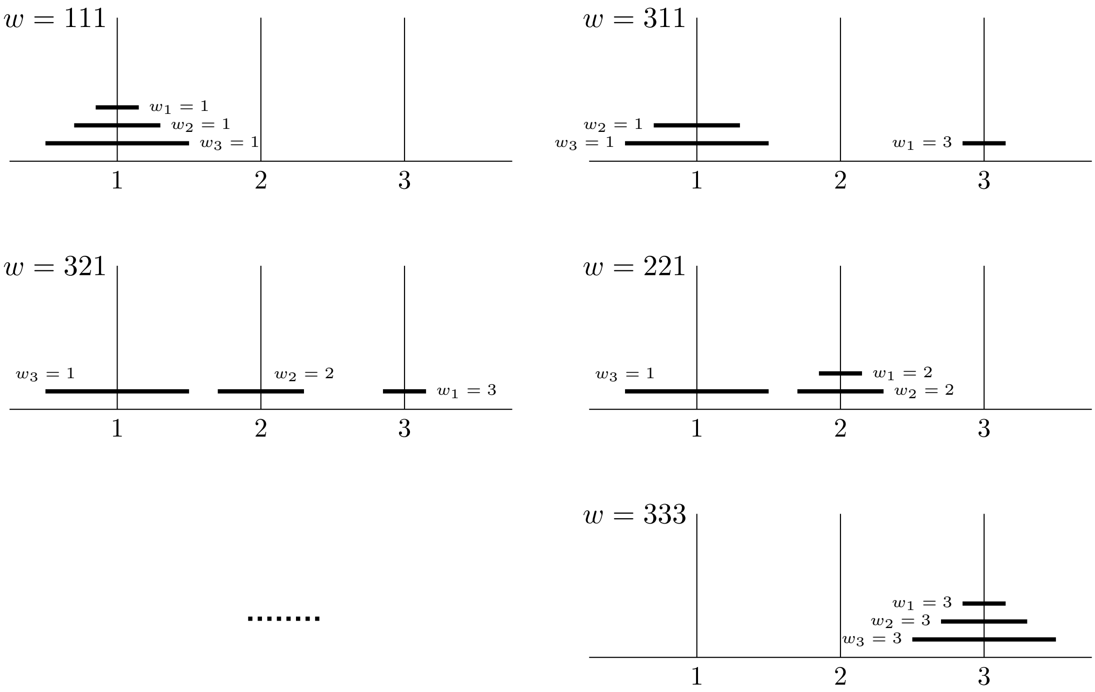


Figure 1.5: Moves in the TH game for $n = 3$.

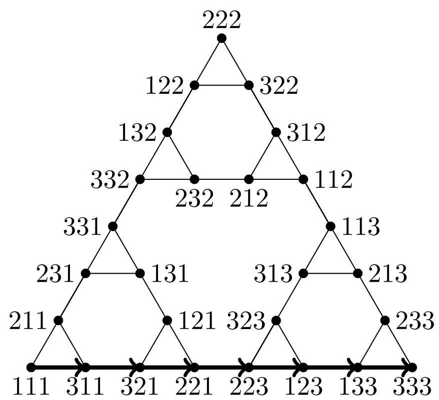


Figure 1.6: The marked edges build the trail corresponding to the moves done in Figure 1.5.

For our purposes we will label the vertices of H^n as the ones of the so-called *Sierpiński graphs* $\mathcal{S}(n,3)$ with vertex set $\mathcal{A}^{n+1}$ and edge set defined recursively as in [30] by

$$E(\mathcal{S}(n+1,3)) = \big\{\{iw, iw'\} : i \in \mathcal{A},\ \{w,w'\} \in E(\mathcal{S}(n,3))\big\} \cup \left\{\{ij\dots j, ji\dots i\} \in \binom{\mathcal{A}^{n+1}}{2}\right\}. \tag{1.0.4}$$

It was proved in [19, Lemma 2] that Sierpiński graphs are isomorphic to Hanoi graphs, thus we can use equivalently this alternative labelling (see Figure 1.7).

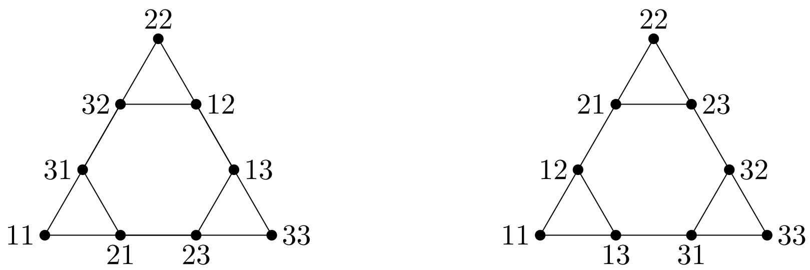


Figure 1.7: Representation of the graphs H^2 and $\mathcal{S}(2,3)$.

We want to regard Hanoi graphs as geometrical objects, treating the vertices as points in $\mathbb{R}^2$ and the edges as straight lines with a determined length. To this purpose, we define the parameter $\alpha > 0$ to be the length of the dotted lines in Figure 1.8. New similar lines appearing over the levels will correspond to lengths that depend on α.

Roughly speaking, if we let α tend to zero, any Hanoi graph H^n will end up to the $(n-1)-$th approximating graph of K, as Figure 1.8 suggests.

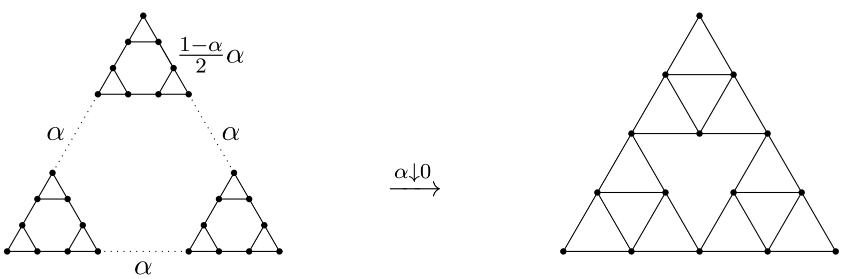


Figure 1.8: H^3 will become Γ^2 as α tends to zero.

Now the following question arrives: if we define a fractal K_α as “limit” of these geometric Hanoi graphs, could we state any convergence of the form

$$\text{“}K_\alpha \xrightarrow{\alpha\downarrow 0} K\text{”?}$$

In order to answer this, we first need to pose the problem in a proper mathematical way, which means specifying the terms “limit” and “convergence”.

Chapter 2 is devoted to the geometric construction of the *Hanoi attractor of parameter* α (see Figure 2.2) denoted by K_α, and the proof of convergence of sequences $(K_{\alpha_k})_{k\in\mathbb{N}_0}$ to K with respect to the Hausdorff metric as well as convergence of the Hausdorff dimension. The most important results of this section have already appeared in [1] by the author and Uta Freiberg, and they are stated in the following theorem.

Theorem 1.0.2. *Let K denote the Sierpiński gasket and for each $k \in \mathbb{N}_0$, let K_{α_k} be the Hanoi attractor of parameter $\alpha_k \in (0, 1/3)$. Then we have:*

(i) For any decreasing sequence $(\alpha_k)_{k\in\mathbb{N}_0}$ with $\alpha_k \downarrow 0$, it holds that

$$h(K_{\alpha_k}, K) \longrightarrow 0 \qquad \text{as } k \to \infty,$$

where h denotes the Hausdorff distance of $\mathbb{R}^2$ between K_{α_k} and K.

(ii) $\dim_{\mathrm{H}} K_{\alpha_k} = \frac{\ln 3}{\ln 2 - \ln(1-\alpha_k)} =: d$ *and* $0 < \mathcal{H}^d(K_{\alpha_k}) < \infty$. *In particular,*

$$\dim_{\mathrm{H}} K_{\alpha_k} \xrightarrow{k\to\infty} \dim_{\mathrm{H}} K.$$

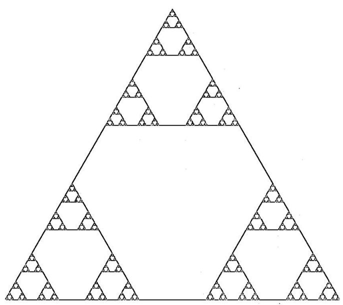


Figure 1.9: Hanoi attractor of parameter α.

In view of this, we became even more interested in the possibility of an analytic convergence of these sequences $(K_{\alpha_k})_{k\in\mathbb{N}_0}$ to K in the sense of convergence of the spectral dimension. To this purpose, Chapter 3 reviews the calculation of the spectral dimension of the Sierpiński gasket. This quantity characterizes the asymptotic behaviour of the *eigenvalue counting function* associated with the Laplacian on K, Δ_K, given by

$$N_N(x) := \#\{\lambda \text{ is a Neumann eigenvalue of } -\Delta_K,\ \lambda \leq x\}$$

respectively

$$N_D(x) := \#\{\lambda \text{ is a Dirichlet eigenvalue of } -\Delta_K,\ \lambda \leq x\},$$

where the eigenvalues are counted with multiplicity.

The *spectral dimension* of K is the number $d_S K > 0$ such that

$$\frac{2\log N_N(x)}{\log x} \asymp d_S K \asymp \frac{2\log N_D(x)}{\log x},$$

i.e. there exist constants $c_1, c_2 > 0$ and $x_0 > 0$ such that

$$c_1 x^{\frac{d_S K}{2}} \leq N_N(x) \leq c_2 x^{\frac{d_S K}{2}} \qquad \forall\, x \geq x_0$$

and the same holds for $N_D(x)$.

Chapter 4 is the core of this work. Here we follow a similar scheme as in Chapter 3: we define a Laplacian on K_α by means of a suitable Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ and afterwards study the asymptotic behaviour of its associated eigenvalue counting function in order to calculate the spectral dimension of the Hanoi attractor K_α. The most important results of this chapter are Theorem 4.2.1, that states

Theorem 1.0.3. *$(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ is a resistance form on K_α,*

and Theorem 4.4.3, where we prove the following estimates for the asymptotic behaviour of the eigenvalue counting function of the Laplacian subject to Neumann (resp. Dirichlet) boundary conditions.

Theorem 1.0.4. *There exist constants $C_{\alpha,1}, C_{\alpha,\beta,1}, C_{\alpha,2}, C_{\alpha,\beta,2} > 0$ depending on α and β, and $x_0 > 0$ such that*

$$C_{\alpha,1} x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,1} x^{1/2} \leq N_D(x) \leq N_N(x) \leq C_{\alpha,2} x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,2} x^{1/2}$$

for all $x \geq x_0$.

The constant β will appear in the construction of a Radon measure on K_α. This theorem allows us to determine the spectral dimension of K_α, which turns out to be $\frac{\ln 9}{\ln 5}$ for all $\alpha \in (0, 1/3)$. Since it coincides with $d_S K$ for all $\alpha \in (0, 1/3)$, we get analytical convergence.

The results referring to the construction of the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ have already been accepted for publication in [2], while the results concerning the asymptotics of the eigenvalue counting function and the spectral dimension will appear in the forthcoming paper [3].

Finally, the last chapter analyses the possible consequences of this result and discusses some ideas and problems for further work.

Chapter 2

Geometric approximation

Recall the parameter α introduced in the previous chapter. For simplicity, we will restrict ourselves to the geometric realisation of the Hanoi graph H^n that fits into an equilateral triangle of side length one and we will consider $\alpha \in (0, 1/3)$.

The aim of this chapter is to define compact sets K_α with a special property: for any monotone decreasing sequence $(\alpha_k)_{k\in\mathbb{N}_0}$ such that $\alpha_k \in (0, 1/3)$ and $\lim_{k\to\infty} \alpha_k = 0$, the sequence $(K_{\alpha_k})_{k\in\mathbb{N}_0}$ converges to the Sierpiński gasket K in the Hausdorff metric. For simplicity, any such sequence $(K_{\alpha_k})_{k\in\mathbb{N}_0}$ will be denoted just by (K_α).

2.1 Hanoi attractors

Let $\mathscr{H}(\mathbb{R}^2)$ denote the space of non-empty compact subsets of $\mathbb{R}^2$ and denote by $|\cdot|$ the Euclidean metric. Given a point $x \in \mathbb{R}^2$ and a compact set $A \in \mathscr{H}(\mathbb{R}^2)$, we define the *distance between the point x and the set A* by

$$d(x, A) := \min\{|x - y| \mid y \in A\}.$$

The minimum is indeed reached since A is compact. Given two non-empty compact sets $A, B \in \mathscr{H}(\mathbb{R}^2)$, we define the *Hausdorff distance between A and B* by

$$h(A, B) := \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon \text{ and } B \subseteq A_\varepsilon\},$$

where $A_\varepsilon := \left\{x \in \mathbb{R}^2 \mid d(x, A) \leq \varepsilon\right\}$ is called the *$\varepsilon-$neighbourhood* of A.

The mapping

$$\begin{aligned} h\colon \mathscr{H}(\mathbb{R}^2) \times \mathscr{H}(\mathbb{R}^2) &\longrightarrow [0, +\infty) \\ (A, B) &\longmapsto h(A, B) \end{aligned}$$

defines a metric on $\mathscr{H}(\mathbb{R}^2)$ and $(\mathscr{H}(\mathbb{R}^2), h)$ is a complete metric space (see [6, Theorem 7.1] for a proof).

A map $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ is called a *contraction* of ratio $c \in (0,1)$ if and only if

$$|S(x) - S(y)| \leq c\,|x-y| \qquad \forall\, x, y \in \mathbb{R}^2.$$

Further, if equality holds, i.e. $|S(x) - S(y)| = c\,|x-y|$ for all $x, y \in \mathbb{R}^2$, then we say that the map S is a *contractive similitude* of ratio c. In this case, we can find a unitary $2 \times 2-$matrix U and a point $x_0 \in \mathbb{R}^2$ such that

$$S(x) = c\,Ux + x_0$$

for each $x \in \mathbb{R}^2$. Note that the image of a compact set under a contraction is again a compact set.

A metric space such as $(\mathbb{R}^2, |\cdot|)$ together with a finite family of contractions $\{S_i\colon \mathbb{R}^2 \to \mathbb{R}^2\}_{i=1}^N$ is called in [6, Definition 7.1] an *iterated function system* (*IFS* for short) and it is denoted by $\{\mathbb{R}^2;\, S_i, i = 1, \ldots, N\}$. Each IFS determines a unique non-empty compact set by considering the mapping

$$\begin{aligned} \mathscr{S}: \mathscr{H}(\mathbb{R}^2) &\longrightarrow \mathscr{H}(\mathbb{R}^2) \\ A \quad &\longmapsto \bigcup_{i=1}^{N} S_i(A). \end{aligned}$$

By [20, Section 5.3], $\mathscr{S}$ is a contraction in the complete metric space $(\mathscr{H}(\mathbb{R}^2), h)$, hence it has a unique fixed point $F \in \mathscr{H}(\mathbb{R}^2)$ satisfying

$$F = \bigcup_{i=1}^{N} S_i(F).$$

This set is called the *attractor* of the IFS $\{\mathbb{R}^2;\, S_i, i = 1, \ldots, N\}$. If the mappings $\{S_i\}_{i=1}^N$ are contractive similitudes, then F is said to be *self-similar.*

Let us now consider the points in $\mathbb{R}^2$

$$p_1 := (0,0), \qquad p_2 := \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad p_3 := (1,0),$$
$$p_4 := \frac{p_2 + p_3}{2}, \qquad p_5 := \frac{p_1 + p_3}{2}, \qquad p_6 := \frac{p_1 + p_2}{2}$$

and define for any fixed $\alpha \in (0, 1/3)$ the mappings

$$\begin{aligned} G_{\alpha,i}\colon \mathbb{R}^2 &\longrightarrow \quad \mathbb{R}^2 \\ x &\longmapsto A_i(x - p_i) + p_i, \qquad\qquad i = 1, \ldots, 6, \end{aligned}$$

where $A_1 = A_2 = A_3 = \frac{1-\alpha}{2}\,\mathrm{I}_2$ and

$$A_4 = \frac{\alpha}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}, \quad A_5 = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_6 = \frac{\alpha}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$

Since $\alpha \in (0, 1/3)$, we have that $0 < \alpha < \frac{1-\alpha}{2} < 1$, thus $G_{\alpha,i}$ is a contraction in $\mathbb{R}^2$ for all $i = 1, \ldots, 6$.

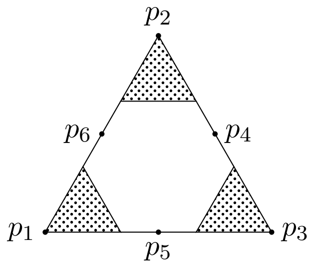


Figure 2.1: $\bigcup_{i=1}^{6} G_{\alpha,i}(\blacktriangle)$, where $\blacktriangle$ denotes the equilateral triangle of side length 1.

If we consider the IFS $\{\mathbb{R}^2; G_{\alpha,i}, i = 1, \ldots, 6\}$, we know that there exists a unique $K_\alpha \in \mathscr{H}(\mathbb{R}^2)$ such that

$$K_\alpha = \bigcup_{i=1}^{6} G_{\alpha,i}(K_\alpha).$$

We call this set the *Hanoi attractor of parameter* α. The name *Hanoi* comes from the fact that it almost looks like the drawing of the graph of the TH game if we had three pegs and “infinitely many discs”.

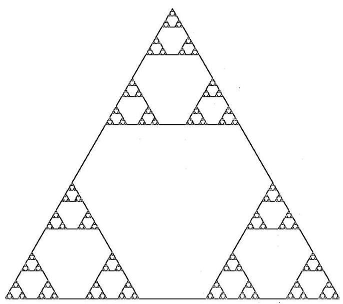


Figure 2.2: Hanoi attractor of parameter α.

Observe that the set K_α is not self-similar because $G_{\alpha,4}, G_{\alpha,5}$ and $G_{\alpha,6}$ are not similitudes. This lack of self-similarity will lead to difficulties in later constructions and proofs.

For the rest of this section, let $\alpha \in (0, 1/3)$ be fixed and let $\mathcal{A}$ denote the alphabet on the three symbols $1, 2$ and 3. Moreover, we define $G_{\alpha,\varnothing} := \mathrm{id}_{\mathbb{R}^2}$ for the empty word $\varnothing$ and write

$$G_{\alpha,w}(x) := G_{\alpha,w_1} \circ G_{\alpha,w_2} \circ \cdots \circ G_{\alpha,w_n}(x)$$

for any word $w = w_1 \cdots w_n \in \mathcal{A}^n$ and $x \in \mathbb{R}^2$.

For each $n \in \mathbb{N}_0$, we define the set $W_{\alpha,n}$ by

$$W_{\alpha,n} := \bigcup_{w \in \mathcal{A}^n} G_{\alpha,w}(\{p_1, p_2, p_3\}).$$

For each $i \in \mathcal{A}$, let us denote by e_i the line segment from $G_{\alpha,j}(p_k)$ to $G_{\alpha,k}(p_j)$, $\{i, j, k\} = \mathcal{A}$ without its endpoints (see Figure 2.3).

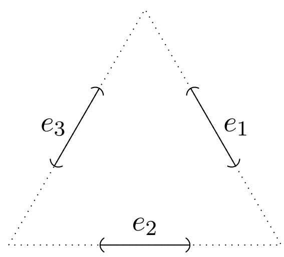


Figure 2.3: The set $J_{\alpha,1}$.

For each $n \in \mathbb{N}_0$, we define the set $J_{\alpha,n}$ by $J_{\alpha,0} := \emptyset$ and

$$J_{\alpha,n} := \bigcup_{m=0}^{n-1} \bigcup_{w \in \mathcal{A}^m} G_{\alpha,w}\left(\bigcup_{i=1}^{3} e_i\right).$$

Note that the sequences $(W_{\alpha,n})_{n \in \mathbb{N}_0}$ and $(J_{\alpha,n})_{n \in \mathbb{N}_0}$ are monotonically increasing, so we can define

$$W_{\alpha,*} := \sup_n W_{\alpha,n} = \bigcup_{n \in \mathbb{N}_0} W_{\alpha,n} \tag{2.1.1}$$

and

$$J_{\alpha} := \sup_n J_{\alpha,n} = \bigcup_{n \in \mathbb{N}_0} J_{\alpha,n}. \tag{2.1.2}$$

Finally we define for each $n \in \mathbb{N}_0$ the sets

$$V_{\alpha,n} := W_{\alpha,n} \cup J_{\alpha,n}$$

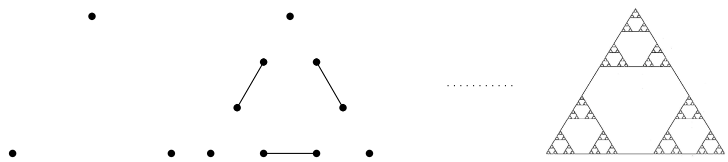


Figure 2.4: $V_{\alpha,0}$, $V_{\alpha,1}$ and K_α.

and since the sequence $(V_{\alpha,n})_{n\in\mathbb{N}_0}$ is also monotonically increasing, we can define

$$V_{\alpha,*} := \sup_n V_{\alpha,n} = \bigcup_{n\in\mathbb{N}_0} V_{\alpha,n}.$$

The following results present some geometric properties of K_α that will be important at a later point.

Lemma 2.1.1. *Let F_α be the unique non-empty compact set such that*

$$F_\alpha = \bigcup_{i=1}^{3} G_{\alpha,i}(F_\alpha). \tag{2.1.3}$$

The Hanoi attractor K_α can be written as the disjoint union

$$K_\alpha = F_\alpha \,\dot{\cup}\, J_\alpha.$$

Proof. (1) $F_\alpha \subseteq K_\alpha$.

Define the mapping

$$\begin{aligned} T\colon \mathscr{H}(\mathbb{R}^2) &\longrightarrow \mathscr{H}(\mathbb{R}^2)\\ B &\longmapsto \bigcup_{i=1}^{3} G_{\alpha,i}(B), \end{aligned}$$

whose unique fixed point is F_α.

We know from [10, Theorem 9.1] that for any starting set $B_0 \in \mathscr{H}(\mathbb{R}^2)$ such that $G_{\alpha,i}(B_0) \subseteq B_0$ for all $i = 1, 2, 3$, the sequence $(B_n)_{n\in\mathbb{N}_0}$ defined by

$$B_n := T^{\circ n}(B_0), \qquad n \in \mathbb{N}_0,$$

converges to F_α in the Hausdorff distance h and

$$F_\alpha = \bigcap_{n\in\mathbb{N}_0} B_n.$$

Further, define the mapping

$$\begin{aligned} \widetilde{T}\colon \mathscr{H}(\mathbb{R}^2) &\longrightarrow \mathscr{H}(\mathbb{R}^2) \\ B \quad &\longmapsto \bigcup_{i=1}^{6} G_{\alpha,i}(B), \end{aligned} \tag{2.1.4}$$

whose fixed point in $\mathscr{H}(\mathbb{R}^2)$ is K_α.

Again by [10, Theorem 9.1], given a starting set $C_0 \in \mathscr{H}(\mathbb{R}^2)$ such that $G_{\alpha,i}(C_0) \subseteq C_0$ for all $i = 1, \dots, 6$, the sequence $(C_n)_{n\in\mathbb{N}_0}$ defined by

$$C_n := \widetilde{T}^{\circ n}(C_0), \qquad n \in \mathbb{N}_0,$$

converges to K_α in the Hausdorff distance h and

$$K_\alpha = \bigcap_{n\in\mathbb{N}_0} C_n. \tag{2.1.5}$$

Now, denote by ▲ the (filled) triangle with vertices p_1, p_2, p_3 and set $B_0 =$ ▲ $= C_0$. Then,

$$F_\alpha = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} T^{\circ n}(B_0) \subseteq \bigcap_{n=1}^{\infty} \widetilde{T}^{\circ n}(B_0) = K_\alpha,$$

as required.

(2) $K_\alpha \setminus F_\alpha = J_\alpha$.

Consider $x \in K_\alpha$, $x \notin F_\alpha$. By definition of F_α, there exists a number $n \in \mathbb{N}$ and a word $w \in \{1, \dots, 6\}^n$ with at least one letter in $\{4, 5, 6\}$, such that

$$x \in G_{\alpha,w}(K_\alpha). \tag{2.1.6}$$

Let w_k, $k \leq n$, be the first letter of w such that $w_k \in \{4, 5, 6\}$ and define $\overline{w} := w_1 w_2 \dots w_{k-1} \in \mathcal{A}^{k-1}$. By definition, $G_{\alpha,w_{k+1}\dots w_n}(K_\alpha) \subseteq K_\alpha$, so we have that

$$x \in G_{\alpha,\overline{w}w_k}(K_\alpha), \tag{2.1.7}$$

and since $G_{\alpha,w_k}(K_\alpha) \subseteq J_{\alpha,1}$, it follows from (2.1.6) and (2.1.7) that

$$x \in G_{\alpha,\overline{w}}(J_{\alpha,1}) \subseteq J_{\alpha,k} \subseteq J_\alpha,$$

and therefore $K_\alpha \setminus F_\alpha \subseteq J_\alpha$.

It remains to prove that $J_\alpha \subseteq K_\alpha \setminus F_\alpha$.

Consider the mapping $\widetilde{T}$ defined in (2.1.4) and denote by △ the (non filled) triangle of vertices p_1, p_2 and p_3. Further, set $B_0 :=$ △ and $C_0 :=$ ▲. It is clear that $J_{\alpha,1} \subseteq B_0$ and $B_0 \subseteq \widetilde{T}(B_0)$, thus $B_0 \subseteq \widetilde{T}^{\circ n}(B_0)$ for all $n \in \mathbb{N}$.

Then we have that

$$J_{\alpha,1} \subseteq \widetilde{T}^{\circ n}(B_0) \subseteq \widetilde{T}^{\circ n}(C_0) =: C_n$$

for all $n \in \mathbb{N}$, which implies that

$$J_{\alpha,1} \subseteq \bigcap_{n\in\mathbb{N}_0} C_n \overset{(2.1.5)}{=} K_\alpha, \tag{2.1.8}$$

and finally we get that

$$J_\alpha = \bigcup_{n\in\mathbb{N}_0} \bigcup_{w\in\mathcal{A}^n} G_{\alpha,w}(J_{\alpha,1}) = \bigcup_{n\in\mathbb{N}_0} \widetilde{T}^{\circ n}(J_{\alpha,1}) \overset{(2.1.8)}{\subseteq} \bigcup_{n\in\mathbb{N}_0} \widetilde{T}^{\circ n}(K_\alpha) = K_\alpha.$$

Thus $J_\alpha \subseteq K_\alpha$ and it only remains to prove that $F_\alpha \cap J_\alpha = \emptyset$.

By contradiction, suppose that there exists $x \in F_\alpha \cap J_\alpha$. By definition of J_α and $W_{\alpha,*}$, we know that $\overline{W}_{\alpha,*} = F_\alpha$ (see [26, Lemma 1.3.11] for a proof) and

$$W_{\alpha,*} \cap J_\alpha = \emptyset, \tag{2.1.9}$$

which implies that x has to belong to the boundary of $W_{\alpha,*}$ and therefore, for any ball of radius $\delta > 0$, $B(x,\delta)$ we have that

$$B(x,\delta) \cap W_{\alpha,*} \neq \emptyset. \tag{2.1.10}$$

Since J_α is the countable union of open sets, it is itself an open set, so if $x \in J_\alpha$, then there exists $\delta' > 0$ such that $x \in B(x,\delta') \subseteq J_\alpha$. This together with (2.1.10) leads to $\emptyset \neq B(x,\delta') \cap W_{\alpha,*} \subseteq J_\alpha \cap W_{\alpha,*}$ contradicting (2.1.9). □

The next proposition shows that the sequence of sets $(V_{\alpha,n})_{n\in\mathbb{N}_0}$ approximates K_α in the Euclidean norm.

Lemma 2.1.2. *The set $V_{\alpha,*}$ is dense in K_α with respect to the Euclidean norm.*

Proof. Note that $V_{\alpha,*}$ may be decomposed as follows

$$V_{\alpha,*} = W_{\alpha,*} \dot{\cup} J_\alpha,$$

where $W_{\alpha,*}$ and J_α were defined in (2.1.1) and (2.1.2).

Since F_α is a self-similar set, we know from [26, Lemma 1.3.11] that $W_{\alpha,*}$ is dense in F_α in the Euclidean norm, i.e.

$$\overline{W}_{\alpha,*} = F_\alpha.$$

Moreover, we know by Lemma 2.1.1 that $K_\alpha = F_\alpha \dot{\cup} J_\alpha$, which implies

$$K_\alpha = F_\alpha \dot{\cup} J_\alpha = \overline{W}_{\alpha,*} \dot{\cup} J_\alpha \subseteq \overline{W}_{\alpha,*} \cup \overline{J}_{\alpha,*} = \overline{V}_{\alpha,*}.$$

On the other hand, K_α is a compact subset of $\mathbb{R}^2$, hence closed with respect to the Euclidean metric and it follows that

$$\overline{V}_{\alpha,*} = \overline{W_{\alpha,*} \cup J_\alpha} \subseteq \overline{K}_\alpha = K_\alpha,$$

as we wanted to prove. □

2.2 Geometric convergence to the Sierpiński gasket

Now that we have defined the Hanoi attractor of parameter $\alpha \in (0, 1/3)$, we consider sequences of Hanoi attractors (K_α), and analyse their geometric behaviour as α tends to zero. The results appearing in this section have already been published in a paper by Uta Freiberg and myself (see [1]).

Recall the definition of the Sierpiński gasket K in (1.0.2) and the approximating sets $V_n := \bigcup_{w \in \mathcal{A}^n} S_w(V_0)$ defined in (1.0.3) for each $n \in \mathbb{N}_0$. We know from [26, Lemma 1.3.11] that the set

$$V_* := \bigcup_{n \in \mathbb{N}_0} V_n \tag{2.2.1}$$

is dense in K with respect to the Euclidean norm, thus the sets V_n approximate K in the Hausdorff metric.

2.2.1 Convergence in the Hausdorff metric

This paragraph is devoted to the proof of the following statement:

Theorem 2.2.1. *It holds that*

$$h(K_\alpha, K) \xrightarrow{\alpha \downarrow 0} 0.$$

In order to prove this, we need some preliminary results.

Lemma 2.2.2. *Let V_* be the set defined in* (2.2.1)*, then we have that*

$$h(V_*, W_{\alpha,*}) \leq \alpha$$

for all $\alpha \in (0, 1/3)$.

Proof. We use the definition of the Hausdorff metric

$$h(V_*, W_{\alpha,*}) = \inf \{\varepsilon > 0 \mid V_* \subseteq (W_{\alpha,*})_\varepsilon \text{ and } W_{\alpha,*} \subseteq (V_*)_\varepsilon\}.$$

(1) $V_* \subseteq (W_{\alpha,*})_\alpha$: we prove by complete induction over m that

$$V_m \subseteq (W_{\alpha,*})_\alpha \qquad \forall\, m \in \mathbb{N}_0.$$

Case $m = 0$: there is nothing to prove because $V_0 = W_{\alpha,0}$.

Assume that $V_m \subseteq (W_{\alpha,*})_\alpha$ holds up to some $m \in \mathbb{N}_0$ and let $x \in V_{m+1} \setminus V_m$. Then, there exist a point $\overline{x} \in V_m$ and a letter $k \in \mathcal{A}$ such that

$$x = S_k(\overline{x}). \tag{2.2.2}$$

Since $\overline{x} \in V_m$, we know from the induction hypotheses that there exists $\overline{y} \in W_{\alpha,*}$ such that

$$|\overline{x} - \overline{y}| \le \alpha. \tag{2.2.3}$$

Consider the point $y := G_{\alpha,k}(\overline{y}) \in W_{\alpha,*}$, where $k \in \mathcal{A}$ is the same as in (2.2.2). Then we get that

$$\begin{aligned} |x - y| &= |S_k(\overline{x}) - G_{\alpha,k}(\overline{y})| = \left| \frac{1}{2}\overline{x} + \frac{1}{2}p_k - \frac{1-\alpha}{2}\overline{y} - \frac{1+\alpha}{2}p_k \right| \\ &= \frac{1}{2}\,|\overline{x} - \overline{y} + \alpha\overline{y} - \alpha p_k| \le \frac{1}{2}\,|\overline{x} - \overline{y}| + \frac{\alpha}{2}\,|\overline{y} - p_k| \\ &\overset{(2.2.3)}{\le} \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha, \end{aligned}$$

and therefore $x \in (W_{\alpha,*})_\alpha$.

Thus we have proved that for every $m \in \mathbb{N}_0$ and for any $x \in V_m$, there exists $y \in W_{\alpha,*}$ such that $|x - y| \le \alpha$, i.e.

$$V_m \subseteq (W_{\alpha,*})_\alpha \qquad \forall\, m \in \mathbb{N}_0.$$

Since $V_* = \bigcup_{m\in\mathbb{N}_0} V_m$, we get

$$V_* \subseteq (W_{\alpha,*})_\alpha,$$

and this holds for all $\alpha \in (0, 1/3)$.

(2) It would remain to prove the inclusion $W_{\alpha,*} \subseteq (V_*)_\alpha$ for all $\alpha \in (0, 1/3)$. The proof of this is analogous to the latter one by simply changing the roles of V_* and $W_{\alpha,*}$. Thus

$$W_{\alpha,m} \subseteq (V_*)_\alpha \qquad \forall\, m \in \mathbb{N}_0,$$

and therefore

$$h(V_*, W_{\alpha,*}) \le \alpha,$$

as we wanted to prove. □

Note that the bound $h(V_*, W_{\alpha,*}) \leq \alpha$ is quite rough, since one could obtain sharper estimates. However, this bound is sufficient for our purposes, as the following corollary shows.

