SPECTRAL ANALYSIS ON INFINITE SIERPINSKI FRACTAFOLDS

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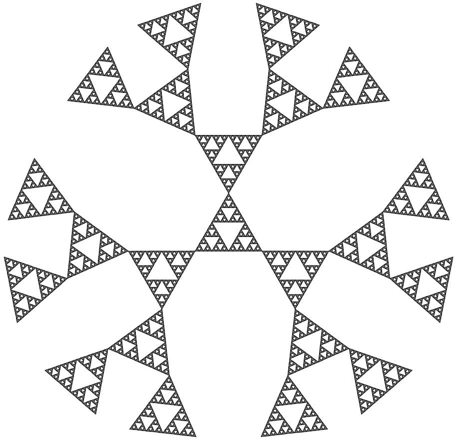

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Abstract. A fractafold, a space that is locally modeled on a specified fractal, is the fractal equivalent of a manifold. For compact fractafolds based on the Sierpiński gasket, it was shown by the first author how to compute the discrete spectrum of the Laplacian in terms of the spectrum of a finite graph Laplacian. A similar problem was solved by the second author for the case of infinite blowups of a Sierpiński gasket, where spectrum is pure point of infinite multiplicity. Both works used the method of spectral decimations to obtain explicit description of the eigenvalues and eigenfunctions. In this paper we combine the ideas from these earlier works to obtain a description of the spectral resolution of the Laplacian for noncompact fractafolds. Our main abstract results enable us to obtain a completely explicit description of the spectral resolution of the fractafold Laplacian. For some specific examples we turn the spectral resolution into a "Plancherel formula". We also present such a formula for the graph Laplacian on the 3-regular tree, which appears to be a new result of independent interest. In the end we discuss periodic fractafolds and fractal fields.

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Contents

1. Introduction

Our aim is a "Plancherel formula":

$$\boldsymbol{P_\lambda f(x) = \int P(\lambda, x, y) f(y) d\mu(y)}$$

$$\boldsymbol{f = \int_{\sigma(-\Delta)} P_\lambda f dm(\lambda)}$$

$$\boldsymbol{-\Delta P_\lambda f = \lambda P_\lambda f}$$

$$\boldsymbol{||f||_2^2 = \int_{\sigma(-\Delta)} ||P_\lambda f||_\lambda^2 dm(\lambda).}$$

Our plan:

find a continuation from graphs to fractafolds.
find the explicit spectral resolution of the graph Laplacian on $\boldsymbol{\Gamma}$;
describe explicitly a Hilbert space of $\boldsymbol{\lambda}$-eigenfunctions with norm $||\quad||_{\lambda}$;

2. Sierpiński fractafolds

2.1. Infinite Sierpiński gaskets.

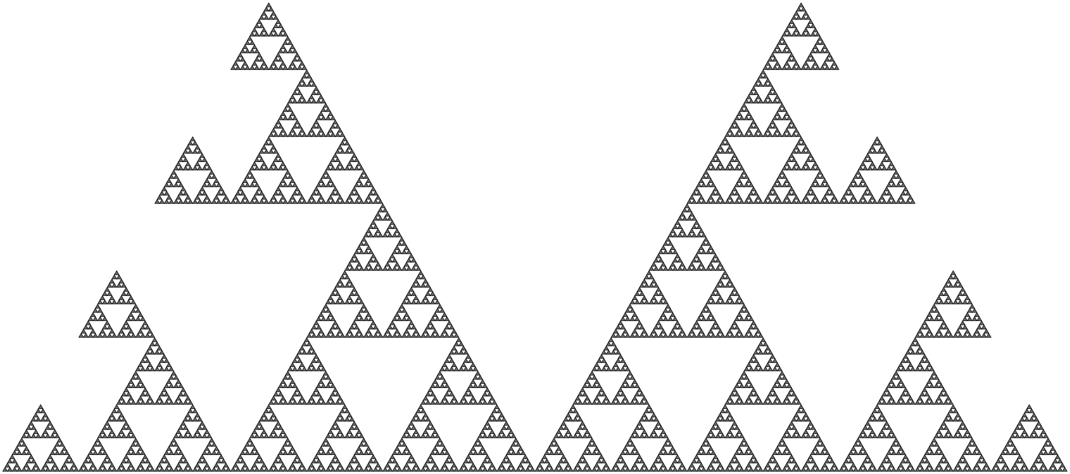


Figure 2.1. A part of an infinite Sierpiński gasket.

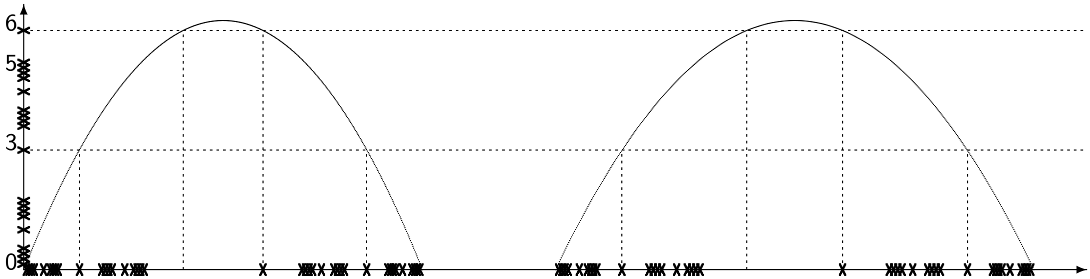


FIGURE 2.2. An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$, the vertical axis contains the spectrum of $\boldsymbol{\sigma}(-\boldsymbol{\Delta}_{\boldsymbol{\Gamma_0}})$ and the horizontal axis contains the spectrum $\boldsymbol{\sigma}(-\boldsymbol{\Delta})$.

Theorem 2.1. On the Barlow-Perkins infinite Sierpiński fractafold the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** and a **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_R)$. [T98, Quint09]

2.2. **Laplacian on the Sierpiński gasket.** Let μ_{SG} be the normalized Hausdorff probability measure on SG.

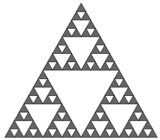


FIGURE 2.3. Sierpiński gasket.

