Hanoi attractors and the Sierpiński gasket: Geometric and analytic convergence

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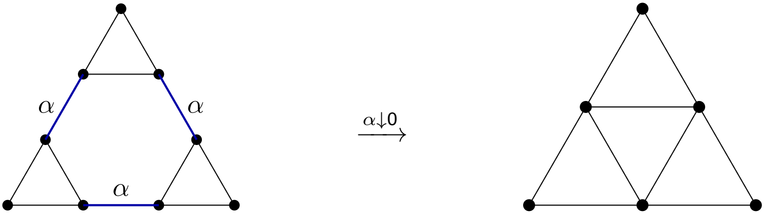
University of Siegen

Siegen, 19 July 2012

Contents

Hanoi and Sierpiński graphs



Hanoi graph　　　　Sierpiński graph

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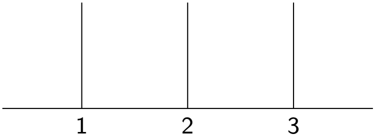

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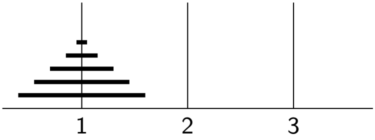

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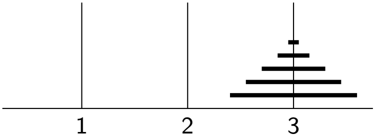

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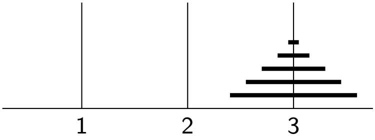

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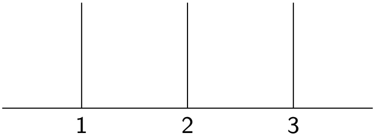
- Each time one and only one disc has to be moved.
- No larger disc lies on a smaller one.



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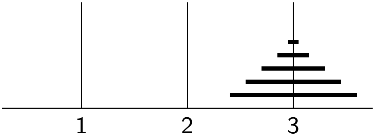

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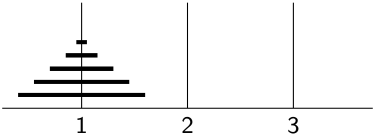

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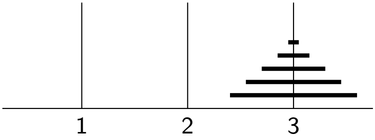

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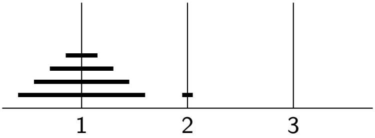

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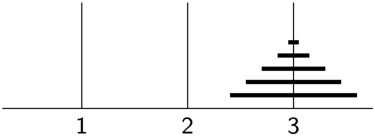

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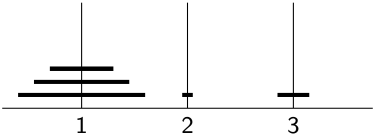

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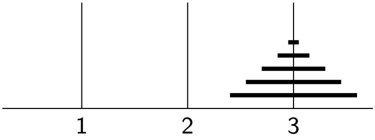

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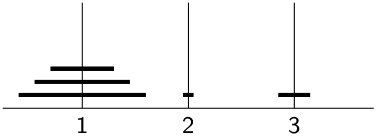

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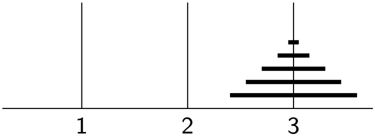

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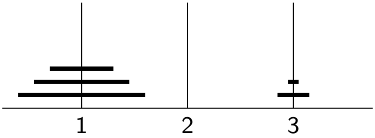

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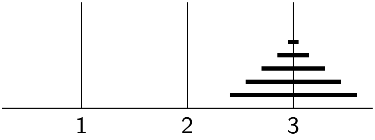

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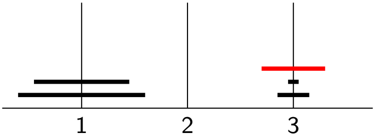

The Hanoi attractor of parameter α

- Set the points $p_1, \dots p_6$.
- For each $i = 1, 2 \dots, 6$, define the maps

$$\begin{aligned} G_{\alpha,i} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto A_i(x - p_i) + p_i, \end{aligned}$$

where

$$A_1 = A_2 = A_3 = \frac{1-\alpha}{2} Id,$$

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$$A_4 = \frac{\alpha}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix} \quad A_5 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A_6 = \frac{\alpha}{4}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$