Corollary 2.2.3.

$$h(V_*, W_{\alpha,*}) \xrightarrow{\alpha\downarrow 0} 0.$$

Lemma 2.2.4. *Let F_α be the set defined in Proposition 2.1.1. It holds that*

$$h(K_\alpha, F_\alpha) \xrightarrow{\alpha\downarrow 0} 0.$$

Proof. In the proof of Lemma 2.1.1, we showed that $F_\alpha \subseteq K_\alpha \; \forall\, \alpha \in (0, 1/3)$, which implies directly that

$$F_\alpha \subseteq (K_\alpha)_\varepsilon \qquad \forall\, \varepsilon > 0.$$

Now we prove that

$$K_\alpha \subseteq (F_\alpha)_{\frac{\alpha}{2}} \qquad \forall\, \alpha \in (0, 1/3). \tag{2.2.4}$$

Let $x \in K_\alpha$. If $x \in F_\alpha$, then we are done. So let us assume, $x \in J_\alpha$. Then, there exists $n \in \mathbb{N}$ and a word $w = w_1 w_2 \dots w_n \in \{1, \dots, 6\}^n$, with at least one letter in $\{4, 5, 6\}$ (remember $x \notin F_\alpha$) such that

$$x = G_{\alpha,w}(K_\alpha).$$

Consider w_k, $k \leq n$ the first letter of w such that $w_k \in \{4, 5, 6\}$ and define $\overline{w} := w_1 w_2 \dots w_{k-1} \in \mathcal{A}^{k-1}$. As in the proof of Lemma 2.1.1,

$$x \in G_{\alpha,\overline{w}w_k}(K_\alpha),$$

so there exists a point $z \in G_{\alpha,w_k}(K_\alpha)$ such that $x = G_{\alpha,\overline{w}}(z)$. By construction (see e.g. Figure 2.5) we can find a point $y \in \{G_{\alpha,i}(p_j), G_{\alpha,j}(p_i)\}$ $i, j \in \mathcal{A}$, $i + j + w_k = 9$ such that

$$|z - y| \leq \frac{\alpha}{2}.$$

Define $\overline{y} := G_{\alpha,\overline{w}}(y) \in F_\alpha$. Since $G_{\alpha,1}, G_{\alpha,2}$ and $G_{\alpha,3}$ are similitudes of ratio $\frac{1-\alpha}{2}$, $G_{\alpha,\overline{w}}$ is a similitude of ratio $\left(\frac{1-\alpha}{2}\right)^{k-1}$ and therefore

$$|G_{\alpha,\overline{w}}(z) - G_{\alpha,\overline{w}}(y)| = \left(\frac{1-\alpha}{2}\right)^{k-1} |z - y|\,.$$

Thus,

$$|x - \overline{y}| = |G_{\alpha,\overline{w}}(z) - G_{\alpha,\overline{w}}(y)| = \left(\frac{1-\alpha}{2}\right)^{k-1} |z - y| \;\; \leq \left(\frac{1-\alpha}{2}\right)^{k-1} \cdot \frac{\alpha}{2} \leq \frac{\alpha}{2},$$

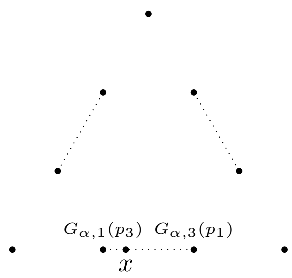


Figure 2.5: For any $x \in G_{\alpha,5}(K_\alpha)$ it holds that $|x - G_{\alpha,1}(p_3)| \leq \frac{\alpha}{2}$ or $|x - G_{\alpha,3}(p_1)| \leq \frac{\alpha}{2}$.

and (2.2.4) holds.

This finally implies

$$h(K_\alpha, F_\alpha) \leq \frac{\alpha}{2} \xrightarrow{\alpha \downarrow 0} 0,$$

as we wanted to prove. □

Now we are ready to prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Since h is a metric, using the triangle inequality we obtain

$$h(K_\alpha, K) \leq h(K_\alpha, F_\alpha) + h(F_\alpha, K). \tag{2.2.5}$$

Moreover, $W_{\alpha,*}$ and V_* are dense in F_α and K respectively, hence

$$h(F_\alpha, K) \leq \underbrace{h(F_\alpha, W_{\alpha,*})}_{=0} + h(W_{\alpha,*}, V_*) + \underbrace{h(V_*, K)}_{=0} = h(W_{\alpha,*}, V_*). \tag{2.2.6}$$

Finally, applying Lemma 2.2.4 and Lemma 2.2.2, it follows from (2.2.5) and (2.2.6) that

$$h(K_\alpha, K) \leq h(K_\alpha, F_\alpha) + h(W_{\alpha,*}, V_*) \xrightarrow{\alpha \downarrow 0} 0,$$

as we wanted to prove. □

2.2.2 Convergence of Hausdorff dimension

In this paragraph we centre our interest in the Hausdorff dimension of the set K_α. In view of Theorem 2.2.1, one expects that the Hausdorff dimension of K_α converges to the Hausdorff dimension of K as α tends to zero. We will show that this is true.

Let us recall some important definitions concerning Hausdorff dimension. Hereby we refer to [10, Chapter 2].

Definition 2.2.5. Let A be a non-empty subset of the $n-$dimensional Euclidean space $\mathbb{R}^n$.

(1) The *diameter* of A is defined as

$$|A| := \sup_{x,y\in A} \{|x-y|\}.$$

(2) A countable collection of sets $\{U_i\}_{i=1}^{\infty}$ is called a $\delta-$*covering* of A if

$$A \subseteq \bigcup_{i=1}^{\infty} U_i \qquad \text{and} \qquad 0 \leq |U_i| \leq \delta.$$

(3) Let $s \geq 0$. For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s \;\mid\; \{U_i\}_{i=1}^{\infty} \text{ is a } \delta-\text{covering of } A \right\}.$$

(4) The limit

$$\mathcal{H}^s(A) = \lim_{\delta\downarrow 0} \mathcal{H}^s_\delta(A)$$

is called the *s-dimensional Hausdorff measure* of A. This limit exists as an element of $[0, +\infty]$ because the sequence $(\mathcal{H}^s_\delta(A))_{\delta>0}$ is monotone and non-decreasing for $\delta \downarrow 0$.

(5) If we consider $\mathcal{H}^s(A)$ as a function of s, there exist a critical value of s where the function jumps down from ∞ to 0 (see [10, pg.31]). This critical value is called the *Hausdorff dimension* of A and it is denoted by $\dim_{\mathrm{H}} A$. It holds that

$$\dim_{\mathrm{H}} A = \inf \{s \geq 0 \mid \mathcal{H}^s(A) = 0\} = \sup \{s \geq 0 \mid \mathcal{H}^s(A) = \infty\}. \tag{2.2.7}$$

Observe that for $s = \dim_{\mathrm{H}} A$, $\mathcal{H}^s(A)$ may be zero, infinite, or some positive finite number.

In the previous section we defined contractions and contractive similitudes. A finite family of contractive similitudes $\left\{S_i \colon \mathbb{R}^2 \to \mathbb{R}^2\right\}_{i=1}^N$ is said to satisfy the *open set condition* (OSC) if there exists a non-empty bounded open set $V \subseteq \mathbb{R}^2$ such that

$$\bigcup_{i=1}^{N} S_i(V) \subseteq V \qquad \text{and} \qquad S_i(V) \cap S_j(V) = \emptyset \quad \forall\, i \neq j.$$

Theorem 2.2.6. *Provided that the family* $\{S_i\}_{i=1}^N$ *with contraction ratios* $c_1, \ldots, c_N \in (0,1)$ *satisfies the (OSC) and* F *is the unique non-empty compact set such that*

$$F = \bigcup_{i=1}^N S_i(F),$$

then $\dim_{\mathrm{H}} F = s$*, where* $s > 0$ *is the unique solution of the equation*

$$\sum_{i=1}^N c_i^s = 1. \tag{2.2.8}$$

Moreover, for this value of s*,* $0 < \mathcal{H}^s(F) < \infty$*.*

Proof. See [20, Section 5.3] for the original proof or [10, Theorem 9.3]. □

Our next goal is to determine the Hausdorff dimension of K_α, but since K_α is not self-similar, we can not apply the formula in (2.2.8) directly to calculate it. To solve this problem, we will use the decomposition proved in Proposition 2.1.1.

Theorem 2.2.7. *Let* $\alpha \in (0, 1/3)$*. The Hausdorff dimension of the Hanoi attractor* K_α *is given by*

$$\dim_{\mathrm{H}} K_\alpha = \frac{\ln(3)}{\ln(2) - \ln(1-\alpha)}.$$

Notice that this result justifies our condition $\alpha \in (0, 1/3)$ since if $\alpha \geq 1/3$, then $\dim_H K_\alpha = 1$.

Proof. By Proposition 2.1.1, $K_\alpha = F_\alpha \dot{\cup} J_\alpha$, and from the countable stability of the Hausdorff dimension, we know that

$$\dim_{\mathrm{H}} K_\alpha = \sup\{\dim_{\mathrm{H}} F_\alpha, \dim_{\mathrm{H}} J_\alpha\}.$$

It is easy to see that F_α satisfies the open set condition (take just V to be the inner of the equilateral triangle with vertices p_1, p_2 and p_3), so we can apply Theorem 2.2.6 to calculate $\dim_{\mathrm{H}} F_\alpha$. This is the unique number $s > 0$ such that

$$\sum_{i=1}^3 \left(\frac{1-\alpha}{2}\right)^s = 1.$$

The solution of this equation is

$$s = \frac{\ln 3}{\ln 2 - \ln(1-\alpha)} = \dim_H F_\alpha.$$

On the other hand, the set J_α is a countable union of segments, i.e. of sets with Hausdorff dimension 1. By countable stability again, we obtain

$$\dim_H J_\alpha = \sup_n \{\dim_H J_{\alpha,n}\} = 1.$$

Thus we get that

$$\dim_{\mathrm{H}} K_\alpha = \max\left\{\frac{\ln(3)}{\ln(2) - \ln(1-\alpha)}, 1\right\}.$$

Since $\alpha \in (0, 1/3)$, it follows that $\frac{\ln 3}{\ln 2 - \ln(1-\alpha)} > 1$ and therefore

$$\dim_{\mathrm{H}} K_\alpha = \frac{\ln 3}{\ln 2 - \ln(1-\alpha)},$$

as we wanted to prove. □

The most important consequence, and the reason for our great interest in this theorem is the following observation.

Corollary 2.2.8.

$$\dim_{\mathrm{H}} K_\alpha \xrightarrow{\alpha \downarrow 0} \frac{\ln 3}{\ln 2} = \dim_{\mathrm{H}} K.$$

2.2.3 Hausdorff measure

Remember that the s-dimensional Hausdorff measure of a Borel set with Hausdorff dimension s can be zero or even infinity. Although we may not use this fact until Chapter 4, it is important to determine if the d_α-dimensional Hausdorff measure of K_α is positive and finite or not. Hereby d_α denotes the Hausdorff dimension of K_α given in Theorem 2.2.7. The following result answers this question.

Theorem 2.2.9. *For all* $\alpha \in (0, 1/3)$,

$$0 < \mathcal{H}^{d_\alpha}(K_\alpha) < \infty.$$

Proof. We apply once more Proposition 2.1.1 and write K_α as the disjoint union $F_\alpha \dot\cup J_\alpha$, where the set F_α was defined as the unique non-empty compact set of $\mathbb{R}^2$ such that

$$F_\alpha = \bigcup_{i=1}^{3} G_{\alpha,i}(F_\alpha).$$

By additivity of the Hausdorff measure, it holds that

$$\mathcal{H}^{d_\alpha}(K_\alpha) = \mathcal{H}^{d_\alpha}(F_\alpha) + \mathcal{H}^{d_\alpha}(J_\alpha). \tag{2.2.9}$$

From the proof of Theorem 2.2.7, we know that $\dim_H J_\alpha = 1$ and since $d_\alpha > 1$, $\mathcal{H}^{d_\alpha}(J_\alpha) = 0$. Thus (2.2.9) becomes

$$\mathcal{H}^{d_\alpha}(K_\alpha) = \mathcal{H}^{d_\alpha}(F_\alpha),$$

and Theorem 2.2.6 implies directly that

$$0 < \mathcal{H}^{d_\alpha}(K_\alpha) < \infty,$$

as we wanted to prove. □

In view of these results, we would like to believe that the Hausdorff measure of K_α also converges to the Hausdorff measure of K, but this still remains a conjecture, since none of those quantities is already known.

Conjecture 2.2.10. *Set* $d_\alpha := \dim_H K_\alpha$ *and* $d := \dim_H K$. *Then it holds that*

$$\mathcal{H}^{d_\alpha}(K_\alpha) \xrightarrow{\alpha \downarrow 0} \mathcal{H}^{d}(K).$$

Chapter 3

Analysis on the Sierpiński gasket K

So far, we have proved convergence of the sequence (K_α) to K as α tends to zero in the Hausdorff metric and convergence of the Hausdorff dimension. We would like to establish next the convergence of the *spectral dimension*, what we could call "analytic" convergence of the sequence (K_α) to K. Dealing with spectral dimension of a set requires the definition of a *Laplacian* or equivalently a *Dirichlet form* on it.

The theory of Dirichlet forms where the underlying space is a fractal started with the construction of Brownian motion on the Sierpiński gasket by Goldstein and Kusuoka [15, 31] and it has been enlarged since then with many results (see for example [5, 24, 25]).

This chapter reviews the construction of the standard Laplacian on K and the computation of its spectral dimension by means of Dirichlet forms. An outline of the theory of Dirichlet forms on Hilbert spaces can be found in the appendix.

3.1 Approximating forms

Recall that a set $F \in \mathscr{H}(\mathbb{R}^2)$ is said to be self-similar if there exists a finite family of contractive similitudes $\{S_i \colon \mathbb{R}^2 \to \mathbb{R}^2\}_{i=1}^N$ such that

$$F = \bigcup_{i=1}^{N} S_i(F).$$

Let $\Sigma := \{1, \ldots, N\}^{\mathbb{N}}$ and define for each $i = 1, \ldots, N$ the mapping

$$\begin{aligned} \sigma_i \colon \quad \Sigma &\longrightarrow \Sigma \\ w_1 w_2 \ldots &\longmapsto i w_1 w_2 \ldots . \end{aligned}$$

The set Σ can be equipped with a metric d given by

$$d(w, \tau) = 2^{\min\{m \mid w_m \neq \tau_m\} - 1}$$

(see [26, Theorem 1.2.2] for a proof) and if there exists a continuous surjection $\pi \colon \Sigma \to F$ such that $S_i \circ \pi = \pi \circ \sigma_i$ on Σ for every $i = 1, \ldots, N$, then $(F, \{1, \ldots, N\}, \{S_i\}_{i=1}^N)$ is called in [25, Definition 1.3] a *self-similar structure*.

For such a structure, its *critical set* $\mathscr{C}(F)$ and its *post critical set* $\mathscr{P}(F)$ are defined by

$$\mathscr{C}(F) := \pi^{-1}\left(\bigcup_{i \neq j} S_i(F) \cap S_j(F) \right) \quad \text{and} \quad \mathscr{P}(F) := \bigcup_{n \geq 1} \sigma^n(\mathscr{C}(F)),$$

where $\sigma \colon \Sigma \to \Sigma$ is the left shift map given by $\sigma(w_1 w_2 w_3 \ldots) = w_2 w_3 \ldots$ and σ^n means the $n-$th iterate of σ.

A self-similar set F is called *post critically finite* (p.c.f. for short) if its post critical set is finite. This corresponds to the mathematical notion of *finitely ramified* fractal.

Hereafter, we consider the self-similar structure $(K, \{1, 2, 3\}, \{S_i\}_{i=1}^3)$ on the Sierpiński gasket K, where the contractive similitudes S_1, S_2 and S_3 were defined in (1.0.1) by

$$\begin{aligned} S_i \colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \frac{x + p_i}{2}, \end{aligned}$$

for $p_1 := (1, 0)$, $p_2 = (1/2, \sqrt{3}/2)$ and $p_3 := (1, 0)$. As in Chapter 2, we set $\mathcal{A} := \{1, 2, 3\}$ to be the alphabet consisting of the three symbols 1, 2 and 3. Moreover, we know from [26, Theorem 1.2.3] that there is a surjection $\pi \colon \mathcal{A}^{\mathbb{N}} \to K$ such that $S_i \circ \pi = \pi \circ i$ for every $i \in \mathcal{A}$.

Moreover, the mappings $S_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ are contractive similitudes of ratio $\frac{1}{2}$ and the critical and post critical set of K are given by

$$\mathscr{C}(K) = \{1\overline{2}, 2\overline{1}, 1\overline{3}, 3\overline{1}, 2\overline{3}, 3\overline{2}\},$$

and

$$\mathscr{P}(K) = \{\overline{1}, \overline{2}, \overline{3}\}$$

respectively, where $\overline{n} = nnn\ldots$ for $n \in \mathbb{N}$. Hence, K is a p.c.f. self-similar set.

The key idea in the construction of the Dirichlet form $(\mathcal{E}_K, \mathcal{D}_K)$ for K is the fact that K can be geometrically approximated by the monotonically increasing sequence of finite sets $(V_n)_{n\in\mathbb{N}_0}$ that were defined in (1.0.3) by $V_0 := \{p_1, p_2, p_3\}$ $(= \pi(\mathscr{P}(K)))$ and

$$V_n := \bigcup_{w\in\mathcal{A}^n} S_w(V_0)$$

for each $n \in \mathbb{N}$. By [26, Lemma 1.3.11], the set $V_* := \bigcup_{n\in\mathbb{N}_0} V_n$ is dense in K with respect to the Euclidean norm. This allows the Dirichlet form $\mathcal{E}_K$ for the whole K to arise as the limit of a sequence of finite bilinear forms $(\mathcal{E}_n)_{n\in\mathbb{N}_0}$ defined for each finite set V_n, $n \in \mathbb{N}_0$.

Definition 3.1.1. Any two points $x, y \in V_n$, $x \neq y$, are called *(Sierpiński) n-neighbours* if there exists $w \in \mathcal{A}^n$ such that $x, y \in S_w(V_0)$. In this case, we write $x \overset{n}{\sim} y$ and $S_w(V_0)$ is called the *$n-$cell* x and y belong to.

For each $n \in \mathbb{N}_0$, we consider the functional $E_n\colon \mathcal{D}_n \to \mathbb{R}$ defined by

$$\begin{cases} \mathcal{D}_n := \{u\colon V_n \to \mathbb{R}\}, \\ E_n[u] := \displaystyle\sum_{x \overset{n}{\sim} y} \big(u(x) - u(y)\big)^2. \end{cases}$$

For any function $u \in \mathcal{D}_n$, the quantity $E_n[u]$ is also known as the *energy of u at level n*. Applying the polarization identity we may define the following bilinear form $E_n\colon \mathcal{D}_n \times \mathcal{D}_n \to \mathbb{R}$,

$$\begin{cases} \mathcal{D}_n := \{u\colon V_n \to \mathbb{R}\}, \\ E_n(u,v) := \dfrac{1}{2}(E_n[u+v] - E_n[u] - E[v]). \end{cases} \tag{3.1.1}$$

Definition 3.1.2. Let $n \in \mathbb{N}_0$ and $u \in \mathcal{D}_n$. The *harmonic extension* of u to V_{n+1} is the function $\tilde{u}\colon V_{n+1} \to \mathbb{R}$ such that

$$E_{n+1}[\tilde{u}] = \min\{E_{n+1}[v] \mid v \in \mathcal{D}_{n+1} \text{ and } v|_{V_n} \equiv u\}.$$

Note that the existence and uniqueness of such an extension follows directly from its construction as the solution of a minimization problem expressed by a linear system of equations (see [39, p.13] for details).

For the sequence of bilinear forms defined in (3.1.1) we would like to have *invariance under harmonic extension*, i.e., if $u \in \mathcal{D}_n$ and $\tilde{u} \in \mathcal{D}_{n+1}$ is its harmonic extension, then it should hold that

$$E_n[u] = E_{n+1}[\tilde{u}].$$

As the following calculation shows, the energy functional E_n does not fulfill this invariance property, so we need to find a factor $\rho_n > 0$ such that if we define $\mathcal{E}_n[u] := \rho_n^{-1} E_n[u]$, then $\mathcal{E}_n$ satisfies it. For each $n \in \mathbb{N}_0$, the factor ρ_n is called *renormalization factor* of the energy E_n.

This factor can be calculated as follows: without loss of generality, consider the function $u_0\colon V_0 \to \mathbb{R}$ whose values are given by $u_0(p_1) = 1$, $u_0(p_2) = 0 = u_0(p_3)$. It is easy to see that $E_0[u_0] = 2$ and if we want $\tilde{u}_1\colon V_1 \to \mathbb{R}$ to be the harmonic extension of u_0, we just need to minimize the quantity

$$E_1[\tilde{u}_1] = \sum_{x \overset{n}{\sim} y} (\tilde{u}_1(x) - \tilde{u}_1(y))^2,$$

where the unknowns are the values of u_1 at $V_1 \setminus V_0$. The property of K being p.c.f. is very important at this point: thanks to it, the set $V_1 \setminus V_0$ is finite and therefore we have to solve a finite number of equations.

Solving this minimization problem leads to

$$E_1[\tilde{u}_1] = \frac{3}{5} E_0[u_0],$$

and an inductive argument shows that

$$E_{n+1}[\tilde{u}] = \frac{3}{5} E_n[u] \qquad \forall\, n \in \mathbb{N}_0,$$

where $\tilde{u}\colon V_{n+1} \to \mathbb{R}$ is the harmonic extension of $u\colon V_n \to \mathbb{R}$ (see [39, p.14] for details).

Thus, given any function $u_0\colon V_0 \to \mathbb{R}$,

$$E_n[\tilde{u}_n] = \left(\frac{3}{5}\right)^n E_0[u_0]$$

for the harmonic extension of u_0 to V_n, $\tilde{u}_n\colon V_n \to \mathbb{R}$. If we now define the bilinear form

$$\mathcal{E}_n(u, v) := \left(\frac{3}{5}\right)^{-n} E_n(u, v), \qquad u, v \in \mathcal{D}_n,$$

then it is invariant under harmonic extension for all $n \in \mathbb{N}_0$ and $\rho_n := \left(\frac{3}{5}\right)^n$ was the factor we were looking for.

An alternative method to obtain this renormalization factor can be found in [39, Section 1.5].

3.2 Dirichlet form on the Sierpiński gasket

Let $d := \dim_{\mathrm{H}} K = \frac{\ln 3}{\ln 2}$ and consider the Hilbert space $L^2(K,\mu)$, where μ is the Borel probability measure given by the normalized d-dimensional Hausdorff measure restricted to K, i.e.

$$\mu(A) := \frac{1}{\mathcal{H}^d(K)} \mathcal{H}^d_{|K}(A), \qquad A \subseteq \mathbb{R}^2 \text{ Borel.}$$

Remark 3.2.1. It is proved in [20] that μ is the unique Borel probability measure with the property

$$\mu(A) = \frac{1}{3}\sum_{i=1}^{3} \mu(S_i^{-1}(A)), \qquad A \subseteq \mathbb{R}^2 \text{ Borel.}$$

Because the measure μ satisfies this property, it is called a *self-similar measure* and the factor $\frac{1}{3}$ may be understood as its weights.

Further, it is proved in [21, Corollary 2] that the sequence of measures $(\mu_n)_{n\in\mathbb{N}_0}$ supported on V_n and defined by

$$\mu_n(A) := \frac{2}{3^{n+1}} \sum_{x\in V_n} \delta_x(A) \qquad \text{for } A \subseteq K \text{ Borel,}$$

where δ_x denotes the Dirac measure at the point x, converges weakly to μ.

The measure μ allows us to consider the Hilbert space $L^2(K,\mu)$ with inner product $(\cdot,\cdot)_\mu$, where we define our desired Dirichlet form.

Given two functions $u, v \colon V_* \to \mathbb{R}$, we define for any $n \in \mathbb{N}_0$

$$\mathcal{E}_n(u,v) := \mathcal{E}_n(u|_{V_n}, v|_{V_n}).$$

Notice that for fixed $u \colon V_* \to \mathbb{R}$, the sequence $(\mathcal{E}_n(u,u))_{n\in\mathbb{N}_0}$ is non-decreasing by construction, so we can define the symmetric bilinear form $\mathcal{E}_K \colon \mathcal{D}_K^* \times \mathcal{D}_K^* \to \mathbb{R}$ by

$$\begin{cases} \mathcal{D}_K^* := \{u \colon V_* \to \mathbb{R} \mid \lim_{n\to\infty} \mathcal{E}_n(u,u) < \infty\}, \\ \mathcal{E}_K(u,v) := \lim_{n\to\infty} \mathcal{E}_n(u,v). \end{cases}$$

This definition only involves functions defined on V_* and we want to consider functions on K. Fortunately, we know from [39, p.19] that any function $u \in \mathcal{D}_K^*$ is Hölder – and therefore uniformly – continuous on V_*. Since V_* is dense in K, any function $u \in C(V_*)$ can be uniquely extended to a continuous function on K. We denote this extension again by u.

Now we can define a symmetric bilinear form $\mathcal{E}_K\colon \widetilde{\mathcal{D}}_K \times \widetilde{\mathcal{D}}_K \to \mathbb{R}$ by

$$\begin{cases} \widetilde{\mathcal{D}}_K := \{u\colon K \to \mathbb{R} \mid u|_{V_*} \in \mathcal{D}^*_K\}, \\ \mathcal{E}_K(u,v) := \lim_{n\to\infty} \mathcal{E}_n(u_{|V_*}, v_{|V_*}). \end{cases}$$

Theorem 3.2.2. *Let $\mathcal{D}_K$ be the completion of $\widetilde{\mathcal{D}}_K$ with respect to the norm*

$$\|u\|_{\mathcal{E}_{K,1}} := \big(\mathcal{E}_K(u,u) + (u,u)_{L^2(K,\mu)}\big)^{1/2}.$$

The pair $(\mathcal{E}_K, \mathcal{D}_K)$ is a local regular Dirichlet form on $L^2(K,\mu)$.

Proof. The original proof can be found in [31, Theorem 4.6] . □

For further properties of this Dirichlet form we refer to [31, 29].

3.3 The weak Laplacian

From the theory of Dirichlet forms (see Appendix) we know that $(\mathcal{E}_K, \mathcal{D}_K)$ has an associated non-positive self-adjoint operator $(L, \mathcal{D}(L))$ such that $\mathcal{D}(L)$ is dense in $\mathcal{D}_K$ and for any $u \in \mathcal{D}(L)$

$$\mathcal{E}_K(u,v) = (-Lu, v)_\mu \qquad \forall\, v \in \mathcal{D}_K.$$

This operator is called the *Laplacian* on K with respect to the measure μ and we will denote it by Δ_μ^N. The superscript N refers to the fact that, if we consider $V_0 = \pi(\mathscr{P}(K))$ as the *boundary* of K, then the normal derivative of u on V_0 (see Definition 3.3.2) is zero for all $u \in \mathcal{D}(\Delta_\mu^N)$. This means, all functions on the domain of the Laplacian fulfil Neumann boundary conditions.

Definition 3.3.1. Let $u \in \mathcal{D}_K$. If there exists a unique continuous function $f \in C(K)$ such that

$$\mathcal{E}_K(u,v) = -\int_K fv\,d\mu \qquad \forall\, v \in \mathcal{D}_K,$$

then we say that $u \in \mathcal{D}(\Delta_\mu^N)$ and for this $u \in \mathcal{D}(\Delta_\mu)$ we set $\Delta_\mu^N u := f$.

Normal derivatives are defined on the boundary in the following way.

Definition 3.3.2. [39, p.38] Let $x \in V_0$ and $u \in C(K)$. The normal derivative of u at the point x is given by

$$\frac{\partial u}{\partial \nu}(x) := \lim_{n\to\infty} \left(\frac{5}{3}\right)^n \sum_{y \overset{n}{\sim} x} \big(u(x) - u(y)\big),$$

whenever the limit exists and is finite.

It can be proven (see [27, Proposition 8.6]) that the functions in the domain of the Laplacian associated to $(\mathcal{E}_K, \mathcal{D}_K)$ directly satisfy Neumann boundary conditions. The Laplacian subject to Dirichlet boundary conditions arises as the operator associated to the following Dirichlet form

$$\begin{cases} \mathcal{D}_K^0 := \{u \in \mathcal{D}_K \mid u|_{V_0} \equiv 0\}, \\ \mathcal{E}_K^0 := \mathcal{E}_K|_{\mathcal{D}_K^0 \times \mathcal{D}_K^0}. \end{cases} \tag{3.3.1}$$

For a first order analysis of these operators, we shall study their spectrum. Thus we are interested in the eigenvalue problem for both Laplacians $-\Delta_\mu^N$ and $-\Delta_\mu^D$ (i.e. Laplacian subject to homogeneous Neumann (resp. Dirichlet) boundary conditions). Both problems turn out to be equivalent to the eigenvalue problem for the respective associated Dirichlet form, as the following shows.

Definition 3.3.3. Let $\lambda \geq 0$ and let $(\mathcal{E}, \mathcal{D})$ be a Dirichlet form in $L^2(K, \mu)$. If there exists a function $u \in \mathcal{D}$, $u \neq 0$, such that

$$\mathcal{E}(u, v) = \lambda(u, v)_\mu \qquad \forall\, v \in \mathcal{D},$$

then λ is called an *eigenvalue* of $\mathcal{E}$ and u is an associated *eigenfunction*.

Proposition 3.3.4. *For $\lambda \in \mathbb{R}$ and $u \in \mathcal{D}_K$,*

$$\mathcal{E}_K(u, v) = \lambda(u, v)_\mu \qquad \forall\, v \in \mathcal{D}_K$$

if and only if $u \in \mathcal{D}(\Delta_\mu^N)$ and $\Delta_\mu^N u = \lambda u$.
Analogously, for $\lambda \in \mathbb{R}$ and $u \in \mathcal{D}_K^0$,

$$\mathcal{E}_K^0(u, v) = \lambda(u, v)_\mu \qquad \forall\, v \in \mathcal{D}_K^0$$

if and only if $u \in \mathcal{D}(\Delta_\mu^D)$ and $\Delta_\mu^D u = \lambda u$.

Proof. See [29, Proposition 5.1], resp. [29, Proposition 5.2]. □

This means that the set of eigenvalues of the Laplacian $-\Delta_\mu^N$ coincides with the set of eigenvalues of the Dirichlet form $(\mathcal{E}_K, \mathcal{D}_K)$ and the same holds for $-\Delta_\mu^D$ and $(\mathcal{E}_K^0, \mathcal{D}_K^0)$.

The following proposition shows that the spectrum of the Laplacian on K subject to Neumann boundary conditions coincides with the set of its eigenvalues. The same results hold for the Dirichlet case just by writing Δ_μ^D and $(\mathcal{E}_K^0, \mathcal{D}_K^0)$ instead of Δ_μ^N and $(\mathcal{E}_K, \mathcal{D}_K)$.

Lemma 3.3.5. *Let $\rho(\Delta_\mu^N)$ denote the resolvent set of Δ_μ^N and for any $\lambda \in \rho(\Delta_\mu^N)$, let $R_\lambda^N := (\lambda - \Delta_\mu^N)^{-1}$ be a resolvent of Δ_μ^N. Then,*

$$\kappa \in \sigma(R_\lambda^N) \Leftrightarrow \frac{1}{\kappa} - \lambda \in \sigma(-\Delta_\mu^N).$$

Proof. See [12, Lemma 6.8]. □

Proposition 3.3.6. *The spectrum of the Laplacian $-\Delta_\mu^N$ consists of countably many non-negative eigenvalues with finite multiplicity and only accumulation point at $+\infty$.*

Proof. Let R_λ^N be a resolvent of Δ_μ^N, $\lambda \in \rho(\Delta_\mu^N)$. Recall that Δ_μ^N is the operator associated with the Dirichlet form $(\mathcal{E}_K, \mathcal{D}_K)$.

Since $\mathcal{E}_K$ is symmetric, we know from the theory of Dirichlet forms that R_λ^N is self-adjoint (see Theorem A.2.12 in Appendix).

On the other hand, we know from [29, Section 5] that the inclusion map from the Hilbert space $(\mathcal{D}_K, \mathcal{E}_{K,1})$ into $L^2(K,\mu)$ is a compact operator. Thus, by [9, Exercise 4.2] R_λ^N is a compact operator.

Now, it follows from the theory of compact self-adjoint operators that R_λ^N has countable many eigenvalues, all non-negative and of finite multiplicity with the only accumulation point at zero (see e.g [7, Theorem 6.8]).

Finally, the assertion follows by Lemma 3.3.5. □

3.4 Spectral asymptotics

The study of the asymptotic behaviour of the spectrum of the Laplacian Δ_μ has great importance from a physical point of view, since it describes the density of states for diffusion or wave propagation that are modelled by differential equations involving Δ_μ. We are interested in the asymptotic order of the eigenvalue distribution of this operator, which is reflected in the asymptotic behaviour of the so called *eigenvalue counting function.*

As we showed in Proposition 3.3.6, the Neumann (resp. Dirichlet) eigenvalues of Δ_μ are non-negative and with finite multiplicity, so it makes sense to "count" them.

Definition 3.4.1. The *eigenvalue counting function* of $-\Delta_\mu^N$ at any point $x \geq 0$ is given by

$$N_N(x) := \#\{\kappa \,|\, \kappa \text{ is an eigenvalue of } -\Delta_\mu^N \text{ with } \kappa \leq x\},$$

where each eigenvalue is counted according to its multiplicity.

The eigenvalue counting function of $-\Delta_\mu^D$ is defined analogously and denoted by $N_D(x)$.

This function can be defined for Dirichlet forms as well: given a Dirichlet form $(\mathcal{E}, \mathcal{D})$, we define its eigenvalue counting function by

$$N(x; \mathcal{E}, \mathcal{D}) := \#\{\kappa \mid \kappa \text{ is an eigenvalue of } \mathcal{E} \text{ with } \kappa \leq x\}.$$

Proposition 3.3.4 therefore says that for all $x \geq 0$, $N(x; \mathcal{E}_K, \mathcal{D}_K) = N_N(x)$ and $N(x; \mathcal{E}_K^0, \mathcal{D}_K^0) = N_D(x)$.

