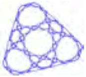
Introduction to Dirichlet forms and diffusions on fractals



Alexander Teplyaev
University of Connecticut

Siegen,
July 19-20, 2012

plan of the talk

- **Part 1. Motivation**

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  - **historical**

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  - **group theory and computer science**

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- **Part 2. Overview of mathematical results:**
  - **diffusions on fractals**
  - **spectral analysis on fractals**

mathematical physics: the spectral dimension of the universe

PRL **95**, 171301 (2005) PHYSICAL REVIEW LETTERS

The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,[1,3,*] J. Jurkiewicz,[2,†] and R. Loll[3,‡]

[1]*The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, D*

[2]*Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian*
Reymonta 4, PL 30-059 Krakow, Poland

[3]*Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Ne*

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We measure the spectral dimension of universes emerging from nonperturbative quantum
defined through state sums of causal triangulated geometries. While four dimensional on large sca
quantum universe appears two dimensional at short distances. We conclude that quantum gravity
''self-renormalizing'' at the Planck scale, by virtue of a mechanism of dynamical dimensional red

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Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.

tral dimension, a diffeomorphism-inv
tained from studying diffusion on the
of geometries. On large scales and wi
curacy, it is equal to four, in agreemen
surements of the large-scale dimensio

other hand, the “short-distance spectral dimension,” obtained by extrapolating Eq. (12) to $\sigma \to 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \tag{15}$$

and thus is compatible with the integer value two.

Numerics

λ_j	$n = 7$	$n = 8$
1	0.0000	0.0000
2	1.0000	1.0000
3	1.0000	1.0000
4	3.2798	3.2798
5	3.2798	3.2798
6	5.2033	5.2032
7	7.8389	7.8386
8	7.8389	7.8386
9	8.9141	8.9139
10	8.9141	8.9139
11	9.4951	9.4950
12	9.4952	9.4950
13	17.5332	17.5326
14	17.5332	17.5327
15	17.6373	17.6366
16	17.6373	17.6366
17	19.8610	19.8607
18	21.7893	21.7882
19	25.7111	25.7089
20	25.7112	25.7091

Table: Hexacarpet renormalized eigenvalues at levels $\mathbf{n = 7}$ and $\mathbf{n = 8}$.

	Level **n**						
c	1	2	3	4	5	6	7
1							
2	1.2801	1.3086	1.3085	1.3069	1.3067	1.3065	1.3064
3	1.2801	1.3086	1.3079	1.3075	1.3066	1.3065	1.3064
4	1.1761	1.3011	1.3105	1.3064	1.3068	1.3065	1.3065
5	1.1761	1.3011	1.3089	1.3074	1.3073	1.3065	1.3065
6	1.0146	1.2732	1.3098	1.3015	1.3067	1.3065	1.3064
7		1.2801	1.3114	1.3055	1.3071	1.3066	1.3065
8		1.2801	1.3079	1.3086	1.3075	1.3067	1.3065
9		1.2542	1.3191	1.2929	1.3056	1.3065	1.3065
10		1.2542	1.3017	1.3089	1.3069	1.3066	1.3065
11		1.2461	1.3051	1.3063	1.3048	1.3065	1.3065
12		1.2461	1.3019	1.3075	1.3068	1.3066	1.3065
13		1.1969	1.6014	1.0590	1.3068	1.3066	1.3065
14		1.1969	1.2972	1.3063	1.3078	1.3066	1.3065
15		1.2026	1.3059	1.3020	1.3060	1.3066	1.3065
16		1.2026	1.2993	1.3074	1.3071	1.3067	1.3065
17		1.1640	1.3655	1.2349	1.3064	1.3066	1.3065
18		1.1755	1.4128	1.2009	1.3069	1.3067	1.3065
19		1.1761	1.5252	1.1171	1.3073	1.3068	1.3066
20		1.1761	1.2988	1.3114	1.3077	1.3068	1.3065

Table: Hexacarpet estimates for resistance coefficient **c** given by $\frac{1}{6}\frac{\lambda_j^n}{\lambda_j^{n+1}}$.