The Laplacian Δ_{SG} on SG is self-adjoint on $L^2(SG, \mu_{SG})$ with appropriate boundary conditions and, using Kigami's resistance (or energy) form,

$$\mathcal{E}(f,f) = \lim_{n\to\infty} \left(\frac{5}{3}\right)^n \sum_{x,y\in V_n, x\sim y} (f(x) - f(y))^2 = -\frac{3}{2}\int_{SG} f\Delta_{SG} f d\mu_{SG}$$

for functions in the corresponding domain of the Laplacian (Dirichlet or Neumann).

Example 2.2. Spectral decimation for the unit interval [0,1]. $\Delta = \Delta_{[0,1]} = \frac{d^2}{dx^2}$ is the standard Laplacian on $[0,1]$, $\mu = \mu_{[0,1]}$ is the Lebesgue measure on $[0,1]$, and

$$\mathcal{E}(f,f) = \lim_{n\to\infty} 2^n \sum \left(f(\tfrac{k}{2^n}) - f(\tfrac{k+1}{2^n})\right)^2 = \int_0^1 (f'(x))^2 dx = -\int_0^1 f \Delta f \, d\mu$$

for functions in the domain of the Dirichlet or Neumann self-adjoint Laplacian. The "eigenfunction extension map" is

$$\psi_{v,\lambda}(x) = \cos(\sqrt{\lambda}\,|x-v|) - \frac{\cos(\sqrt{\lambda})}{\sin(\sqrt{\lambda})} \sin(\sqrt{\lambda}\,|x-v|)$$

where v is 0 or 1. See [Post2008].

To compute the spectrum of $-\Delta_{[0,1]}$ one can use the spectral decimation method with inverse iterations of the polynomial

$$R(z) = z(4-z).$$

Each positive eigenvalue can be written as

$$\lambda = \lim_{m\to\infty} 4^m \lambda_m$$

for a sequence $\{\lambda_m\}_{m=m_0}^{\infty}$ such that

$$\lambda_m = R(\lambda_{m+1})$$

and $\boldsymbol{\lambda}_{m_0} \in \{0, 4\}$. Then

$$\mathfrak{R}(z) = \lim_{k\to\infty} R^{\circ k}(4^{-k} z) = 2 - 2\cos(\sqrt{z})$$

satisfies the functional equation $R(\mathfrak{R}(z)) = \mathfrak{R}(4z)$ and $\sigma(-\Delta_{[0,1]}) \subset \mathfrak{R}^{-1}\{0, 4\}$.

2.3. **Sierpiński gasket: spectral decimation and the eigenfunction extension map.** Fukushima-Shima-Stichartz-T [FS, St03, St06book, T98]: Each positive eigenvalue for $-\Delta_{SG} u = \lambda u$ can be written as

$$\lambda = \lim_{m\to\infty} 5^m \lambda_m = 5^{m_0} \lim_{k\to\infty} 5^k \lambda_{k+m_0}$$

for a sequence $\{\lambda_m\}_{m=m_0}^{\infty}$ such that $\lambda_m = R(\lambda_{m+1})$ and $\lambda_{m_0} \in \{2,5,6\}$ where

$$R(z) = z(5-z).$$

With solutions of Poincare functional equations

$$\mathfrak{R}(z) = \lim_{k\to\infty} R^{\circ k}(5^{-k} z) \qquad R(\mathfrak{R}(z)) = \mathfrak{R}(5z).$$

we obtain

$$\Sigma_D = 5\left(\mathfrak{R}^{-1}\{2,5\} \cup 5\mathfrak{R}^{-1}\{5\} \bigcup_{m_0=2}^{\infty} 5^{m_0}\mathfrak{R}^{-1}\{3,5\}\right)$$

and

$$\Sigma_N = \{0\} \cup 5\left(\mathfrak{R}^{-1}\{3\} \cup \bigcup_{m_0=1}^{\infty} 5^{m_0}\mathfrak{R}^{-1}\{3,5\}\right).$$

The explicitly computed multiplicities grow exponentially fast.

If we define

$$\Sigma_{ext} = 5\left(\mathfrak{R}^{-1}\{2\} \cup \bigcup_{m=0}^{\infty} 5^m \mathfrak{R}^{-1}\{5\}\right) \subset \mathfrak{R}^{-1}\{0,6\}.$$

then

Proposition 2.3. For any $v \in \partial SG$ and any complex number $\lambda \notin \Sigma_{ext}$ there is a unique continuous function $\psi_{v,\lambda}(\cdot) : SG \to \mathbb{R}$, called the **eigenfunction extension map**, such that $\psi_{v,\lambda}(v) = 1$, $\psi_{v,\lambda}$ vanishes at the other two boundary points, and the pointwise eigenfunction equation $-\Delta\psi_{v,\lambda}(x) = \lambda\psi_{v,\lambda}(x)$ holds at every point $x \in SG \backslash \partial SG$.

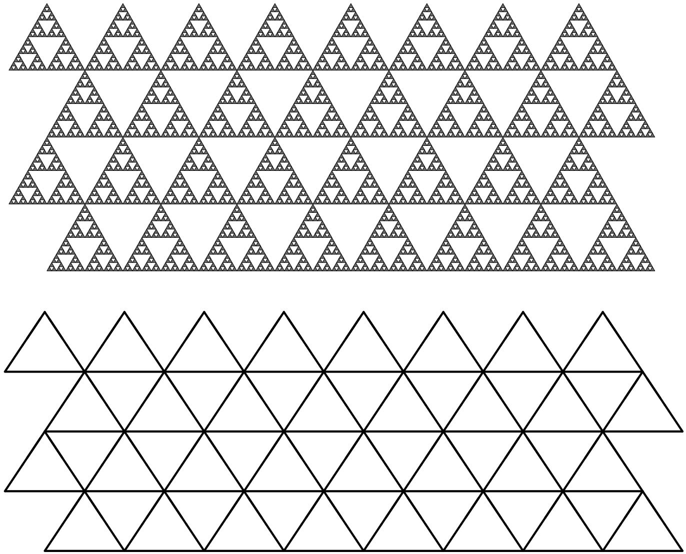


Figure 3.1. A part of the periodic triangular lattice finitely ramified Sierpiński fractal field and the graph $\mathbf{\Gamma_0}$.

