The asypmtotic beaviour of the eigenvalues of the Laplacian on irregular or random Cantor-like fractals

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Definition[1]

Let μ be a finite Borel-measure on $[a,b]$ and let $f:[a,b]\to\mathbb{R}$. A function $h\in L_2([a,b],\mu)$ is called the *μ-derivative of f*, iff

$$f(x)=f(a)+\int_{[a,x]} h\,d\mu \qquad \text{for all } x\in[a,b].$$

Proposition

The μ-derivative is unique in $L_2(\mu)$ and is denoted by $\frac{df}{d\mu}$.

We denote $\frac{df}{d\lambda}$ by f' (λ is the Lebesgue measure).

[1] Freiberg: *Analytical properties of measure geometric Krein-Feller-operators on the real line*, Math. Nachr. 260 (2003)

Definition

Let ν and μ be atomless, finite Borel-measures on $[a, b]$.

- $H^1(\nu) := \{f : [a, b] \to \mathbb{R} | \frac{df}{d\nu} \text{ exists}\}$
- $H^2(\nu, \mu) := \{f \in H^1(\nu) : \frac{df}{d\nu} \in H^1(\mu)\}$

Definition

We define the operator Δ^μ on $L_2([a, b], \mu)$ by

$$\Delta^\mu f = \frac{d}{d\mu} f', \quad f \in H^2(\lambda, \mu).$$

Basic properties

Proposition

- *Let $f, g \in H^1(\mu)$. Then $fg \in H^1(\mu)$ and*

$$\frac{d}{d\mu}(fg) = \frac{df}{d\mu}g + f\frac{dg}{d\mu}.$$

- *Let $f \in H^2(\lambda, \mu)$ and $g \in H^1(\lambda)$. Then*

$$\int_{[a,b]} (\Delta^\mu f) g \, d\mu = f'g\Big|_a^b - \int_a^b f'(x)g'(x)\, dx.$$

Basic properties

Proposition

- *Let* $g\colon [a,b] \to \mathbb{R}$ *invertable and* $f \in H^1\big([g(a), g(b)], \mu\big)$*.*
  Then $f \circ g \in H^1(g^{-1}\mu)$ *and*

$$\frac{d}{d(g^{-1}\mu)}(f \circ g) = \frac{df}{d\mu} \circ g.$$

  $g^{-1}\mu$ *denotes the image measure of* μ *with respect to* g^{-1}*, that is,* $g^{-1}\mu(A) = \mu\big(g(A)\big)$ *for every Borel-set* $A \subseteq [a,b]$*.*

1. Derivatives with respect to μ
2. Scale irregular Cantor-like fractals
3. Scaling properties
4. Spectral asymptotics

Scale irregular Cantor fractals

Suppose for every $k \in \{1, \dots, K\}$ ($K \in \mathbb{N}$) we are given an IFS $\mathcal{S}^{(k)} = (S_1^{(k)}, \dots, S_{N_k}^{(k)})$ by

$$S_i^{(k)}(x) := r_i^{(k)} x + c_i^{(k)}, \quad x \in [a, b], \quad i = 1, \dots, N_k,$$

with $r_i^{(k)}$ and $c_i^{(k)}$ s.th.

$$a = S_1^{(k)}(a) < S_1^{(k)}(b) < S_2^{(k)}(a) < \dots < S_{N_k}^{(k)}(b) = b.$$

A sequence $\xi = (\xi_1, \xi_2, \dots)$ with $\xi_i \in \{1, \dots, K\}$ is called *environment sequence*[1].

[1] Barlow, Hambly: *Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets*, Ann. Inst. Henri Poincaré, 33 (1997), no. 5

Construction of the fractal

Set

$$W_n^{(\xi)} := \{1, \ldots, N_{\xi_1}\} \times \{1, \ldots, N_{\xi_2}\} \times \cdots \times \{1, \ldots, N_{\xi_n}\},$$
$$S_w^{(\xi)} := S_{w_1}^{(\xi_1)} \circ S_{w_2}^{(\xi_2)} \circ \cdots \circ S_{w_n}^{(\xi_n)} \text{ for } w = (w_1, \ldots, w_n) \in W_n^{(\xi)},$$

and

$$\begin{aligned} K_0 &= [a, b], \\ K_1 &= S_1^{(\xi_1)}([a, b]) \cup \cdots \cup S_{N_{\xi_1}}^{(\xi_1)}([a, b]), \\ &\vdots \\ K_n^{(\xi)} &= \bigcup_{w \in W_n^{(\xi)}} S_w^{(\xi)}([a, b]). \end{aligned}$$