Conjecture

We conjecture that

1. *on the Strichartz hexacarpet there exists a unique self-similar local regular conservative Dirichlet form* $\mathcal{E}$ *with resistance scaling factor* $\rho \approx$*1.304 and the Laplacian scaling factor* $\tau = 6\rho$*;*
2. *the simple random walks on the repeated barycentric subdivisions of a triangle, with the time renormalized by* τ^{n}*, converge to the diffusion process, which is the continuous symmetric strong Markov process corresponding to the Dirichlet form* $\mathcal{E}$*;*
3. *this diffusion process satisfies the sub-Gaussian heat kernel estimates and elliptic and parabolic Harnack inequalities, possibly with logarithmic corrections, corresponding to the Hausdorff dimension* $\frac{\log(6)}{\log(2)} \approx 2.58$ *and the spectral dimension* $2\frac{\log(6)}{\log(\tau)} \approx 1.74$*;*
4. *the spectrum of the Laplacian has spectral gaps in the sense of Strichartz;*
5. *the spectral zeta function has a meromorphic continuation to* $\mathbb{C}$*.*

Early (physics) results on spectral analysis on fractals

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

Early results on diffusions on fractals

Sheldon Goldstein, *Random walks and diffusions on fractals.* Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time $\mathbf{n}$ is at $\mathbf{X(n)}$, on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension $\mathbf{d}$ we show that, as $\mathbf{L} \to \infty$, the process $\mathbf{Y_L(t) \equiv X([(d+3)^L t])/2^L}$ converges in distribution to a diffusion on the Sierpin'ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'. In particular,

$$|\mathbf{X(n)}| \sim \mathbf{n}^{\gamma},$$

where $\gamma = \mathbf{(\ln 2)/\ln(d+3))} \leqslant \mathbf{2}$.

Shigeo Kusuoka, *A diffusion process on a fractal.* Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

- M.T. Barlow, E.A. Perkins, *Brownian motion on the Sierpinski gasket.* (1988)
- M. T. Barlow, R. F. Bass, *The construction of Brownian motion on the Sierpiński carpet.* Ann. Inst. Poincaré Probab. Statist. (1989)
- S. Kusuoka, *Dirichlet forms on fractals and products of random matrices.* (1989)
- T. Lindstrøm, *Brownian motion on nested fractals.* Mem. Amer. Math. Soc. **420**, 1989.
- J. Kigami, *A harmonic calculus on the Sierpiński spaces.* (1989)
- J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988) Cambridge Univ. Press, 1992.
- M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket.* (1992)
- J. Kigami, *Harmonic calculus on p.c.f. self–similar sets.* Trans. Amer. Math. Soc. **335** (1993)
- J. Kigami and M. L. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals.* Comm. Math. Phys. **158** (1993)

Main classes of fractals considered

- $[0, 1]$

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- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- metric measure Dirichlet spaces, possibly with heat kernel estimates (MMD+HKE)

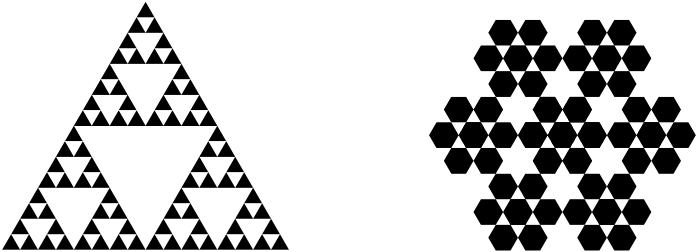

Figure: Sierpiński gasket and Lindstrøm snowflake (nested fractals), p.c.f., finitely ramified)

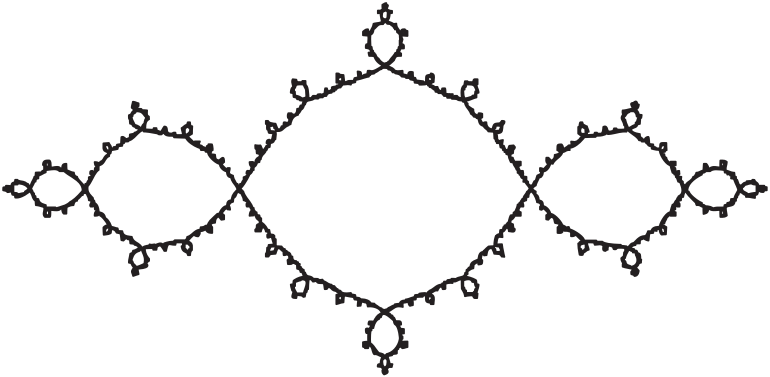

Figure: The basilica Julia set, the Julia set of $\mathbf{z}^2 - \mathbf{1}$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al., Rogers-T.).