Theorem 3.4.2. *There exist constants $C_1, C_2 > 0$ and $x_0 > 0$ such that*

$$C_1 x^{\frac{d_S}{2}} \leq N_N(x) \leq C_2 x^{\frac{d_S}{2}} \qquad \forall\, x \geq x_0.$$

The exponent d_S is called spectral dimension *of the Sierpiński gasket K and for this particular case it holds that*

$$d_S = \frac{\ln 9}{\ln 5}.$$

The same holds for the function $N_D(x)$ with the same number d_S.

Proof. See [29, Theorem 2.4]. □

The crucial step in the proof of this theorem is the following lemma:

Lemma 3.4.3. *For all $x \geq 0$ we have*

$$\sum_{i=1}^{3} N_D\left(\frac{x}{5}\right) \leq N_D(x) \leq N_N(x) \leq \sum_{i=1}^{3} N_N\left(\frac{x}{5}\right) \tag{3.4.1}$$

and

$$N_D(x) \leq N_N(x) \leq N_D(x) + \#V_0. \tag{3.4.2}$$

Proof. See [29, Lemma 2.3]. □

On showing inequalities (3.4.1) and (3.4.2), one applies the *scaling property* of the Dirichlet form $(\mathcal{E}_K, \mathcal{D}_K)$ and the so called *Dirichlet-Neumann bracketing* method. Since both are quite important results, we explain them below.

The scaling property of the Dirichlet form $(\mathcal{E}_K, \mathcal{D}_K)$ is stated as follows:

Lemma 3.4.4. *For each $u, v \in \mathcal{D}_K$ it holds that*

$$\mathcal{E}_K(u, v) = \frac{5}{3} \sum_{i=1}^{3} \mathcal{E}_K(u \circ S_i, u \circ S_i).$$

Proof. See [29, Lemma 6.1]. □

This allows us to define a new Dirichlet form $(\widetilde{\mathcal{E}}_K, \widetilde{\mathcal{D}}_K)$ on $L^2(K,\mu)$, given by

$$\begin{cases} \widetilde{\mathcal{D}}_K := \{u\colon K\setminus V_1 \to \mathbb{R} \mid \forall\, i=1,2,3,\ \exists\, u_i \in \mathcal{D}_K \text{ s. t. } u_{i|_{K\setminus V_0}} = u\circ S_{i|_{K\setminus V_0}}\}, \\ \widetilde{\mathcal{E}}_K(u,v) := \frac{5}{3}\sum_{i=1}^{3} \mathcal{E}_K(u_i,v_i). \end{cases}$$

Proposition 3.4.5. *The pair $(\widetilde{\mathcal{E}}_K, \widetilde{\mathcal{D}}_K)$ has the following properties:*

(i) $\mathcal{D}(\mathcal{E}_K) \subseteq \widetilde{\mathcal{D}}_K$ *and* $\mathcal{E}_K = \widetilde{\mathcal{E}}_K|_{\mathcal{D}_K\times\mathcal{D}_K}$.

(ii) $(\widetilde{\mathcal{E}}_K, \widetilde{\mathcal{D}}_K)$ *is a local regular Dirichlet form on* $L^2(K,\mu)$.

(iii) The inclusion map $\widetilde{\mathcal{D}}_K \hookrightarrow L^2(K,\mu)$ *is a compact operator.*

(iv) $N(x;\widetilde{\mathcal{E}}_K, \widetilde{\mathcal{D}}_K) = \sum_{i=1}^{3} N\left(\frac{x}{5};\mathcal{E}_K,\mathcal{D}_K\right)$.

Proof. See [29, Proposition 6.2]. □

The method of Dirichlet-Neumann bracketing is based in the *max-min principle* and it was introduced in [8, Chapter VI.1]. This method relates the eigenvalue counting function of different Dirichlet forms, leading to the following monotonicity principle stated in [32, Proposition 4.2]: Given two Dirichlet forms $(\mathcal{E}_K,\mathcal{D}_K)$ and $(\mathcal{E}^0_K,\mathcal{D}^0_K)$ such that $\mathcal{D}^0_K$ is a closed subspace of $\mathcal{D}_K$ and $\mathcal{E}^0_K = \mathcal{E}_K|_{\mathcal{D}^0_K\times\mathcal{D}^0_K}$, it holds that

$$N(x;\mathcal{E}^0_K,\mathcal{D}^0_K) \le N(x;\mathcal{E}_K,\mathcal{D}_K) \qquad \forall\, x\ge 0.$$

Equivalently, this means

$$N_D(x) \le N_N(x) \qquad \forall\, x \ge 0,$$

which is the second inequality in (3.4.1).

Remark 3.4.6. This result can also be obtained using the so–called *minimax principle* or *variational method*, that can be found in [9, Chapter 4]. We will make use of it in the next chapter.

Due to Proposition 3.4.5 $(i)-(ii)$, we can also apply the Dirichlet-Neumann bracketing to the Dirichlet forms $(\mathcal{E}_K,\mathcal{D}_K)$ and $(\widetilde{\mathcal{E}}_K, \widetilde{\mathcal{D}}_K)$ to get

$$N(x;\mathcal{E}_K,\mathcal{D}_K) \le N(x;\widetilde{\mathcal{E}}_K, \widetilde{\mathcal{D}}_K).$$

It follows from Proposition 3.4.5 (iv) that

$$N(x;\mathcal{E}_K,\mathcal{D}_K) \le \sum_{i=1}^{3} N\left(\frac{x}{5};\mathcal{E}_K,\mathcal{D}_K\right),$$

which is equivalent to

$$N_N(x) \le \sum_{i=1}^{3} N_N\left(\frac{x}{5}\right),$$

so the last inequality in (3.4.1) is proved. The inequality

$$\sum_{i=1}^{3} N_D\left(\frac{x}{5}\right) \le N_D(x)$$

is obtained analogously by considering the pair of Dirichlet forms $(\mathcal{E}_K^0,\mathcal{D}_k^0)$ and $(\widetilde{\mathcal{E}}_K^0,\widetilde{\mathcal{D}}_k^0)$, where

$$\begin{cases} \widetilde{\mathcal{D}}_K^0 := \{u \in \mathcal{D}_K^0 \mid u|_{V_1} = 0\}, \\ \widetilde{\mathcal{E}}_K^0(u,v) := \mathcal{E}_K^0|_{\widetilde{\mathcal{D}}_K^0 \times \widetilde{\mathcal{D}}_K^0}. \end{cases}$$

Finally, inequality (3.4.2) assures that Theorem 3.4.2 holds for both functions $N_N(x)$ and $N_D(x)$ with the same exponent d_S.

Remark 3.4.7. It is proved in [29, Theorem 2.4] that the spectral dimension d_S is the unique positive number such that

$$\sum_{i=1}^{3} \left(\frac{1}{5}\right)^{\frac{d_S}{2}} = 1.$$

The number $\frac{1}{5}$ is the product $\frac{3}{5} \cdot \frac{1}{3}$, where $\frac{3}{5}$ is the renormalization factor of the energy $\mathcal{E}_K$ and $\frac{1}{3}$ is the contraction factor of the self-similar measure μ.

All results obtained in this chapter will be compared in the next one with their analogue in the case of K_α.

Chapter 4

Analysis on the Hanoi attractor K_α

This section is devoted to the establishment of a calculus for any Hanoi attractor K_α, $\alpha \in (0, 1/3)$. The construction is based on the *theory of resistance and Dirichlet forms* and it follows an analogous scheme as the case of the Sierpiński gasket: First, we construct a sequence of renormalized bilinear forms $(\mathcal{E}_{\alpha,n})_{n\in\mathbb{N}_0}$, where each $\mathcal{E}_{\alpha,n}$ is defined for the approximating set $V_{\alpha,n}$. Secondly, we obtain a resistance form $\mathcal{E}_{K_\alpha}$ for the Hanoi attractor K_α by taking the limit of $\mathcal{E}_{\alpha,n}$ as $n \to \infty$. Afterwards, we introduce a Radon probability measure μ_α on K_α, so that $\mathcal{E}_{K_\alpha}$ becomes a Dirichlet form on $L^2(K_\alpha, \mu_\alpha)$ with an associated *Laplacian* on K_α. Finally, we study the asymptotic behaviour of the distribution of the eigenvalues of this Laplacian and determine the spectral dimension of K_α.

The results concerning the construction of the Dirichlet form will appear in the paper by Uta Freiberg and myself, [2], while the calculation of the spectral dimension will appear in the forthcoming paper [3].

4.1 Approximating forms

In this paragraph we define a sequence of bilinear forms $(\mathcal{E}_{\alpha,n})_{n\in\mathbb{N}_0}$ that will lead to the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$. These bilinear forms (together with their corresponding domain) have been proved to be also Dirichlet forms in [2].

We work with the alphabet $\mathcal{A} := \{1, 2, 3\}$, the IFS $\{\mathbb{R}^2; G_{\alpha,i}, i = 1, \dots, 6\}$ associated to K_α and the sets $W_{\alpha,n}$, $J_{\alpha,n}$ and $V_{\alpha,n}$, defined in Section 2.1 by

$$W_{\alpha,n} := \bigcup_{w\in\mathcal{A}^n} G_{\alpha,w}(\{p_1, p_2, p_3\}), \quad J_{\alpha,n} := \bigcup_{m=0}^{n-1} \bigcup_{w\in\mathcal{A}^m} G_{\alpha,w}(\cup_{i=1}^3 e_i)$$

and $V_{\alpha,n} := W_{\alpha,n} \cup J_{\alpha,n}$ for each $n \in \mathbb{N}_0$.

In addition, we denote by $\mathcal{J}_{\alpha,n}$ the set of connected components of $J_{\alpha,n}$ for each $n \in \mathbb{N}_0$ and $\mathcal{J}_\alpha := \bigcup_{n\in\mathbb{N}} \mathcal{J}_{\alpha,n}$.

Since all results presented in this chapter hold for any $\alpha \in (0, 1/3)$, we will no longer mention this condition explicitly. Moreover, notice that the set $V_{\alpha,0} = \{p_1, p_2, p_3\}$ is independent of α, thus we will denote it from now on just by V_0 and consider it as the *boundary* of K_α.

4.1.1 Non-renormalized forms

In order to construct the sequence $(\mathcal{E}_{\alpha,n})_{n\in\mathbb{N}_0}$, we start by defining a quite simple bilinear form acting on functions $u\colon V_{\alpha,n} \to \mathbb{R}$ for each $n \in \mathbb{N}_0$.

Definition 4.1.1. Any two points $x, y \in V_{\alpha,n}$, $x \neq y$, are called *(Sierpiński) (α, n)-neighbours* if there exists $w \in \mathcal{A}^n$ such that $x, y \in G_{\alpha,w}(V_0)$. In this case we write $x \overset{\alpha,n}{\sim} y$ and $G_{\alpha,w}(V_0)$ is called the $n-$*cell* x and y belong to.

Definition 4.1.2. For any $n \in \mathbb{N}_0$ and $x \in W_{\alpha,n}$ we denote by w^x the unique word of length $n+1$, $w_1^x \ldots w_{n+1}^x \in \mathcal{A}^{n+1}$ such that $x = G_{\alpha,w_1^x\ldots w_n^x}(p_{w_{n+1}^x})$. The *Sierpiński $\alpha-$graph* Γ_α^n is defined by

$$\begin{cases} V(\Gamma_\alpha^n) = W_{\alpha,n}, \\ E(\Gamma_\alpha^n) = \{\{x,y\} \mid x \overset{\alpha,n}{\sim} y\}, \end{cases}$$

and the *Hanoi $\alpha-$graph* H_α^n by

$$\begin{cases} V(H_\alpha^n) = W_{\alpha,n}, \\ E(H_\alpha^n) = \{\{x,y\} \mid \{w^x, w^y\} \in E(\mathcal{S}(n,3))\}, \end{cases}$$

where $\mathcal{S}(n,3)$ is the Sierpiński graph considered in (1.0.4).

Definition 4.1.3. Define $\mathcal{D}_0 := \{u\colon V_0 \to \mathbb{R}\}$ and for each $n \in \mathbb{N}$, set

$$\mathcal{D}_{\alpha,n} := \{u\colon V_{\alpha,n} \to \mathbb{R} \,|\, u \text{ continuous and } u|_e \in W^{1,2}(e, dx)\ \forall\, e \in \mathcal{J}_{\alpha,n}\}.$$

The functional $E_{\alpha,n}\colon \mathcal{D}_{\alpha,n} \to \mathbb{R}$ is defined by

$$E_{\alpha,n}[u] := \sum_{x \overset{\alpha,n}{\sim} y} (u(x)-u(y))^2 + \int_{J_{\alpha,n}} |\nabla u|^2\, dx,$$

and we call $E_{\alpha,n}[u]$ the *energy of u at level n*. Further, the functionals $E^d_{\alpha,n}, E^c_{\alpha,n}\colon \mathcal{D}_{\alpha,n} \to \mathbb{R}$ defined by

$$E^d_{\alpha,n}[u] := \sum_{x \overset{\alpha,n}{\sim} y} (u(x)-u(y))^2 \tag{4.1.1}$$

and

$$E^c_{\alpha,n}[u] := \int_{J_{\alpha,n}} |\nabla u|^2 \, dx, \tag{4.1.2}$$

are called the *discrete* resp. *continuous part* of $E_{\alpha,n}$.

The integral expression in (4.1.2) has to be understood as follows: For each $e = (a_e, b_e) \in \mathcal{J}_{\alpha,n}$, let $\varphi_e \colon [0,1] \to \mathbb{R}^2$ be the curve parametrization of the line segment joining a_e and b_e, i.e.

$$\varphi_e(t) := (b_e - a_e) \cdot t + a_e.$$

Then, for any function $u \in \mathcal{D}_{\alpha,n}$,

$$E^c_{\alpha,n}[u] = \int_{J_{\alpha,n}} |\nabla u|^2 \, dx = \sum_{e \in \mathcal{J}_{\alpha,n}} \int_e |\nabla u|^2 \, dx := \sum_{e \in \mathcal{J}_{\alpha,n}} \frac{1}{b_e - a_e} \int_0^1 \left|(u \circ \varphi_e)'\right|^2 \, dt.$$

4.1.2 Harmonic extension

Once we have defined the energy functional $E_{\alpha,n}$, we are interested in which functions are energy-minimizers (also called *harmonic functions*) and in particular in how to extend any given function $u \in \mathcal{D}_0$ to a function $\tilde{u} \in \mathcal{D}_{\alpha,n}$ minimizing its energy at every level up to n.

Definition 4.1.4. Let $u \in \mathcal{D}_0$. The *harmonic extension at level* $n+1$ of u is the unique function $\tilde{u} \in \mathcal{D}_{\alpha,n+1}$ that satisfies

$$E_{\alpha,k}[\tilde{u}] = \inf\{E_{\alpha,k}[v] \mid v \in \mathcal{D}_{\alpha,k} \text{ and } v|_{V_0} \equiv u\}$$

for all $0 \leq k \leq n+1$.

Note that this is well defined by the following propositions. Once we have analysed the case $n = 0$, the harmonic extension of any given function at any level $n \in \mathbb{N}$ will be obtained by a straightforward process of iteration.

Proposition 4.1.5. *For any function $u \in \mathcal{D}_0$, the infimum*

$$\inf\{E_{\alpha,1}[v] \mid v \in \mathcal{D}_{\alpha,1} \textit{ and } v|_{V_0} \equiv u\}$$

is attained by a unique function $\tilde{u} \in \mathcal{D}_{\alpha,1}$ given on $W_{\alpha,1}$ by

$$\tilde{u}_1(G_{\alpha,i}(p_j)) = \frac{2+3\alpha}{5+3\alpha} u(p_i) + \frac{2}{5+3\alpha} u(p_j) + \frac{1}{5+3\alpha} u(p_k) \tag{4.1.3}$$

for any $i \in \mathcal{A}$, $\{i,j,k\} = \mathcal{A}$, and linear interpolation on $J_{\alpha,1}$.

Proof. Without loss of generality, we may assume that the function $u_0 \in \mathcal{D}_0$ is given by

$$u_0(p_1) = 1, \quad u_0(p_2) = 0, \quad u_0(p_3) = 0.$$

Let us denote by $\tilde{u}_1$ the function we want to determine. If we look at the expression of the energy $E_{\alpha,1}[\tilde{u}_1]$, we see that its continuous part,

$$E^c_{\alpha,1}[\tilde{u}_1] = \sum_{e\in\mathcal{J}_{\alpha,1}} \int_e |\nabla \tilde{u}_1|^2\, dx,$$

is minimized when each of the integrals $\int_e |\nabla \tilde{u}_1|^2\, dx$ does.

This can be easily obtained by extending the function $\tilde{u}_1|_{W_{\alpha,1}}$ linearly to $J_{\alpha,1}$, i.e. by defining

$$\tilde{u}_1|_e(x) := \frac{\tilde{u}_1(b_e) - \tilde{u}_1(a_e)}{b_e - a_e} \cdot x + \frac{\tilde{u}_1(a_e)b_e - \tilde{u}_1(b_e)a_e}{b_e - a_e}$$

at each $x \in e = (a_e, b_e) = (G_{\alpha,i}(p_j), G_{\alpha,j}(p_i)) \subseteq J_{\alpha,1}$, $i \neq j$ (see Figure 4.1).

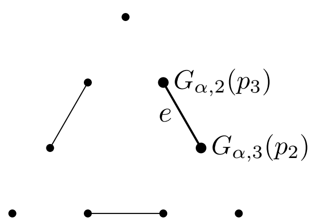


Figure 4.1: Harmonic extension $\tilde{u}_1$.

The integral expression becomes

$$\int_e |\nabla \tilde{u}_1|^2\, dx = \frac{(\tilde{u}_1(b_e) - \tilde{u}_1(a_e))^2}{|b_e - a_e|},$$

and all we need to obtain the total energy $E_{\alpha,1}[\tilde{u}_1]$ is the value of $\tilde{u}_1$ on $W_{\alpha,1}$.

Due to the definition of u_0 and the symmetry of $V_{\alpha,1}$ we have that

$$\begin{aligned}
\tilde{u}_1(G_{\alpha,1}(p_2)) &= \tilde{u}_1(G_{\alpha,1}(p_3)) = x,\\
\tilde{u}_1(G_{\alpha,2}(p_1)) &= \tilde{u}_1(G_{\alpha,3}(p_1)) = y,\\
\tilde{u}_1(G_{\alpha,2}(p_3)) &= \tilde{u}_1(G_{\alpha,3}(p_1)) = z,
\end{aligned}$$

where x, y and z are the values we are looking for (see Figure 4.2).

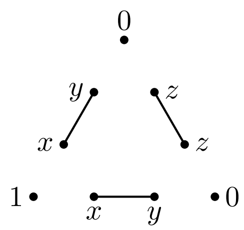


Figure 4.2: Values of $\tilde{u}_1$ in $W_{\alpha,1}$.

Let us define

$$c_{pq}^{\alpha,1} := \begin{cases} 1, & \text{if } \{p,q\} \in E(\Gamma_\alpha^1), \\ \alpha^{-1}, & \text{if } \{p,q\} \in E(H_\alpha^1) \setminus E(\Gamma_\alpha^1), \end{cases}$$

where Γ_α^1 and H_α^1 are the Sierpiński (resp. Hanoi) $\alpha-$graph with vertex set $W_{\alpha,1}$ of Definition 4.1.2.

Now, the energy of the harmonic extension $\tilde{u}_1$ can be expressed as the sum

$$E_{\alpha,1}[\tilde{u}_1] = \sum_{\{p,q\}\in E(H_\alpha^1)} c_{pq}^{\alpha,1}(\tilde{u}_1(p) - \tilde{u}_1(q))^2$$

and solving the minimization problem

$$E_{\alpha,1}[\tilde{u}_1] = \min\{E_{\alpha,1}[v] \mid v \in \mathcal{D}_{\alpha,1} \text{ such that } v|_{V_0} \equiv u_0\}$$

is equivalent to minimizing the quantity

$$E_{\alpha,1}[\tilde{u}_1] = 2\big((1-x)^2 + y^2 + z^2 + (y-z)^2\big) + \frac{2}{\alpha}(x-y)^2.$$

This is easily obtained by solving

$$\frac{\partial E_1[\tilde{u}_1]}{\partial x} = \frac{\partial E_1[\tilde{u}_1]}{\partial y} = \frac{\partial E_1[\tilde{u}_1]}{\partial z} = 0,$$

which leads to the linear system

$$\begin{cases} (1+\alpha^{-1})x &= \alpha^{-1}y + 1 \\ (2+\alpha^{-1})y &= \alpha^{-1}x + z \\ 2z &= y. \end{cases} \tag{4.1.4}$$

Note that the uniqueness of the extension follows from the fact that this linear system has a unique solution, which can be calculated directly and which is given by

$$x = \frac{2+3\alpha}{5+3\alpha}, \quad y = \frac{2}{5+3\alpha}, \quad z = \frac{1}{5+3\alpha}. \tag{4.1.5}$$

Because of the symmetry, we get the same solution if we put the value 1 at any of the vertices of V_0, so if the initial values $u_0 \colon V_0 \to \mathbb{R}$ are

$$u_0(p_1) = a, \quad u_0(p_2) = b, \quad u_0(p_3) = c,$$

the harmonic extension $\tilde{u}_1$ satisfies the following "rule":

$$\tilde{u}_1(p) = \frac{2+3\alpha}{5+3\alpha}a + \frac{2}{5+3\alpha}b + \frac{1}{5+3\alpha}c,$$

for a point p whose nearest vertex in V_0 has the function value a, the second b and the furthest c as Figure 4.3 shows. This proves the proposition.

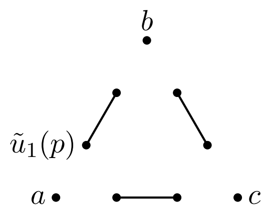


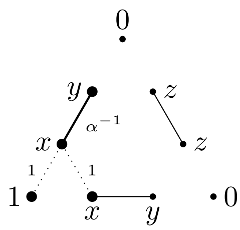
Figure 4.3: The extension $\tilde{u}_1$ at $p \in W_{\alpha,1}$ for an arbitrary u_0.

□

The expression given in (4.1.3) may be considered as a kind of "extension algorithm", where α is the length of the lines in $\mathcal{J}_{\alpha,1}$.

The equalities in (4.1.4) express the "weighted mean value property" that the function value at each point is the weighted average of the function values of the neighbouring points in the Hanoi graph H_α^1. For example, the point $p := G_{\alpha,1}(p_2)$ has three neighbours, $G_{\alpha,1}(p_1)$, $G_{\alpha,1}(p_3)$ and $G_{\alpha,2}(p_1)$, where the function takes the values $1, x$ and y respectively (see Figure below). The value of the function at this point therefore satisfies

$$\tilde{u}_1(p) := x = \frac{x \cdot 1 + y \cdot \alpha^{-1} + 1 \cdot 1}{1 + 1 + \alpha^{-1}}.$$



Our next step consists in constructing the harmonic extension to the next level of any given function $u \in \mathcal{D}_{\alpha,n}$. The extension to an arbitrary level will be obtained using the following induction argument: if we know how to construct the harmonic extension from level n to $n+1$ for any $n \in \mathbb{N}_0$, then we just need to iterate the process as many times as necessary until we reach the desired level. Note that this is only possible because K_α is p.c.f. and therefore we always get a finite number of equations to solve.

Proposition 4.1.6. *Define $d_{\alpha,0} := 0$ and for each $n \in \mathbb{N}$*

$$d_{\alpha,n} := |G_{\alpha,wj}(p_k) - G_{\alpha,wk}(p_j)|\,, \tag{4.1.6}$$

where $w \in \mathcal{A}^{n-1}$, $j,k \in \mathcal{A}$ and $j \neq k$. Then, for any function $u \in \mathcal{D}_{\alpha,n}$, the infimum

$$\inf\{E_{\alpha,n+1}[v] \mid v \in \mathcal{D}_{\alpha,n+1} \text{ and } v|_{V_{\alpha,n}} \equiv u\}$$

is attained by a unique function $\tilde{u} \in \mathcal{D}_{\alpha,n+1}$ given on $W_{\alpha,n+1}$ by

$$\begin{aligned}\tilde{u}(G_{\alpha,wi}(p_j)) = {}& \frac{2+3d_{\alpha,n}}{5+3d_{\alpha,n}} u\big(G_{\alpha,wi}(p_i)\big) \\ & + \frac{2}{5+3d_{\alpha,n}} u\big(G_{\alpha,wj}(p_j)\big) + \frac{1}{5+3d_{\alpha,n}} u\big(G_{\alpha,wk}(p_k)\big)\end{aligned}$$

for each $wi \in \mathcal{A}^{n+1}$, $\{i,j,k\} = \mathcal{A}$, and linear interpolation on $J_{\alpha,n+1} \setminus J_{\alpha,n}$.

Proof. First, note that $d_{\alpha,n}$ coincides with the length of the shortest lines in $\mathcal{J}_{\alpha,n}$ and equals $\alpha\left(\frac{1-\alpha}{2}\right)^{n-1}$.

We define for each edge $\{p,q\} \in E(H^1_\alpha)$ its so–called *conductance*

$$c^{\alpha,n}_{pq} := \begin{cases} 1 & \text{if } \{p,q\} \in E(\Gamma^1_\alpha), \\ d^{-1}_{\alpha,n} & \text{if } \{p,q\} \in E(H^1_\alpha) \setminus E(\Gamma^1_\alpha), \end{cases}$$

and for any $u\colon W_{\alpha,1} \to \mathbb{R}$ we write

$$\widetilde{E}_{\alpha,n}[u] := \sum_{\{p,q\}\in E(H^1_\alpha)} c^{\alpha,n}_{pq}\big(u(p)-u(q)\big)^2. \tag{4.1.7}$$

Given any function $u \in \mathcal{D}_{\alpha,n}$, the finite energy of any extension $v\colon V_{\alpha,n} \to \mathbb{R}$ is given by

$$\begin{aligned}E_{\alpha,n+1}[v] = E^c_{\alpha,n}[v] + \sum_{w\in\mathcal{A}^n} \sum_{x \overset{\alpha,1}{\sim} y} \big(v\circ G_{\alpha,w}(x) - v\circ G_{\alpha,w}(y)\big)^2 \\ + \int_{J_{\alpha,n+1}\setminus J_{\alpha,n}} |\nabla u|^2\,dx,\end{aligned}$$

where the first term of this sum is already known because $v|_{V_{\alpha,n}} \equiv u$. Since the harmonic extension $\tilde{u}$ minimizes energy, it will be defined on $J_{\alpha,n+1} \setminus J_{\alpha,n}$

as the linear extension of $\tilde{u}|_{W_{\alpha,n+1}}$. In this way, the integral expression above is minimized and it becomes

$$\begin{aligned}\int_{J_{\alpha,n+1}\setminus J_{\alpha,n}} |\nabla\tilde{u}|^2\,dx &= \sum_{e\in\mathcal{J}_{\alpha,n+1}\setminus\mathcal{J}_{\alpha,n}} \int_e |\nabla\tilde{u}|^2\,dx \\ &= \sum_{e\in\mathcal{J}_{\alpha,n+1}\setminus\mathcal{J}_{\alpha,n}} d_{\alpha,n+1}^{-1}\big(u(b_e)-u(a_e)\big)^2, \\ &= \sum_{w\in\mathcal{A}^n}\ \sum_{\{x,y\}\in E(H_\alpha^1)\setminus E(\Gamma_\alpha^1)} d_{\alpha,n+1}^{-1}\big(u\circ G_{\alpha,w}(x)-u\circ G_{\alpha,w}(y)\big)^2.\end{aligned}$$

Using the notation introduced in (4.1.7), the finite energy at level $n+1$ of the harmonic extension can be written as

$$E_{\alpha,n+1}[\tilde{u}] = E_{\alpha,n}^c[u] + \sum_{w\in\mathcal{A}^n} \widetilde{E}_{\alpha,n+1}[\tilde{u}\circ G_{\alpha,w}],$$

hence minimizing $E_{\alpha,n+1}[\tilde{u}]$ is equivalent to minimizing $\widetilde{E}_{\alpha,n+1}[\tilde{u}\circ G_{\alpha,w}]$ for each $w\in\mathcal{A}^n$. In this way we have reduced the initial minimization problem to $|\mathcal{A}^n| = 3^n$ local minimization problems of the sort we solved for the case $n=0$.

Applying the harmonic extension algorithm obtained in Proposition 4.1.5, we get that the harmonic extension $\tilde{u}\colon V_{\alpha,n+1}\to\mathbb{R}$ is given at each $G_{\alpha,w}(W_{\alpha,1})$, $w\in\mathcal{A}^n$, by

$$\begin{aligned}\tilde{u}\circ G_{\alpha,w}\big(G_i(p_j)\big) &= \frac{2+3d_{\alpha,n}}{5+3d_{\alpha,n}} u\big(G_{\alpha,wi}(p_i)\big) \\ &\quad + \frac{2}{5+3d_{\alpha,n}} u\big(G_{\alpha,wj}(p_j)\big) + \frac{1}{5+3d_{\alpha,n}} u\big(G_{\alpha,wk}(p_k)\big),\end{aligned}$$

and extended linearly on $G_{\alpha,w}(J_{\alpha,1})$ (see Figure 4.4).

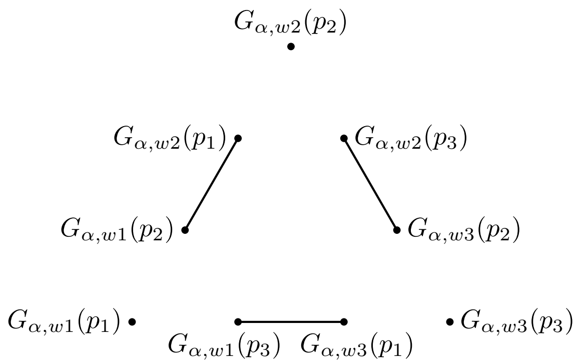


Figure 4.4: The set $G_{\alpha,w}(V_{\alpha,1})$.

Finally, since

$$\bigcup_{w\in\mathcal{A}^n}(G_{\alpha,w}(W_{\alpha,1})\cup G_{\alpha,w}(J_{\alpha,1}))=\bigcup_{w\in\mathcal{A}^n}G_{\alpha,w}(V_{\alpha,1})=V_{\alpha,n+1}\setminus J_{\alpha,n},$$

the function $\tilde{u}$ minimizing energy is defined for all $x\in V_{\alpha,n+1}$, as was to be proven. The uniqueness of the extension follows from the same argument as in Proposition 4.1.5. □

4.1.3 Renormalization factor

So far we have defined the bilinear form $E_{\alpha,n}$ just by "glueing" its discrete and continuous part $E^d_{\alpha,n}$ and $E^c_{\alpha,n}$. This means that, until now, both parts of the energy have been independent of each other. However, since we want the energy functionals to be *invariant under harmonic extension*, we still have to renormalize them. This renormalization is precisely what correlates $E^d_{\alpha,n}$ and $E^c_{\alpha,n}$.

Definition 4.1.7. A sequence of functionals $\{\mathcal{E}_n\colon\mathcal{D}_n\to\mathbb{R}\}_{n\in\mathbb{N}_0}$ is said to be *invariant under harmonic extension* if for any $u\in\mathcal{D}_n$ and its harmonic extension $\tilde{u}\in\mathcal{D}_{n+1}$ it holds that

$$\mathcal{E}_n[u]=\mathcal{E}_{n+1}[\tilde{u}].$$

Further, if $\{E_n\colon\mathcal{D}_n\to\mathbb{R}\}_{n\in\mathbb{N}_0}$ is another sequence of functionals and for each $n\in\mathbb{N}_0$ we can write $\mathcal{E}_n[u]=\rho_n^{-1}E_n[u]$ for some number $\rho_n>0$, then ρ_n is called the *renormalization factor* of E_n.

In our case, we need to introduce some notation before computing this factor.

Define for each $n\in\mathbb{N}$ the quantities

$$r^d_{\alpha,n}:=\frac{15}{(5+3d_{\alpha,n})^2}\quad\text{and}\quad r^c_{\alpha,n}:=\frac{(1-\alpha)}{2}r^d_{\alpha,n}\,, \tag{4.1.8}$$

where $d_{\alpha,n}$ was defined in (4.1.6).

In order to clarify calculations, we adopt in this section the following matrix form notation

$$E_{\alpha,n}[u]:=\begin{pmatrix}E^d_n[u]\\E^c_n[u]\end{pmatrix}\qquad\text{for } u\in\mathcal{D}_{\alpha,n}.$$

Lemma 4.1.8. *Let $u_0\in\mathcal{D}_0$ and denote by $\tilde{u}_n\in\mathcal{D}_{\alpha,n}$ its harmonic extension at level $n\in\mathbb{N}$. Then, for any $w\in\mathcal{A}^{n-1}$ we have that $\tilde{u}_n\circ G_{\alpha,w}\in\mathcal{D}_{\alpha,1}$ and*

$$E_{\alpha,1}[\tilde{u}_n\circ G_{\alpha,w}]=\frac{(5+3d_{\alpha,n})^2}{15+18d_{\alpha,n}}E_{\alpha,0}[\tilde{u}_n\circ G_{\alpha,w}]. \tag{4.1.9}$$

Further, for any $n \geq 2$ and $w \in \mathcal{A}^{n-2}$ it holds that $\tilde{u}_n \circ G_{\alpha,w} \in \mathcal{D}_{\alpha,2}$ and

$$E_{\alpha,2}[\tilde{u}_n \circ G_{\alpha,w}] = \begin{pmatrix} r^d_{\alpha,n} & 0 \\ 0 & 1 + r^c_{\alpha,n} \end{pmatrix} E_{\alpha,1}[\tilde{u}_n \circ G_{\alpha,w}]. \tag{4.1.10}$$

Proof. The equality (4.1.9) for $n = 1$ follows directly from Proposition 4.1.5: without loss of generality, we can consider again the function $u_0 \in \mathcal{D}_0$ given by $u_0(p_1) = 1$, $u_0(p_2) = 0 = u_0(p_3)$, so that $E_{\alpha,0}[\tilde{u}_1] = E_{\alpha,0}[u_0] = 6$. If we substitute in the expression of the energy

$$E_{\alpha,1}[\tilde{u}_1] = \sum_{\{x,y\} \in E(H^1_\alpha)} c^{\alpha,1}_{xy} (u(x) - u(y))^2$$

the values given by the extension algorithm in (4.1.3), we get

$$E_{\alpha,1}[\tilde{u}_1] = \frac{15}{(5 + 3d_{\alpha,1})^2} E_{\alpha,0}[\tilde{u}_1] + 2\frac{9d_{\alpha,1}}{(5 + 3d_{\alpha,1})^2} = \frac{15 + 18d_{\alpha,1}}{(5 + 3d_{\alpha,1})^2} E_{\alpha,0}[\tilde{u}_1],$$

as we wanted to prove.