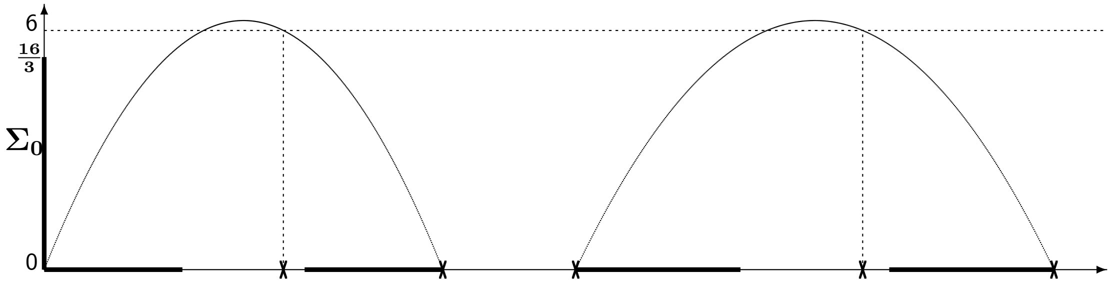


FIGURE 3.2. Computation of the spectrum on the triangular lattice finitely ramified Sierpiński fractal field.

Proposition 3.1. The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum.

The **absolutely continuous spectrum is** $\mathfrak{R}^{-1}[0, \frac{16}{3}]$**.**

The **pure point spectrum** consists of two infinite series of eigenvalues of infinite multiplicity. The series $5\mathfrak{R}^{-1}\{3\} \subsetneq \mathfrak{R}^{-1}\{6\}$ consists of isolated eigenvalues, and the series $5\mathfrak{R}^{-1}\{5\} = \mathfrak{R}^{-1}\{0\}\backslash\{0\}$ is at the gap edges of the a.c. spectrum. The eigenfunction with compact support are complete in the p.p. spectrum. The spectral resolution is given in the main theorem.

Remark 3.2. ***Note that on a periodic graph, linear combinations of compactly supported eigenfunctions are dense in an eigenspace.***

(see [Kuchment05, Theorem 8], [Kuchment93], [KuchmentPost, Lemma 3.5])

The computation of compactly supported 5- and 6- series eigenfunctions is discussed in detail in [St03, T98], and the eigenfunctions with compact support are complete in the corresponding eigenspaces. In particular, [St03, T98] show that any 6-series finitely supported eigenfunction on $\mathbf{\Gamma}_{n+1}$ is the continuation of any finitely supported function on $\mathbf{\Gamma}_{n}$, and the corresponding continuous eigenfunction on the Sierpiński fractafold $\mathfrak{F}$ can be computed using the eigenfunction extension map on fractafolds (see Subsection 6.2). Similarly, any 5-series finitely supported eigenfunction on $\mathbf{\Gamma}_{n+1}$ can be described by a cycle of triangles (homology) in $\mathbf{\Gamma}_{n}$, and the corresponding continuous eigenfunction on the Sierpiński fractafold $\mathfrak{F}$ is computed using the eigenfunction extension map on fractafolds.

Example 3.3. *The Ladder Fractafold.*

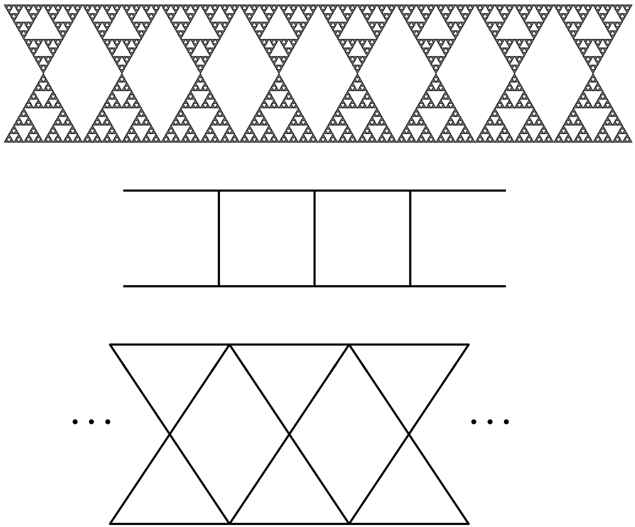


FIGURE 3.3. The graphs $\mathbf{\Gamma}$ and $\mathbf{\Gamma_0}$ for the Ladder Fractafold

It is easy to see that the spectrum of $-\mathbf{\Delta_\Gamma}$ *is* $[\mathbf{0}, \mathbf{6}]$, $-\mathbf{\Delta_{\Gamma_0}}$ *has absolutely continuous spectrum* $[\mathbf{0}, \mathbf{6}]$ *with multiplicity* $\mathbf{2}$ *in* $[\mathbf{0}, \mathbf{2}]$ *and* $[\mathbf{4}, \mathbf{6}]$ *and multiplicity* $\mathbf{4}$ *in* $[\mathbf{2}, \mathbf{4}]$.

Example 3.4. *The Honeycomb Fractafold.*

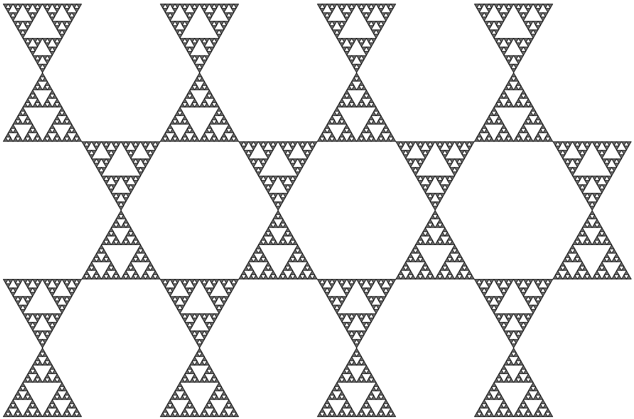
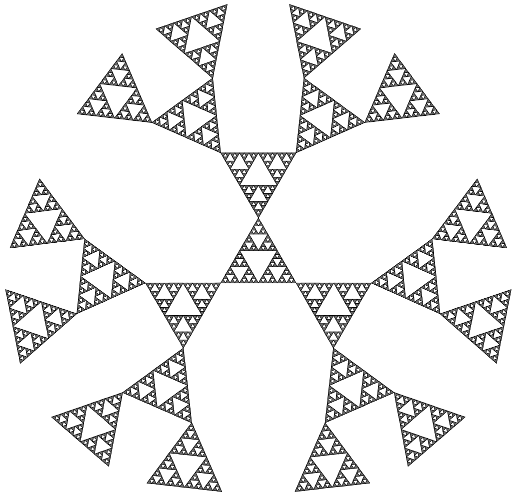


FIGURE 3.4. A part of the infinite periodic Sierpiński fractafold based on the hexagonal (honeycomb) lattice.