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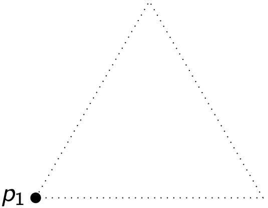

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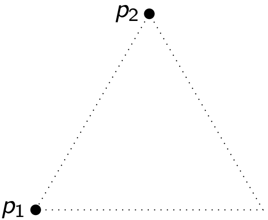

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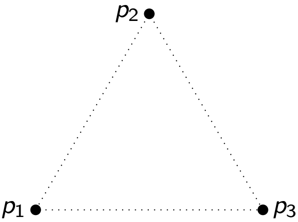

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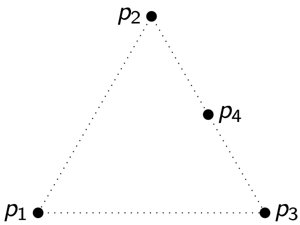

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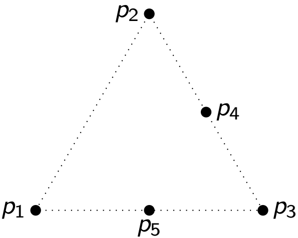

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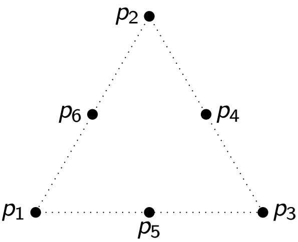

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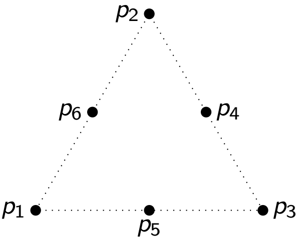

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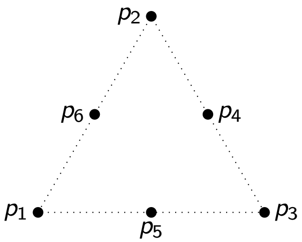

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Let $0 < \alpha < 1/3$ and consider the family of contractions $\{G_{\alpha,i}\}_{i=1}^{6}$. The unique non-empty compact set K_α such that

$$K_\alpha = \bigcup_{i=1}^{6} G_{\alpha,i}(K_\alpha),$$

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K_α is not self-similar, since $G_{\alpha,4}$, $G_{\alpha,5}$ and $G_{\alpha,6}$ are not similarities.

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Notations

- Let $\mathcal{A} := \{1, 2, 3\}$ and for each word $w = w_1 w_2 \cdots w_n \in \mathcal{A}^n$, denote

$$G_{\alpha,w}(x) := G_{\alpha,w_1} \circ G_{\alpha,w_2} \circ \cdots \circ G_{\alpha,w_n}(x), \quad x \in \mathbb{R}^2.$$

- Define $W_{\alpha,0} := \{p_1, p_2, p_3\}$ and

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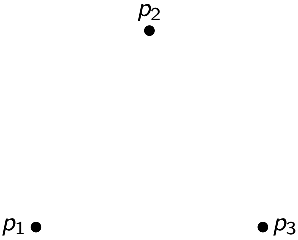

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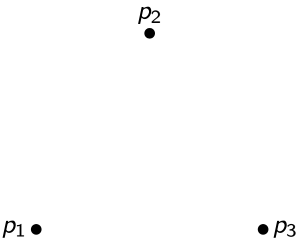

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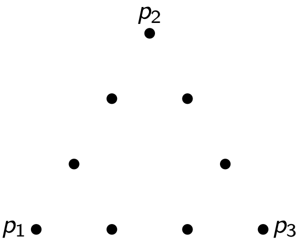

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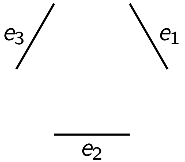

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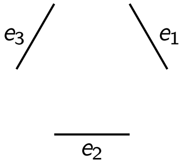

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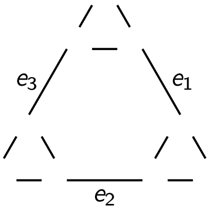

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Notations

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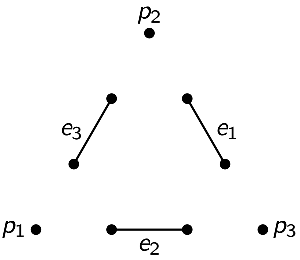

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The set $V_{\alpha,*} := \bigcup_{n\geq 0} V_{\alpha,n}$ *is dense in* K_α *with respect to the Euclidean metric.*

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Let F_α *be the unique non-empty compact subset of* $\mathbb{R}^2$ *such that*

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Geometric properties of K_α

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Let K denote the Sierpiński gasket.