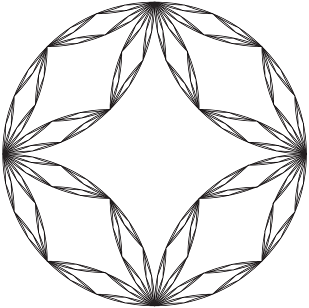

Figure: Diamond fractals, non-p.c.f., but finitely ramified

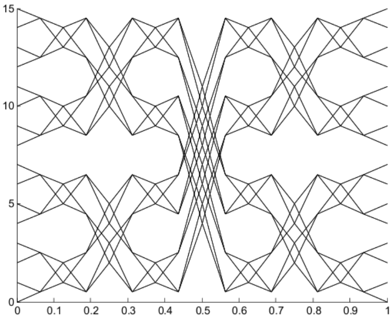

Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

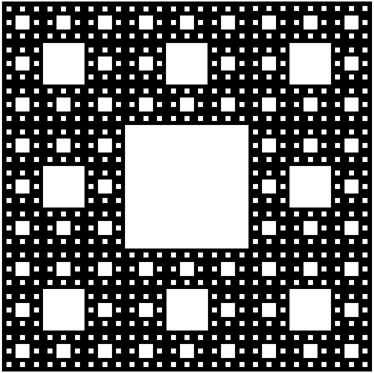

Figure: Sierpiński carpet, infinitely ramified

Existence, uniqueness, heat kernel estimates

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

$$\text{distance} \sim \sqrt{\text{time}}$$

"Einstein space–time relation for Brownian motion"

De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;

Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with **Ricci** $\geqslant$ **0**:

$$\mathbf{p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)}$$

$$\mathbf{distance \sim \sqrt{time}}$$

Brownian motion on $\mathbb{R}^{\mathbf{d}}$: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{ct^{1/2}}$.

Anomalous diffusion: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{o(t^{1/2})}$, or (in regular enough situations),

$$\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| \approx \mathbf{t^{1/d_w}}$$

with $\mathbf{d_w} > \mathbf{2}$.

Here $\mathbf{d_w}$ is the so-called **walk dimension** (should be called **"walk index"** perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$\mathbf{p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)}$$

$$\mathbf{distance \sim (time)^{\frac{1}{d_w}}}$$

$$\mathbf{d_H} = \text{Hausdorff dimension}$$

$$\frac{\mathbf{1}}{\gamma} = \mathbf{d_w} = \text{“walk dimension” } (\gamma\text{=diffusion index})$$

$$\frac{\mathbf{2d_H}}{\mathbf{d_w}} = \mathbf{d_S} = \text{“spectral dimension” (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

Theorem (Barlow, Bass, Kumagai (2006)).

Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the sub-Gaussian **heat kernel estimates are stable under rough isometries**, *i.e. under maps that preserve distance and energy up to scalar factors.*

Gromov-Hausdorff + energy

Theorem. (Barlow, Bass, Kumagai, T. (1989–2010).) On any fractal in the class of generalized Sierpiński carpets (includes cubes in $\mathbb{R}^{\mathbf{d}}$) there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries.

Therefore there there is a unique corresponding symmetric Markov process and a unique Laplacian. Moreover, the Markov process is Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

Main difficulties:

If it is not a cube in $\mathbb{R}^{\mathbf{n}}$, then

- $\mathbf{d_S} < \mathbf{d_H}$, $\mathbf{d_w} > \mathbf{2}$
- the energy measure and the Hausdorff measure are mutually singular;

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- Lipschitz functions are not of finite energy;
- in fact, we can not compute any functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.

Theorem. (Grigor'yan and Telcs, also [BBK])