4. The Tree Fractafold



In this section we study in detail the spectrum of the Laplacian on the tree fractafold whose cell graph Γ is the 3-regular tree. In a sense this example is the "universal covering space" of all the other examples.

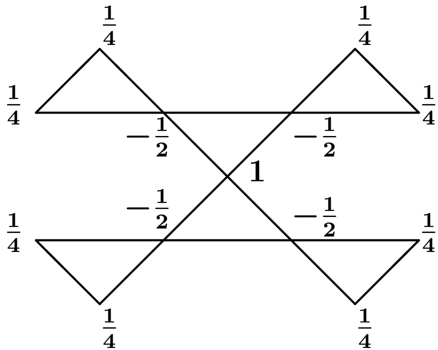


FIGURE 4.1. Values of $\sqrt{3}F_z$ (the center point is z).

It is easy to see from the 6-eigenvalue equation that F_z is the unique (up to a constant multiple) function in E_6 that is radial about z (a function of $d(x,z)$). Let $\tilde{P}_6(x,y) = \frac{1}{\sqrt{3}}F_x(y) = \frac{1}{3}(-\frac{1}{2})^{d(x,y)}$ and define $\tilde{P}_6F(x) = \sum_y \tilde{P}_6(x,y)F(y)$.

Theorem 4.1. *$\tilde{P}_6$ is the orthogonal projection $\ell^2(\Gamma_0) \to E_6$; $\{F_z\}$ is not an orthonormal basis of E_6, since $< F_z, F_y >= \sqrt{3}F_z(y)$, but it is a tight frame*

$$\sum_z | < F, F_z > |^2 = 3||F||^2_{\ell^2(\Gamma_0)}.$$

The solution of problem $(\boldsymbol{a})$ is due to Cartier [Cartier]. We outline the solution following [F-TN].

Definition 4.2. *Let* $\boldsymbol{z} \in \mathbb{C}$ *with* $\mathbf{2}^{\mathbf{2}\boldsymbol{z}-\mathbf{1}} \neq \mathbf{1}$. *Let* $\boldsymbol{c}(\boldsymbol{z}) = \frac{1}{3}\frac{2^{1-z}-2^{z-1}}{2^{-z}-2^{z-1}}$, $\boldsymbol{c}(\mathbf{1}-\boldsymbol{z}) = \frac{1}{3}\frac{2^{-z}-2^{z}}{2^{-z}-2^{z-1}}$ *and* $\boldsymbol{\varphi_z(n) = c(z)2^{-nz} + c(1-z)2^{-n(1-z)}}$.

Remark 4.3. *Note that* $\boldsymbol{c(z)}$ *and* $\boldsymbol{c(1-z)}$ *are characterized by the identities* $\boldsymbol{c(z)+c(1-z) = 1}$ *and* $\boldsymbol{c(z)2^{-z} + c(1-z)2^{z-1} = c(z)2^{z} + c(1-z)2^{1-z}}$ *which imply* $\boldsymbol{\varphi_z(0) = 1}$ *and* $\boldsymbol{\varphi_z(1) = \varphi_z(-1)}$.

Theorem 4.4. *For any fixed* $\boldsymbol{y} \in \boldsymbol{\Gamma}$, *let* $\boldsymbol{f_y(x) = \varphi_z(d(x,y))}$. *Then*

$$-\boldsymbol{\Delta}_{\boldsymbol{\Gamma}} \boldsymbol{f_y} = (\mathbf{3} - \mathbf{2}^{\boldsymbol{z}} - \mathbf{2}^{\mathbf{1}-\boldsymbol{z}})\boldsymbol{f_y}$$

and $\boldsymbol{f_y}$ *may be characterized as the unique* $(\mathbf{3} - \mathbf{2}^{\boldsymbol{z}} - \mathbf{2}^{\mathbf{1}-\boldsymbol{z}})$*-eigenfunction that is radial about* $\boldsymbol{y}$ *and satisfying* $\boldsymbol{f_y(y) = 1}$.

Theorem 4.5. *For any* $\boldsymbol{F} \in \boldsymbol{\ell}^{\mathbf{2}}(\boldsymbol{\Gamma}_{\mathbf{0}})$ *we have the explicit spectral resolution*

$$\boldsymbol{F} = \tilde{\boldsymbol{P}}_{\mathbf{6}}\boldsymbol{F} + \int_{\boldsymbol{\Sigma}} \tilde{\boldsymbol{P}}_{\boldsymbol{\lambda}}\boldsymbol{F}\,\boldsymbol{dm}(\boldsymbol{\lambda})$$

for

$$\tilde{\boldsymbol{P}}_{\boldsymbol{\lambda}}\boldsymbol{F}(\boldsymbol{x}) = \frac{\mathbf{1}}{\mathbf{3}(\mathbf{6}-\boldsymbol{\lambda})}\sum_{\boldsymbol{y}} \psi_{\frac{1}{2}+it}(\boldsymbol{d}(\boldsymbol{x},\boldsymbol{y}))\boldsymbol{F}(\boldsymbol{y}).$$

An explicit Plancherel formula on Γ is given in terms of the modified mean inner product

$$< f, g >_M = \lim_{N \to \infty} \frac{1}{N} \sum_{d(x,x_0) \leq N} f(x)\overline{g(x)}.$$

We deal with eigenspaces for which the limit exists and is independent of the point x_0. This is not the usual mean on Γ, since the cardinality of the ball $\{x : d(x, x_0) \leq N\}$ is $O(2^n)$, but it is tailor made for functions of growth rate $O(2^{-d(x,x_0)/2})$, which is exactly the growth rate of our **generalized** eigenfunctions.