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$$\lim_{\alpha\downarrow 0} dim_{\mathrm{H}} K_\alpha = dim_{\mathrm{H}} K.$$

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Construction of a Laplacian on K_α

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Use theory of Dirichlet forms:

1. Define a Dirichlet form $(E_{\alpha,n}, \mathcal{D}_{\alpha,n})$ on $V_{\alpha,n}$ for each $n \geq 0$.
2. Renormalize $(E_{\alpha,n}, \mathcal{D}_{\alpha,n})$ to get invariance under harmonic extension and denote it by $(\mathcal{E}_{\alpha,n}, \mathcal{D}_{\alpha,n})$, $n \geq 0$.
3. Define the symmetric bilinear form $\mathcal{E}_{K_\alpha}$ as the "limit" of $\mathcal{E}_{\alpha,n}$ and look for a suitable domain $\mathcal{D}_{K_\alpha}$, such that $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ is a Dirichlet form.
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$$L^2(V_{\alpha,n}, \mu_{\alpha,n}) := \ell^2(W_{\alpha,n}) \oplus L^2(J_{\alpha,n}, \mu^c_{\alpha,n}).$$

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For any $x, y \in W_{\alpha,n}$, we say that x and y are *n-neighbours* (and write $x \overset{\alpha,n}{\sim} y$), iff $\exists\ w \in \mathcal{A}^n$ such that $x, y \in G_{\alpha,w}(V_{\alpha,0})$.

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The bilinear form $\mathcal{E}_{\alpha,n} \colon \mathcal{D}_{\alpha,n} \times \mathcal{D}_{\alpha,n} \to \mathbb{R}$ is said to be invariant under harmonic extension (i.u.h.e. for short) if

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- For any $n \geq 0$, $\rho_{\alpha,n} =?$
- Calculate explicitly $\rho_{\alpha,0}, \rho_{\alpha,1}, \rho_{\alpha,2}$ and iterate the process.

2. Renomalization (I)

Calculation of $\rho_{\alpha,0}, \rho_{\alpha,1}$:

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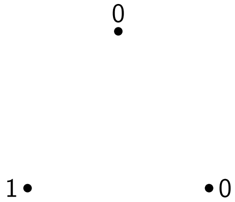

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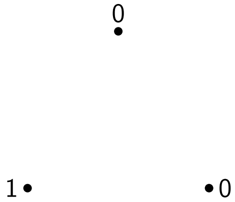

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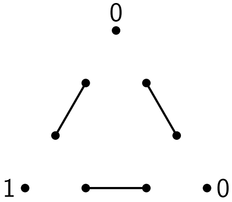

Let u_1 be the harmonic extension of u_0.

$$E_{\alpha,1}(u_1, u_1) = \sum_{x \overset{\alpha,\mathbf{1}}{\sim} y} (u_1(x) - u_1(y))^2 + \int_{J_{\alpha,\mathbf{1}}} |\nabla u_1|^2 \, \mu^c_{\alpha,1}(dx)$$

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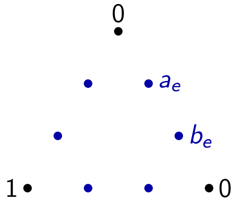

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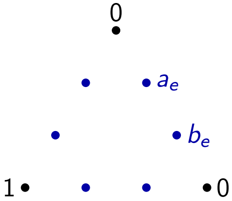
$$= \sum_{x \overset{\alpha,\mathbf{1}}{\sim} y} (u_1(x) - u_1(y))^2 + \sum_{e \in J_{\alpha,\mathbf{1}}} d^{-1}_{\alpha,1} (u_1(a_e) - u_1(b_e))^2$$

2. Renomalization (I)

Calculation of $\rho_{\alpha,0}, \rho_{\alpha,1}$:

- W.l.o.g. $u_0(p_1) = 1,\ u(p_2) = u(p_3) = 0.$



$$E_{\alpha,0}(u_0, u_0) = \sum_{x \overset{\alpha,\mathbf{0}}{\sim} y} (u_0(x) - u_0(y))^2 = 2,$$

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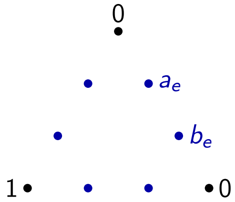

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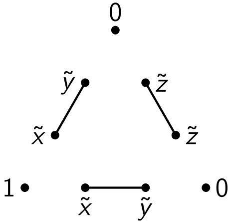
to minimize.