Given an arbitrary $n > 1$, we obtain the very same calculations just replacing $d_{\alpha,1}$ by $d_{\alpha,n}$ as in Proposition 4.1.6.

Let us now prove equality (4.1.10) for the case $n = 2$. We consider the function $\tilde{u}_1 \in \mathcal{D}_{\alpha,1}$ being defined on $W_{\alpha,1}$ as described in Figure 4.5 and by linear interpolation on $J_{\alpha,1}$.

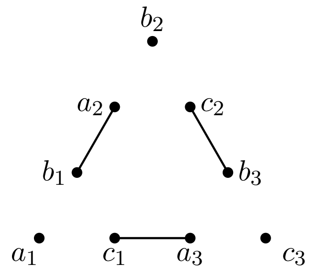


Figure 4.5: Values of $\tilde{u}_1$ in $W_{\alpha,1}$.

Note that this choice of $\tilde{u}_1$ is completely general because we know that $\tilde{u}_1 \in \mathcal{D}_{\alpha,1}$ is the harmonic extension of a function $u_0 \in \mathcal{D}_0$ and it therefore has to be linear on $J_{\alpha,1}$.

Let $\tilde{u}_2 \colon V_{\alpha,2} \to \mathbb{R}$ be the harmonic extension of $\tilde{u}_1$, whose energy at level 2 can be written using the expression (4.1.7) as

$$\begin{aligned}E_{\alpha,2}[\tilde{u}_2] &= \sum_{i=1}^{3} \widetilde{E}_{\alpha,2}[\tilde{u}_2 \circ G_{\alpha,i}] + E^c_{\alpha,1}[\tilde{u}_2] \\ &= \sum_{i=1}^{3} \widetilde{E}_{\alpha,2}[\tilde{u}_2 \circ G_{\alpha,i}] + \frac{1}{d_{\alpha,1}}\big((a_2-b_1)^2+(a_3-c_1)^2+(c_2-b_3)^2\big).\end{aligned} \tag{4.1.11}$$

The quantities to be minimized are thus

$$\widetilde{E}_{\alpha,2}[\tilde{u}_2 \circ G_{\alpha,i}] = \sum_{\{x,y\}\in E(H^1_\alpha)} c^{\alpha,2}_{xy}\big(\tilde{u}_2 \circ G_{\alpha,i}(x) - \tilde{u}_2 \circ G_{\alpha,i}(y)\big)^2$$

for each $i \in \mathcal{A}$. By construction of $\tilde{u}_2 \circ G_{\alpha,i}$ we get that

$$\begin{aligned}\widetilde{E}_{\alpha,2}[\tilde{u}_2 \circ G_{\alpha,i}] &= \frac{15}{(5+3d_{\alpha,2})^2} E_{\alpha,0}[\tilde{u}_2 \circ G_{\alpha,i}] \\ &\quad + \frac{9d_{\alpha,2}}{(5+3d_{\alpha,2})^2}((a_i-b_i)^2+(a_i-c_i)^2+(b_i-c_i)^2),\end{aligned}$$

thus (4.1.11) becomes

$$\begin{aligned}E_{\alpha,2}[\tilde{u}_2] &= \frac{15}{(5+3d_{\alpha,2})^2}\sum_{i=1}^{3} E_{\alpha,0}[\tilde{u}_2 \circ G_{\alpha,i}] \\ &\quad + \frac{9d_{\alpha,2}}{(5+3d_{\alpha,2})^2}\sum_{i=1}^{3}((a_i-b_i)^2+(a_i-c_i)^2+(b_i-c_i)^2) \\ &\quad + \frac{1}{d_{\alpha,1}}\big((a_2-b_1)^2+(a_3-c_1)^2+(c_2-b_3)^2\big).\end{aligned} \tag{4.1.12}$$

On the other hand, we know that $\tilde{u}_1$ is the harmonic extension of a function u_0 defined in V_0, i.e. the values of $\tilde{u}_1$ on $V_{\alpha,1} \setminus V_0$ can be expressed in terms of a_1, b_2 and c_3 as follows:

$$\begin{aligned}b_1 &= \frac{2+3d_{\alpha,1}}{5+3d_{\alpha,1}}a_1 + \frac{2}{5+3d_{\alpha,1}}b_2 + \frac{1}{5+3d_{\alpha,1}}c_3 \\ a_2 &= \frac{2+3d_{\alpha,1}}{5+3d_{\alpha,1}}b_2 + \frac{2}{5+3d_{\alpha,1}}a_1 + \frac{1}{5+3d_{\alpha,1}}c_3 \\ c_2 &= \frac{2+3d_{\alpha,1}}{5+3d_{\alpha,1}}b_2 + \frac{2}{5+3d_{\alpha,1}}c_3 + \frac{1}{5+3d_{\alpha,1}}a_1 \\ b_3 &= \frac{2+3d_{\alpha,1}}{5+3d_{\alpha,1}}c_3 + \frac{2}{5+3d_{\alpha,1}}b_2 + \frac{1}{5+3d_{\alpha,1}}a_1 \\ a_3 &= \frac{2+3d_{\alpha,1}}{5+3d_{\alpha,1}}c_3 + \frac{2}{5+3d_{\alpha,1}}a_1 + \frac{1}{5+3d_{\alpha,1}}b_2 \\ c_1 &= \frac{2+3d_{\alpha,1}}{5+3d_{\alpha,1}}a_1 + \frac{2}{5+3d_{\alpha,1}}c_3 + \frac{1}{5+3d_{\alpha,1}}b_2.\end{aligned} \tag{4.1.13}$$

Substituting this values in (4.1.12), we obtain

$$
\begin{aligned}
E_{\alpha,2}[\tilde{u}_2] &= \frac{15}{(5+3d_{\alpha,2})^2}\sum_{i=1}^{3} E_{\alpha,0}[\tilde{u}_2 \circ G_{\alpha,i}] \\
&+ \frac{9d_{\alpha,2}}{(5+3d_{\alpha,2})^2}\frac{15}{(5+3d_{\alpha,1})^2}\left((b_2-a_1)^2+(c_3-a_1)^2+(b_2-c_3)^2\right) \\
&+ \frac{9d_{\alpha,1}}{(5+3d_{\alpha,1})^2}\left((b_2-a_1)^2+(c_3-a_1)^2+(b_2-c_3)^2\right) \\
&= \frac{15}{(5+3d_{\alpha,2})^2}E^d_{\alpha,1}[\tilde{u}_2] + \left(1+\frac{15(1-\alpha)}{2(5+3d_{\alpha,2})^2}\right)E^c_{\alpha,1}[\tilde{u}_2] \\
&= \begin{pmatrix} r^d_{\alpha,2} & 0 \\ 0 & 1+r^c_{\alpha,2}\end{pmatrix} E_{\alpha,1}[\tilde{u}_2],
\end{aligned}
$$

as we wanted to show.

Note that this proves the result for all $n > 2$: substitute in this last proof $\tilde{u}_2 \circ G_{\alpha,i}$ by $\tilde{u}_n \circ G_{\alpha,wi}$, where $w \in \mathcal{A}^{n-2}$, and $d_{\alpha,2}$ by $d_{\alpha,n}$ $\left(= \left(\frac{1-\alpha}{2}\right)^{n-2} d_{\alpha,2}\right)$. This substitution comes from the fact that $G_{\alpha,w}(V_{\alpha,2})$ is a $\left(\frac{1-\alpha}{2}\right)^{n-2}$ −times smaller "copy" of $V_{\alpha,2}$. □

Proposition 4.1.9. *For each $n \in \mathbb{N}_0$ and $u \in \mathcal{D}_{\alpha,n+1}$ it holds that*

$$
E_{\alpha,n+1}[u] = \sum_{i=1}^{3}\begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{1-\alpha}\end{pmatrix} E_{\alpha,n}[u \circ G_{\alpha,i}] + \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix} E_{\alpha,1}[u].
$$

Proof. Decomposing the energy into its discrete and continuous part, we have to prove that

$$
E^d_{\alpha,n+1}[u] = \sum_{i=1}^{3} E^d_{\alpha,n}[u \circ G_{\alpha,i}]. \tag{4.1.14}
$$

and

$$
E^c_{\alpha,n+1}[u] = \frac{2}{1-\alpha}\sum_{i=1}^{3} E^c_{\alpha,n}[u \circ G_{\alpha,i}] + E^c_{\alpha,1}[u]. \tag{4.1.15}
$$

On one hand, for any $(\alpha, n+1)-$neighbours $x, y \in W_{\alpha,n}$, there exists $iw \in \mathcal{A}^{n+1}$ such that $x, y \in G_{\alpha,iw}(V_0)$ and $x', y' \in G_{\alpha,w}(V_0)$ such that $x = G_{\alpha,i}(x')$, $y = G_{\alpha,i}(y')$, i.e. $x' \overset{\alpha,n}{\sim} y'$.

From the definition of $E^d_{\alpha,n+1}[u]$ in (4.1.1) we obtain

$$\begin{aligned} E^d_{\alpha,n+1}[u] &= \sum_{x \overset{\alpha,n+1}{\sim} y} (u(x)-u(y))^2 \\ &= \sum_{i=1}^{3} \sum_{G_{\alpha,i}(x') \overset{\alpha,n}{\sim} G_{\alpha,i}(y')} \Big(u(G_{\alpha,i}(x')) - u(G_{\alpha,i}(y'))\Big)^2 \\ &= \sum_{i=1}^{3} \sum_{x' \overset{\alpha,n}{\sim} y'} \Big(u\circ G_{\alpha,i}(x') - u\circ G_{\alpha,i}(y')\Big)^2 \\ &= \sum_{i=1}^{3} E^d_{\alpha,n}[u\circ G_{\alpha,i}], \end{aligned}$$

hence (4.1.14) holds.

On the other hand, since $J_{\alpha,n+1} = \bigcup_{i=1}^{3} G_{\alpha,i}(J_{\alpha,n}) \cup J_{\alpha,1}$, it follows from the definition of $E^c_{\alpha,n+1}[u]$ in (4.1.2) that

$$\begin{aligned} E^c_{\alpha,n+1}[u] &= \int_{J_{\alpha,n+1}} |\nabla u|^2\, dx \\ &= \sum_{i=1}^{3} \int_{G_i(J_{\alpha,n})} |\nabla u|^2\, dx + \int_{J_{\alpha,1}} |\nabla u|^2\, dx. \end{aligned}$$

Applying the change of variables $x = G_i(y)$, we have that $dx = \frac{1-\alpha}{2} dy$ and

$$\nabla(u\circ G_i) = \nabla u(G_i) G_i' = \frac{1-\alpha}{2} \nabla u(G_i).$$

By plugging this in the expression above we get

$$\begin{aligned} E^c_{n+1}[u] &= \sum_{i=1}^{3} \frac{2}{1-\alpha} \int_{J_{\alpha,n}} |\nabla(u\circ G_{\alpha,i})|^2\, dy + \int_{J_{\alpha,1}} |\nabla u|^2\, dx \\ &= \frac{2}{1-\alpha} \sum_{i=1}^{3} E^c_{\alpha,n}[u\circ G_{\alpha,i}] + E^c_{\alpha,1}[u], \end{aligned}$$

as asserted in (4.1.15). □

Corollary 4.1.10. *Let $n \in \mathbb{N}_0$ and $u \in \mathcal{D}_{\alpha,n+1}$. Then,*

$$\begin{aligned} E_{\alpha,n+1}[u] = \sum_{w\in\mathcal{A}^m} &\begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^m \end{pmatrix} E_{\alpha,n+1-m}[u\circ G_{\alpha,w}] \\ &+ \sum_{k=0}^{m-1} \sum_{w\in\mathcal{A}^k} \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^k \end{pmatrix} E_{\alpha,1}[u\circ G_{\alpha,w}] \end{aligned}$$

holds for $0 \le m \le n$.

Proof. The case $m = 1$ is Proposition 4.1.9 and the assertion follows by induction over m. □

The next theorem determines the renormalization factor at each level $n \in \mathbb{N}_0$. Observe that, since we are using matrix notation, this "factor" turns out to be a $2\times 2-$ matrix.

Theorem 4.1.11. *Let $u_0\colon V_0 \to \mathbb{R}$ and denote by $\tilde{u}_n \in \mathcal{D}_{\alpha,n}$ its harmonic extension at level $n \in \mathbb{N}$. Then,*

(i) for $n = 0$,

$$E_{\alpha,0}[\tilde{u}_1] = \begin{pmatrix} \rho_{\alpha,0} & 0 \\ 0 & 0 \end{pmatrix} E_{\alpha,1}[\tilde{u}_1], \tag{4.1.16}$$

where $\rho_{\alpha,0} := \frac{(5+3d_{\alpha,1})^2}{15+18d_{\alpha,1}}$.

(ii) for $n \in \mathbb{N}$,

$$E_{\alpha,n}[\tilde{u}_n] = \begin{pmatrix} \rho^d_{\alpha,n} & 0 \\ 0 & \rho^c_{\alpha,n} \end{pmatrix} E_{\alpha,1}[\tilde{u}_1], \tag{4.1.17}$$

where

$$\rho^d_{\alpha,n} := \begin{cases} 1, & \text{for } n = 1, \\ \prod_{i=2}^n r^d_{\alpha,i}, & \text{for } n \geq 2, \end{cases}$$

and

$$\rho^c_{\alpha,n} := \begin{cases} 1, & \text{for } n = 1, \\ 1 + \sum_{j=2}^n \prod_{i=2}^j r^c_{\alpha,i}, & \text{for } n \geq 2, \end{cases}$$

where $r^d_{\alpha,i}$ and $r^c_{\alpha,i}$ were defined in (4.1.8) for each $i \in \mathbb{N}$.

Proof. Note that the equality (4.1.16) has already been proven in Lemma 4.1.8.

We prove the formula in (4.1.17) by induction over $n \in \mathbb{N}$: the case $n = 1$ is trivial since $\rho^d_{\alpha,1} = \rho^c_{\alpha,1} = 1$ by definition.

In order to prove the case $n + 1$, consider $u_1 \in \mathcal{D}_{\alpha,1}$ and let $\tilde{u}_{n+1} \in \mathcal{D}_{\alpha,n+1}$ be its harmonic extension at level $n + 1$. From the expression obtained in Corollary 4.1.10, we have that

$$\begin{aligned} E_{\alpha,n+1}[\tilde{u}_{n+1}] = &\sum_{w\in\mathcal{A}^{n-1}} \begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^{n-1} \end{pmatrix} E_{\alpha,2}[\tilde{u}_{n+1}\circ G_{\alpha,w}] \\ &+ \sum_{k=0}^{n-2}\sum_{w\in\mathcal{A}^k} \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^{k} \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1}\circ G_{\alpha,w}], \end{aligned} \tag{4.1.18}$$

and by Lemma 4.1.8,

$$E_{\alpha,2}[\tilde{u}_{n+1}\circ G_{\alpha,w}] = \begin{pmatrix} r^d_{\alpha,n+1} & 0 \\ 0 & 1 + r^c_{\alpha,n+1} \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1}\circ G_{\alpha,w}]$$

holds for all $w \in \mathcal{A}^{n-1}$.

On substituting this in (4.1.18) we obtain

$$E_{\alpha,n+1}[\tilde{u}_{n+1}] = \begin{pmatrix} r^d_{\alpha,n+1} & 0 \\ 0 & 1 + r^c_{\alpha,n+1} \end{pmatrix} \sum_{w \in \mathcal{A}^{n-1}} \begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^{n-1} \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1} \circ G_{\alpha,w}]$$
$$+ \sum_{k=0}^{n-2} \sum_{w \in \mathcal{A}^k} \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^k \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1} \circ G_{\alpha,w}], \qquad (4.1.19)$$

and again by Corollary 4.1.10 we can replace the first summation in (4.1.19) by

$$E_{\alpha,n}[\tilde{u}_{n+1}] - \sum_{k=0}^{n-2} \sum_{w \in \mathcal{A}^k} \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^k \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1} \circ G_{\alpha,w}],$$

which leads to

$$E_{\alpha,n+1}[\tilde{u}_{n+1}] = \begin{pmatrix} r^d_{\alpha,n+1} & 0 \\ 0 & 1 + r^c_{\alpha,n+1} \end{pmatrix} E_{\alpha,n}[\tilde{u}_{n+1}]$$
$$- \begin{pmatrix} r^d_{\alpha,n+1} & 0 \\ 0 & 1 + r^c_{\alpha,n+1} \end{pmatrix} \sum_{k=0}^{n-2} \sum_{w \in \mathcal{A}^k} \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^k \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1} \circ G_{\alpha,w}]$$
$$+ \sum_{k=0}^{n-2} \sum_{w \in \mathcal{A}^k} \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^k \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1} \circ G_{\alpha,w}]$$
$$= \begin{pmatrix} r^d_{\alpha,n+1} & 0 \\ 0 & 1 + r^c_{\alpha,n+1} \end{pmatrix} E_{\alpha,n}[\tilde{u}_{n+1}]$$
$$- \sum_{k=0}^{n-2} \sum_{w \in \mathcal{A}^k} \begin{pmatrix} 0 & 0 \\ 0 & r^c_{\alpha,n+1} \left(\frac{2}{1-\alpha}\right)^k \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1} \circ G_{\alpha,w}]. \qquad (4.1.20)$$

The last term of the expression above can be rewritten using Corollary 4.1.10 as

$$E^c_{\alpha,n-1}[\tilde{u}_{n+1}] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} E_{\alpha,n-1}[\tilde{u}_{n+1}] = \sum_{k=0}^{n-2} \sum_{w \in \mathcal{A}^k} \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{2}{1-\alpha}\right)^k \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1} \circ G_{\alpha,w}],$$

and substituting this in (4.1.20) we obtain

$$E_{\alpha,n+1}[\tilde{u}_{n+1}] = \begin{pmatrix} r^d_{\alpha,n+1} & 0 \\ 0 & 1 + r^c_{\alpha,n+1} \end{pmatrix} E_{\alpha,n}[\tilde{u}_{n+1}]$$
$$- \begin{pmatrix} 0 & 0 \\ 0 & r^c_{\alpha,n+1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} E_{\alpha,n-1}[\tilde{u}_{n+1}].$$

Applying the induction hypotheses we get

$$\begin{aligned}
E_{\alpha,n+1}[\tilde{u}_{n+1}] &= \begin{pmatrix} r^d_{\alpha,n+1} & 0 \\ 0 & 1+r^c_{\alpha,n+1} \end{pmatrix} \begin{pmatrix} \prod_{i=2}^n r^d_{\alpha,i} & 0 \\ 0 & 1+\sum_{j=2}^n \prod_{i=2}^j r^c_{\alpha,i} \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1}] \\
&- \begin{pmatrix} 0 & 0 \\ 0 & r^c_{\alpha,n+1} \end{pmatrix} \begin{pmatrix} \prod_{i=2}^{n-1} r^d_{\alpha,i} & 0 \\ 0 & 1+\sum_{j=2}^{n-1} \prod_{i=2}^j r^c_{\alpha,i} \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1}] \\
&= \begin{pmatrix} \prod_{i=2}^{n+1} r^d_{\alpha,i} & 0 \\ 0 & (1+r^c_{\alpha,n+1})\left(1+\sum_{j=2}^{n-1} \prod_{i=2}^j r^c_{\alpha,i} + \prod_{i=2}^n r^c_{\alpha,i}\right) \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1}] \\
&- \begin{pmatrix} 0 & 0 \\ 0 & r^c_{\alpha,n+1}\left(1+\sum_{j=2}^{n-1} \prod_{i=2}^j r^c_{\alpha,i}\right) \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1}] \\
&= \begin{pmatrix} \prod_{i=2}^{n+1} r^d_{\alpha,i} & 0 \\ 0 & 1+\sum_{j=2}^{n-1} \prod_{i=2}^j r^c_{\alpha,i} + \prod_{i=2}^n r^c_{\alpha,i} + \prod_{i=2}^{n+1} r^c_{\alpha,i} \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1}] \\
&= \begin{pmatrix} \prod_{i=2}^{n+1} r^d_{\alpha,i} & 0 \\ 0 & 1+\sum_{j=2}^{n+1} \prod_{i=2}^j r^c_{\alpha,i} \end{pmatrix} E_{\alpha,1}[\tilde{u}_{n+1}],
\end{aligned}$$

as we wanted to prove. □

This last theorem allows us to define the renormalization factor $\rho_{\alpha,n}$ at level $n \in \mathbb{N}_0$ by

$$\rho_{\alpha,n} := \begin{pmatrix} \rho^d_{\alpha,n} & 0 \\ 0 & \rho^c_{\alpha,n} \end{pmatrix},$$

and the renormalized energy functional $\mathcal{E}_{\alpha,n} \colon \mathcal{D}_{\alpha,n} \to \mathbb{R}$ is therefore given by

$$\mathcal{E}_{\alpha,n}[u] := \rho^{-1}_{\alpha,n} E_{\alpha,n}[u].$$

Applying the polarization identity leads to the renormalized symmetric bilinear form $\mathcal{E}_{\alpha,n} \colon \mathcal{D}_{\alpha,n} \times \mathcal{D}_{\alpha,n} \to \mathbb{R}$,

$$\mathcal{E}_{\alpha,n}(u,v) := \frac{1}{2}\left(\mathcal{E}_{\alpha,n}[u+v] - \mathcal{E}_{\alpha,n}[u] - \mathcal{E}_{\alpha,n}[v]\right).$$

which can also be written as

$$\mathcal{E}_{\alpha,n}(u,v) := (\rho^d_{\alpha,n})^{-1} E^d_{\alpha,n}(u,v) + (\rho^c_{\alpha,n})^{-1} E^c_{\alpha,n}(u,v).$$

We call $\mathcal{E}^d_{\alpha,n}(u,v) := (\rho^d_{\alpha,n})^{-1} E^d_{\alpha,n}(u,v)$ and $\mathcal{E}^c_{\alpha,n}(u,v) := (\rho^c_{\alpha,n})^{-1} E^c_{\alpha,n}(u,v)$ the *discrete* and *continuous* part of $\mathcal{E}_{\alpha,n}$.

Note that the expression $\rho^{-1}_{\alpha,n}$ is consistent in its matrix form because

$$\rho^{-1}_{\alpha,n} = \begin{pmatrix} \rho^d_{\alpha,n} & 0 \\ 0 & \rho^c_{\alpha,n} \end{pmatrix}^{-1} = \begin{pmatrix} \left(\rho^d_{\alpha,n}\right)^{-1} & 0 \\ 0 & \left(\rho^c_{\alpha,n}\right)^{-1} \end{pmatrix}.$$

The bilinear form $\mathcal{E}_{\alpha,n}$ has now the property of being invariant under harmonic extension: given $u_0 \colon V_0 \to \mathbb{R}$, its harmonic extension $\tilde{u}_n \in \mathcal{D}_{\alpha,n}$ at level $n \in \mathbb{N}$ satisfies

$$\begin{aligned}\mathcal{E}_{\alpha,n}(\tilde{u}_n, \tilde{u}_n) :=& \rho_n^{-1} E_{\alpha,n}(\tilde{u}_n, \tilde{u}_n) = \rho_n^{-1} \rho_n E_{\alpha,1}(\tilde{u}_n, \tilde{u}_n) \\ =& E_{\alpha,1}(\tilde{u}_1, \tilde{u}_1) = \mathcal{E}_{\alpha,1}(\tilde{u}_1, \tilde{u}_1).\end{aligned}$$

Hence

$$\mathcal{E}_{\alpha,0}(u_0, u_0) = \mathcal{E}_{\alpha,1}(\tilde{u}_1, \tilde{u}_1) = \mathcal{E}_{\alpha,2}(\tilde{u}_2, \tilde{u}_2) = \ldots = \mathcal{E}_{\alpha,n}(\tilde{u}_n, \tilde{u}_n) = \ldots$$

which implies that for all $n \in \mathbb{N}_0$,

$$\mathcal{E}_{\alpha,n+1}(\tilde{u}_{n+1}, \tilde{u}_{n+1}) = \min\{\mathcal{E}_{\alpha,n+1}(v, v) \mid v \in \mathcal{D}_{\alpha,n+1}, \text{ and } v|_{V_{\alpha,n}} \equiv \tilde{u}_n\}.$$

Remark 4.1.12. It is important to notice that the continuous and discrete part are no more independent of each other because both renormalization factors $\rho^d_{\alpha,n}$ and $\rho^c_{\alpha,n}$ depend on the whole energy $E_{\alpha,n}$, and therefore on both $E^d_{\alpha,n}$ and $E^c_{\alpha,n}$. These factors have also the following properties.

(1) The sequence $(r^d_{\alpha,i})_{i\in\mathbb{N}}$ is monotonically increasing and $r^d_{\alpha,i} \xrightarrow{i\to\infty} \frac{3}{5}$. Thus the sequence $(\rho^d_{\alpha,n})_{n\in\mathbb{N}}$ is monotonically decreasing and it is majorized by $\left(\left(\frac{3}{5}\right)^{n-1}\right)_{n\in\mathbb{N}}$.

(2) The sequence $(r^c_{\alpha,i})_{i\geq 2}$ is monotonically increasing and $r^c_{\alpha,i} \xrightarrow{i\to\infty} \frac{3(1-\alpha)}{10}$. This implies that the sequence $(r^c_{\alpha,i})_{i\geq 2}$ is bounded from below by 1 and from above by $\frac{3(1-\alpha)}{10}$. Hence the sequence $(\rho^c_{\alpha,n})_{n\in\mathbb{N}}$ is monotonically increasing and it holds that

$$\begin{aligned}\rho^c_{\alpha,n} &= 1 + \sum_{j=2}^{n} \prod_{i=2}^{j} r^c_{\alpha,i} \leq 1 + \sum_{j=2}^{n} \prod_{i=2}^{j} \frac{3(1-\alpha)}{10} \\ &= 1 + \sum_{j=2}^{n} \left(\frac{3(1-\alpha)}{10}\right)^{j-1} = \sum_{k=0}^{n-1} \left(\frac{3(1-\alpha)}{10}\right)^{k} \\ &\xrightarrow{n\to\infty} \sum_{k=0}^{\infty} \left(\frac{3(1-\alpha)}{10}\right)^{k} < \infty\end{aligned}$$

because $\frac{3(1-\alpha)}{10} < 1$, and we can write $\Theta_\alpha := \lim_{n\to\infty} \rho^c_{\alpha,n}$. This quantity will appear in later calculations.

4.2 Resistance and Dirichlet form on the Hanoi attractor K_α

Once we have defined the sequence of approximating forms $(\mathcal{E}_{\alpha,n})_{n\in\mathbb{N}_0}$, we proceed to establish a Dirichlet form for the whole fractal by using the theory of *resistance forms* developed by Kigami in [26, Chapter 2] and deeply studied in the case of p.c.f. self-similar fractals in [27, 40]. A short review of the basic notions of this theory is given in the Appendix and we refer to [28, Chapter 3] for more details.

We start constructing a resistance form on K_α whose associated resistance metric is compatible with the original (Euclidean) topology of K_α. This resistance form will induce our desired Dirichlet form on K_α.

4.2.1 Resistance form

Resistance forms are analytic structures for sets of functions defined on a metric space that do not require any measure on the space. While developing our analysis on K_α, we begin with the most general possible structure (a resistance form) and end up with a very specific one (the Laplacian on K_α).

For ease of notation, we write $\mathcal{E}_{\alpha,n}[u] := \mathcal{E}_{\alpha,n}(u,u)$ for any $u \in \mathcal{D}_{\alpha,n}$ and $n \in \mathbb{N}_0$, and for each function $u\colon V_{\alpha,*} \to \mathbb{R}$, we define its energy at level $n \in \mathbb{N}$ by

$$\mathcal{E}_{\alpha,n}[u] := \mathcal{E}_{\alpha,n}[u|_{V_{\alpha,n}}].$$

It follows from the definition of $\mathcal{E}_{\alpha,n}$ that the sequence $(\mathcal{E}_{\alpha,n}[u])_{n\in\mathbb{N}_0}$ is non-decreasing, so we may consider the (non-trivial) functional $\mathcal{E}_{K_\alpha}\colon \mathcal{F}^*_{K_\alpha} \to \mathbb{R}$ given by

$$\begin{cases} \mathcal{F}^*_{K_\alpha} := \{u\colon V_{\alpha,*} \to \mathbb{R} \mid \lim\limits_{n\to\infty} \mathcal{E}_{\alpha,n}[u] < \infty\}, \\ \mathcal{E}_{K_\alpha}[u] := \lim\limits_{n\to\infty} \mathcal{E}_{\alpha,n}[u] < \infty. \end{cases}$$

Let us now define $C(J_\alpha) := \{u\colon V_{\alpha,*} \to \mathbb{R} \mid u \in C([a_e,b_e])\ \forall\, e \in \mathcal{J}_\alpha\}$. Since $V_{\alpha,*}$ is dense in K_α by Lemma 2.1.2 and by Proposition 4.2.2 any function $u \in \mathcal{F}^*_{K_\alpha} \cap C(J_\alpha)$ is Hölder -and therefore uniformly- continuous on $V_{\alpha,*}$, u can be uniquely extended to a continuous function on K_α. We denote this extension again by u and set

$$\mathcal{F}_{K_\alpha} := \{u\colon K_\alpha \to \mathbb{R} \mid u|_{V_{\alpha,*}} \in \mathcal{F}^*_{K_\alpha} \cap C(J_\alpha),\, \mathcal{E}_{K_\alpha}[u] < \infty\}.$$

The rest of the paragraph is devoted to the proof of the following theorem.

Theorem 4.2.1. *$(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ is a resistance form on K_α.*

Before proving this we need a previous result.

Proposition 4.2.2. *Every function in $\mathcal{F}_{K_\alpha}$ is continuous on K_α.*

Proof. Since know from 2.1.2 that $V_{\alpha,*}$ is dense in K_α with respect to the Euclidean norm, it suffices to show continuity on $V_{\alpha,*}$. Consider $u \in \mathcal{F}_{K_\alpha}$ and $x, y \in V_{\alpha,*}$.

1. If $x, y \in W_{\alpha,n}$ are $(\alpha, n)-$neighbours, then $|x-y| = \left(\frac{1-\alpha}{2}\right)^n$ and

$$\left(\rho^d_{\alpha,n}\right)^{-1} |u(x) - u(y)|^2 \leq \mathcal{E}^d_{K_\alpha}[u] \leq \mathcal{E}_{K_\alpha}[u],$$

which implies that

$$\begin{aligned} |u(x) - u(y)| &\leq \left(\rho^d_{\alpha,n}\right)^{1/2} \mathcal{E}^{1/2}_{K_\alpha}[u] \leq \left(\frac{3}{5}\right)^{1/2} \mathcal{E}^{1/2}_{K_\alpha}[u] \\ &= \mathcal{E}^{1/2}_{K_\alpha}[u]\, |x-y|^{l_\alpha}, \end{aligned}$$

where $l_\alpha := \frac{\ln 3 - \ln 5}{2(\ln(1-\alpha) - \ln 2)}$.

2. If $x, y \in W_{\alpha,n}$ are not neighbours we proceed as follows: Consider a chain of points $x_n, y_{n+1}, x_{n+2}, y_{n+2}, \ldots, x_{n+k-1}, y_{n+k} \in V_{\alpha,*}$ such that $x_{n+j}, y_{n+j+1} \in W_{\alpha,n+j}$ are $(\alpha, n+j+1)$-neighbours and $(y_{n+j+1}, x_{n+j+1}) \in \mathcal{J}_{\alpha,n+j} \setminus \mathcal{J}_{\alpha,n+j-1}$ for each $0 \leq j \leq k-1$ (see Figure 4.6).

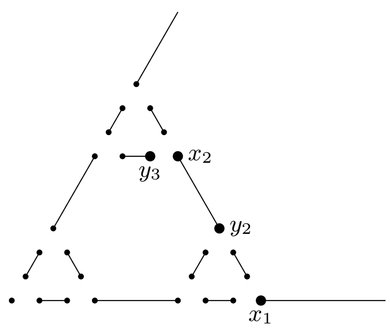


Figure 4.6: Chain with $x_1 \in W_{\alpha,1}$, $y_2, x_2 \in W_{\alpha,2}$ and $y_3 \in W_{\alpha,3}$.