Theorem 4.6. *Suppose f has finite support. Then*

$$< P_\lambda f, f > = 12b(\lambda)^{-1} < P_\lambda f, P_\lambda f >_M$$

and

$$||f||^2_{\ell^2(\Gamma)} = \int_\Sigma < P_\lambda f, P_\lambda f >_M 12b(\lambda)^{-1} d\mu(\lambda).$$

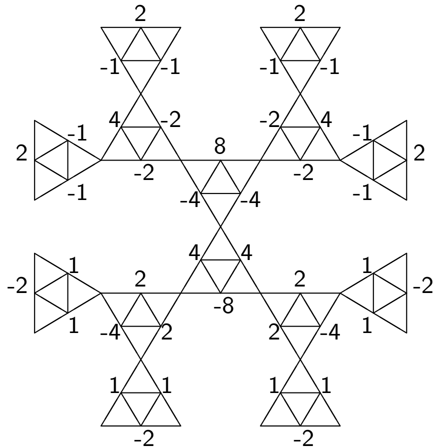


FIGURE 4.2. A part of $\mathbf{\Gamma_1}$ with a 5-eigenfunction (values not shown are equal to zero).

5. General infinite fractafolds and graphs

Let $\boldsymbol{\Gamma}$ be the cell graph, an arbitrary infinite 3-regular graph. Then $\boldsymbol{\Sigma} = \boldsymbol{\sigma}(-\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}) \subset [\mathbf{0}, \mathbf{6}]$, and for $\boldsymbol{\mu} - \boldsymbol{a.e.\lambda}$

$$-\boldsymbol{\Delta}_{\boldsymbol{\Gamma}} \boldsymbol{P}_{\boldsymbol{\lambda}}(\cdot, \boldsymbol{b}) = \boldsymbol{\lambda} \boldsymbol{P}_{\boldsymbol{\lambda}}(\cdot, \boldsymbol{b})$$

$$\boldsymbol{P}_{\boldsymbol{\lambda}} \boldsymbol{f}(\boldsymbol{a}) = \sum_{\boldsymbol{b} \in \boldsymbol{\Gamma}} \boldsymbol{P}_{\boldsymbol{\lambda}}(\boldsymbol{a}, \boldsymbol{b}) \boldsymbol{f}(\boldsymbol{b})$$

$$\boldsymbol{f} = \int_{\boldsymbol{\Sigma}} \boldsymbol{P}_{\boldsymbol{\lambda}} \boldsymbol{f} \, \boldsymbol{d\mu}(\boldsymbol{\lambda})$$

$$-\boldsymbol{\Delta}_{\boldsymbol{\Gamma}} \boldsymbol{P}_{\boldsymbol{\lambda}} \boldsymbol{f} = \boldsymbol{\lambda} \boldsymbol{P}_{\boldsymbol{\lambda}} \boldsymbol{f},$$

Let Γ_0 denote the 4-regular edge graph of Γ and

$$\tilde{P}_\lambda(x,y) = \frac{1}{6-\lambda}\sum_{a\in x}\sum_{b\in y} P_\lambda(a,b)$$

(there are 4 terms in the sum).

Theorem 5.1. *The spectral resolution of* $-\Delta_{\Gamma_0}$ *is given by*

$$F = \tilde{P}_6 F + \int_\Sigma \tilde{P}_\lambda F d\mu(\lambda)$$

where

$$-\Delta_{\Gamma_0}\tilde{P}_\lambda F = \lambda \tilde{P}_\lambda F$$

for $\mu - a.e.\lambda$, *and*

$$\tilde{P}_\lambda F(x) = \sum_{y\in\Gamma_0} \tilde{P}_\lambda(x,y)F(y).$$

In particular, $\sigma(-\Delta_{\Gamma_0}) = \Sigma$ *or* $\Sigma \cup \{6\}$.

Problems:

(a) Find an explicit formula for $\boldsymbol{P_\lambda(a,b)}$;
(b) Give an explicit description of the projection operator $\boldsymbol{\tilde{P}_6}$;
(c) Find an explicit description of the generalized eigensapce $\boldsymbol{\xi_\lambda}$ and its inner product, and transfer this to $\boldsymbol{\tilde{\xi_\lambda}}$ of $\boldsymbol{\Gamma_0}$.

Conjecture 5.2. *For* $\boldsymbol{\mu - a.e.\lambda}$ *there exists a Hilbert space of* $\boldsymbol{\lambda}$*-eigenfunctions* $\boldsymbol{\xi_\lambda}$ *with inner product* $\boldsymbol{<,>_\lambda}$ *such that* $\boldsymbol{P_\lambda f \in \xi_\lambda}$ *for* $\boldsymbol{\mu - a.e.\lambda}$ *for every* $\boldsymbol{f \in \ell^2(\Gamma)}$*, and*

$$\boldsymbol{< P_\lambda f, f > = < P_\lambda f, P_\lambda f >_\lambda .}$$

Moreover a similar statement holds for $\boldsymbol{< \tilde{P}_\lambda F, F > .}$

6. Technical details

6.1. Underlying graph assumptions and Sierpiński fractafolds. Let $\Gamma_0 = (V_0, E_0)$ be a finite or infinite graph. To define a Sierpiński fractafold, we assume that Γ_0 is a 4-regular graph which is a union of complete graphs of 3 vertices. It can be said that Γ_0 is a regular 3-hyper-graph in which every vertex belongs to two hyper-edges.

We define a Sierpiński fractafold $\mathfrak{F}$ by replacing each cell of Γ_0 by a copy of SG.

6.2. Eigenfunction extension map on fractafolds. For any function f_0 on Γ_0 (and any λ as above), we define the eigenfunction extension map by

$$\Psi_\lambda f_0(x) = \sum_{v \in V_0} f_0(v)\psi_{v,\lambda}(x).$$

By definition, $f = \Psi_\lambda f_0$ is a continuous extension of f_0 to the Sierpiński fractafold $\mathfrak{F}$ which is a pointwise solution to the eigenvalue equation

$$-\Delta\psi_{v,\lambda}(x) = \lambda\psi_{v,\lambda}(x)$$

for all $x \in \mathfrak{F}\backslash V_0$. $\Psi_\lambda : \ell^2(V_0) \to L^2(\mathfrak{F}, \mu)$ is a bounded linear operator for any $\lambda \notin \mathfrak{R}^{-1}\{2, 5, 6\}$, and its adjoint $\Psi_\lambda^* : L^2(\mathfrak{F}, \mu) \to \ell^2(V_0)$ is

$$\left(\Psi_\lambda^* g\right)(v) = \int_{\mathfrak{F}} g(x)\psi_{v,\lambda}(x)d\mu(x).$$