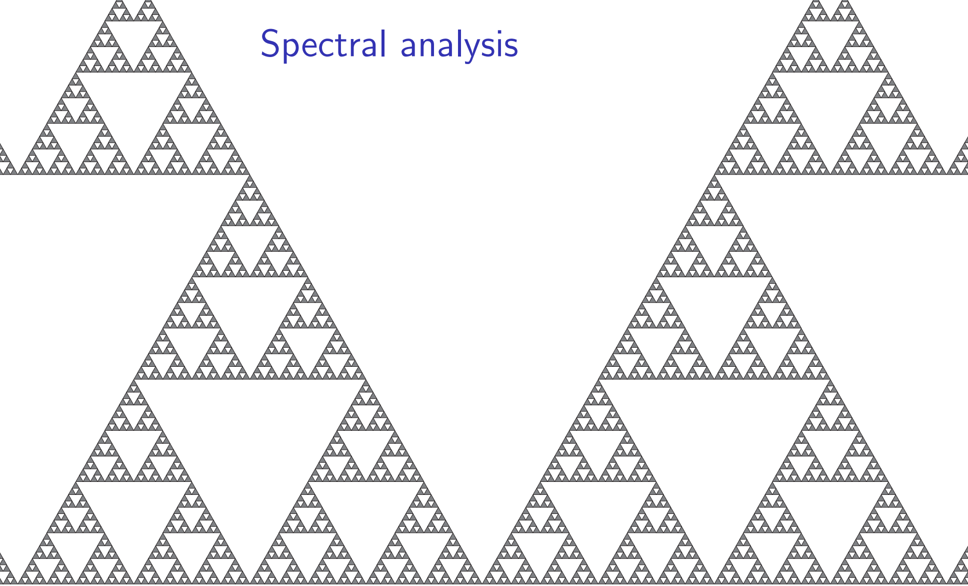
On a MMD space the following are equivalent

- **(VD)**, **(EHI)** and **(RES)**
- **(VD)**, **(EHI)** and **(ETE)**
- **(PHI)**
- **(HKE)**

and the constants in each implication are effective.

Abbreviations: Metric Measure Dirichlet spaces, Volume Doubling, Elliptic Harnack Inequality, Exit Time Estimates, Parabolic Harnack Inequality, Heat Kernel Estimates.

Spectral analysis



A part of an infinite Sierpiński gasket.

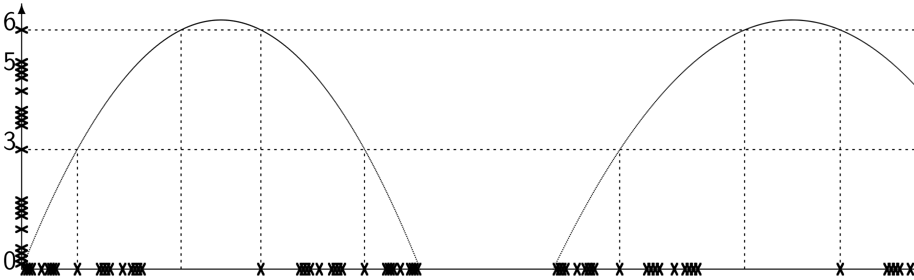
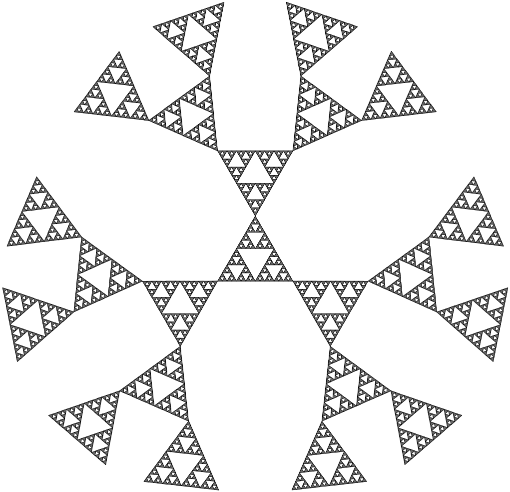
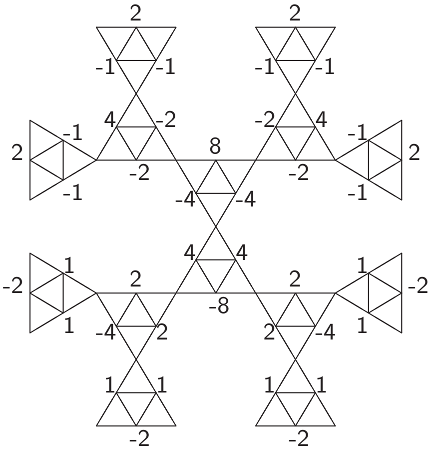


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

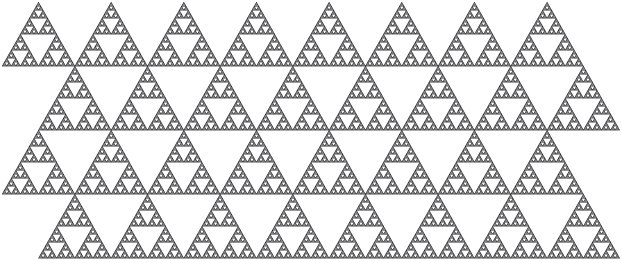
Theorem. (Béllissard 1988, T. 1998, Quint 2009)
On the infinite Sierpiński gasket the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** and a **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_\mathsf{R})$.