2. Renormalization (I)

- Solution of the minimization problem:

$$\tilde{x} = \frac{2+3d_{\alpha,1}}{5+3d_{\alpha,1}}, \quad \tilde{y} = \frac{2}{5+3d_{\alpha,1}}, \quad \tilde{z} = \frac{1}{5+3d_{\alpha,1}}.$$



- Substituting in the expression of $E_{\alpha,1}(u_1,u_1)$:

$$E_{\alpha,1}(u_1,u_1) = \left(\frac{15}{(5+3d_{\alpha,1})^2} + \frac{18d_{\alpha,1}}{(5+3d_{\alpha,1})^2}\right) E_{\alpha,0}(u_0,u_0)$$

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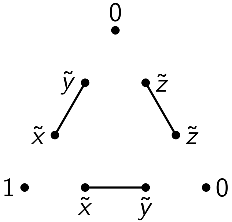

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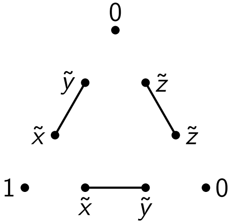

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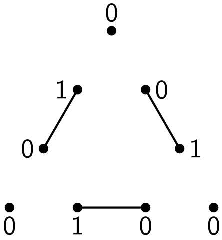

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$$= \sum_{i=1}^{3} \widetilde{E}_{\alpha,1}(u_2 \circ G_{\alpha,i}, u_2 \circ G_{\alpha,i}) + \frac{3}{d_{\alpha,1}},$$

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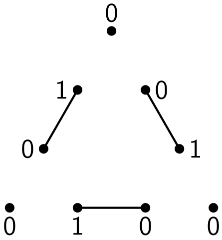

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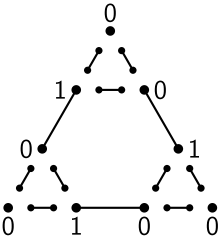

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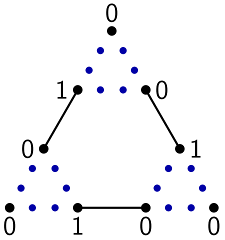

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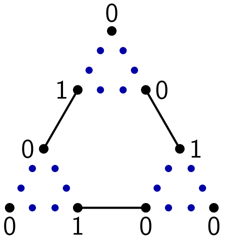

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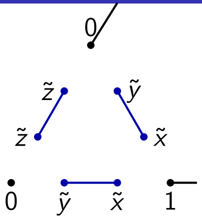
2. Renomalization (I)

- Solution of the minimization problem (use $n=1$):

$$\tilde{x} = \frac{2+3\frac{d_{\alpha,2}}{\beta}}{5+3\frac{d_{\alpha,2}}{\beta}}, \quad \tilde{y} = \frac{2}{5+3\frac{d_{\alpha,2}}{\beta}}, \quad \tilde{z} = \frac{1}{5+3\frac{d_{\alpha,2}}{\beta}}.$$



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$$E_{\alpha,2}(u_2,u_2) = \frac{15}{\left(5+3\frac{d_{\alpha,2}}{\beta}\right)^2} E^d_{\alpha,1}(u_1,u_1) + \left(\frac{9d_{\alpha,2}d_{\alpha,1}}{\beta\left(5+3\frac{d_{\alpha,2}}{\beta}\right)^2} + 1\right) E^c_{\alpha,1}(u_1,u_1)$$

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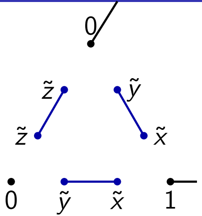

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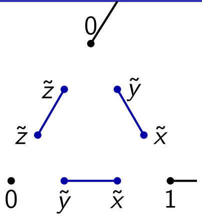

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Theorem 3.4.