6.3. **Spectral decomposition (resolution of the identity).** Let the self-adjoint discrete Laplacian $\boldsymbol{\Delta_{\Gamma_0}}$ on $\boldsymbol{\Gamma_0}$ have a spectral decomposition

$$-\boldsymbol{\Delta_{\Gamma_0}} = \int_{\sigma(-\boldsymbol{\Delta_{\Gamma_0}})} \boldsymbol{\lambda} dE_{\boldsymbol{\Gamma_0}}(\boldsymbol{\lambda})$$

$$-\boldsymbol{\Delta_{\Gamma_0}} f_0(v) = \int_{\sigma(-\boldsymbol{\Delta_{\Gamma_0}})} \boldsymbol{\lambda} \sum_{u \in V_0} P_{\boldsymbol{\Gamma_0}}(\boldsymbol{\lambda}, u, v) f_0(u) dm_{\boldsymbol{\Gamma_0}}(\boldsymbol{\lambda}).$$

We define

$$M(\boldsymbol{\lambda}) = \prod_{m=1}^{\infty} \frac{(1 - \frac{1}{5}\boldsymbol{\lambda}_m)(1 - \frac{1}{2}\boldsymbol{\lambda}_m)}{(1 - \frac{1}{6}\boldsymbol{\lambda}_m)(1 - \frac{2}{5}\boldsymbol{\lambda}_m)}$$

where $\boldsymbol{\lambda} = \lim_{m\to\infty} 5^m \boldsymbol{\lambda}_m$ and $\boldsymbol{\lambda}_m = R(\boldsymbol{\lambda}_{m+1})$. This function does not depend on the fractafold, but only on the Sierpiński gasket.

Let

$$\boldsymbol{\Sigma}'_\infty = 5 \left(\bigcup_{m=1}^{\infty} 5^m \mathfrak{R}^{-1}\{3,5\} \right) \subsetneq \boldsymbol{\Sigma}_\infty = 5 \left(\mathfrak{R}^{-1}\{2\} \cup \bigcup_{m=0}^{\infty} 5^m \mathfrak{R}^{-1}\{3,5\} \right).$$

Theorem 6.1. The Laplacian Δ is self-adjoint and

$$\mathfrak{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma'_\infty \subset \sigma(-\Delta) \subset \mathfrak{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma_\infty.$$

Moreover, the spectral decomposition $-\Delta = \int\limits_{\sigma(-\Delta)} \lambda dE(\lambda)$ can be written as

$$-\Delta = \int\limits_{\mathfrak{R}^{-1}(\sigma(-\Delta_{\Gamma_0}))\setminus\Sigma_\infty} \lambda M(\lambda)\Psi^*_\lambda d\Big(E_{\Gamma_0}(\mathfrak{R}(\lambda))\Big)\Psi_\lambda + \sum_{\lambda\in\Sigma_\infty} \lambda\, E\{\lambda\}.$$

Here $E\{\lambda\}$ denotes the eigenprojection if λ is an eigenvalue. All eigenvalues and eigenfunctions of Δ can be computed by the spectral decimation method. Furthermore, the Laplacian Δ on the Sierpiński fractafold $\mathfrak{F}$ has the spectral decomposition of the form

$$-\Delta f(x) = \int\limits_{\mathfrak{R}^{-1}(\sigma(-\Delta_{\Gamma_0}))\setminus\Sigma_\infty} \lambda \left(\int_{\mathfrak{F}} P(\lambda, x, y) f(y) d\mu(y)\right) dm(\lambda) + \sum_{\lambda\in\Sigma_\infty} \lambda\, E\{\lambda\} f(x)$$

where $m = m_{\Gamma_0} \circ \mathfrak{R}$ and

$$P(\lambda, x, y) = M(\lambda) \sum_{u,v\in V_0} \psi_{v,\lambda}(x)\psi_{u,\lambda}(y) P_{\Gamma_0}(\mathfrak{R}(\lambda), u, v).$$

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VECTOR ANALYSIS ON FRACTALS AND APPLICATIONS

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Abstract. The paper surveys some recent results concerning vector analysis on fractals. We start with a local regular Dirichlet form and use the framework of **1**-forms and derivations introduced by Cipriani and Sauvageot to set up some elements of a related vector analysis in weak and non-local formulation. This allows to study various scalar and vector valued linear and non-linear partial differential equations on fractals that had not been accessible before. Subsequently a stronger (localized, pointwise or fiberwise) version of this vector analysis can be developed, which is related to previous work of Kusuoka, Kigami, Eberle, Strichartz, Hino, **Ionescu, Rogers, Röckner**, and the authors.

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Contents

1. Introduction

$$\mathbf{div}(\boldsymbol{a}(\boldsymbol{\nabla u})) = \boldsymbol{f} \tag{1.1}$$

$$\boldsymbol{\Delta u} + \boldsymbol{b}(\boldsymbol{\nabla u}) = \boldsymbol{f} \tag{1.2}$$

$$\boldsymbol{i}\frac{\boldsymbol{\partial u}}{\boldsymbol{\partial t}} = (-\boldsymbol{i\nabla} - \boldsymbol{A})^{\mathbf{2}}\boldsymbol{u} + \boldsymbol{Vu}. \tag{1.3}$$

$$(1.4)\qquad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u - \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases}$$