If there exists some $k > 1$ such that $x := x_n \in W_{\alpha,n}$ and $y := y_{n+k} \in W_{\alpha,n+k} \setminus W_{\alpha,n+k-1}$, then, $|x_{n+j} - y_{n+j+1}| = \left(\frac{1-\alpha}{2}\right)^{n+j+1}$ and

$|y_{n+j}-x_{n+j}| = \alpha\left(\frac{1-\alpha}{2}\right)^{n+j-1}$ and we get that

$$\begin{aligned}
|u(x)-u(y)| &\le \sum_{j=0}^{k-1}|u(x_{n+j})-u(y_{n+j+1})| + \sum_{j=1}^{k-1}|u(y_{n+j})-u(x_{n+j})| \\
&\le \mathcal{E}_{K_\alpha}^{1/2}[u]\sum_{j=0}^{k-1}|x_{n+j}-y_{n+j+1}|^{l_\alpha} + \Theta_\alpha^{1/2}\,\mathcal{E}_{K_\alpha}^{1/2}[u]\sum_{j=1}^{k}|y_{n+j}-x_{n+j}|^{1/2} \\
&= \mathcal{E}_{K_\alpha}^{1/2}[u]\left(\frac{1-\alpha}{2}\right)^{(n+1)l_\alpha}\sum_{j=0}^{k-1}\left(\frac{1-\alpha}{2}\right)^{l_\alpha j} \\
&\quad + \Theta_\alpha^{1/2}\,\mathcal{E}_{K_\alpha}^{1/2}[u]\alpha^{1/2}\left(\frac{1-\alpha}{2}\right)^{\frac{n-1}{2}}\sum_{j=1}^{k}\left(\frac{1-\alpha}{2}\right)^{j/2}.
\end{aligned}$$

Since $l_\alpha < 1/2$, $\alpha\left(\frac{1-\alpha}{2}\right)^{-1} < 1$ and $\left(\frac{1-\alpha}{2}\right)^{l_\alpha} < 1$, we get that

$$\begin{aligned}
|u(x)-u(y)| &\le \mathcal{E}_{K_\alpha}^{1/2}[u]\left(\frac{1-\alpha}{2}\right)^{nl_\alpha}\sum_{j=0}^{k-1}\left(\frac{1-\alpha}{2}\right)^{l_\alpha j} \\
&\quad + \Theta_\alpha^{1/2}\,\mathcal{E}_{K_\alpha}^{1/2}[u]\left(\frac{1-\alpha}{2}\right)^{nl_\alpha}\sum_{j=1}^{k}\left(\frac{1-\alpha}{2}\right)^{l_\alpha j} \\
&\le (1+\Theta_\alpha^{1/2})\mathcal{E}_{K_\alpha}^{1/2}[u]\left(\frac{1-\alpha}{2}\right)^{nl_\alpha}\sum_{j=0}^{\infty}\left[\left(\frac{1-\alpha}{2}\right)^{l_\alpha}\right]^j \\
&= (1+\Theta_\alpha^{1/2})\mathcal{E}_{K_\alpha}^{1/2}[u]\left[1-\left(\frac{1-\alpha}{2}\right)^{l_\alpha}\right]^{-1}\left(\frac{1-\alpha}{2}\right)^{nl_\alpha}.
\end{aligned}$$

Finally, $\left(\frac{1-\alpha}{2}\right)^n \le |x-y|$ because $y \notin W_{\alpha,n}$ by assumption, hence, if we set $C := (1+\Theta_\alpha^{1/2})\left[1-\left(\frac{1-\alpha}{2}\right)^{l_\alpha}\right]^{-1}$, we obtain

$$|u(x)-u(y)| \le C\mathcal{E}_{K_\alpha}^{1/2}[u]\,|x-y|^{l_\alpha}.$$

In the case $k=0$, i.e. $x,y \in W_{\alpha,n}\setminus W_{\alpha,n-1}$ are not (α,n)-neighbours, we can join them by at most two such chains, say $x := x_n,\dots,y_{n+k}$ and $y := x'_n,\dots,y'_{n+k}$ for some $k\in\mathbb{N}$ and an extra segment (y_{n+k},y'_{n+k}) of length $\alpha\left(\frac{1-\alpha}{2}\right)^{n+k-1}$ (in the case that $y_{n+k}\neq y'_{n+k}$) so that we get

$$\begin{aligned}
|u(x)-u(y)| &\le |u(x)-u(y_{n+k})| + \left|u(y_{n+k})-u(y'_{n+k})\right| + \left|u(y'_{n+k})-u(y)\right| \\
&\le 2C\mathcal{E}_{K_\alpha}^{1/2}[u]\left(\frac{1-\alpha}{2}\right)^{(n+1)l_\alpha} + \Theta_\alpha^{1/2}\,\mathcal{E}_{K_\alpha}^{1/2}[u]\alpha^{1/2}\left(\frac{1-\alpha}{2}\right)^{\frac{n+k-1}{2}}
\end{aligned}$$

and by using again the fact that $l_\alpha < 1/2$, $\alpha\left(\frac{1-\alpha}{2}\right)^{-1} < 1$, $k \geq 1$ and $|x-y| > \left(\frac{1-\alpha}{2}\right)^{(n+1)}$, we obtain

$$\begin{aligned}|u(x)-u(y)| &\leq (2C+\Theta_\alpha^{1/2})\mathcal{E}_{K_\alpha}^{1/2}[u]\left(\frac{1-\alpha}{2}\right)^{(n+1)l_\alpha}\\ &\leq (2C+\Theta_\alpha^{1/2})\mathcal{E}_{K_\alpha}^{1/2}[u]\,|x-y|^{l_\alpha}.\end{aligned}$$

3. If x, y belong to the same component $e \in \mathcal{J}_{\alpha,n}$ for some $n \in \mathbb{N}$, u is in particular continuous on e so we have that

$$\begin{aligned}|u(x)-u(y)|^2 = \left|\int_x^y \nabla u\,dx\right|^2 &\overset{C-S}{\leq} \int_x^y |\nabla u|^2\,dx \cdot |x-y|\\ &\leq \int_e |\nabla u|^2\,dx \cdot |x-y|,\end{aligned}$$

and therefore

$$\begin{aligned}\left(\rho_{\alpha,n}^c\right)^{-1}|u(x)-u(y)|^2 &\leq \left(\rho_{\alpha,n}^c\right)^{-1}\int_e |\nabla u|^2\,dx\cdot|x-y|\\ &\leq |x-y|\left(\rho_{\alpha,n}^c\right)^{-1}\int_{J_\alpha}|\nabla u|^2\,dx\\ &\leq \mathcal{E}_{K_\alpha}[u]\,|x-y|,\end{aligned}$$

which leads to

$$\begin{aligned}|u(x)-u(y)| &\leq \left(\rho_{\alpha,n}^c\right)^{1/2}\mathcal{E}_{K_\alpha}^{1/2}[u]\,|x-y|^{1/2}\\ &\leq \Theta_\alpha^{1/2}\,\mathcal{E}_{K_\alpha}^{1/2}[u]\,|x-y|^{1/2}. \qquad (4.2.1)\end{aligned}$$

The same calculations apply if $x \in e \in \mathcal{J}_{\alpha,n}$ and $y \in W_{\alpha,n}$ is one of its endpoints.

4. If $x, y \in J_{\alpha,n} \setminus J_{\alpha,n-1}$ do not belong to the same edge, then there exists $e_1, e_2 \in J_{\alpha,n}$ such that $x \in e_1$, $y \in e_2$. Now we can join both points as follows: consider $x' \in W_{\alpha,n}$ the nearest endpoint of e_1 to x, and $y' \in W_{\alpha,n}$ the nearest in e_2 to y. Then, by an analogous calculation as the previous case we have

$$\begin{aligned}|u(x)-u(y)| &\leq \left|u(x)-u(x')\right| + \left|u(x')-u(y')\right| + \left|u(y')-u(y)\right|\\ &\leq (2C+3\Theta_\alpha^{1/2})\mathcal{E}_{K_\alpha}^{1/2}[u]\,|x-y|^{l_\alpha}. \qquad (4.2.2)\end{aligned}$$

Now, choosing $\tilde{C} := 2C + 3\Theta_\alpha^{1/2}$, it follows from inequalities (4.2.1)–(4.2.2) that

$$|u(x)-u(y)| \leq \tilde{C}\,\mathcal{E}_{K_\alpha}^{1/2}[u]\,|x-y|^{l_\alpha}$$

for all $x, y \in W_{\alpha,*}$, hence u is uniformly Hölder-continuous.

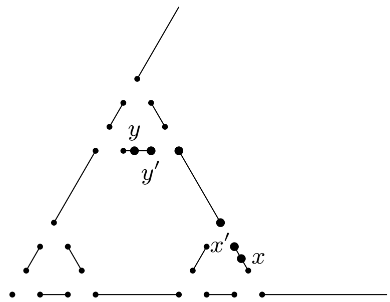


Figure 4.7: Chain with $x \in e_1$, $y \in e_2$.

5. Finally, the case when $x \in J_{\alpha,n}$ and $y \in W_{\alpha,n}$ follows by combining the two last cases.

□

Proof of theorem 4.2.1.

(RF1) $\mathcal{F}_{K_\alpha}$ is a linear subspace of $\{u\colon K_\alpha \to \mathbb{R}\}$ and $\mathcal{E}_{K_\alpha}$ is a non-negative quadratic form on $\mathcal{F}_{K_\alpha}$. Moreover, it follows from the definition of $\mathcal{E}_{\alpha,n}$ that $0 = \mathcal{E}_{K_\alpha}[u] = \lim_{n\to\infty} \mathcal{E}_{\alpha,n}[u]$ if and only if $u \equiv$ const, which implies that $\mathcal{F}_{K_\alpha}$ contains constants.

(RF2) Define the equivalence relation on $\mathcal{F}_{K_\alpha}$ by $u \sim v \;\Leftrightarrow\; u-v \equiv$ const and consider the space $(\mathcal{F}_{K_\alpha}/_\sim, \mathcal{E}_{K_\alpha})$. We prove now that this is a Hilbert space.

All properties of $\mathcal{E}_{K_\alpha}$ for being an inner product are satisfied by definition except that $\mathcal{E}_{K_\alpha}[u] = 0 \;\Leftrightarrow\; u \equiv 0$. This follows from the fact that $\mathcal{E}_{K_\alpha}[u] = 0$ if and only if $u \equiv$ const on K_α and constants are the zero class in $\mathcal{F}_{K_\alpha}/_\sim$.

In order to prove that $(\mathcal{F}_{K_\alpha}/_\sim, \mathcal{E}_{K_\alpha})$ is complete, we identify the set $\mathcal{F}_{K_\alpha}/_\sim$ with the set $\mathcal{R}_\alpha := \{u \in \mathcal{F}_{K_\alpha} \,|\, u(p_1) = 0\}$ by the isomorphism

$$\begin{array}{ccc} \mathcal{F}_{K_\alpha}/_\sim & \longrightarrow & \mathcal{R}_\alpha \\ u & \longmapsto & u - u(p_1). \end{array}$$

Let $(u_m)_{m\in\mathbb{N}_0}$ be a Cauchy sequence in $\mathcal{R}_\alpha$, $\mathcal{E}_{K_\alpha}[u_n - u_m] \xrightarrow{n,m\to\infty} 0$. For all $x \in W_{\alpha,*}$, $(u_n(x))_{n\in\mathbb{N}_0}$ is a Cauchy sequence on $\mathbb{R}$ and therefore convergent, so its limit exists.

On the other hand, by definition of $\mathcal{E}_{K_\alpha}$, we know that for each edge $e = (a_e, b_e) \in \mathcal{J}_\alpha$, $u_n|_e \in W^{1,2}(e, dx)$ and

$$\|\nabla u_n|_e - \nabla u_m|_e\|_2 \le \mathcal{E}_{K_\alpha}^{1/2}[u_n - u_m] \xrightarrow{m,n\to\infty} 0, \tag{4.2.3}$$

hence $(\nabla u_n|_e)_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $L^2(e,dx)$ and it converges. Thus, there exists $v^e \in L^2(e,dx)$ such that $\|\nabla u_n|_e - v^e\|_2 \xrightarrow{n\to\infty} 0$ and we may choose v^e to be continuous.

Now, since $a_e \in W_{\alpha,*}$, we can set $u(a_e) := \lim_{n\to\infty} u_n(a_e)$ and define

$$u^e(x) := \int_{a_e}^{x} v^e\, dx + u(a_e) \qquad \forall\, x \in (a_e, b_e).$$

We claim that the limit we were looking for is (the unique extension to K_α of) the function $u\colon V_{\alpha,*} \to \mathbb{R}$ defined by

$$u(x) := \begin{cases} \lim_{n\to\infty} u_n(x), & x \in W_{\alpha,*}, \\ u^e(x), & x \in e \in \mathcal{J}_\alpha. \end{cases} \tag{4.2.4}$$

Indeed, since $u_n \in \mathcal{R}_\alpha$, u_n is continuous, so applying (4.2.3) and the definition of u we get

$$\begin{aligned}
|u(b_e) - u_n(b_e)| &= \left| \int_{a_e}^{b_e} v^e\, dx + u(a_e) - u_n(b_e) \right| \\
&\leq \left| \int_{a_e}^{b_e} v^e\, dx + u_n(a_e) - u_n(b_e) \right| + |u(a_e) - u_n(a_e)| \\
&= \left| \int_{a_e}^{b_e} v^e\, dx - \int_{a_e}^{b_e} \nabla u_n\, dx \right| + |u(a_e) - u_n(a_e)| \\
&= \left| \int_{a_e}^{b_e} (v^e - \nabla u_n)\, dx \right| + |u(a_e) - u_n(a_e)| \\
&\leq \int_{a_e}^{b_e} |v^e - \nabla u_n|\, dx + |u(a_e) - u_n(a_e)| \\
&\leq |a_e - b_e|^{1/2}\, \|v^e - \nabla u_n|_e\|_2 + |u(a_e) - u_n(a_e)| \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Hence $u \in C(K_\alpha)$. Moreover,

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{E}_{K_\alpha}[u_n - u] &= \lim_{n\to\infty} \lim_{k\to\infty} \left(\rho^d_{\alpha,k}\right)^{-1} \sum_{x \overset{\alpha,k}{\sim} y} \big(u(x) - u_n(x) - u(y) + u_n(y)\big)^2 \\
&\quad + \left(\rho^c_{\alpha,k}\right)^{-1} \sum_{e\in\mathcal{J}_{\alpha,k}} \int_e |\nabla u^e - \nabla u_n|^2\, dx \\
&= \lim_{k\to\infty} \left(\rho^d_{\alpha,k}\right)^{-1} \sum_{x \overset{\alpha,k}{\sim} y} \lim_{n\to\infty} \big(u(x) - u_n(x) - u(y) + u_n(y)\big)^2 \\
&\quad + \lim_{k\to\infty} \left(\rho^c_{\alpha,k}\right)^{-1} \sum_{e\in\mathcal{J}_{\alpha,k}} \lim_{n\to\infty} \|v^e - \nabla u_n|_e\|_2^2 \\
&= 0,
\end{aligned}$$

and $u \in \mathcal{F}_{K_\alpha}$.

Finally, $u(p_1) = \lim_{n\to\infty} u_n(p_1) = 0$ by definition, so $u \in \mathcal{R}_\alpha$ and we are done.

(RF3) If $x, y \in K_\alpha$ are such that $x \neq y$, we can construct a function $u \in \mathcal{F}_{K_\alpha}$ with $u(x) \neq u(y)$ in the following manner: without loss of generality, we can consider $x, y \in V_{\alpha,n}$ for some $n \in \mathbb{N}_0$. Since $x \neq y$, there exists a neighbourhood of x, $B_\varepsilon(x)$ such that $y \in V_{\alpha,n} \setminus B_\varepsilon(x) (=: V_\varepsilon)$. Define $u_n \colon V_{\alpha,n} \to \mathbb{R}$ by $u_n(x) := 1$, $u_n|_{V_\varepsilon} \equiv 0$. If $x \in e \in \mathcal{J}_{\alpha,n}$, define u_n on $e = (a_e, b_e)$ to be some smooth function with $u_n(a_e) = 0 = u_n(b_e)$ and $u(x) = 1$.

Clearly, $u_n \in \mathcal{D}_{\alpha,n}$ because $\mathcal{E}_{\alpha,n}[u_n] < \infty$, thus by defining $u \colon K_\alpha \to \mathbb{R}$ to be the harmonic extension of u_n, we get that $u \in \mathcal{F}_{K_\alpha}$, $u(x) = 1$ and $u(y) = 0$, so in particular $u(x) \neq u(y)$, as desired.

(RF4) We have to prove that for any $x, y \in K_\alpha$

$$\sup\left\{\frac{|u(x)-u(y)|^2}{\mathcal{E}_{K_\alpha}[u]} \mid u \in \mathcal{F}_{K_\alpha},\, \mathcal{E}_{K_\alpha}[u] < \infty\right\} < \infty.$$

From Proposition 4.2.2 we know that there exists $\tilde{C} > 0$ such that

$$|u(x)-u(y)| \leq \tilde{C}\,\mathcal{E}_{K_\alpha}[u]\,|x-y|^{l_\alpha} \qquad \forall\, u \in \mathcal{F}_{K_\alpha},\ x, y \in K_\alpha.$$

Hence,

$$\sup\left\{\frac{|u(x)-u(y)|^2}{\mathcal{E}_{K_\alpha}[u]} \mid u \in \mathcal{F}_{K_\alpha},\, \mathcal{E}_{K_\alpha}[u] < \infty\right\} \leq \tilde{C}\,|x-y|^{2l_\alpha} < \infty \tag{4.2.5}$$

and the assertion is proved.

(RF5) $\mathcal{E}_{K_\alpha}$ satisfies the Markov property. From the definition of $\mathcal{E}_{\alpha,n}$, $n \in \mathbb{N}_0$, we see that for any $v \in \mathcal{D}_{\alpha,n}$, $0 \vee v \wedge 1 \in \mathcal{D}_{\alpha,n}$ and

$$\mathcal{E}_{\alpha,n}[0 \vee v \wedge 1] \leq \mathcal{E}_{\alpha,n}[v].$$

This implies that

$$\mathcal{E}_{K_\alpha}[0 \vee u \wedge 1] = \lim_{n\to\infty} \mathcal{E}_{\alpha,n}[0 \vee u|_{V_{\alpha,n}} \wedge 1] \leq \lim_{n\to\infty} \mathcal{E}_{\alpha,n}[u] = \mathcal{E}_{K_\alpha}[u] < \infty,$$

thus in particular $0 \vee u \wedge 1 \in \mathcal{F}_{K_\alpha}$ and we are done.

□

By [27, Proposition 2.10], the supremum in (RF4) is in fact the maximum because $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ is a resistance form. The function $R\colon K_\alpha \times K_\alpha \to [0, +\infty]$ defined by

$$R(x,y) := \max\left\{\frac{|u(x)-u(y)|^2}{\mathcal{E}_{K_\alpha}[u]} \mid u \in \mathcal{F}_{K_\alpha},\, \mathcal{E}_{K_\alpha}[u] < \infty\right\} \tag{4.2.6}$$

is thus a distance in K_α, which is called the *resistance metric* on K_α associated with $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$. It follows immediately (see [27] for a proof) that

$$|u(x)-u(y)|^2 \le \mathcal{E}_{K_\alpha}[u]R(x,y) \qquad \forall\, x,y \in K_\alpha,\, u \in \mathcal{F}_{K_\alpha},$$

and therefore any function $u \in \mathcal{F}_{K_\alpha}$ is continuous with respect to the resistance metric.

The metric R also satisfies the following property, that is crucial for the construction of the Dirichlet form.

Lemma 4.2.3. *The resistance metric R defined in* (4.2.6) *is compatible with the original topology of* $(K_\alpha, |\cdot|)$.

Proof. We follow the standard proof in [4, Proposition 7.18]. On the one hand, given a sequence $(x_n)_{n\in\mathbb{N}_0} \subseteq K_\alpha$ that converges to $x \in K_\alpha$ with respect to the Euclidean metric, it follows from (4.2.5) that

$$R(x_n, x) \le \tilde{C}\,|x_n - x|^{2l_\alpha} \xrightarrow{n\to\infty} 0$$

thus $(x_n)_{n\in\mathbb{N}}$ converges with respect to the resistance metric too.

On the other hand, let $(x_n)_{n\in\mathbb{N}_0} \subseteq K_\alpha$ converge to some $x \in K_\alpha$ with respect to the resistance metric. Then, $\forall\, \varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}_0$ such that $R(x_n, x) < \varepsilon$ for all $n \ge n_0$.

Now, for each $\varepsilon > 0$ we can consider a function $u \in \mathcal{F}_{K_\alpha}$ such that $u(x) = 1$ and $\operatorname{supp}(u) \subseteq B_\varepsilon(x)$. Then we have that

$$R(x,y) > \frac{1}{\mathcal{E}_{K_\alpha}[u]} > 0 \qquad \forall\, y \in K_\alpha \setminus B_\varepsilon(x),$$

hence there exists $n_0 \in \mathbb{N}_0$ such that $x_n \in B_\varepsilon(x)$ for all $n \ge n_0$, which means that $(x_n)_{n\in\mathbb{N}_0}$ converges with respect to the Euclidean norm too. This finishes the proof. □

4.2.2 Dirichlet form

In this section, we use the resistance form $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ to obtain our desired Dirichlet form. Dirichlet forms are defined in the context of a Banach (in our case Hilbert) space, therefore we need to introduce a locally finite measure μ_α on K_α at this point. The choice of such measure is of course not unique and we will not specify it until the next section, when this is required for the study of the associated Laplacian.

Let $L^2(K_\alpha,\mu_\alpha)$ be the Hilbert space associated with a Radon measure μ_α on K_α and define for any $u,v\in L^2(K_\alpha,\mu_\alpha)$

$$\mathcal{E}_{K_\alpha,1}(u,v) := \mathcal{E}_{K_\alpha}(u,v) + \int_{K_\alpha} uv\, d\mu_\alpha. \tag{4.2.7}$$

It follows from [26, Theorem 2.4.1] that $(\mathcal{F}_{K_\alpha}\cap L^2(K_\alpha,\mu_\alpha),\mathcal{E}^{1/2}_{K_\alpha,1})$ is a Hilbert space, so we can define $\mathcal{D}_{K_\alpha}$ as the closure of $C_0(K_\alpha)\cap\mathcal{F}_{K_\alpha}$ with respect to the norm $\mathcal{E}_{K_\alpha,1}$.

The main result of this paragraph is the following theorem.

Theorem 4.2.4. *$(\mathcal{E}_{K_\alpha},\mathcal{D}_{K_\alpha})$ is a local regular Dirichlet form on $L^2(K_\alpha,\mu_\alpha)$.*

Before proving this, we need to prove some previous results.

Proposition 4.2.5. *For any $u\in\mathcal{D}_{K_\alpha}$ and $i\in\mathcal{A}$, denote $u_i:=u\circ G_{\alpha,i}$. Then,*

$$\mathcal{E}_{K_\alpha}(u,v) = \sum_{i=1}^{3}\left(\frac{5}{3}\mathcal{E}^d_{K_\alpha}(u_i,v_i) + \frac{2}{1-\alpha}\mathcal{E}^c_{K_\alpha}(u_i,v_i)\right) + \Theta_\alpha^{-1}E^c_{\alpha,1}(u,v) \tag{4.2.8}$$

for all $u,v\in\mathcal{D}_{K_\alpha}$.

Proof. By Proposition 4.1.9 we have that

$$E^d_{\alpha,n+1}[u] = \sum_{i=1}^{3} E^d_{\alpha,n}[u\circ G_{\alpha,i}]$$

and

$$E^c_{\alpha,n+1}[u] = \frac{2}{1-\alpha}\sum_{i=1}^{3} E^c_{\alpha,n}[u\circ G_{\alpha,i}] + E^c_{\alpha,1}[u],$$

hence

$$\begin{aligned}
\mathcal{E}_{\alpha,n+1}(u,v) &= \left(\rho^d_{\alpha,n+1}\right)^{-1} E^d_{\alpha,n+1}(u,v) + \left(\rho^c_{\alpha,n+1}\right)^{-1} E^c_{\alpha,n+1}(u,v) \\
&= \left(\rho^d_{\alpha,n+1}\right)^{-1} \sum_{i=1}^{3} E^d_{\alpha,n}(u_i,v_i) \\
&+ \left(\rho^c_{\alpha,n+1}\right)^{-1} \sum_{i=1}^{3} \frac{2}{1-\alpha} E^c_{\alpha,n}(u_i,v_i) + \left(\rho^c_{\alpha,n+1}\right)^{-1} E^c_{\alpha,1}(u,v) \\
&= \frac{\rho^d_{\alpha,n}}{\rho^d_{\alpha,n+1}} \sum_{i=1}^{3} \mathcal{E}^d_{\alpha,n}(u_i,v_i) \\
&+ \frac{\rho^c_{\alpha,n}}{\rho^c_{\alpha,n+1}} \sum_{i=1}^{3} \frac{2}{1-\alpha} \mathcal{E}^c_{\alpha,n}(u_i,v_i) + \left(\rho^c_{\alpha,n+1}\right)^{-1} E_{\alpha,1}(u,v). \quad (4.2.9)
\end{aligned}$$

Now, note that

$$\frac{\rho^d_{\alpha,n}}{\rho^d_{\alpha,n+1}} = \frac{\prod_{i=1}^{n} r^d_{\alpha,i}}{\prod_{i=1}^{n+1} r^d_{\alpha,i}} = \frac{1}{r^d_{\alpha,n+1}} = \frac{\left(5+3\alpha\left(\frac{1-\alpha}{2}\right)^n\right)^2}{15} \xrightarrow{n\to\infty} \frac{5}{3},$$

and since $(\rho^c_{\alpha,n})_{n\in\mathbb{N}}$ is a convergent sequence, we have that

$$\frac{\rho^c_{\alpha,n}}{\rho^c_{\alpha,n+1}} \xrightarrow{n\to\infty} 1.$$

Letting $n \to \infty$ in both sides of the equality (4.2.9), we obtain (4.2.8). $\square$

Corollary 4.2.6. *For any $m \in \mathbb{N}_0$ and $u, v \in \mathcal{D}_{K_\alpha}$*

$$\begin{aligned}
\mathcal{E}_{K_\alpha}(u,v) &= \sum_{w\in\mathcal{A}^m} \left(\frac{5}{3}\right)^m \mathcal{E}^d_{K_\alpha}(u\circ G_{\alpha,w}, v\circ G_{\alpha,w}) \\
&+ \sum_{w\in\mathcal{A}^m} \left(\frac{2}{1-\alpha}\right)^m \mathcal{E}^c_{K_\alpha}(u\circ G_{\alpha,w}, v\circ G_{\alpha,w}) \\
&+ \Theta_\alpha^{-1} \sum_{k=0}^{m-1} \left(\frac{2}{1-\alpha}\right)^k \sum_{w\in\mathcal{A}^k} E^c_{\alpha,1}(u\circ G_{\alpha,w}, v\circ G_{\alpha,w}).
\end{aligned}$$

Lemma 4.2.7. *K_α is a compact set with respect to the resistance metric R.*

Proof. Let $\overline{V}^R_{\alpha,*}$ denote the closure of $V_{\alpha,*}$ with respect to the resistance metric and consider the mappings $G_{\alpha,i}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the IFS associated to K_α. We prove that they are contractions on the complete metric space $(\overline{V}^R_{\alpha,*}, R)$, i.e. for all $i = 1, \dots, 6$ there exists $r_i \in (0,1)$ such that

$$R(G_{\alpha,i}(x), G_{\alpha,i}(y)) \le r_i R(x,y) \qquad \forall\, x,y \in V_{\alpha,*}.$$

On one hand we know by Proposition 4.2.5 that for any $u \in \mathcal{D}_{K_\alpha}$

$$\mathcal{E}_{K_\alpha}[u] = \frac{5}{3}\sum_{i=1}^{3}\mathcal{E}_{K_\alpha}[u\circ G_{\alpha,i}] + \frac{1+5\alpha}{3(1-\alpha)}\sum_{i=1}^{3}\mathcal{E}^c_{K_\alpha}[u\circ G_{\alpha,i}] + \Theta_\alpha^{-1}E^c_{\alpha,1}[u],$$

which implies that

$$\mathcal{E}_{K_\alpha}[u] \geq \frac{5}{3}\mathcal{E}_{K_\alpha}[u\circ G_{\alpha,i}] \qquad \forall\, i=1,2,3.$$

This leads to

$$\frac{|u(G_{\alpha,i}(x)) - u(G_{\alpha,i}(y))|^2}{\mathcal{E}_{K_\alpha}[u]} \leq \frac{3}{5}\frac{|u\circ G_{\alpha,i}(x) - u\circ G_{\alpha,i}(y)|^2}{\mathcal{E}_{K_\alpha}[u\circ G_{\alpha,i}]} \qquad \forall\, u\in\mathcal{D}_{K_\alpha},$$

and since $u\circ G_{\alpha,i}\in\mathcal{D}_{K_\alpha}$, we have that

$$R(G_{\alpha,i}(x), G_{\alpha,i}(y)) \leq \frac{3}{5}R(x,y) \qquad \forall\, x,y\in V_{\alpha,*}.$$

Hence $G_{\alpha,i}$ are contractions with respect to the resistance metric with contraction factor $r_i = \frac{3}{5}$ for $i=1,2,3$.

On the other hand, for each $i = 4,5,6$, there exists $e \in \mathcal{J}_{\alpha,1}$ such that $\operatorname{supp}(u\circ G_{\alpha,i}) \subseteq e \in \mathcal{J}_{\alpha,1}$. Thus,

$$\begin{aligned}
\mathcal{E}_{K_\alpha}[u\circ G_{\alpha,i}] &= \Theta_\alpha^{-1}\int_{J_{\alpha,1}} |\nabla(u\circ G_{\alpha,i})|^2\, dx \\
&= \Theta_\alpha^{-1}\int_e |\nabla u(G_{\alpha,i}(x))|^2\, \alpha^2 dx \\
&= \Theta_\alpha^{-1}\int_{G_{\alpha,i}(e)} |\nabla u(y)|^2\, \alpha^2\frac{dy}{\alpha} \qquad\qquad (y = G_{\alpha,i}(x)) \\
&\leq \alpha\Theta_\alpha^{-1}\sum_{e\in J_\alpha}\int_e |\nabla u|^2\, dy \\
&\leq \alpha\mathcal{E}_{K_\alpha}[u].
\end{aligned}$$

and we obtain

$$R(G_{\alpha,i}(x), G_{\alpha,i}(y)) \leq \alpha R(x,y) \qquad \forall\, x,y\in V_{\alpha,*}.$$

Thus, $G_{\alpha,i}$ is also a contraction with respect to the resistance form with contraction factor $r_i = \alpha$ for $i=4,5,6$ and $\{G_{\alpha,i}\}_{i=1}^6$ is a family of contractions on the complete metric space $(\overline{V}^R_{\alpha,*}, R)$. Hence there exists a unique non-empty compact set $\widetilde{K}_\alpha \subseteq \overline{V}^R_{\alpha,*}$ such that

$$\widetilde{K}_\alpha = \bigcup_{i=1}^{6} G_{\alpha,i}(\widetilde{K}_\alpha).$$

From [26, Lemma 1.3.11], the set $S := \bigcup_{n\in\mathbb{N}_0}\bigcup_{w\in\{1,\dots,6\}^n} G_{\alpha,w}(V_0)$ is dense in $\widetilde{K}_\alpha$ with respect to R and dense in K_α with respect to the Euclidean metric, thus by Lemma 4.2.3

$$\widetilde{K}_\alpha = \overline{S}^R = \overline{S}^{|\cdot|} = K_\alpha,$$

hence K_α is compact with respect to the resistance metric as we wanted to prove. □

Remark 4.2.8. Note that this Lemma implies that $C_0(K_\alpha) = C(K_\alpha)$, so Lemma 4.2.2 and (RF2) in Theorem 4.2.1 imply that in this case we actually have $\mathcal{D}_{K_\alpha} = \mathcal{F}_{K_\alpha}$.

Proof of Theorem 4.2.4. On one hand, it follows from Lemma 4.2.7 that $\mathbb{C}_0(K_\alpha) = C(K_\alpha)$ and $\mathcal{D}_{K_\alpha} = \mathcal{F}_{K_\alpha}$. On the other hand, it follows from [28, Corollary 6.4] that $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ is regular, hence $\mathcal{D}_{K_\alpha}$ is dense in $C(K_\alpha)$ with respect to the uniform norm. Due to the properties of the measure μ_α, we know from classical analysis (see [38, Thm.3.14.] for a proof) that $C(K_\alpha)$ is dense in $L^2(K_\alpha, \mu_\alpha)$. Hence $\mathcal{D}_{K_\alpha}$ is also dense in $L^2(K_\alpha, \mu_\alpha)$ and by [28, Theorem 9.4], $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ is a regular Dirichlet form.

Let us now consider $u, v \in \mathcal{D}_{K_\alpha}$ such that $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$. Since $\mathrm{supp}(u)$ and $\mathrm{supp}(v)$ are compact sets, there exists $n \in \mathbb{N}$ such that for all $w \in \mathcal{A}^n$, either $\mathrm{supp}(u) \cap G_{\alpha,w}(K_\alpha) = \emptyset$ or $\mathrm{supp}(v) \cap G_{\alpha,w}(K_\alpha) = \emptyset$. By Corollary 4.2.6 we get that $\mathcal{E}_{K_\alpha}(u, v) = 0$, hence the form is local. □

4.3 Laplacian on K_α

It is a known fact from the theory of Dirichlet forms that $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ defines a Laplacian on K_α in the weak sense as follows.

Definition 4.3.1. The *Laplacian on* K_α is the unique non-positive self-adjoint operator $\Delta_{\mu_\alpha} \colon \mathcal{D}(\Delta_{\mu_\alpha}) \to L^2(K_\alpha, \mu_\alpha)$ such that $\mathcal{D}(\Delta_{\mu_\alpha})$ is a dense subset of $\mathcal{D}_{K_\alpha}$ and for any $u \in \mathcal{D}(\Delta_{\mu_\alpha})$

$$\mathcal{E}_{K_\alpha}(u, v) = (-\Delta_{\mu_\alpha} u, v)_{\mu_\alpha} \qquad \forall\, v \in \mathcal{D}_{K_\alpha}.$$

We say that $u \in \mathcal{D}(\Delta_{\mu_\alpha})$ if and only if $u \in \mathcal{D}_{K_\alpha}$ and there exists $f \in L^2(K_\alpha, \mu_\alpha)$ such that

$$\mathcal{E}_{K_\alpha}(u, v) = -\int_{K_\alpha} f v \, d\mu_\alpha \qquad \forall\, v \in \mathcal{D}_{K_\alpha}.$$

In this case, we define $\Delta_{\mu_\alpha} u := f$.

4.3.1 A Borel regular measure on K_α

From the definition of the Laplacian Δ_{μ_α}, it is clear that this operator strongly depends on the measure μ_α and in general there is no *canonical* choice of it. The measure constructed here has been chosen in this particular manner for technical reasons. We will discuss this point in the outlook after this chapter.