Let $n \geq 1$, $u_1 \in \mathcal{D}_{\alpha,1}$ *and denote by* $\tilde{u}_n \in \mathcal{D}_{\alpha,n}$ *its harmonic extension. Then it holds that*

$$E_{\alpha,n}(\tilde{u}_n, \tilde{u}_n) = (\rho^d_{\alpha,n})^{-1} E^d_{\alpha,n}(u_1, u_1) + (\rho^c_{\alpha,n})^{-1} E^c_{\alpha,1}(u_1, u_1),$$

where

$$\rho^d_{\alpha,n} := \begin{cases} 1, & \text{for } n = 1, \\ \prod_{i=2}^n r^d_{\alpha,i}, & \text{for } n \geq 2, \end{cases} \tag{1}$$

and

$$\rho^c_{\alpha,n} := \begin{cases} 1, & \text{for } n = 1, \\ \sum_{j=2}^n \prod_{i=2}^j r^c_{\alpha,i}, & \text{for } n \geq 2, \end{cases} \tag{2}$$

for some numbers $r^d_{\alpha,i}$ *and* $r^c_{\alpha,i}$, *depending on* β, $d_{\alpha,i}$, $i = 2, \dots n$.

Corollary 3.5.

The pair $(\mathcal{E}_{\alpha,n}, \mathcal{D}_{\alpha,n})$ *is a Dirichlet form on* $L^2(V_{\alpha,n}, \mu_{\alpha,n})$.

3. Dirichlet form on K_α

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Construction of the Radon measure μ_α on K_α:

- From Proposition 2.1 we know that $K_\alpha = F_\alpha \dot\cup J_\alpha$.
- Define the self-similar measure μ_α^d on F_α by

$$\mu_\alpha^d(A) := \frac{1}{\mathcal{H}^{d_\alpha}(F_\alpha)} \mathcal{H}^{d_\alpha}_{|F_\alpha}(A), \quad A \subseteq \mathbb{R}^2 \text{ Borel},$$

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- Define the Hilbert space

$$L^2(K_\alpha,\mu_\alpha)=L^2(F_\alpha,\mu_\alpha^d)\oplus L^2(J_\alpha,\mu_\alpha^c).$$

- Define the bilinear form

$$\mathcal{E}_{K_\alpha}(u,v):=\lim_{n\to\infty}\left(\mathcal{E}^d_{\alpha,n}(u,v)+\mathcal{E}^c_{\alpha,n}(u,v)\right),\quad u,v\colon K_\alpha\to\mathbb{R}.$$

- Set $\widetilde{\mathcal{D}}_\alpha:=\{u\colon K_\alpha\to\mathbb{R}:\ \mathcal{E}_{K_\alpha}(u,u)<\infty\}$. Consider its completion w.r.t. the norm

$$\left(\mathcal{E}_{K_\alpha}(u,u)+\|u\|^2_{L^2(K_\alpha,\mu_\alpha)}\right)^{1/2},$$

and denote it by $\mathcal{D}_{K_\alpha}$.

Theorem 3.6.

The pair $(\mathcal{E}_{K_\alpha},\mathcal{D}_{K_\alpha})$ is a local regular Dirichlet form on $L^2(K_\alpha,\mu_\alpha)$.

3. Dirichlet form on K_α

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$$L^2(K_\alpha, \mu_\alpha) = L^2(F_\alpha, \mu_\alpha^d) \oplus L^2(J_\alpha, \mu_\alpha^c).$$

- Define the bilinear form
$$\mathcal{E}_{K_\alpha}(u, v) := \lim_{n\to\infty} \left(\mathcal{E}^d_{\alpha,n}(u, v) + \mathcal{E}^c_{\alpha,n}(u, v)\right), \quad u, v\colon K_\alpha \to \mathbb{R}.$$

- Set $\widetilde{\mathcal{D}}_\alpha := \{u\colon K_\alpha \to \mathbb{R} : \mathcal{E}_{K_\alpha}(u, u) < \infty\}$. Consider its completion w.r.t. the norm
$$\left(\mathcal{E}_{K_\alpha}(u, u) + \|u\|^2_{L^2(K_\alpha,\mu_\alpha)}\right)^{1/2},$$
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The pair $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ is a local regular Dirichlet form on $L^2(K_\alpha, \mu_\alpha)$.

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The pair $(\mathcal{E}_{K_\alpha}, \mathcal{D}_{K_\alpha})$ is a local regular Dirichlet form on $L^2(K_\alpha, \mu_\alpha)$.

Proposition 3.7.

There exists $D(-\Delta_N^{\mu_\alpha})$ dense subset of $\mathcal{D}_{K_\alpha}$ and for each $u \in D(-\Delta_N^{\mu_\alpha})$, there exists $f \in L^2(K_\alpha, \mu_\alpha)$ such that

$$\mathcal{E}_{K_\alpha}(u, v) = \int_{K_\alpha} fv \, \mu_\alpha(dx), \qquad \textit{for all } v \in \mathcal{D}_{K_\alpha}.$$

We write $-\Delta_N^{\mu_\alpha} u := f$.