2. Navier-Stokes equations

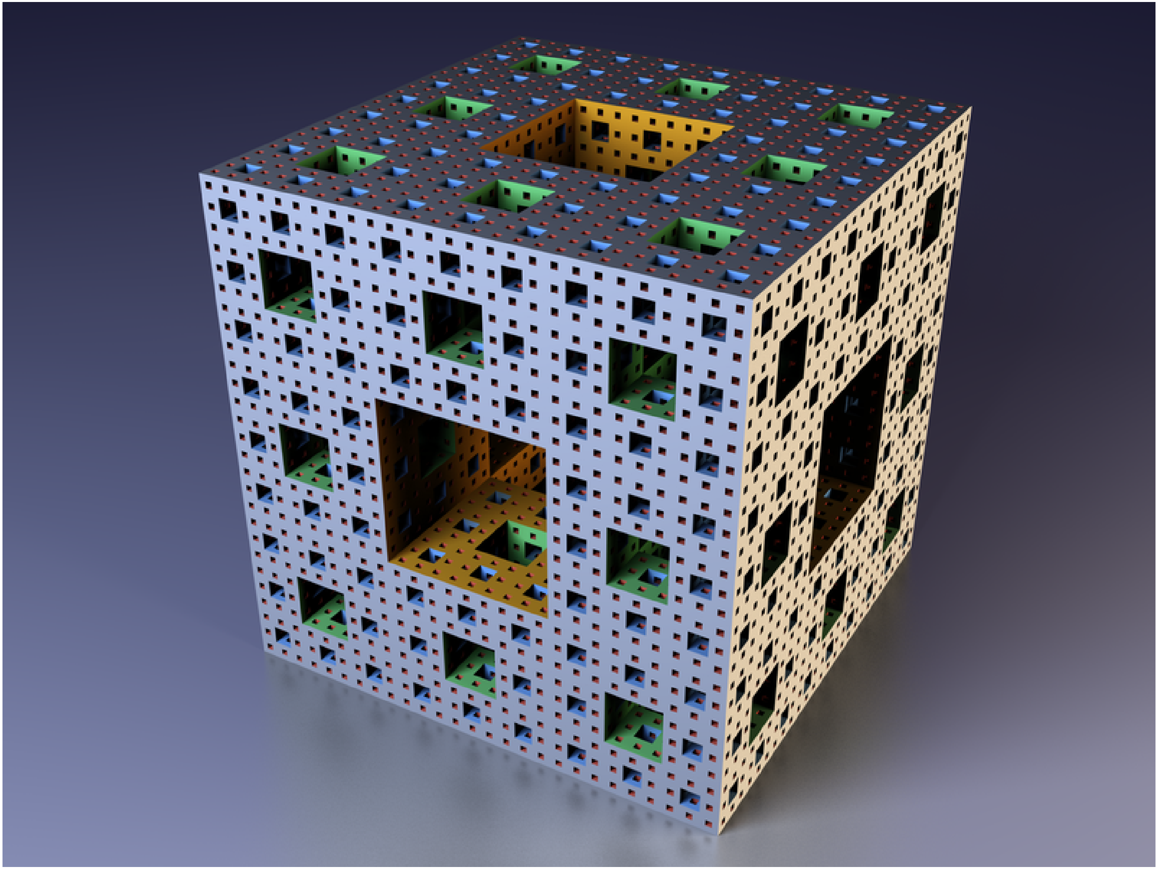
Assume that the space X is *compact, connected and topologically one-dimensional* of arbitrarily large Hausdorff and spectral dimensions.

Theorem 2.1 (The Hodge theorem). *A* 1*-form* $\omega \in \mathcal{H}$ *is harmonic if and only if it is in* $(Im\, \partial)^{\perp}$*, that is* $\operatorname{div} \omega = 0$*.*

Using the classical identity $\frac{1}{2}\nabla |u|^2 = (u \cdot \nabla) u + u \times \operatorname{curl} u$ we obtain

$$(2.1)\qquad \begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\partial \Gamma_{\mathcal{H}}(u) - \Delta_1 u + \partial p = 0 \\ \partial^* u = 0. \end{cases}$$

Theorem 2.2. *Any weak solution* u *of (2.1) is harmonic and stationary, i.e.* u *is independent of* $t \in [0, \infty)$*. Given an initial condition* u_0 *the corresponding weak solution is uniquely determined.*



Theorem 2.3. *Assume that points have positive capacity (i.e. we have a resistance form in the sense of Kigami) and the topological dimension is one. Then a nontrivial solution to (2.1) exists if and only if the first Čech cohomology* $\check{\boldsymbol{H}}^{\mathbf{1}}(\boldsymbol{X})$ *of* $\boldsymbol{X}$ *is nontrivial.*

Remark 2.4. *We conjecture that any set that carries a regular resistance form is a topologically one-dimensional space when equipped with the associated resistance metric.*

$$\mathcal{E}^{a,V}(f,g) = \langle(-i\partial - a)f, (-i\partial - a)g\rangle_{\mathcal{H}} + \langle fV, g\rangle_{L_2(X,m)}, \quad f,g \in \mathcal{C}_{\mathbb{C}},$$

Theorem 3.1. *Let $a \in \mathcal{H}_\infty$ and $V \in L_\infty(X,m)$.*

(i) *The quadratic form $(\mathcal{E}^{a,V}, \mathcal{F}_{\mathbb{C}})$ is closed.*

(ii) *The self-adjoint non-negative definite operator on $L_{2,\mathbb{C}}(X,m)$ uniquely associated with $(\mathcal{E}^{a,V}, \mathcal{F}_{\mathbb{C}})$ is given by*

$$H^{a,V} = (-i\partial - a)^*(-i\partial - a) + V,$$

and the domain of the operator A is a domain of essential self-adjointness for $H^{a,V}$.

Note: related Dirac operator is well defined and self-adjoint

$$D = \begin{pmatrix} 0 & -i\partial^* \\ -i\partial & 0 \end{pmatrix}$$

4. Dirichlet forms and energy measures

Let X be a locally compact separable metric space and m a Radon measure on X such that each nonempty open set is charged positively. We assume that $(\mathcal{E}, \mathcal{F})$ is a symmetric local regular Dirichlet form on $L_2(X, m)$ with core $\mathcal{C} := \mathcal{F} \cap C_0(X)$. Endowed with the norm $\|f\|_{\mathcal{C}} := \mathcal{E}(f)^{1/2} + \sup_X |f|$ the space $\mathcal{C}$ becomes an algebra and in particular,

$$\mathcal{E}(fg)^{1/2} \leq \|f\|_{\mathcal{C}} \, \|g\|_{\mathcal{C}}\,, \quad f, g \in \mathcal{C}, \tag{4.1}$$

see [16]. For any $g, h \in \mathcal{C}$ we can define a finite signed Radon measure $\Gamma(g, h)$ on X such that

$$2 \int_X f \, d\Gamma(g, h) = \mathcal{E}(fg, h) + \mathcal{E}(fh, g) - \mathcal{E}(gh, f)\ , \quad f \in \mathcal{C},$$

the *mutual energy measure* of g and h. By approximation we can also define the mutual energy measure $\Gamma(g, h)$ for general $g, h \in \mathcal{F}$. Note that Γ is symmetric and bilinear, and $\Gamma(g) \geq 0$, $g \in \mathcal{F}$. For details we refer the reader to [28]. We provide some examples.