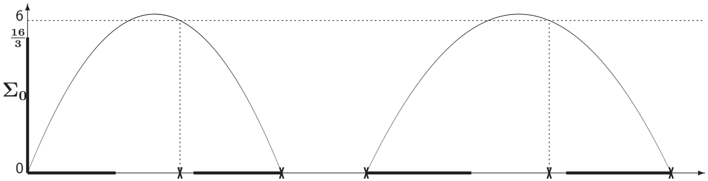


The Tree Fractafold.



An eigenfunction on the Tree Fractafold.





Theorem. (Strichartz, T. 2010) The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum. The **absolutely continuous spectrum is $\mathfrak{R}^{-1}[0, \frac{16}{3}]$.** The **pure point spectrum** consists of two infinite series of eigenvalues of infinite multiplicity. The spectral resolution is given in the main theorem.

More on motivations and connections to other areas: Cheeger, Heinonen, Koskela, Shanmugalingam, Tyson

J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces,* Geom. Funct. Anal. **9** (1999) J. Heinonen, *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001. J. Heinonen, *Nonsmooth calculus,* Bull. Amer. Math. Soc. (N.S.) **44** (2007) J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*. J. Anal. Math. 85 (2001)

In this paper the authors give a definition for the class of Sobolev functions from a metric measure space into a Banach space. They characterize Sobolev classes and study the absolute continuity in measure of Sobolev mappings in the "borderline case". Specifically, the authors prove that the validity of a Poincaré inequality for mappings of a metric space is independent of the target Banach space; they obtain embedding theorems and **Lipschitz approximation** of Sobolev functions; they also prove that pseudomonotone Sobolev mappings in the "borderline case" are absolutely continuous in measure, which is a generalization of the existing results by Y. G. Reshetnyak [Sibirsk. Mat. Zh. 28 (1987)] and by J. Malý and O. Martio [J. Reine Angew. Math. 458 (1995)]. The authors show that quasisymmetric homeomorphisms belong to a Sobolev space of

Remark: what are dimensions of the Sierpiński gasket?

- $\frac{\log 3}{\log \frac{5}{3}} \approx \mathbf{2.15} \ =$ Hausdorff dimension in effective resistance metric
- $\mathbf{2} \ =$ geometric, linear dimension
- $\frac{\log 3}{\log 2} \approx \ \mathbf{1.58} \ =$ usual Hausdorff (Minkowsky, box, self-similarity) dimension in Euclidean coordinates (geodesic metric)
- $\frac{2\log 3}{\log 5} \approx \mathbf{1.37} \ =$ usual spectral dimension
- **...** **...** **...** $=$
  there are several Lyapunov exponent type dimensions related to harmonic functions and harmonic coordinates (Kajino, Ionescu-Rogers-T)
- $\mathbf{1} \ =$ topological dimension, martingale dimension
- $\frac{2\log 2}{\log 5} \approx \mathbf{0.86} \ =$ polynomial spectral co-dimension ?

still open math problems on fractals (with some progress made)

- Existence of self-similar diffusions on finitely ramified fractals? on any self-similar fractals? on limit sets of self-similar groups? Is there a natural diffusion on any connected set with a finite Hausdorff measure (Béllissard)?

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- Resolvent and $\mathbf{e^{iHt}}$ estimates (Rogers)?

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- Resolvent and $\mathbf{e^{iHt}}$ estimates (Rogers)?
- PDEs involving derivatives, such as the Navier-Stokes equation?

still open math problems on fractals (with some progress made)

- Existence of self-similar diffusions on finitely ramified fractals? on any self-similar fractals? on limit sets of self-similar groups? Is there a natural diffusion on any connected set with a finite Hausdorff measure (Béllissard)?
- Spectral analysis on finitely ramified fractals but with few symmetries, such as Julia sets (Rogers-T), and infinitely ramified fractals (Joe Chen)? Meromorphic spectral zeta function (Steinhurst-T, Kajino)?
- Distributions or generalized functions (Rogers-Strichartz)?
- Resolvent and $\mathbf{e}^{\mathbf{iHt}}$ estimates (Rogers)?
- PDEs involving derivatives, such as the Navier-Stokes equation?
- Derivatives on fractals; differential geometry of fractals (Rogers-Ionescu-T, Cipriani-Guido-Isola-Sauvageot, **Hinz**)?