- The index N refers to Neumann boundary conditions in the domain $D(-\Delta_N^{\mu_\alpha})$. For the case of Dirichlet boundary conditions, define $\mathcal{D}_{K_\alpha,0} := \{u \in \mathcal{D}_{K_\alpha} \, : \, u|_{V_{\alpha,\mathbf{0}}} \equiv 0\}$ and $\mathcal{E}_{K_\alpha,0} := \mathcal{E}_{K_\alpha}|_{\mathcal{D}_{K_\alpha,\mathbf{0}} \times \mathcal{D}_{K_\alpha,\mathbf{0}}}$ and prove $(\mathcal{E}_{K_\alpha,0}, \mathcal{D}_{K_\alpha,0})$ is a local regular Dirichlet form.

- How do the spectra $\sigma(-\Delta_N)$, $\sigma(-\Delta_D)$ behave asymptotically? $\longrightarrow$ eigenvalue counting function $\longrightarrow$ spectral dimension.

4. The Laplacian on K_α

Proposition 3.7.

There exists $D(-\Delta_N^{\mu_\alpha})$ dense subset of $\mathcal{D}_{K_\alpha}$ and for each $u \in D(-\Delta_N^{\mu_\alpha})$, there exists $f \in L^2(K_\alpha, \mu_\alpha)$ such that

$$\mathcal{E}_{K_\alpha}(u, v) = \int_{K_\alpha} fv\, \mu_\alpha(dx), \qquad \text{for all } v \in \mathcal{D}_{K_\alpha}.$$

We write $-\Delta_N^{\mu_\alpha} u := f$.

- The index N refers to Neumann boundary conditions in the domain $D(-\Delta_N^{\mu_\alpha})$. For the case of Dirichlet boundary conditions, define $\mathcal{D}_{K_\alpha,0} := \{u \in \mathcal{D}_{K_\alpha} \,:\, u|_{V_{\alpha,\mathbf{0}}} \equiv 0\}$ and $\mathcal{E}_{K_\alpha,0} := \mathcal{E}_{K_\alpha}|_{\mathcal{D}_{\boldsymbol{K_\alpha,0}} \times \mathcal{D}_{\boldsymbol{K_\alpha,0}}}$ and prove $(\mathcal{E}_{K_\alpha,0}, \mathcal{D}_{K_\alpha,0})$ is a local regular Dirichlet form.

- How do the spectra $\sigma(-\Delta_N)$, $\sigma(-\Delta_D)$ behave asymptotically?
  $\longrightarrow$ eigenvalue counting function $\longrightarrow$ spectral dimension.

Eigenvalue counting function

Consider the case of Neumann boundary conditions.

Definition 3.8.

The eigenvalue counting function of $-\Delta_N^{\mu_\alpha}$ at $x \geq 0$ is given by

$$N_N(x) := \#\{\lambda \in \sigma(-\Delta_N^{\mu_\alpha});\ \lambda \leq x\},$$

according to multiplicity.

This definition only makes sense if the eigenvalues of $-\Delta_N^{\mu_\alpha}$ have finite multiplicity!

Conjecture 3.9.

The inclusion map from the Hilbert space $\left(\mathcal{D}_{K_\alpha}, \left(\mathcal{E}_{K_\alpha} + \|\cdot\|^2_{L^2(K_\alpha,\mu_\alpha)}\right)^{1/2}\right)$ *into* $L^2(K_\alpha, \mu_\alpha)$ *is a compact operator.*

Spectral dimension

Conjecture 4.1.

There exists constants $c_{\alpha,1}, c_{\alpha,2}, c_{\alpha,3}, c_{\alpha,4} > 0$ *and* $x_0 > 0$ *such that*

$$c_{\alpha,1}x^{1/2} + c_{\alpha,2}x^{\ln 3/\ln 5} \leq N_N(x) \leq c_{\alpha,3}x^{1/2} + c_{\alpha,4}x^{\ln 3/\ln 5}, \quad x \geq x_0.$$

Therefore $d_S(K_\alpha) = \frac{\ln 9}{\ln 5}$ *for all* $\alpha \in (0, 1/3)$.

This would imply that the spectral dimension of the Hanoi attractors K_α, where $\alpha \in (0, 1/3)$, coincides with the spectral dimension of the Sierpiński gasket K.

Thank you for your attention!