Examples

(i) *Dirichlet forms on Euclidean domains*. Let $X = \Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and

$$\mathcal{E}(f,g) = \int_\Omega \nabla f \nabla g \, dx, \quad f,g \in C^\infty(\Omega).$$

If $H_0^1(\Omega)$ denotes the closure of $C^\infty(\Omega)$ with respect to the scalar product $\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + \langle f,g \rangle_{L_2(\Omega)}$, then $(\mathcal{E}, H_0^1(\Omega))$ is a local regular Dirichlet form on $L_2(\Omega)$. The mutual energy measure of $f,g \in H_0^1(\Omega)$ is given by $\nabla f \nabla g dx$.

(ii) *Dirichlet forms on Riemannian manifolds*. Let $X = M$ be a smooth compact Riemannian manifold and

$$\mathcal{E}(f,g) = \int_M \langle df, dg \rangle_{T^*M} \; dvol, \quad f,g \in C^\infty(M).$$

Here $dvol$ denotes the Riemannian volume measure. Similarly as in (i) the closure of $\mathcal{E}$ in $L_2(M, dvol)$ yields a local regular Dirichlet form. The mutual energy measure of two energy finite functions f,g is given by $\langle df, dg \rangle_{T^*M} \; dvol$.

(iii) *Dirichlet forms induced by resistance forms on fractals*.

5. 1-FORMS AND VECTOR FIELDS

Consider $\mathcal{C} \otimes \mathcal{B}_b(X)$, where $\mathcal{B}_b(X)$ is the space of bounded Borel functions on X with the symmetric bilinear form

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} := \int_X bd \, d\Gamma(a,c), \tag{5.1}$$

$a \otimes b, c \otimes d \in \mathcal{C} \otimes \mathcal{B}_b(X)$, let $\|\cdot\|_{\mathcal{H}}$ denote the associated seminorm on $\mathcal{C} \otimes \mathcal{B}_b(X)$ and write

Define *space of differential* 1*-forms on* X

$$\mathcal{H} = \mathcal{C} \otimes \mathcal{B}_b(X)/ker \ \|\cdot\|_{\mathcal{H}}$$

we

The space $\mathcal{H}$ becomes a bimodule if we declare the algebras $\mathcal{C}$ and $\mathcal{B}_b(X)$ to act on it as follows: For $a \otimes b \in \mathcal{C} \otimes \mathcal{B}_b(X)$, $c \in \mathcal{C}$ and $d \in \mathcal{B}_b(X)$ set

$$c(a \otimes b) := (ca) \otimes b - c \otimes (ab) \tag{5.2}$$

and

$$(a \otimes b)d := a \otimes (bd). \tag{5.3}$$

A *derivation operator* $\partial : \mathcal{C} \to \mathcal{H}$ can be defined by setting

$$\partial f := f \otimes 1.$$

It obeys the Leibniz rule,

$$\partial(fg) = f\partial g + g\partial f, \quad f, g \in \mathcal{C}, \tag{5.4}$$

and is a bounded linear operator satisfying

$$\|\partial f\|_{\mathcal{H}}^2 = \mathcal{E}(f), \quad f \in \mathcal{C}. \tag{5.5}$$

On Euclidean domains and on smooth manifolds the operator ∂ coincides with the classical exterior derivative (in the sense of L_2-differential forms). Details can be found in [21, 22, 39, 40, 46].

Being Hilbert, $\mathcal{H}$ is self-dual. We therefore regard 1-forms also as *vector fields* and ∂ as the *gradient operator*. Let $\mathcal{C}^*$ denote the dual space of $\mathcal{C}$, normed by

$$\|w\|_{\mathcal{C}^*} = \sup\left\{|w(f)| : f \in \mathcal{C}, \|f\|_{\mathcal{C}} \leq 1\right\}.$$

Given $f, g \in \mathcal{C}$, consider the functional

$$u \mapsto \partial^*(g\partial f)(u) := -\langle \partial u, g\partial f\rangle_{\mathcal{H}} = -\int_X g \, d\Gamma(u, f)$$

on $\mathcal{C}$. It defines an element $\partial^*(g\partial f)$ of $\mathcal{C}^*$, to which we refer as the *divergence of the vector field* $g\partial f$.

Lemma 5.1. *The divergence operator ∂^* extends continuously to a bounded linear operator from $\mathcal{H}$ into $\mathcal{C}^*$ with $\|\partial^* v\|_{\mathcal{C}^*} \leq \|v\|_{\mathcal{H}}$, $v \in \mathcal{H}$. We have*

$$\partial^* v(u) = -\langle \partial u, v\rangle_{\mathcal{H}}$$

for any $u \in \mathcal{C}$ and any $v \in \mathcal{H}$.

The Euclidean identity

$$\boldsymbol{div}\,(\boldsymbol{g}\,\boldsymbol{grad}\,\boldsymbol{f}) = \boldsymbol{g}\Delta \boldsymbol{f} + \nabla \boldsymbol{f} \nabla \boldsymbol{g}$$

has a counterpart in terms of ∂ and ∂^*. Let $(A, dom\ A)$ denote the infinitesimal $L_2(X,\mu)$-generator of $(\mathcal{E},\mathcal{F})$.

Lemma 5.2. *We have*

$$\partial^*(g\partial f) = gAf + \Gamma(f,g)\ ,$$

for any simple vector field $g\partial f$, $f, g \in \mathcal{C}$, *and in particular,* $Af = \partial^*\partial f$ *for* $f \in \mathcal{C}$.

Corollary 5.3. *The domain* $dom\ \partial^*$ *agrees with the subspace*

$$\{v \in \mathcal{H} : v = \partial f + w : f \in dom\ A\ ,\ \ w \in ker\ \partial^*\}\,.$$

For any $v = \partial f + w$ *with* $f \in dom\ A$ *and* $w \in ker\ \partial^*$ *we have* $\partial^* v = Af$.

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