From now on, we fix the following Radon probability measure μ_α on K_α.

Let $\lambda(\cdot)$ denote the $1-$dimensional Hausdorff measure, set $d_\alpha := \dim_H K_\alpha$ and consider some positive number β satisfying

$$0 < \beta < \beta^{1/2} < \frac{2}{3(1-\alpha)}. \tag{4.3.1}$$

On the one hand, let μ_α^d be the Radon measure on $\mathbb{R}^2$ with support on F_α (recall the definition of this set from Lemma 2.1.1) given by the normalized $d_\alpha-$dimensional Hausdorff measure

$$\mu_\alpha^d(A) := \frac{1}{\mathcal{H}^{d_\alpha}(F_\alpha)}\mathcal{H}^{d_\alpha}_{|F_\alpha}(A), \qquad A \subseteq \mathbb{R}^2 \text{ Borel.}$$

On the other hand, let μ_α^c be the the Radon measure on $\mathbb{R}^2$ with support on J_α defined by

$$\mu_\alpha^c(A) := \frac{1}{\tilde{\mu}^c_{\alpha,\beta}(J_\alpha)}\tilde{\mu}^c_{\alpha,\beta}(A) \qquad \text{for } A \subseteq \mathbb{R}^2 \text{ Borel,}$$

where

$$\tilde{\mu}^c_{\alpha,\beta}(A) := \sum_{e\in\mathcal{J}_\alpha} \beta_e \lambda(A\cap e)$$

and $\beta_e := \beta^k$ for $e \in \mathcal{J}_{\alpha,k+1} \setminus \mathcal{J}_{\alpha,k}$, β being the constant chosen in (4.3.1).

By Proposition 2.1.1, K_α can be decomposed into the disjoint union $F_\alpha \,\dot\cup\, J_\alpha$, so we may define the Radon measure on $\mathbb{R}^2$ as the sum

$$\mu_\alpha(A) := \frac{1}{2}\left(\mu_\alpha^d(A\cap F_\alpha) + \mu_\alpha^c(A\cap J_\alpha)\right) \qquad \text{for } A \subseteq \mathbb{R}^2 \text{ Borel.}$$

Note that $\operatorname{supp}(\mu_\alpha) = K_\alpha$ and $\mu_\alpha(K_\alpha) = 1$.

Remark 4.3.2.

(1) The choice of β in the definition of $\tilde{\mu}^c_{\alpha,\beta}$ ensures that $\tilde{\mu}^c_{\alpha,\beta}(J_\alpha) < \infty$. We have that

$$\begin{aligned}\tilde{\mu}^c_{\alpha,\beta}(J_\alpha) &= \sum_{k=0}^{\infty} \beta^k \sum_{e\in\mathcal{J}_{\alpha,k+1}\setminus\mathcal{J}_{\alpha,k}} \lambda(e) = \sum_{k=0}^{\infty} \beta^k \sum_{e\in\mathcal{J}_{\alpha,k+1}\setminus\mathcal{J}_{\alpha,k}} \alpha\left(\frac{1-\alpha}{2}\right)^k \\ &= \sum_{k=0}^{\infty} \beta^k 3^{k+1}\alpha\left(\frac{1-\alpha}{2}\right)^k = 3\alpha\sum_{k=0}^{\infty}\left(\frac{3\beta(1-\alpha)}{2}\right)^k\end{aligned}$$

and since $\frac{3\beta(1-\alpha)}{2} < 1$, this quantity is finite. The measure μ_α is thus locally finite.

(2) Let $\mathcal{A}^* := \bigcup_{n\in\mathbb{N}} \mathcal{A}^n$ be the set of all words on the alphabet $\mathcal{A}$ of finite length. The measure μ_α is a Borel probability measure and belongs to the set

$$\mathcal{M}(K_\alpha) := \left\{\mu \;\Big|\; \begin{array}{c}\mu \text{ is a probability measure on } K_\alpha,\ \mu(\{x\}) = 0\ \forall\, x\in K_\alpha,\\ \mu(G_{\alpha,w}(K_\alpha)) > 0 \text{ and } \mu(G_{\alpha,w}(V_0)) = 0 \text{ for any } w\in\mathcal{A}^*\end{array}\right\}.$$

(3) We will prove in Lemma 4.4.5 that μ_α is an *elliptic* measure, i.e. there exists $\gamma\in(0,\infty)$ such that $\mu_\alpha(G_{\alpha,wi}(K_\alpha)) \geq \gamma\mu_\alpha(G_{\alpha,w}(K_\alpha))$ for all $w\in\mathcal{A}^*$, $i\in\mathcal{A}$.

(4) For each $w\in\mathcal{A}^*$, define $\mu^w_\alpha := \frac{1}{\mu_\alpha(G_{\alpha,w}(K_\alpha))}\mu_\alpha\circ G_{\alpha,w}$. We have that $\mu^w_\alpha\in\mathcal{M}(K_\alpha)$ and for any Borel measurable function $u\colon K_\alpha\to\mathbb{R}$,

$$\begin{aligned}\int_{K_\alpha} u\circ G_{\alpha,w}\, d\mu^w_\alpha(x) &= \frac{1}{\mu_\alpha(G_{\alpha,w}(K_\alpha))}\int_{K_\alpha} u\circ G_{\alpha,w}\, d\mu_\alpha(G_{\alpha,w}(x)) \\ &= \frac{1}{\mu_\alpha(G_{\alpha,w}(K_\alpha))}\int_{G_{\alpha,w}(K_\alpha)} u\, d\mu_\alpha.\end{aligned}$$

4.3.2 Normal derivatives and Gauss-Green formula

In this paragraph we would like to establish a Gauss-Green formula for our Laplacian Δ_{μ_α} as we have in the classical case

$$\int_\Omega \nabla u\nabla v\, dx = -\int_\Omega (\Delta u)v\, dx + \int_{\partial\Omega}\frac{\partial u}{\partial\nu}v\, d\sigma$$

for $\Omega\subseteq\mathbb{R}$ open and $d\sigma$ the surface measure on $\partial\Omega$. Since the boundary of K_α consists of just three points, we expect something of the form

$$\mathcal{E}_{K_\alpha}(u,v) = -\int_{K_\alpha}(\Delta_{\mu_\alpha}u)v\, d\mu_\alpha + \sum_{x\in V_0}\frac{\partial u}{\partial\nu}(x)v(x),$$

where $\frac{\partial u}{\partial\nu}$ denotes the normal derivative of u on the boundary V_0 which is given in [39, Definition 2.3.1] as follows.

Definition 4.3.3. Let $x \in V_0$ and $u \in \mathcal{D}_{K_\alpha}$. The *normal derivative* of u at x is defined by

$$\frac{\partial u}{\partial \nu}(x) = \lim_{n\to\infty} \left(\rho^d_{\alpha,n}\right)^{-1} \sum_{y \overset{\alpha,n}{\sim} x} (u(x) - u(y)),$$

whenever the limit exists.

The next theorem shows how this definition fits into the Gauss-Green formula for Δ_{μ_α}.

Theorem 4.3.4. *Let $u \in \mathcal{D}(\Delta_{\mu_\alpha})$. Then, $\frac{\partial u}{\partial \nu}(x)$ exists for all $x \in V_0$ and the Gauss-Green formula*

$$\mathcal{E}_{K_\alpha}(u, v) = -\int_{K_\alpha} (\Delta_{\mu_\alpha} u) v \, d\mu_\alpha + \sum_{x \in V_0} \frac{\partial u}{\partial \nu} u(x) v(x)$$

holds for all $v \in \mathcal{D}_{K_\alpha}$.

Proof. The proof can be found in [39, Theorem 2.3.2], but we include a more detailed version for completeness. Without loss of generality, we may choose $v \in \mathcal{D}_{K_\alpha}$ with $v(p_1) = 1$ and $v(p_2) = 0 = v(p_3)$.

Then, for all $n \in \mathbb{N}_0$ we have that

$$\begin{aligned}
\mathcal{E}_{\alpha,n}(u, v) &= \mathcal{E}^d_{\alpha,n}(u, v) + \mathcal{E}^c_{\alpha,n}(u, v) \\
&= \left(\rho^d_{\alpha,n}\right)^{-1} \sum_{\substack{y \overset{\alpha,n}{\sim} x \\ x \in V_0}} \big(u(x) - u(y)\big)v(x) - \big(u(x) - u(y)\big)v(y) \\
&\quad + \left(\rho^d_{\alpha,n}\right)^{-1} \sum_{\substack{y \overset{\alpha,n}{\sim} x \\ x,y \notin V_0}} \big(u(x) - u(y)\big)\big(v(x) - v(y)\big) + \mathcal{E}^c_{\alpha,n}(u, v) \\
&= \left(\rho^d_{\alpha,n}\right)^{-1} \sum_{y \overset{\alpha,n}{\sim} p_1} \big(u(p_1) - u(y)\big) + \mathcal{E}^d_{\alpha,n}(u, v_0) + \mathcal{E}^c_{\alpha,n}(u, v_0),
\end{aligned}$$

where $v_0 \equiv v$ on $K_\alpha \setminus V_0$ and $v_0 \equiv 0$ on V_0.

Taking the limit $n \to \infty$ in both sides we get

$$\mathcal{E}_{K_\alpha}(u, v) = \lim_{n\to\infty} \left(\rho^d_{\alpha,n}\right)^{-1} \sum_{y \overset{\alpha,n}{\sim} p_1} \big(u(p_1) - u(y)\big) + \mathcal{E}_{K_\alpha}(u, v_0),$$

which implies that

$$\frac{\partial u}{\partial \nu}(p_1) = \mathcal{E}_{K_\alpha}(u, v) - \mathcal{E}_{K_\alpha}(u, v_0) < \infty,$$

proving the existence of the normal derivative $\frac{\partial u}{\partial \nu}$ at p_1 and therefore on V_0.
Finally, we may argue analogously for an arbitrary $v \in \mathcal{D}_{K_\alpha}$ and get

$$\mathcal{E}_{\alpha,n}(u,v) = \left(\rho^d_{\alpha,n}\right)^{-1} \sum_{\substack{y \overset{\alpha,n}{\sim} x \\ x \in V_{\alpha,0}}} \big(u(x) - u(y)\big)v(x) + \mathcal{E}^d_{\alpha,n}(u,v_0) + \mathcal{E}^c_{\alpha,n}(u,v_0)$$

for all $n \geq 0$.
Taking the limit $n \to \infty$ in both sides of the equality we obtain

$$\begin{aligned}\mathcal{E}_{K_\alpha}(u,v) &= \sum_{x \in V_0} v(x) \lim_{n\to\infty} \left(\rho^d_{\alpha,n}\right)^{-1} \sum_{y \overset{\alpha,n}{\sim} x} \big(u(x) - u(y)\big) + \mathcal{E}_{K_\alpha}(u,v_0) \\ &= \sum_{x \in V_0} \frac{\partial u}{\partial \nu}(x)v(x) - \int_{K_\alpha} (\Delta_{\mu_\alpha} u) v_0 \, d\mu_\alpha.\end{aligned}$$

Since $v = v_0$ μ_α–a.e. in K_α, last equality becomes the Gauss-Green formula

$$\mathcal{E}_{K_\alpha}(u,v) = -\int_{K_\alpha} (\Delta_{\mu_\alpha} u) v \, d\mu_\alpha + \sum_{x \in V_0} \frac{\partial u}{\partial \nu}(x)v(x).$$

□

Note that since the Laplacian Δ_{μ_α} is the operator associated to $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$, we know that for any $u \in \mathcal{D}(\Delta_{\mu_\alpha})$

$$\mathcal{E}_{K_\alpha}(u,v) = -(\Delta_{\mu_\alpha} u, v)_{\mu_\alpha} = -\int_{K_\alpha} (\Delta_{\mu_\alpha} u) v \, d\mu_\alpha \qquad \forall\, v \in \mathcal{D}_{K_\alpha},$$

so it follows directly from this last theorem that

$$\sum_{x \in V_0} \frac{\partial u}{\partial \nu}(x)v(x) = 0 \qquad \forall\, v \in \mathcal{D}_{K_\alpha},$$

hence

$$\frac{\partial u}{\partial \nu}(x) = 0 \qquad \forall\, x \in V_0.$$

The set V_0 was defined to be the *boundary* of K_α, therefore last equality means that all functions in the domain of Δ_{μ_α} have normal derivative zero on the boundary, i.e. they satisfy homogeneous Neumann boundary conditions. Thus from now on we will adopt the notation $\Delta^N_{\mu_\alpha}$ for the Laplacian associated to $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$.

The Laplacian subject to Dirichlet boundary conditions is defined analogously as the non-positive self-adjoint operator $\Delta^D_{\mu_\alpha}$ associated to the Dirichlet form $(\mathcal{E}^0_{K_\alpha}, \mathcal{D}^0_{K_\alpha})$ given by

$$\begin{cases} \mathcal{D}^0_{K_\alpha} := \{u \in \mathcal{D}_{K_\alpha} \mid u|_{V_0} \equiv 0\}, \\ \mathcal{E}^0_{K_\alpha} := \mathcal{E}_{K_\alpha}|_{\mathcal{D}^0_{K_\alpha} \times \mathcal{D}^0_{K_\alpha}}. \end{cases}$$

Note that the domain $\mathcal{D}^0_{K_\alpha}$ sets explicitly Dirichlet boundary conditions and Theorem 4.3.4 holds trivially for $\Delta^D_{\mu_\alpha}$ because $v = 0$ on V_0 for all $v \in \mathcal{D}^0_{K_\alpha}$ and therefore

$$\sum_{x\in V_0} \frac{\partial u}{\partial \nu}(x)v(x) = 0 \qquad \forall\, u \in \mathcal{D}(\Delta^D_{\mu_\alpha}).$$

In the same way as we did for $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$, one can prove that $(\mathcal{E}^0_{K_\alpha}, \mathcal{D}^0_{K_\alpha})$ is a resistance form and all results that derive from this fact hold in this case too.

4.3.3 Green's function

In this paragraph we define the Green function for the Laplacian with Neumann (resp. Dirichlet) boundary conditions by means of resistance forms. Hereby we refer to [27, Section 4].

Definition 4.3.5. Let V be a finite set. For any point $x \in V$, the characteristic function $\chi^V_x : V \to \{0, 1\}$ is defined as

$$\chi^V_x(y) := \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 4.3.6. *Let $V \subseteq V_{\alpha,*}$ be a finite set. Then for any $u_0 \colon V \to \mathbb{R}$, there exists a unique $\tilde{u} \in \mathcal{D}_{K_\alpha}$ such that $\tilde{u}|_V = u_0$ and*

$$\mathcal{E}_{K_\alpha}[\tilde{u}] = \min\{\mathcal{E}_{K_\alpha}[v] \mid v \in \mathcal{D}_{K_\alpha} \text{ and } v|_V = u_0\}.$$

Proof. See[26, Lemma 2.3.5]. □

We denote the harmonic extension $\tilde{u}$ appearing in the above proposition by $h_V(u)$.

Definition 4.3.7. For any finite subset $V \subseteq V_{\alpha,*}$ define $\mathscr{H}_V := \mathrm{Im}(h_V)$. If $\tilde{u} = h_V(u_0)$ for $u_0 \colon V \to \mathbb{R}$, then $\tilde{u} \in \mathscr{H}_V$ is called the *$V-$harmonic function with boundary value* u_0 with respect to $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$. Also, $h_V(\chi^V_x)$ is denoted by ψ^V_x for any $x \in V$.

Note that $\mathscr{H}_V$ is spanned by $\{\psi^V_x\}_{x\in V}$, so for any $u \in \mathscr{H}_V$ we can write $u = \sum_{x\in V} u(x)\psi^V_x$.

From now on, we fix the set V to be V_0 and write $\psi_x := \psi^{V_0\cup x}_x$ for each $x \in V_{\alpha,*}$.

Proposition 4.3.8. *The space $\mathscr{H}_{V_0}$ of harmonic functions on $L^2(K_\alpha, \mu_\alpha)$ with respect to $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ has dimension 3.*

Proof. This follows directly from the fact that $\mathscr{H}_{V_0}$ is spanned by $\{\psi_x^{V_0}\}_{x\in V_0}$ and $\#V_0 = 3$. □

If we define

$$\begin{aligned} P_{V_0}\colon \mathcal{F}_{K_\alpha} &\longrightarrow \mathscr{H}_{V_0} \\ u &\longmapsto h_{V_0}(u|_{V_0}), \end{aligned}$$

any $u \in \mathcal{F}_{K_\alpha}$ can be thus written as

$$u = \underbrace{P_{V_0}u}_{\in \mathscr{H}_{V_0}} + \underbrace{(u - P_{V_0}u)}_{\in \mathcal{F}^0_{K_\alpha}},$$

where $\mathcal{F}^0_{K_\alpha} := \{u \in \mathcal{F}_{K_\alpha} \,|\, u|_{V_0} \equiv 0\}$. Thus $\mathcal{F}_{K_\alpha}$ (or equivalently $\mathcal{D}_{K_\alpha}$) may be decomposed into the sum $\mathcal{F}_{K_\alpha} = \mathscr{H}_{V_0} \oplus \mathcal{F}^0_{K_\alpha}$. Moreover, by [27, Lemma 2.20], $\mathcal{E}_{K_\alpha}(u, v) = 0$ for any $u \in \mathscr{H}_{V_0}$ and $v \in \mathcal{F}^0_{K_\alpha}$, so $\mathscr{H}_{V_0}$ may be seen as the "orthogonal complement" of $\mathcal{F}^0_{K_\alpha}$ (and vice versa) with respect to $\mathcal{E}_{K_\alpha}$, although this is not an inner product on $\mathcal{F}_{K_\alpha}$.

Definition 4.3.9. Let the function $\tilde{g}\colon V_{\alpha,*} \times V_{\alpha,*} \to \mathbb{R}$ be defined by

$$\tilde{g}(x, y) := \begin{cases} 0, & \text{if } x \in V_0, \\ (\mathcal{E}_{K_\alpha}[\psi_x])^{-1}\, \psi_x(y), & \text{otherwise .} \end{cases} \tag{4.3.2}$$

The extension of this function to K_α, $g\colon K_\alpha \times K_\alpha \to \mathbb{R}$ is called the *Green function* of the resistance form $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ with boundary V_0.

This function is symmetric and $g(x, y) = 0$ if $x \in V_0$ or $y \in V_0$. Further properties can be read in [27, Proposition 4.2].

Proposition 4.3.10. *For each $x \in K_\alpha$, let $g^x(y) := g(x, y)$ be the Green function defined in* (4.3.2)*. Then,*

$$\mathcal{E}_{K_\alpha}(g^x, u) = u(x) - \sum_{y \in V_0} u(y)\psi_y(x)$$

for any $u \in \mathcal{D}_{K_\alpha}$.

Proof. Notice that $\mathcal{D}^0_{K_\alpha} = \mathcal{F}^0_{K_\alpha}$. We prove first that

$$\mathcal{E}_{K_\alpha}(g^x, u) = u(x) \qquad \forall\, u \in \mathcal{D}^0_{K_\alpha} \text{ and } x \in K_\alpha. \tag{4.3.3}$$

We may suppose that $x \notin V_0$ (otherwise the result is trivial) and define $V_0^x := V_0 \cup \{x\}$.

Then,

$$\begin{aligned}\mathcal{E}_{K_\alpha}(g^x,u) &= \mathcal{E}_{K_\alpha}[\psi_x]^{-1}\mathcal{E}_{K_\alpha}(\psi_x,u)\\ &= \mathcal{E}_{K_\alpha}[\psi_x]^{-1}\left(\mathcal{E}_{K_\alpha}(\psi_x, P_{V_0^x}u + u - P_{V_0^x}u)\right).\end{aligned}$$

Since $u - P_{V_0^x}u \equiv 0$ in V_0^x and $\psi_x \in \mathscr{H}_{V_0^x}$, it follows that

$$\mathcal{E}_{K_\alpha}(\psi_x, u - P_{V_0^x}u) = 0$$

so we get

$$\begin{aligned}\mathcal{E}_{K_\alpha}(g^x,u) &= \mathcal{E}_{K_\alpha}[\psi_x]^{-1}\mathcal{E}_{K_\alpha}(\psi_x,P_{V_0^x}u)\\ &= \mathcal{E}_{K_\alpha}[\psi_x]^{-1}\mathcal{E}_{K_\alpha}\big(\psi_x,\sum_{y\in V_0^x}u(y)\psi_y\big)\\ &= \mathcal{E}_{K_\alpha}[\psi_x]^{-1}\mathcal{E}_{K_\alpha}(\psi_x,u(x)\psi_x)\\ &= u(x),\end{aligned}$$

as we wanted to prove.

Now, given any function $u \in \mathcal{D}_{K_\alpha}$ we have that

$$\mathcal{E}_{K_\alpha}(g^x,u) = \mathcal{E}_{K_\alpha}(g^x, P_{V_0}u + u - P_{V_0}u).$$

Since $\psi_x \in \mathcal{D}^0_{K_\alpha}$ and $P_{V_0}u \in \mathscr{H}_{V_0}$, it follows that

$$\mathcal{E}_{K_\alpha}(\psi_x, P_{V_0}u) = 0,$$

and therefore

$$\mathcal{E}_{K_\alpha}(g^x,u) = \mathcal{E}_{K_\alpha}(g^x, u - P_{V_0}u).$$

Finally, $u - P_{V_0}u \in \mathcal{D}^0_{K_\alpha}$ and we obtain from (4.3.3)

$$\begin{aligned}\mathcal{E}_{K_\alpha}(g^x,u) &= \mathcal{E}_{K_\alpha}(g^x,u-P_{V_0}u)\\ &= u(x) - P_{V_0}u(x)\\ &= u(x) - \sum_{y\in V_0}u(y)\psi_y(x),\end{aligned}$$

as required. □

As a consequence, the last result of this paragraph shows that g is the reproducing kernel of the form $(\mathcal{E}^0_{K_\alpha}, \mathcal{D}^0_{K_\alpha})$.

Theorem 4.3.11. *Given $f \in C(K_\alpha)$, there exists a unique $u \in \mathcal{D}(\Delta^D_{\mu_\alpha})$ such that*

$$\Delta^D_{\mu_\alpha}u = f$$

and this function u is given by

$$u(x) = -\int_{K_\alpha} f(y)g(x,y)\,d\mu_\alpha(y).$$

Proof. Suppose there exist $u, u' \in \mathcal{D}(\Delta^D_{\mu_\alpha})$ such that $\Delta^D_{\mu_\alpha} u = \Delta^D_{\mu_\alpha} u'$. Then $\mathcal{E}^0_{K_\alpha}(u - u', v) = 0$ for all $v \in \mathcal{D}^0_{K_\alpha}$, which implies that $u = u'$.

From the definition of $\Delta^D_{\mu_\alpha} u$ we know that

$$\mathcal{E}_{K_\alpha}(u, v) = -\int_{K_\alpha} f v \, d\mu_\alpha \qquad \forall\, v \in \mathcal{D}^0_{K_\alpha}.$$

In particular, if we take $v = g^x$, by Proposition 4.3.10 we obtain that

$$\mathcal{E}_{K_\alpha}(u, g^x) = u(x) - \sum_{y \in V_0} u(y)\psi_y(x),$$

and since $u \in \mathcal{D}^0_{K_\alpha}$, $u = 0$ on V_0, which leads to

$$u(x) = \mathcal{E}_{K_\alpha}(u, g^x) = \int_{K_\alpha} f(y) g(x, y) \, d\mu_\alpha(y),$$

as we wanted to prove. □

4.3.4 The spectrum of the Laplacian

We already announced that the theory of resistance forms is a very useful tool in order to characterise the spectra $\sigma(-\Delta^N_{\mu_\alpha})$ and $\sigma(-\Delta^D_{\mu_\alpha})$, as the following theorem shows.

Theorem 4.3.12. *The operator $-\Delta^N_{\mu_\alpha}$ has pure point spectrum consisting of countable many non-negative eigenvalues with finite multiplicity and only accumulation point at $+\infty$. The same holds for the operator $-\Delta^D_{\mu_\alpha}$.*

Proof. By Lemma 4.2.7, K is compact in the resistance metric R. Hence it follows from [28, Lemma 9.7] that the inclusion map

$$(\mathcal{D}_{K_\alpha}, \mathcal{E}_{K_\alpha,1}) \hookrightarrow L^2(K_\alpha, \mu_\alpha)$$

is a compact operator. This together with the fact that $-\Delta^N_{\mu_\alpha}$ is a non-negative self-adjoint operator on $L^2(K_\alpha, \mu_\alpha)$ implies that the operator $-\Delta^N_{\mu_\alpha}$ has a compact resolvent (see [9, Exercise 4.2]). Thus by [9, Theorem 4.5.1], $\sigma(-\Delta^N_{\mu_\alpha})$ is a countable set, all eigenvalues have finite multiplicity and the only accumulation point is $+\infty$.

The same arguments writing $\mathcal{D}^0_{K_\alpha}$ instead of $\mathcal{D}_{K_\alpha}$ hold and prove the theorem for $-\Delta^D_{\mu_\alpha}$. □

4.4 Spectral dimension

In the last paragraph we showed that $-\Delta^N_{\mu_\alpha}$ and $-\Delta^D_{\mu_\alpha}$ are self-adjoint operators with a discrete spectrum $\{\kappa_i\}_{i=1}^\infty$ whose eigenvalues have finite multiplicity and an only accumulation point at $+\infty$. This property allows us to ask ourselves about the distribution of these eigenvalues in $[0,+\infty)$. To this purpose we study the asymptotic behaviour of the *eigenvalue counting function* associated to each operator.

Definition 4.4.1. Let H be a Hilbert space and $L\colon \mathcal{D}(L) \to H$ be some densely defined operator. The *eigenvalue counting function* of L is defined (if possible) for each $x \geq 0$ as

$$N_L(x) := \#\{\kappa \mid \kappa \text{ eigenvalue of } L \text{ and } \kappa \leq x\}$$

counted with multiplicity.

In our case, we write $N_N(x) := N_{-\Delta^N_{\mu_\alpha}}(x)$ and $N_D(x) := N_{-\Delta^D_{\mu_\alpha}}(x)$.

Remark 4.4.2. Let H be a Hilbert space and $(\mathcal{E},\mathcal{D})$ a Dirichlet form on H. We say that $\kappa \in \mathbb{R}$ is an *eigenvalue of* $\mathcal{E}$ with eigenfunction $u \in \mathcal{D}$ if and only if $\mathcal{E}(u,v) = \kappa(u,v)$ for all $v \in \mathcal{D}$. The eigenvalue counting function can therefore be defined also for a Dirichlet form $(\mathcal{E},\mathcal{D})$ on a Hilbert space H as

$$N(x;\mathcal{E},\mathcal{D}) := \#\{\kappa \mid \kappa \text{ eigenvalue of } \mathcal{E} \text{ and } \kappa \leq x\}$$

for any $x \geq 0$.

Moreover, we know from [32, Proposition 4.2] that $N_N(x) = N(x;\mathcal{E}_{K_\alpha},\mathcal{D}_{K_\alpha})$ and $N_D(x) = N(x;\mathcal{E}^0_{K_\alpha},\mathcal{D}^0_{K_\alpha})$.

Our interest in this section is the *spectral dimension* of K_α, that describes the asymptotic scaling in the eigenvalue counting function and it is defined as the number $d_S(K_\alpha) > 0$ such that

$$\frac{2\log N_N(x)}{\log x} \asymp d_S(K_\alpha) \asymp \frac{2\log N_D(x)}{\log x}. \tag{4.4.1}$$

The following estimate of the eigenvalue counting function is therefore crucial to determine $d_S(K_\alpha)$.

Theorem 4.4.3. *There exist constants* $C_{\alpha,1}, C_{\alpha,\beta,1}, C_{\alpha,2}, C_{\alpha,\beta,2} > 0$ *depending on* α *and* β*, and* $x_0 > 0$ *such that*

$$C_{\alpha,1}x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,1}x^{1/2} \leq N_D(x) \leq N_N(x) \leq C_{\alpha,2}x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,2}x^{1/2} \tag{4.4.2}$$

for all $x \geq x_0$.

The proof of this result will be divided into several lemmas and it follows mainly the ideas of Kajino in [23], based on the *minimax principle* (also called *variational principle*) for the eigenvalues of non-negative self-adjoint operators.

4.4.1 Preliminaries

In this paragraph we prove some technical results that will be used in the lemmas leading to Theorem 4.4.3. As usual, we work with the alphabet $\mathcal{A}$ and the set of words of finite length, $\mathcal{A}^* = \bigcup_{n\in\mathbb{N}} \mathcal{A}^n$. Moreover, given any word $w \in \mathcal{A}^*$, we write $K_{\alpha,w} := G_{\alpha,w}(K_\alpha)$.

Lemma 4.4.4. *For any $m \in \mathbb{N}_0$ and $w \in \mathcal{A}^m$ it holds that*

$$\mu_\alpha(K_{\alpha,w}) = \frac{1}{2}\left(\frac{1}{3^m} + \left(\beta\frac{1-\alpha}{2}\right)^m\right).$$

Proof. Since $G_{\alpha,w}$ is a contraction with factor $\left(\frac{1-\alpha}{2}\right)^m$ and $\left(\frac{1-\alpha}{2}\right)^{d_\alpha} = \frac{1}{3}$, by definition of μ_α^d we have that

$$\mu_\alpha^d(G_{\alpha,w}(F_\alpha)) = \left(\frac{1}{3}\right)^m \mu_\alpha^d(F_\alpha). \tag{4.4.3}$$

Further, by definition of μ_α^c and since the length of the largest lines in $G_{\alpha,w}(J_\alpha)$ is $\alpha\left(\frac{1-\alpha}{2}\right)^m$, we get that

$$\begin{aligned}\tilde{\mu}_\alpha^c(G_{\alpha,w}(J_\alpha)) &= \sum_{k=1}^\infty 3^k\alpha\left(\frac{1-\alpha}{2}\right)^{m+k-1}\beta^{m+k-1}\\ &= 3\alpha\left(\frac{1-\alpha}{2}\right)^m\beta^m\sum_{k=0}^\infty\left(\frac{3\beta(1-\alpha)}{2}\right)^k\\ &= \left(\frac{1-\alpha}{2}\right)^m\beta^m\tilde{\mu}_\alpha^c(J_\alpha). \end{aligned}\tag{4.4.4}$$

Hence, $\mu_\alpha^c(G_{\alpha,w}(J_\alpha)) = \frac{1}{\tilde{\mu}_\alpha^c(J_\alpha)}\tilde{\mu}_\alpha^c(G_{\alpha,w}(J_\alpha)) = \left(\frac{1-\alpha}{2}\right)^m\beta^m$.

Finally, applying (4.4.3) and (4.4.4) we obtain

$$\begin{aligned}\mu_\alpha(K_{\alpha,w}) &= \frac{1}{2}\left(\mu_\alpha^d(G_{\alpha,w}(F_\alpha)) + \mu_\alpha^c(G_{\alpha,w}(J_\alpha))\right)\\ &= \frac{1}{2}\left(\frac{1}{3^m} + \left(\beta\frac{1-\alpha}{2}\right)^m\right),\end{aligned}$$

as we wanted to prove. ∎

Lemma 4.4.5. *The measure μ_α is elliptic.*

Proof. Consider $w \in \mathcal{A}^*$ and $i \in \mathcal{A}$. Then, $w \in \mathcal{A}^m$ for some $m \in \mathbb{N}_0$ and since $\frac{1}{3} \geq \beta\frac{1-\alpha}{2}$, we have that

$$\begin{aligned}\mu_\alpha(K_{\alpha,wi}) &= \frac{1}{2}\left(\frac{1}{3^{m+1}} + \left(\beta\frac{1-\alpha}{2}\right)^{m+1}\right)\\ &\geq \beta\frac{1-\alpha}{4}\left(\frac{1}{3^m} + \left(\beta\frac{1-\alpha}{2}\right)^{m}\right)\\ &= \beta\frac{1-\alpha}{2}\mu_\alpha(K_{\alpha,w}).\end{aligned}$$

Hence choosing $\gamma := \beta\frac{1-\alpha}{2} \in (0,1)$ we have that $\mu_\alpha(K_{\alpha,wi}) \geq \gamma\mu_\alpha(K_{\alpha,w})$ for any $w \in \mathcal{A}^*$ and $i \in \mathcal{A}$, as we wanted to prove. □

We finish this paragraph with a definition and a remark that connect directly with the beginning of the proof of Theorem 4.4.3.

Definition 4.4.6. For any non-empty set $U \subseteq K_\alpha$, we define

$$\mathcal{C}_U := \{u \in \mathcal{D}_{K_\alpha} : \operatorname{supp}(u) \subseteq U\}, \qquad \mathcal{D}_U := \overline{\mathcal{C}}_U,$$

where the closure is taken with respect to $\mathcal{E}_{K_\alpha,1}$, and write $\mathcal{E}_U := \mathcal{E}_{K_\alpha}|_{\mathcal{D}_U\times\mathcal{D}_U}$. The pair $(\mathcal{E}_U, \mathcal{D}_U)$ is called the *part of the Dirichlet form* $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ *on* U.

Remark 4.4.7. Since $u \equiv 0$ μ_α–a.e. on $K_\alpha \setminus U$ for any $u \in \mathcal{D}_U$, we can regard $\mathcal{D}_U$ as a subspace of $L^2(U, \mu_\alpha|_U)$. In the case when $U \subseteq K_\alpha$ is open, we know from [14, Theorem 4.4.3] that $(\mathcal{E}_U, \mathcal{D}_U)$ is a Dirichlet form on $L^2(U, \mu_\alpha|_U)$. We denote by H_U the non-negative self-adjoint operator on $L^2(U, \mu_\alpha|_U)$ associated to $(\mathcal{E}_U, \mathcal{D}_U)$.

4.4.2 Spectral asymptotics of the Laplacian

This section is devoted to the proof of Theorem 4.4.3, that we divide into two parts: the lower and the upper bound.

Upper bound of (4.4.2)

First of all, note that for any $m \in \mathbb{N}$, $J_{\alpha,m}$ is an open set, hence we know from Remark 4.4.7 that $(\mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}})$ is a Dirichlet form on $L^2(J_{\alpha,m}, \mu_\alpha|_{J_{\alpha,m}})$. Moreover, since $J_{\alpha,m}$ is just a finite union of 1–dimensional open intervals, it follows from the definition of μ_α that in fact $\mathcal{D}_{J_{\alpha,m}}$ can be identified with the Sobolev space $\bigoplus_{e\in\mathcal{J}_{\alpha,m}} H_0^1(e, dx)$.

Lemma 4.4.8. *For each $m \in \mathbb{N}$, the non-negative self-adjoint operator $H_{J_{\alpha,m}}$ associated with the Dirichlet form $(\mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}})$ on $L^2(J_{\alpha,m}, \mu_\alpha|_{J_{\alpha,m}})$ has compact resolvent. Further, there exists a constant $C_{\alpha,\beta,2} > 0$ depending on α and β, and $x_0 > 0$ such that*

$$N_{J_{\alpha,m}}(x) \leq C_{\alpha,\beta,2} x^{1/2} \tag{4.4.5}$$

for all $x \geq x_0$.

Proof. Note that the operator $H_{J_{\alpha,m}}$ is nothing but the classical Laplacian Δ restricted to the 1−dimensional subset $J_{\alpha,m}$ which has compact resolvent since $J_{\alpha,m}$ is of finite length.

Let us now prove the inequality (4.4.5). Let $u \in \mathcal{D}_{J_{\alpha,m}}$ be an eigenfunction of $(\mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}})$ with eigenvalue κ, i.e.

$$\mathcal{E}_{J_{\alpha,m}}(u, v) = \kappa (u, v)_{\mu_\alpha|_{J_{\alpha,m}}} \qquad \forall\, v \in \mathcal{D}_{J_{\alpha,m}}. \tag{4.4.6}$$

Now consider $e \in \mathcal{J}_{\alpha,m}$ and for any $h \in H_0^1(e, dx)$ define

$$\tilde{h}(x) := \begin{cases} h(x), & \text{if } x \in e, \\ 0, & \text{if } x \in J_{\alpha,m} \setminus e. \end{cases}$$

Then, $\tilde{h} \in \mathcal{D}_{J_{\alpha,m}}$ and we get from (4.4.6) that

$$\begin{aligned} \int_e \nabla u \nabla h \, dx &= \Theta_\alpha \lim_{n\to\infty} \left(\rho^c_{\alpha,n}\right)^{-1} \sum_{e \in \mathcal{J}_{\alpha,m}} \int_e \nabla u \nabla \tilde{h} \, dx \\ &= \Theta_\alpha \mathcal{E}_{J_{\alpha,m}}(u, \tilde{h}) = \Theta_\alpha \kappa (u, \tilde{h})_{\mu_\alpha|J_{\alpha,m}} \\ &= \Theta_\alpha \kappa \int_e uh \, d\mu_\alpha(x) \\ &= \Theta_\alpha \kappa \beta_e \int_e uh \, dx, \end{aligned}$$

where $\beta_e = \beta^n$ for $e \in \mathcal{J}_{\alpha,n+1} \setminus \mathcal{J}_{\alpha,n}$ and $\Theta_\alpha = \lim_{n\to\infty} \rho^c_{\alpha,n}$.

Thus,

$$\int_e \nabla u \nabla h \, dx = \kappa \Theta_\alpha \beta_e \int_e uh \, dx \qquad \forall\, h \in H_0^1(e, dx),$$

which implies that $\kappa\Theta_\alpha\beta_e$ is an eigenvalue of the classical Laplacian $-\Delta$ on $L^2(e, dx)$ subject to Dirichlet boundary conditions with eigenfunction $u|_e$.

Conversely, it is easy to see that if for any $e \in \mathcal{J}_\alpha$, $\kappa\Theta_\alpha\beta_e$ is an eigenvalue of the classical Laplacian $-\Delta$ on $L^2(e, dx)$ subject to Dirichlet boundary conditions with eigenfunction $u \in H_0^1(e, dx)$, then κ is an eigenvalue of $(\mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}})$ with eigenfunction

$$\tilde{u}(x) := \begin{cases} u(x), & x \in e, \\ 0, & x \in J_{\alpha,m} \setminus e. \end{cases}$$

Since $J_{\alpha,n}$ is the disjoint union of all its connected components e, if we denote by $N_e(x)$ the eigenvalue counting function of the classical $-\Delta|_e$ subject to Dirichlet boundary conditions for each $e \in \mathcal{J}_{\alpha,n}$, we have that

$$N_{J_{\alpha,m}}(x) = \sum_{e\in\mathcal{J}_{\alpha,m}} N_e\left(\beta_e\Theta_\alpha x\right). \tag{4.4.7}$$

Since these components $e \in \mathcal{J}_{\alpha,m}$ are 1−dimensional, we know from Weyl's theorem [42] that

$$N_e(x) = \frac{\lambda(e)}{\pi}x^{1/2} + o(x^{1/2}) \qquad \text{as } x \to \infty$$

for each $e \in \mathcal{J}_{\alpha,m}$ and in particular, there exists $\tilde{c}_1 > 0$ (independent of e) and $x_0 > 0$ such that

$$N_e(\beta_e\Theta_\alpha x) \leq \tilde{c}_1\frac{(\beta_e\Theta_\alpha)^{1/2}}{\pi}\lambda(e)x^{1/2} \qquad \forall\, x \in\geq x_0.$$

By substituting this in (4.4.7) we get

$$N_{J_{\alpha,m}}(x) \leq \sum_{e\in\mathcal{J}_{\alpha,m}} \tilde{c}_1\frac{(\beta_e\Theta_\alpha)^{1/2}}{\pi}\lambda(e)x^{1/2} \leq \frac{\tilde{c}_1\Theta_\alpha^{1/2}}{\pi}x^{1/2}\sum_{e\in\mathcal{J}_\alpha}\beta_e^{1/2}\lambda(e),$$

and since β_e was chosen in (4.3.1) so that $\sum_{e\in\mathcal{J}_\alpha}\beta_e^{1/2}\lambda(e) < \infty$, by setting

$$C_{\alpha,\beta,2} := \frac{\tilde{c}_1\Theta_\alpha^{1/2}}{\pi}\sum_{e\in\mathcal{J}_\alpha}\beta_e^{1/2}\lambda(e),$$

the assertion is proved. □

Let us now define for each $m \in \mathbb{N}$ the set $K_{\alpha,m} := \bigcup_{w\in\mathcal{A}^m} K_{\alpha,w}$ and consider the pair $(\mathcal{E}_{K_{\alpha,m}}, \mathcal{D}_{K_{\alpha,m}})$ given by

$$\begin{cases} \mathcal{D}_{K_{\alpha,m}} := (\mathcal{D}_{J_{\alpha,m}})^\perp, \\ \mathcal{E}_{K_{\alpha,m}} := \mathcal{E}_{K_\alpha}|_{\mathcal{D}_{K_{\alpha,m}}\times\mathcal{D}_{K_{\alpha,m}}}, \end{cases} \tag{4.4.8}$$

where $(\mathcal{D}_{J_{\alpha,m}})^\perp$ denotes the orthogonal complement of $\mathcal{D}_{J_{\alpha,m}}$ with respect to the inner product $\mathcal{E}_{K_\alpha,1}$ defined in (4.2.7). By definition, $u \equiv 0$ μ_α−a.e. on $K_\alpha \setminus K_{\alpha,m} (= J_{\alpha,m})$ for all $u \in \mathcal{D}_{K_{\alpha,m}}$, hence $\mathcal{D}_{K_{\alpha,m}}$ can be regarded as a subspace of $L^2(K_{\alpha,m}, \mu_\alpha|_{K_{\alpha,m}})$.

Lemma 4.4.9. *The pair* $(\mathcal{E}_{K_{\alpha,m}}, \mathcal{D}_{K_{\alpha,m}})$ *defined in* (4.4.8) *is a Dirichlet form on* $L^2(K_{\alpha,m}, \mu_\alpha|_{K_{\alpha,m}})$.

Proof. First we show that $\mathcal{D}_{K_{\alpha,m}}$ is a dense subspace of $L^2(K_{\alpha,m}, \mu_\alpha|_{K_{\alpha,m}})$. Any function $u \in L^2(K_{\alpha,m}, \mu_\alpha|_{K_{\alpha,m}})$ may be extended by zero to a function $\tilde{u} \in L^2(K_\alpha, \mu_\alpha)$ that can be approximated in the L^2-norm by a sequence $(\tilde{u}_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{K_\alpha}$ such that $\tilde{u}_n = \tilde{v}_n + \tilde{w}_n$, where $\tilde{v}_n \in \mathcal{D}_{K_{\alpha,m}}$ and $\tilde{w}_n \in \mathcal{D}_{J_{\alpha,m}}$ for each $n \in \mathbb{N}$.

Since $\operatorname{supp}(\tilde{u}) \subseteq K_{\alpha,m}$ and $\operatorname{supp}(\tilde{v}_n) \subseteq J_{\alpha,m}$, we have that

$$\|\tilde{u} - \tilde{u}_n\|^2_{L^2(K_\alpha)} = \int_{K_{\alpha,m}} |u - \tilde{v}_n|^2 \, d\mu_\alpha + \int_{J_{\alpha,m}} |\tilde{w}_n|^2 \, d\mu_\alpha,$$

thus $\|\tilde{w}_n\|_{L^2(K_\alpha)} = \|\tilde{w}_n\|_{L^2(J_{\alpha,m})} \xrightarrow{n\to\infty} 0$ and hence

$$\begin{aligned}\|u - \tilde{v}_n\|_{L^2(K_{\alpha,m})} &= \|\tilde{u} - \tilde{v}_n\|_{L^2(K_{\alpha,m})} \\ &\leq \|\tilde{u} - \tilde{u}_n\|_{L^2(K_\alpha)} + \|\tilde{w}_n\|_{L^2(K_\alpha)} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Therefore, $(\tilde{v}_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}_{K_{\alpha,m}}$ approximates u in the corresponding L^2-norm.

Secondly, $(\mathcal{D}_{K_{\alpha,m}}, \mathcal{E}_{K_\alpha,1})$ is a Hilbert space because $\mathcal{D}_{K_{\alpha,m}}$ is a closed subspace of $\mathcal{D}_{K_\alpha}$.

Finally, the Markov property is inherited from the form $\mathcal{E}_{K_\alpha}$ and we are done. □

Although the following lemma may not seem very special, it is in fact essential for the proof of Theorem 4.4.3: here we use the idea of decomposing K_α (and therefore the domain of the Laplacian) into two distinguished pieces where we have a better control of the eigenvalues.

Lemma 4.4.10. *For any $m \in \mathbb{N}_0$, let $H_{J_{\alpha,m}}$ be the non-negative self-adjoint operator on $L^2(J_{\alpha,m}, \mu_\alpha|_{J_{\alpha,m}})$ associated with the Dirichlet form $(\mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}})$ and let $H_{K_{\alpha,m}}$ be the non-negative self-adjoint operator on $L^2(K_{\alpha,m}, \mu_\alpha|_{K_{\alpha,m}})$ associated with the Dirichlet form $(\mathcal{E}_{K_{\alpha,m}}, \mathcal{D}_{K_{\alpha,m}})$. Then, $H_{J_{\alpha,m}}$ and $H_{K_{\alpha,m}}$ have compact resolvent and for $N_{J_{\alpha,m}}(x) := N(x; \mathcal{E}_{J_{\alpha,m}}, \mathcal{D}_{J_{\alpha,m}}) = N_{H_{J_{\alpha,m}}}(x)$ and $N_{K_{\alpha,m}}(x) := N(x; \mathcal{E}_{K_{\alpha,m}}, \mathcal{D}_{K_{\alpha,m}}) = N_{H_{K_{\alpha,m}}}(x)$, we have that*

$$N_N(x) \leq N_{K_{\alpha,m}}(x) + N_{J_{\alpha,m}}(x)$$

holds for any $x \in [0, \infty)$.

Proof. The statements about compactness of the resolvent are proved in Lemma 4.4.8 and Lemma 4.4.13 respectively.

On one hand, we have that $L^2(K_\alpha, \mu_\alpha) = L^2(K_{\alpha,m}, \mu_\alpha|_{K_{\alpha,m}}) \oplus L^2(J_{\alpha,m}, \mu_\alpha|_{J_{\alpha,m}})$ because $K_{\alpha,m} \cap J_{\alpha,m} = \emptyset$ and by definition of $(\mathcal{E}_{K_{\alpha,m}}, \mathcal{D}_{K_{\alpha,m}})$ we have that $\mathcal{E}_{K_\alpha} = \mathcal{E}_{K_{\alpha,m}} \oplus \mathcal{E}_{J_{\alpha,m}}$.

On the other hand, $\mathcal{D}_{K_\alpha} \subseteq \mathcal{D}_{K_{\alpha,m}} \oplus \mathcal{D}_{J_{\alpha,m}}$ and it follows from [32, Proposition 4.2, Lemma 4.2] that

$$N_N(x) \leq N(x; \mathcal{E}_{K_\alpha}, \mathcal{D}_{K_{\alpha,m}} \oplus \mathcal{D}_{J_{\alpha,m}}) = N_{K_{\alpha,m}}(x) + N_{J_{\alpha,m}}(x),$$

as we wanted to prove. □

We recall now the following result from spectral theory of self-adjoint operators.

Lemma 4.4.11. *Let $(\mathcal{E}, \mathcal{D})$ be a Dirichlet form on a Hilbert space H and let A be non-negative self-adjoint operator on H associated with it. Further, define*

$$\kappa(L) := \sup\left\{\mathcal{E}(u,u) \mid u \in L,\, \|u\|_H = 1\right\}, \quad L \subseteq \mathcal{D} \textit{ subspace},$$

and

$$\kappa_n := \inf\{\kappa(L) \mid L \textit{ subspace of } \mathcal{D},\, \dim L = n\}.$$

If the sequence $\{\kappa_n\}_{n=1}^\infty$ is unbounded, then the operator A has compact resolvent.

Proof. This follows from [9, Theorem 4.5.3], the converse of [9, Theorem 4.5.2] and [9, Corollary 4.2.3]. □

The proof of the next lemma will make use of the following so–called *uniform Poincaré inequality.*

Definition 4.4.12. A Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(K_\alpha, \mu_\alpha)$ is said to satisfy the *uniform Poincaré inequality* if and only if $\exists\, C_P > 0$ such that for any $w \in \mathcal{A}^*$ and all $u \in \{u \in L^2(K_\alpha, \mu_\alpha^w) \mid \exists\, v \in \mathcal{D} \cap C(K_\alpha),\ u \equiv v \circ G_{\alpha,w}\}$

$$\mathcal{E}(u,u) \geq C_P \int_{K_\alpha} \left|u - \overline{u}^{\mu_\alpha^w}\right|^2 d\mu_\alpha^w,$$

where μ_α^w is the measure defined in Remark 4.3.2-(5) and $\overline{u}^{\mu_\alpha^w} := \int_{K_\alpha} u\, d\mu_\alpha^w$.

In our case, the uniform Poincaré inequality holds for the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ by [23, Proposition 4.4] because $\mathcal{F}_{K_\alpha} = \mathcal{D}_{K_\alpha} \subseteq C(K_\alpha)$ and $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ is a resistance form whose associated resistance metric is compatible with the original (Euclidean) topology of K_α by Lemma 4.2.3.

Lemma 4.4.13. *Let $m \in \mathbb{N}_0$ and define*

$$\kappa(L) := \sup\left\{\mathcal{E}_{K_{\alpha,m}}[u] \mid u \in L,\, \int_{K_{\alpha,m}} |u|^2 = 1\right\}, \quad L \subseteq \mathcal{D}_{K_{\alpha,m}} \textit{ subspace},$$
$$\kappa_n := \inf\{\kappa(L) \mid L \textit{ subspace of } \mathcal{D}_{K_{\alpha,m}},\, \dim L = n\}.$$

Then, there exists a constant $C_P > 0$ such that

$$\kappa_{3^m+1} \geq 5^m C_P. \tag{4.4.9}$$

In particular, the non-negative self-adjoint operator on $L^2(K_{\alpha,m}, \mu_\alpha|_{K_{\alpha,m}})$ associated with $(\mathcal{E}_{K_{\alpha,m}}, \mathcal{D}_{K_{\alpha,m}})$ has compact resolvent.

Proof. The last assertion follows from Lemma 4.4.11 in view of inequality (4.4.9). Let us thus prove it. Define

$$\widetilde{\mathcal{D}}_{K_{\alpha,m}} := \left\{ u \in L^2(K_{\alpha,m}, \mu_\alpha|_{K_{\alpha,m}}) \;\middle|\; \begin{array}{c} \exists\, v \in \mathcal{D}_{K_{\alpha,m}},\ u \circ G_{\alpha,w} = v \circ G_{\alpha,w} \\ \text{on } K_{\alpha,m}\ \forall\, w \in \mathcal{A}^m \end{array} \right\},$$

$$\widetilde{\mathcal{E}}_{K_{\alpha,m}}[u] := \sum_{w \in \mathcal{A}^m} \left(\frac{5}{3}\right)^m \mathcal{E}^d_{K_\alpha}[u \circ G_{\alpha,w}] + \left(\frac{2}{1-\alpha}\right)^m \mathcal{E}^c_{K_\alpha}[u \circ G_{\alpha,w}],$$

and

$$\tilde{\kappa}(L) := \sup\left\{ \widetilde{\mathcal{E}}_{K_{\alpha,m}}[u] \mid u \in L,\ \int_{K_{\alpha,m}} |u|^2 = 1 \right\}, \quad L \subseteq \widetilde{\mathcal{D}}_{K_{\alpha,m}} \text{ subspace},$$
$$\tilde{\kappa}_n := \inf\{\tilde{\kappa}(L) \mid L \text{ subspace of } \widetilde{\mathcal{D}}_{K_{\alpha,m}},\ \dim L = n\}.$$

It is clear that $\mathcal{D}_{K_{\alpha,m}} \subseteq \widetilde{\mathcal{D}}_{K_{\alpha,m}}$ and by Corollary 4.2.6, $\widetilde{\mathcal{E}}_{K_{\alpha,m}}$ coincides with $\mathcal{E}_{K_{\alpha,m}}$ on $\mathcal{D}_{K_{\alpha,m}}$. Hence the minimax principle implies that $\kappa_n \geq \tilde{\kappa}_n$ for all $n \in \mathbb{N}_0$.

Let us consider $L_0 := \{\sum_{w \in \mathcal{A}^m} a_w \mathbb{1}_{K_{\alpha,m}} \mid a_w \in \mathbb{R}\}$, which is a 3^m-dimensional subspace of $\widetilde{\mathcal{D}}_{K_{\alpha,m}}$. Note that $\widetilde{\mathcal{E}}_{K_{\alpha,m}}|_{L_0 \times L_0} \equiv 0$. Now, let $L \subseteq \widetilde{\mathcal{D}}_{K_{\alpha,m}}$ be a (3^m+1)-dimensional subspace of $\widetilde{\mathcal{D}}_{K_{\alpha,m}}$ and set $\widetilde{L} := L_0 + L$. The bilinear form $\widetilde{\mathcal{E}}_{K_{\alpha,m}}$ on $\widetilde{L}$ is associated with a non-negative self-adjoint operator A satisfying $\widetilde{\mathcal{E}}_{K_{\alpha,m}}(u,v) = \int_{K_{\alpha,m}} (Au)v\, d\mu_\alpha$ for all $u, v \in \widetilde{L}$.

By the theory of finite-dimensional real symmetric matrices, the (3^m+1)-th smallest eigenvalue of A is given by

$$\kappa_A := \inf\{\tilde{\kappa}(L') \mid L' \text{ is a subspace of } \widetilde{L},\ \dim L' = 3^m + 1\}.$$

Let $u_A \in \widetilde{L}$ be the eigenfunction corresponding to the eigenvalue κ_A and normalize it so that $\int_{K_{\alpha,m}} |u_A|^2\, d\mu_\alpha = 1$. Note that this function is orthogonal to L_0, so we can apply the Poincaré inequality to it. Now, since $\frac{3}{5} < \frac{2}{1-\alpha}$

and $3^m\mu_\alpha(K_{\alpha,w})<1$ for all $w\in\mathcal{A}^m$, we have that

$$\begin{aligned}
\tilde{\kappa}(L) &\geq \kappa_A = \kappa_A \int_{K_{\alpha,m}} |u_A|^2\,d\mu_\alpha = \widetilde{\mathcal{E}}_{K_{\alpha,m}}[u_A]\\
&= \sum_{w\in\mathcal{A}^m}\left(\frac{5}{3}\right)^m \mathcal{E}^d_{K_\alpha}[u_A\circ G_{\alpha,w}] + \left(\frac{2}{1-\alpha}\right)^m \mathcal{E}^c_{K_\alpha}[u_A\circ G_{\alpha,w}]\\
&\geq \sum_{w\in\mathcal{A}^m}\left(\frac{5}{3}\right)^m \mathcal{E}_{K_\alpha}[u_A\circ G_{\alpha,w}]\\
&\geq \left(\frac{5}{3}\right)^m \sum_{w\in\mathcal{A}^m} C_P\int_{K_\alpha} |u_A\circ G_{\alpha,w}|^2\,d\mu_\alpha^w\\
&= \left(\frac{5}{3}\right)^m \sum_{w\in\mathcal{A}^m} \frac{C_P}{\mu_\alpha(K_{\alpha,w})}\int_{K_{\alpha,w}} |u_A|^2\,d\mu_\alpha\\
&\geq \frac{5^m C_P}{3^m\max_{w\in\mathcal{A}^m}\{\mu_\alpha(K_{\alpha,w})\}}\int_{K_{\alpha,w}} |u_A|^2\,d\mu_\alpha\\
&\geq 5^m C_P.
\end{aligned}$$

It follows that $\kappa_{3^m+1}\geq 5^m C_P$, as we wanted to prove. ☐

Proposition 4.4.14. *There exist $C_{\alpha,2}, C_{\alpha,\beta,2}>0$ depending on α and β, and $x_0>0$ such that*

$$N_N(x)\leq C_{\alpha,2}x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,2}x^{1/2}$$

for all $x\in[x_0,\infty)$.

Proof. Let $x_0>C_P$ and $x\in[x_0,\infty)$. Then we can choose $m\in\mathbb{N}$ such that $C_P 5^{m-1}\leq x< C_P5^m$. From Lemma 4.4.13 we know that

$$\kappa_{3^m+1}\geq 5^m C_P > x,$$

hence $N_{K_{\alpha,m}}(x)\leq 3^m\leq C_{\alpha,2}x^{\frac{\ln 3}{\ln 5}}$, where $C_{\alpha,2}:=3C_P^{-\frac{\ln 3}{\ln 5}}$. Finally, by Lemma 4.4.10 and Lemma 4.4.8 we obtain

$$N_N(x)\leq C_{\alpha,2}x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,2}x^{1/2},$$

as we wanted to prove. ☐

Lower bound

Let us write $K_\alpha^0 := K_\alpha\setminus V_0$ and $K^0_{\alpha,w}:=G_{\alpha,w}(K^0_\alpha)$ for any $w\in\mathcal{A}^*$ and set $K^0_{\alpha,m}:=\bigcup_{w\in\mathcal{A}^m}K^0_{\alpha,w}$.

Lemma 4.4.15. *Let $m \in \mathbb{N}$. For any $w \in \mathcal{A}^m$, the operators $H_{K^0_{\alpha,w}}$ and $H_{K^0_{\alpha,m}\cup J_{\alpha,m}}$ have compact resolvent and for any $x > 0$ we have that*

$$\sum_{w\in\mathcal{A}^m} N_{K^0_{\alpha,w}}(x) + N_{J_{\alpha,m}}(x) = N_{K^0_{\alpha,m}\cup J_{\alpha,m}}(x) \leq N_D(x). \tag{4.4.10}$$

Proof. Note that by definition, $\mathcal{D}_U \subseteq \mathcal{D}_{K^0_\alpha}$ and $\mathcal{E}_U = \mathcal{E}_{K^0_\alpha}|_{U\times U}$ for both $U \in \{\mathcal{D}_{K^0_{\alpha,m}}, \mathcal{D}_{K^0_{\alpha,m}\cup J_{\alpha,m}}\}$ and any $m \in \mathbb{N}$. Since $H_{K^0_\alpha}$ has compact resolvent by Theorem 4.3.12, the minimax principle implies that the operators $H_{K^0_{\alpha,w}}$ and $H_{K^0_{\alpha,m}\cup J_{\alpha,m}}$ also have compact resolvent and the inequality in (4.4.10) holds.

Let us now prove the equality

$$N_{K^0_{\alpha,m}\cup J_{\alpha,m}}(x) = N_{J_{\alpha,m}}(x) + N_{K^0_{\alpha,m}}(x) \tag{4.4.11}$$

Let $u \in \mathcal{D}_{J_{\alpha,m}}$. Since $K_\alpha \setminus J_{\alpha,m} \subseteq K_{\alpha,m}$, $L_\alpha := \operatorname{supp}_{K_\alpha}(u) \cap K_\alpha \subseteq J_{\alpha,m}$ and therefore $u \cdot \mathbb{1}_{J_{\alpha,m}} \in C(K_\alpha)$ and $\operatorname{supp}_{K_\alpha}(u \cdot \mathbb{1}_{J_{\alpha,m}}) \subseteq J_{\alpha,m}$. Since L_α is compact and $J_{\alpha,m}$ is open, we know by [14, Exercise 1.4.1] that we can find a function $\varphi \in \mathcal{D}_{K_\alpha}$ such that $\varphi \geq 0$, $\varphi|_{L_\alpha} \equiv 1$ and $\varphi|_{K_{\alpha,m}} \equiv 0$. Then, $u \cdot \mathbb{1}_{J_{\alpha,m}} = u\varphi \in \mathcal{D}_{J_{\alpha,m}}$ and $u \cdot \mathbb{1}_{J_{\alpha,m}} \in \mathcal{C}_{J_{\alpha,m}}$ (recall Definition 4.4.6).

Similarly, if $u \in \mathcal{D}_{K^0_{\alpha,m}}$ and $\widetilde{L}_\alpha := \operatorname{supp}_{K_\alpha}(u) \cap K_\alpha \subseteq K^0_{\alpha,m}$, we can find $\psi \in \mathcal{D}_{K_\alpha}$ such that $\psi \geq 0$, $\psi|_{\widetilde{L}_\alpha} \equiv 1$ and $\psi|_{J_{\alpha,m}} \equiv 0$. Thus $u \cdot \mathbb{1}_{K_{\alpha,m}} = u\psi \in \mathcal{D}_{K^0_{\alpha,m}}$ and we have that $\mathcal{C}_{K^0_{\alpha,m}\cup J_{\alpha,m}} = \mathcal{C}_{K_{\alpha,m}} \oplus \mathcal{C}_{J^0_{\alpha,m}}$, both spaces being orthogonal to each other with respect to $\mathcal{E}_{K_\alpha}$ and the inner product of $L^2(K_\alpha, \mu_\alpha)$. Taking the closure with respect to $\mathcal{E}_{K_\alpha,1}$ we get that $\mathcal{D}_{K^0_{\alpha,m}\cup J_{\alpha,m}} = \mathcal{D}_{K^0_{\alpha,m}} \oplus \mathcal{D}_{J_{\alpha,m}}$, where both spaces keep being orthogonal to each other. Hence (4.4.11) follows.

It remains to prove that

$$N_{K^0_{\alpha,m}}(x) = \sum_{w\in\mathcal{A}^m} N_{K^0_{\alpha,w}}(x). \tag{4.4.12}$$

In this case we argue as above: for any $w \in \mathcal{A}^m$ and $u \in \mathcal{D}_{K^0_{\alpha,w}}$, since $K_{\alpha,m} \setminus K^0_{\alpha,w} = G_{\alpha,w}(V_0) \cup \bigcup_{w'\in\mathcal{A}^m\setminus\{w\}} K_{\alpha,w}$, we have that $L'_\alpha := K_{\alpha,w} \cap \operatorname{supp}_{K_\alpha}(u) \subseteq K^0_{\alpha,m}$ and $\operatorname{supp}_{K_\alpha}(u \cdot \mathbb{1}_{K^0_{\alpha,w}}) \subseteq K^0_{\alpha,w}$. Again, L'_α is compact and $K^0_{\alpha,w}$ is open, so we find $\varphi_w \in \mathcal{D}_{K_\alpha}$ such that $\varphi_w \geq 0$, $\varphi_w|_{L'_\alpha} \equiv 1$ and $\varphi_w|_{K_{\alpha,m}\setminus K^0_{\alpha,w}} \equiv 0$. Then, $u \cdot \mathbb{1}_{K^0_{\alpha,w}} = u\varphi_w \in \mathcal{D}_{K^0_{\alpha,m}}$, hence $u \cdot \mathbb{1}_{K^0_{\alpha,w}} \in \mathcal{C}_{K^0_{\alpha,w}}$ and $\mathcal{C}_{K^0_{\alpha,m}} = \bigoplus_{w\in\mathcal{A}^m} \mathcal{C}_{K^0_{\alpha,w}}$, where $\mathcal{C}_{K^0_{\alpha,w}}$ are orthogonal to each other with respect to both $\mathcal{E}_{K_{\alpha,m}}$ and the inner product of $L^2(K_{\alpha,m}, \mu_\alpha|_{K_{\alpha,m}})$. Taking the closure with respect to $\mathcal{E}_{K_\alpha,1}$ we get $\mathcal{D}_{K^0_{\alpha,m}} = \bigoplus_{w\in\mathcal{A}^m} \mathcal{D}_{K^0_{\alpha,w}}$, with all $\mathcal{D}_{K^0_{\alpha,w}}$ again orthogonal to each other so (4.4.12) follows. □

Lemma 4.4.16. *Let $m \in \mathbb{N}$. There exists a constant $C_{\alpha,\beta,1} > 0$ depending on α and β, and $x_0 > 0$ such that*

$$C_{\alpha,\beta,1} x^{1/2} \leq N_{J_{\alpha,m}}(x)$$

for all $x \geq x_0$.

Proof. We know from the proof of Lemma 4.4.8 that

$$N_{J_{\alpha,m}}(x) = \sum_{e \in \mathcal{J}_{\alpha,m}} N_e\left(\beta_e \Theta_\alpha x\right), \tag{4.4.13}$$

and

$$N_e(x) = \frac{\lambda(e)}{\pi} x^{1/2} + o(x^{1/2}) \qquad \text{as } x \to \infty$$

for all $e \in \mathcal{J}_{\alpha,m}$. Thus, there exists $\tilde{c}_2 \in (0, \infty)$ (independent of e) and $x_0 > 0$ such that

$$\tilde{c}_1 \frac{(\beta_e \Theta_\alpha)^{1/2}}{\pi} \lambda(e) x^{1/2} \leq N_e(\beta_e \Theta_\alpha x) \qquad \forall\, x \geq x_0.$$

By substituting this in (4.4.13) we get

$$N_{J_{\alpha,m}}(x) \geq \sum_{e \in \mathcal{J}_{\alpha,m}} \tilde{c}_1 \frac{(\beta_e \Theta_\alpha)^{1/2}}{\pi} \lambda(e) x^{1/2} \geq \frac{\tilde{c}_1 \Theta_\alpha^{1/2}}{\pi} x^{1/2} \sum_{e \in \mathcal{J}_{\alpha,m}} \beta_e^{1/2} \lambda(e),$$

and setting $C_{\alpha,\beta,1} := \frac{\tilde{c}_1 (\Theta_\alpha \beta \alpha)^{1/2}}{\pi}$, the assertion is proved. □

For the proof of the next lemma we need to introduce the following identification mapping. Recall the IFS $\{\mathbb{R}^2; S_i, i = 1, 2, 3\}$ associated with K and $\{\mathbb{R}^2; G_{\alpha,i}, i = 1, \ldots, 6\}$ associated with K_α as well as the sets V_* and $W_{\alpha,*}$ defined in (2.2.1) and (2.1.1). Further, we know that for any $x \in W_{\alpha,*}$, there exists a word $w^x \in \mathcal{A}^*$ such that $x = G_{\alpha,w^x}(p_i)$ for some $p_i \in V_0$, so we can define

$$\begin{aligned} \mathcal{I} \colon W_{\alpha,*} &\longrightarrow V_* \\ x \;\; &\longmapsto S_{w^x}(p_i). \end{aligned}$$

This mapping allows us to construct functions in $\mathcal{D}_{K_\alpha}$ from functions in $\mathcal{D}_K$ (the domain of the Dirichlet form defined in Theorem 3.2.2) as follows: for any $u \in \mathcal{D}_K$, we define the function $u_\alpha \colon V_{\alpha,*} \to \mathbb{R}$ by

$$u_\alpha(x) := \begin{cases} u \circ \mathcal{I}(x), & x \in W_{\alpha,*}, \\ u \circ \mathcal{I}(a_e), & x \in [a_e, b_e],\ e \in \mathcal{J}_\alpha, \end{cases} \tag{4.4.14}$$

which is well defined since $\mathcal{I}(a_e) = \mathcal{I}(b_e)$ for all $e \in \mathcal{J}_\alpha$. If $(\mathcal{E}_K, \mathcal{D}_K)$ denotes the Dirichlet form associated with K, then

$$\begin{aligned}\mathcal{E}_{K_\alpha}[u_\alpha] &= \lim_{n\to\infty} \left(\rho^d_{\alpha,n}\right)^{-1} E^d_{\alpha,n}[u_\alpha] + \left(\rho^c_{\alpha,n}\right)^{-1} E^c_{\alpha,n}[u_\alpha] \\ &= \lim_{n\to\infty} \frac{3^n}{5^n \rho^d_{\alpha,n}} \left(\frac{3}{5}\right)^{-n} E^d_{\alpha,n}[u]\end{aligned}$$

and since

$$L := \lim_{n\to\infty} \frac{3^n}{5^n \rho^d_{\alpha,n}} \leq \lim_{n\to\infty} \left(1 + \frac{\alpha}{5}\left(\frac{1-\alpha}{2}\right)^{n-1}\right)^{2n} < \infty$$

because $1 + \frac{\alpha}{5}\left(\frac{1-\alpha}{2}\right)^{n-1} \leq 1 + \frac{\alpha/5}{n}$ for sufficiently large n, we get that

$$\mathcal{E}_{K_\alpha}[u_\alpha] = \mathcal{E}^d_{K_\alpha}[u_\alpha] = L \cdot \mathcal{E}_K[u] < \infty,$$

hence $u_\alpha \in \mathcal{D}_{K_\alpha}$.

Lemma 4.4.17. *Let $m \in \mathbb{N}$. There exists $C_D \in (0,\infty)$ such that for all $w \in \mathcal{A}^m$*

$$\kappa_1(K^0_{\alpha,w}) := \inf_{\substack{u \in \mathcal{C}_{K^0_{\alpha,w}} \\ u \neq 0}} \left\{ \frac{\mathcal{E}_{K_\alpha}[u]}{\|u\|^2_{L^2(K^0_{\alpha,m})}} \right\} \leq 5^m C_D. \tag{4.4.15}$$

Proof. Let $v \in \mathcal{A}^*$ such that $S_v(K) \subseteq K \setminus V_0$ and consider $u \in \mathcal{D}^0_K$ a function such that $\mathrm{supp}(u) \subseteq K \setminus V_0$ and $u \equiv 1$ on $S_v(K)$ (such a function exists by [14, Exercise 1.4.1] because $S_v(K)$ is compact and $K \setminus V_0$ is open). The function $u_\alpha \in \mathcal{D}^0_{K_\alpha}$ defined as in (4.4.14) has by construction the property that $u_\alpha \equiv 1$ on $K_{\alpha,v}$. Now, for any $w \in \mathcal{A}^m$ we define the function

$$u^w(x) := \begin{cases} u_\alpha \circ G^{-1}_{\alpha,w}(x), & x \in K^0_{\alpha,w}, \\ 0, & x \in K_\alpha \setminus K^0_{\alpha,w}. \end{cases}$$

Then, $u^w \in \mathcal{C}_{K^0_{\alpha,w}}$ and by Corollary 4.2.6 we have that

$$\begin{aligned}\mathcal{E}_{K_\alpha}[u^w] = &\left(\frac{5}{3}\right)^m \sum_{w' \in \mathcal{A}^m} \mathcal{E}^d_{K_\alpha}[u^w \circ G_{\alpha,w'}] + \left(\frac{2}{1-\alpha}\right)^m \sum_{w' \in \mathcal{A}^m} \mathcal{E}^c_{K_\alpha}[u^w \circ G_{\alpha,w'}] \\ &+ \Theta^{-1} \sum_{k=0}^{m-1} \left(\frac{2}{1-\alpha}\right)^k \sum_{w' \in \mathcal{A}^k} \mathcal{E}^c_{\alpha,1}[u^w \circ G_{\alpha,w'}] \\ &= \left(\frac{5}{3}\right)^m \mathcal{E}^d_{K_\alpha}[u^w \circ G_{\alpha,w}] = \left(\frac{5}{3}\right)^m \mathcal{E}^d_{K_\alpha}[u_\alpha] \\ &= \left(\frac{5}{3}\right)^m L\mathcal{E}_K[u].\end{aligned} \tag{4.4.16}$$

On the other hand,

$$\begin{aligned}\int_{K_\alpha} |u^w(x)|^2\, d\mu_\alpha(x) &= \int_{K^0_{\alpha,m}} \left|u_\alpha \circ G^{-1}_{\alpha,w}(x)\right|^2 d\mu_\alpha(x)\\ &= \int_{K_\alpha} |u_\alpha|^2\, d\mu_\alpha(G_{\alpha,w}(y))\\ &\geq \int_{K_{\alpha,v}} d\mu_\alpha(G_{\alpha,w}(y))\\ &= \mu_\alpha(G_{\alpha,w}(K_{\alpha,v})) = \mu_\alpha(K_{\alpha,wv}),\end{aligned}$$

and since μ_α is elliptic by Lemma 4.4.5, there exists $\gamma \in (0,\infty)$ such that $\mu_\alpha(K_{\alpha,vw}) \geq \gamma^{|v|}\mu_\alpha(K_{\alpha,w})$, thus

$$\int_{K_\alpha} |u^w(x)|^2\, d\mu_\alpha(x) \geq \gamma^{|v|}\mu_\alpha(K_{\alpha,w}). \tag{4.4.17}$$

From inequalities (4.4.16) and (4.4.17) and the fact that $3^m\mu_\alpha(K_{\alpha,w}) > \frac{1}{2}$, we obtain

$$\begin{aligned}\inf_{\substack{u\in\mathcal{C}_{K^0_{\alpha,w}}\\ u\neq 0}} \left\{\frac{\mathcal{E}_{K_\alpha}[u]}{\|u\|^2_{L^2(K^0_{\alpha,m})}}\right\} &\leq \frac{\mathcal{E}_{K_\alpha}[u^w]}{\int_{K^0_{\alpha,w}} |u^w|^2\, d\mu_\alpha} \leq \frac{\mathcal{E}_{K_\alpha}[u^w]}{\gamma^{|v|}\mu_\alpha(K_{\alpha,w})}\\ &= \frac{\left(\frac{5}{3}\right)^m L\mathcal{E}_K[u]}{\gamma^{|v|}\mu_\alpha(K_{\alpha,w})} = \frac{5^m L\mathcal{E}_K[u]}{3^m\gamma^{|v|}\mu_\alpha(K_{\alpha,w})}\\ &\leq C_D 5^m,\end{aligned}$$

where $C_D := \frac{2L\mathcal{E}_K[u]}{\gamma^{|v|}}$ is independent of w. Thus, inequality (4.4.15) follows for all $w \in \mathcal{A}^m$. □

Now we are ready to prove the lower bound of Theorem 4.4.3.

Proposition 4.4.18. *There exist $C_{\alpha,1}, C_{\alpha,\beta,1} > 0$ depending on α and β, and $x_0 > 0$ such that*

$$C_{\alpha,1}x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,1}x^{1/2} \leq N_D(x)$$

for all $x \geq x_0$.

Proof. For $x \geq C_D$, choose $m \in \mathbb{N}_0$ such that $C_D 5^m \leq x < C_D 5^{m+1}$. We know from Lemma 4.4.17 that

$$\kappa_1(K^0_{\alpha,w}) \leq C_D 5^m \qquad \forall\, w \in \mathcal{A}^m,$$

which implies that $N_{K^0_{\alpha,w}}(x) \geq 1$ for all $w \in \mathcal{A}^m$.

By Lemmas 4.4.15 and 4.4.16 we get that

$$\begin{aligned} N_D(x) &\geq \sum_{w\in\mathcal{A}^m} N_{K^0_{\alpha,w}}(x) + N_{J_{\alpha,m}}(x) \geq \#\mathcal{A}^m + C_{\alpha,\beta,1}x^{1/2} \\ &\geq C_{\alpha,1}x^{\frac{\ln 3}{\ln 5}} + C_{\alpha,\beta,1}x^{1/2}, \end{aligned}$$

where $C_{\alpha,1} := \frac{1}{3}C_D^{\frac{-\ln 3}{\ln 5}}$. □

Finally, we are ready to prove Theorem 4.4.3

Proof of Theorem 4.4.3. First note that the Dirichlet form $(\mathcal{E}^0_{K_\alpha}, \mathcal{D}^0_{K_\alpha})$ corresponds to $(\mathcal{E}_{K^0_\alpha}, \mathcal{D}_{K^0_\alpha})$ in the notation of Definition 4.4.6.

By Theorem 4.3.12, its associated non-negative self-adjoint operator on $L^2(K_\alpha, \mu_\alpha)$ has compact resolvent and since $\mathcal{D}_{K^0_\alpha} \subseteq \mathcal{D}_{K_\alpha}$ and $\mathcal{E}_{K^0_\alpha} = \mathcal{E}_{K_\alpha}|_{\mathcal{D}_{K^0_\alpha}\times\mathcal{D}_{K^0_\alpha}}$, it follows from the minimax principle that $N_D(x) \leq N_N(x)$ for any $x \geq 0$.

Finally, consider $x_0 > \max\{C_P, C_D\}$. Then Propositions 4.4.14 and 4.4.18 provide the first and third inequality and the theorem is proved. □

Remark 4.4.19. In the discussion of Section 4.3.3, we showed the decomposition $\mathcal{D}_{K_\alpha} = \mathscr{H}_{V_0} \oplus \mathcal{D}^0_{K_\alpha}$, where $\mathscr{H}_{V_0}$ denoted the space of harmonic functions. By Proposition 4.3.8, this is a 3-dimensional subspace and therefore

$$\dim(\mathcal{D}_{K_\alpha}/\mathcal{D}_{K^0_\alpha}) = 3,$$

hence by [29, Corollary 4.7], we also have that

$$N_D(x) \leq N_N(x) \leq N_D(x) + 3.$$

Corollary 4.4.20. *For any $\alpha \in (0, 1/3)$, it holds that*

$$d_S(K_\alpha) = \frac{2\ln 3}{\ln 5} = d_S(K).$$

Proof. Looking to the definition of spectral dimension in (4.4.1), the assertion follows from Theorem 4.4.3 and the previous Remark. □

Chapter 5

Consequences and further research

The most important conclusion of this work is Corollary 4.4.20, telling us that the spectral dimension of the Hanoi attractor coincides with the one of the Sierpiński gasket for all $\alpha \in (0, 1/3)$. This means that, although both the Sierpiński gasket and Hanoi attractors have been proved to be geometrically different in chapter 2, they coincide as analytic objects. One could say that we can distinguish them if we see them, but we could not do so if we would just "hear" them.

Measure

It is important to remark once again that the spectral dimension of K_α strongly depends on the choice of the measure μ_α because the operators $\Delta^N_{\mu_\alpha}$ and $\Delta^D_{\mu_\alpha}$ also depend on it. It would be therefore interesting to study Dirichlet forms induced by the resistance form $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ on different L^2-spaces and investigate the influence of the choice of the measure on the spectral properties of the corresponding Laplacian. In this way one could eventually answer the question *which measure is the best for what purposes?*

Heat kernel estimates

Since $(\mathcal{E}_{K_\alpha}, \mathcal{F}_{K_\alpha})$ is a regular resistance form, it follows from [28, Theorem 10.4] that the induced Dirichlet form $(\mathcal{E}^0_{K_\alpha}, \mathcal{D}^0_{K_\alpha})$ (the one associated with Dirichlet boundary conditions) has a jointly continuous heat kernel $p(t, x, y)\colon (0, \infty) \times K_\alpha \times K_\alpha \to [0, \infty)$ and in particular one has the on-diagonal estimate

$$p(t, x, x) \leq \frac{2\overline{R}(x, A)}{t} + \frac{\sqrt{2}}{\mu_\alpha(A)} \qquad \forall\, x \in K_\alpha,\ t > 0,$$

where $\overline{R}(x,A) := \sup\{R(x,y)\,|\,y \in A\}$ and A is any Borel set satisfying $0 < \mu_\alpha(A) < \infty$.

It would be thus very interesting to study if one can find more (and better) estimates for this kernel. In particular, since the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ is local, we know from [14, Theorem 7.2.1] that there exists an associated diffusion process. In such cases one is interested in so–called *Li-Yau type* (sub-)Gaussian estimates because they are closely related to the concept of *walk dimension*, that we discuss afterwards. This kind of estimates are of the form

$$p(t,x,y) \asymp \frac{C_1}{\mu(B_d(x,t^{1/\delta}))} \exp\left(-C_2\left(\frac{d(x,y)^\delta}{t}\right)^{1/(1-\delta)}\right),$$

where μ is a regular measure on K_α, $B_d(x,r)$ denotes the ball of radius $r > 0$ with respect to the distance d and $1 < \delta < 2$.

One of the principal assumptions needed is the *volume doubling property* for the measure μ, i.e.

$$\mu(B_d(x,2r) \leq C\mu(B_d(x,r))$$

for some constant $C \in (0,\infty)$.

In the case that this does *not* hold for the measure μ_α, which would make finding this sort of estimate almost hopeless, one could substitute the Euclidean distance by the distance induced by the resistance metric in order to get the volume doubling property and restore the hope for Li-Yau type estimates. Eventually, one could answer the question, *under what distance and measure do we have Li-Yau type heat kernel estimates associated to* $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$*?*

Note that a change in the distance can carry a change in the Hausdorff dimension of the set K_α. It would be thus interesting to answer the question: *does the Hausdorff dimension of* K_α *(with respect to the resistance metric) converge to the Hausdorff dimension of* K_α*?*

Walk dimension and Einstein relation

As we already pointed out, the local regular Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ has an associated diffusion process $(X_t)_{t\geq 0}$. The space-time relation of this process is given by the so–called *walk dimension*, defined as

$$d_w K_\alpha = \lim_{r\downarrow 0} \frac{\log \mathbb{E}^x[T_{B_d(x,r)}]}{\log r},$$

where $T_{B_d(x,r)}$ denotes the (random) time the process needs to exit a ball of radius r centred at $x \in K_\alpha$. If the set is sufficiently homogeneous, this limit is independent of x.

Moreover, if one has Gaussian estimates for the heat kernel, this dimension coincides with the parameter δ of the estimate (see [17, Example 3.2] for the case of the Sierpiński gasket).

Spectral dimension and walk dimension are in general related by the so–called *Einstein relation*

$$d_S d_w = 2d_H,$$

where d_H denotes the Hausdorff dimension of the set. This relation shows the connection between three fundamental points of view on a set, namely analysis, probability theory and geometry.

The Einstein relation has not yet been proven to hold in general but it is known to be truth in the case of the Sierpiński gasket (see e.g. [13]). The case of Hanoi attractors seems to be quite interesting because of the fact that

$$d_H(K_\alpha) < d_S(K_\alpha) \qquad \forall\, \alpha \in (1-\frac{2}{\sqrt{5}}, \frac{1}{3}).$$

In case this relation holds, then we get $d_w K_\alpha < 2$ for $\alpha \in (1-\frac{2}{\sqrt{5}}, \frac{1}{3})$. This would mean that the diffusion process associated to the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ for these α's moves faster than two-dimensional Brownian motion and we would be facing a super-diffusive process on a set that can be embedded into the Euclidean space, turning K_α into a superconductor.

Generalization

Since it was possible to apply the ideas of [23] also in an non self-similar case, one could investigate if and what conditions could be modified in order to generalize these results.

Further topics

Hanoi attractors are closely related to the Sierpiński gasket, so it would also be interesting to look at these fractals in *harmonic coordinates* and study their relationship with the Sierpińksi gasket in harmonic coordinates, studied in [40] as an example of a self-similar space where it is possible to develop the concept of a *weak gradient*. In this case, the associated Dirichlet form can be expressed as the integral of the norm squared of the gradient with respect to an energy measure.

Finally, I would like to mention that this thesis has produced a joint work (still in process) on *quantum graphs* with D. Kelleher and A. Teplyaev from the University of Connecticut, due to the fact that the continuous part of the Dirichlet form $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ suggests the study of Dirichlet forms on quantum graphs with fractal nature.

Appendix A

Theoretical background

A.1 Resistance forms

This section makes a short review of the basic notions on resistance forms that are used in chapter 4. This theory was established by Kigami in [26, Chapter 2] but we will mostly refer to his most recent work [28], where definitions and results have been improved.

A resistance form is a quite general analytic structure, for which just a metric space is needed. We therefore consider a metric space (X, d) and denote by $\ell(X)$ the space of all real valued functions $u\colon X \to \mathbb{R}$. Moreover, we use the notation $u \vee v := \max\{u, v\}$ and $u \wedge v := \min\{u, v\}$ for any $u, v \in \ell(X)$.

Definition A.1.1. A pair $(\mathcal{E}, \mathcal{F})$ is called a *resistance form* on X if the following properties are satisfied:

(RF1) $\mathcal{F}$ is a linear subspace of $\ell(X)$ containing constants. $\mathcal{E}$ is a non-negative symmetric bilinear form on $\mathcal{F}$ and for all $u \in \mathcal{F}$, $\mathcal{E}(u, u) = 0$ if and only if $u \equiv$ const.

(RF2) If we consider the space $\mathcal{F}/_\sim$, where $u \sim v$ if and only if $u - v \equiv$ const for all $u, v \in \mathcal{F}$, then $(\mathcal{F}/_\sim, \mathcal{E})$ is a Hilbert space.

(RF3) For any $x, y \in X$, $x \neq y$, there exists a function $u \in \mathcal{F}$ such that $u(x) \neq u(y)$.

(RF4) For any $x, y \in X$,

$$R(x, y) := \sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u, u)} \mid u \in \mathcal{F},\, \mathcal{E}(u, u) > 0 \right\}$$

is finite.

(RF5) For any $u \in \mathcal{F}$, the function $\tilde{u} := 0 \vee u \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\tilde{u}, \tilde{u}) \leq \mathcal{E}(u, u)$.

Proposition A.1.2. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form. The supremum in (RF4) is indeed the maximum and $R\colon X \times X \to [0, +\infty)$ defines a metric on X.*

Proof. See [26, Theorem 2.3.4] □

The metric R is called the *resistance metric* on X associated with the resistance form $(\mathcal{E}, \mathcal{F})$ and by (RF4), the following result holds.

Proposition A.1.3. *Let $(\mathcal{E}, \mathcal{F})$ be a resistance form. Then, for any $u \in \mathcal{F}$,*

$$|u(x) - u(y)|^2 \leq \mathcal{E}(u, u) R(x, y) \qquad \forall\, x, y \in X.$$

Proof. See [26, Theorem 2.3.4]. □

This proposition says that any function in $\mathcal{F}$ is continuous with respect to the resistance metric R.

One of the most important features of a resistance form is that, if we consider the metric space (X, R) to be separable and equip it with a finite regular measure μ, then the resistance form $(\mathcal{E}, \mathcal{F})$ induces a Dirichlet form $(\mathcal{E}, \mathcal{D})$ in the following way:

Definition A.1.4. Let $\mathcal{E}_1$ be the symmetric bilinear form defined by

$$\mathcal{E}_1(u, v) := \mathcal{E}(u.v) + \int_X uv \, d\mu \qquad \forall\, u, v \in \mathcal{F} \cap L^2(X, \mu).$$

Then we have the following facts.

Lemma A.1.5. *$(\mathcal{F} \cap L^2(X, \mu), \mathcal{E}_1^{1/2})$ is a Hilbert space.*

Proof. See [26, Theorem 2.4.1]. □

Theorem A.1.6. *Let $C_0(X)$ denote the set of continuous functions on X with compact support and let $(\mathcal{E}, \mathcal{F})$ be a resistance form. Further, let $\mathcal{D}$ be the closure of $\mathcal{F} \cap C_0(X)$ with respect to $\mathcal{E}_1$. Then, $(\mathcal{E}, \mathcal{D})$ is a Dirichlet form.*

Proof. See [28, Theorem 9.4]. □

A.2 (Brief) introduction to the theory of Dirichlet forms

The theory of Dirichlet forms is a subject of great interest because it combines analysis and probability by closing a circle of connections between *resolvents*, *semigroups*, *generators* and *coercive bilinear forms*. In this paragraph we give an outline of the most important facts. Hereby we refer mostly to [34, 14].

Although this theory works in a more general setting (on Banach spaces), since we are dealing with Dirichlet forms induced by a resistance metric, we will present all results in terms of a real Hilbert space $L^2(X,\mu)$, where (X,d,μ) is a metric measurable space. We denote the associated inner product by $(\cdot,\cdot)_\mu$ and its associated norm by $\|\cdot\|_2 := (\cdot,\cdot)_\mu^{1/2}$.

A pair $(L,\mathcal{D}(L))$ is called a *linear operator* on $L^2(X,\mu)$ if $\mathcal{D}(L)$ is a dense linear subspace of $L^2(X,\mu)$ and $L\colon \mathcal{D}(L)\to L^2(X,\mu)$ is a linear map. L is said to be *continuous* (or *bounded*) in $L^2(X,\mu)$ if

$$\|L\| := \sup\{\|Lu\|_2 \mid u\in L^2(X,\mu),\ \|u\|_2\leq 1\} < \infty.$$

Definition A.2.1. The *resolvent set* of a linear operator $(L,\mathcal{D}(L))$ is defined as

$$\rho(L) := \{\lambda\in\mathbb{R} \mid (\lambda - L) \text{ is a bijection with bounded inverse}\}.$$

Moreover, the set $\sigma(L) := \mathbb{R}\setminus\rho(L)$ is called the *spectrum of* L and the set $\{(\lambda-L)^{-1} : \lambda\in\rho(L)\}$ is called the *resolvent of* L.

Definition A.2.2. A family of linear operators $\{R_\lambda\colon \mathcal{D}(R_\lambda)\to L^2(X,\mu)\}_{\lambda>0}$ with $\mathcal{D}(R_\lambda)=L^2(X,\mu)$ for all $\lambda\in(0,\infty)$ is said to be a *strongly continuous resolvent* if

(i) $\lim_{\lambda\to\infty}\lambda R_\lambda u = u$ for all $u\in L^2(X,\mu)$ (*strong continuity*),

(ii) $\|\lambda R_\lambda u\|_2 \leq \|u\|_2$ for all $\lambda>0$, $u\in L^2(X,\mu)$ (*contraction property*),

(iii) $R_\lambda - R_\kappa = (\kappa-\lambda)R_\lambda R_\kappa$ for all $\lambda,\kappa>0$ (*first resolvent equation).*

As its name suggests, a strongly continuous resolvent is closely related to the resolvent set of some linear operator. This relationship is given in the following result:

Proposition A.2.3. *Let $\{R_\lambda\}_{\lambda>0}$ be a strongly continuous resolvent on $L^2(X,\mu)$. Then there exists a unique linear operator $(L,\mathcal{D}(L))$ such that $(0,\infty)\subseteq\rho(L)$ and $R_\lambda=(\lambda-L)^{-1}$ for all $\lambda>0$. Moreover, L is closed and densely defined.*

Proof. See [34, Proposition 1.5]. □

This operator $(L, \mathcal{D}(L))$ is called the *generator* of $\{R_\lambda\}_{\lambda>0}$ and it can be characterized as follows:

Definition A.2.4. Let $\{R_\lambda\}_{\lambda>0}$ be a strongly continuous resolvent on $L^2(X,\mu)$. The linear operator $(L, \mathcal{D}(L))$ defined by

$$\begin{cases} \mathcal{D}(L) := R_\lambda(L^2(X,\mu)), \\ \quad Lu := (\lambda - R_\lambda)^{-1}u, \end{cases}$$

for some $\lambda > 0$, is called the (infinitesimal) generator of $\{R_\lambda\}_{\lambda>0}$.

Note that the inverse of R_λ exists for all $\lambda > 0$: suppose there exists a function $u \in L^2(X,\mu)$ such that $R_\lambda u = 0$. Then, it follows from the resolvent equation that $R_\kappa u = 0$ for all $\kappa > 0$ and from the strong continuity we get that $u = \lim_{\lambda\to\infty} \lambda R_\lambda u = 0$ and the existence of R_λ^{-1} on $\mathcal{D}(L)$ is assured.

Definition A.2.5. A family $\{T_t\}_{t>0}$ of linear operators with $\mathcal{D}(T_t) = L^2(X,\mu)$ for all $t > 0$ is called a *strongly continuous contraction semigroup* if

(i) $T_{t+s} = T_t T_s$, for all $t, s > 0$ *(semigroup property)*,

(ii) $\|T_t u\|_2 \leq \|u\|_2$ for all $t > 0$, $u \in L^2(X,\mu)$ *(contraction property)*,

(iii) $\lim_{t\to 0} T_t u = u$ for all $u \in L^2(X,\mu)$ *(strong continuity)*.

Strongly continuous semigroups have also an associated generator:

Definition A.2.6. Let $\{T_t\}_{t>0}$ be a strongly continuous contraction semigroup. The linear operator $(L, \mathcal{D}(L))$ defined by

$$\begin{cases} \mathcal{D}(L) := \{u \in L^2(X,\mu) \mid \exists \lim_{t\to 0} \dfrac{T_t u - u}{t}\}, \\ \quad Lu := \lim_{t\to 0} \dfrac{T_t u - u}{t} \qquad u \in \mathcal{D}(L), \end{cases}$$

is called the *(infinitesimal) generator* of $\{T_t\}_{t>0}$.

The following theorem, originally due to E. Hille [16] and K. Yosida [43], gives us a necessary and sufficient condition to decide if a densely defined linear operator is the generator of any strongly continuous contraction semigroup.

Theorem A.2.7. *Let $(L, \mathcal{D}(L))$ be a linear operator such that $\mathcal{D}(L)$ is dense in $L^2(X,\mu)$. Then, L is the generator of a strongly continuous contraction semigroup if and only if*

(i) $(Lu, u)_\mu \leq 0$ for all $u \in \mathcal{D}(L)$ (negative semi-definite), and

(ii) $(\lambda - L)(\mathcal{D}(L)) = L^2(X,\mu)$ *for some* $\lambda > 0$.

Proof. See [37, Theorem X.48]. □

Remark A.2.8. The above theorem holds because $L^2(X,\mu)$ is a real Hilbert space. For the general case of Banach spaces, (i) and (ii) should be replaced by $(0,\infty) \subseteq \rho(L)$ and $\|\lambda(\lambda - L)^{-1}\| \leq 1$ respectively (see [34, Theorem 1.12] for a proof).

Next proposition gives us the first relationship between strongly continuous contraction semigroups and resolvents.

Proposition A.2.9. *Let* $\{T_t\}_{t>0}$ *be a strongly continuous contraction semigroup with generator* $(L, \mathcal{D}(L))$. *Then* L *is closed,* $\mathcal{D}(L)$ *is dense in* $L^2(X,\mu)$, $(0,\infty) \subseteq \rho(L)$ *and* $\{R_\lambda\}_{\lambda>0}$ *is a strongly continuous contraction resolvent, where*

$$R_\lambda u = \int_0^\infty e^{-\lambda s} T_s u \, ds, \quad f \in L^2(X,\mu),\ \lambda > 0. \tag{A.2.1}$$

Proof. See [34, Prop.1.10] □

Note that (A.2.1) means that R_λ is the Laplace transform of T_t. Thus, given a strongly continuous contraction resolvent, we can obtain its associated strongly continuous contraction semigroup by taking the inverse Laplace transform of each R_λ.

Finally, we introduce Dirichlet forms and their connection with resolvents and semigroups.

Let $\mathcal{D}$ be a linear subspace of $L^2(X,\mu)$ and $\mathcal{E} \colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ a bilinear map. For $\beta > 0$ we define

$$\mathcal{E}_\beta(u,v) := \mathcal{E}(u,v) + \beta(u,v)_\mu, \quad u,v \in \mathcal{D}.$$

If $\mathcal{E}$ is *positive semi-definite* (i.e., $\mathcal{E}(u,u) \geq 0$ for all $u \in \mathcal{D}$) and *symmetric* (i.e. $\mathcal{E}(u,v) = \mathcal{E}(v,u)$ for all $u,v \in \mathcal{D}$), then the norms $\mathcal{E}_\beta(\cdot,\cdot)^{1/2}$ on $\mathcal{D}$ for any $\beta > 0$ are equivalent.

Definition A.2.10. A symmetric bilinear form $\mathcal{E} \colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ is said to be *closed on* $L^2(X,\mu)$ if $\mathcal{D}$ is a dense linear subspace of $L^2(X,\mu)$ and $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1^{1/2})$ is a complete space.

Definition A.2.11. A symmetric closed bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$ such that for all $u \in \mathcal{D}$, $0 \vee u \wedge 1 \in \mathcal{D}$, and $\mathcal{E}(0 \vee u \wedge 1, 0 \vee u \wedge 1) \leq \mathcal{E}(u,u)$ (*Markov property*), is called a *symmetric Dirichlet form.*

Thus, if we are given a pair $(\mathcal{E}, \mathcal{D})$ and we want to check if it is a symmetric Dirichlet form, we have to prove this three properties:

(DF1) $\mathcal{D}$ is dense in $L^2(X, \mu)$ with respect to the norm $\|\cdot\|_2$.

(DF2) $(\mathcal{D}, \mathcal{E}_1(\cdot,\cdot)^{1/2})$ is a complete space.

(DF3) $(\mathcal{E}, \mathcal{D})$ satisfies the Markov property.

The next results show how strongly continuous contraction resolvents and semigroups are related to Dirichlet forms. Note that they hold even more generally for any coercive closed symmetric form.

Theorem A.2.12. *Let $(\mathcal{E}, \mathcal{D})$ be a Dirichlet form on $L^2(X, \mu)$. Then there exists a unique strongly continuous contraction resolvent $\{R_\lambda\}_{\lambda>0}$ on $L^2(X, \mu)$ such that $\mathcal{D} \subseteq L^2(X, \mu)$ and*

$$\mathcal{E}_\lambda(R_\lambda u, v) = (u, v)_\mu = \mathcal{E}_\lambda(u, R_\lambda v), \tag{A.2.2}$$

for all $u \in L^2(X, \mu)$, $v \in \mathcal{D}$ and $\lambda > 0$. In particular, R_λ is self-adjoint for all $\lambda > 0$.

Proof. See [34, Theorem 2.8]. □

Theorem A.2.13. *Let $(L, \mathcal{D}(L))$ be the generator of the strongly continuous contraction resolvent on $L^2(X, \mu)$, $\{R_\lambda\}_{\lambda>0}$, with the property that for each $\lambda > 0$ there exists $C_\lambda > 0$ such that*

$$|(R_\lambda u, v)| \leq C_\lambda \mathcal{E}_\lambda(u, u)^{1/2} \mathcal{E}_\lambda(v, v)^{1/2} \qquad \forall\, u, v \in L^2(X, \mu).$$

Define

$$\mathcal{E}(u, v) := (-Lu, v)_\mu, \qquad u, v \in \mathcal{D}(L)$$

and let $\mathcal{D}$ be the completion of $\mathcal{D}(L)$ with respect to the norm $\mathcal{E}_1^{1/2}$. If we denote again by $\mathcal{E}$ the unique bilinear extension of $\mathcal{E}$ to $\mathcal{D}$ which is continuous with respect to $\mathcal{E}_1^{1/2}$, then $(\mathcal{E}, \mathcal{D})$ is a Dirichlet form and for each $u \in \mathcal{D}$,

$$\mathcal{E}(u, v) = (-Lu, v)_\mu, \qquad \forall\, v \in \mathcal{D}.$$

Furthermore, $\{R_\lambda\}_{\lambda>0}$ and $(\mathcal{E}, \mathcal{D})$ are related by (A.2.2).

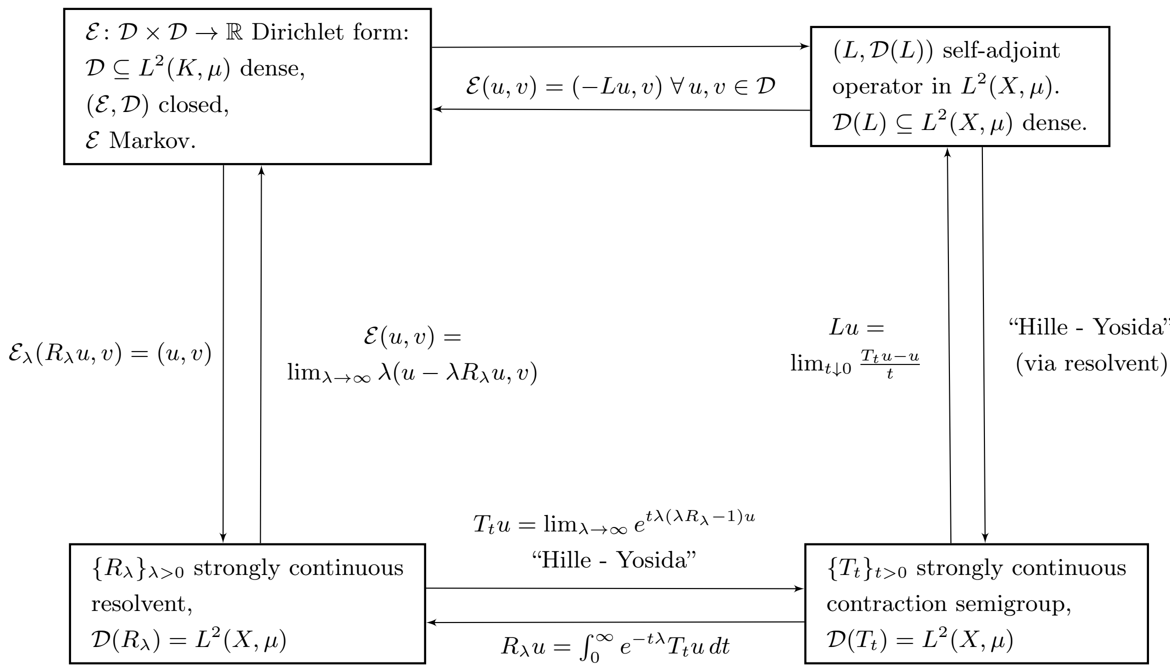
Proof. See [34, Theorem 2.15]. □

We finish this paragraph with two more important definitions concerning Dirichlet forms and a diagram resuming all the connections shown between Dirichlet forms, self-adjoint operators, strongly continuous resolvents and strongly continuous contraction semigroups.

Definition A.2.14. Let $(\mathcal{E}, \mathcal{D})$ be a Dirichlet form.

(1) $(\mathcal{E}, \mathcal{D})$ is called *regular* if $\mathcal{D} \cap C_0(X)$ is dense in $\mathcal{D}$ with respect to the norm $\mathcal{E}_1^{1/2}$ and $\mathcal{D} \cap C_0(X)$ is dense in $C_0(X)$ with respect to the uniform norm.

(2) $(\mathcal{E}, \mathcal{D})$ is called *local* if for all $u, v \in \mathcal{D}$ such that $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \emptyset$ we have that $\mathcal{E}(u, v) = 0$.



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