Construction of the fractal

Then

$$K^{(\xi)} := \bigcap_{i=1}^{\infty} K_i^{(\xi)}.$$

By construction,

$$K^{(\xi)} = \bigcup_{i=1}^{N_{\xi_1}} S_i^{(\xi_1)}\left(K^{(\theta\xi)}\right),$$

where $\theta\xi = (\xi_2, \xi_3, \dots)$ denotes a left shift.

Definition of the measure $\mu^{(\xi)}$

For each $k \in \{1, \ldots, K\}$, let $m_1^{(k)}, \ldots, m_{N_k}^{(k)} \in (0,1)$ with $\sum_{i=1}^{N_k} m_i^{(k)} = 1$. We construct a measure $\mu^{(\xi)}$ on $[a,b]$ with support $K^{(\xi)}$ such that

$$\mu^{(\xi)} = \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} S_i^{(\xi_1)} \mu^{(\theta\xi)},$$

where $S_i^{(\xi_1)} \mu^{(\theta\xi)}$ is the image measure of $\mu^{(\theta\xi)}$ through $S_i^{(\xi_1)}$.

Lemma

$$S_i^{(\xi_1)^{-1}} \mu^{(\xi)} = m_i^{(\xi_1)} \mu^{(\theta\xi)}, \qquad \textit{for all } i = 1, \ldots, N_{\xi_1}.$$

Derivation with respect to $\mu^{(\xi)}$

If $f \in H^2(\lambda, \mu^{(\xi)})$, then

$$f'(x) = f'(a) + \int_a^x \Delta^{\mu^{(\xi)}} f \, d\mu^{(\xi)}, \qquad x \in [a, b].$$

That means, f' is constant on each component of $[a, b] \setminus K^{(\xi)}$.

Lemma

If $f \in H^2(\lambda, \mu^{(\xi)})$, then $f \circ S_i^{(\xi_1)} \in H^2(\lambda, \mu^{(\theta\xi)})$ for $i = 1, \ldots, N_{\xi_1}$ and

$$\Delta^{\mu^{(\theta\xi)}}(f \circ S_i^{(\xi_1)}) = m_i^{(\xi_1)} r_i^{(\xi_1)} \left(\Delta^{\mu^{(\xi)}} f\right) \circ S_i^{(\xi_1)}.$$

Definition

1. We define

$$\mathcal{E}(f,g)=\int_a^b f'(x)g'(x)\,dx, \qquad f,g\in H^1(\lambda)$$

2. We put

$$\mathcal{F}^{(\xi)}=\{f\in H^1(\lambda)\colon\ f' \text{ is constant on each component of } [a,b]\setminus K^{(\xi)}\}.$$

Remark

$$H^2(\lambda,\mu^{(\xi)})\subseteq\mathcal{F}^{(\xi)}\subseteq H^1(\lambda)\subseteq L_2([a,b],\mu^{(\xi)}).$$

Proposition

For any environment sequence ξ, $(\mathcal{E}, \mathcal{F}^{(\xi)})$ is a Dirichlet Form on $L_2([a,b], \mu^{(\xi)})$.

Lemma

Let $f \in H^2(\lambda, \mu)$ and $g \in H^1(\lambda)$. Then

$$\mathcal{E}(f,g) = -\left\langle \Delta^\mu f, g \right\rangle_{L_2(\mu)} + f'(b)g(b) - f'(a)g(a).$$

Eigenvalues

Proposition

For $k \in \mathbb{R}$ and $f \in \mathcal{F}^{(\xi)}$,

$$\mathcal{E}(f, g) = k\langle f, g\rangle_{L_2(\mu^{(\xi)})} \quad \text{for all } g \in \mathcal{F}^{(\xi)}$$

if and only if $f \in H^2(\lambda, \mu^{(\xi)})$ and

$$\Delta^{\mu^{(\xi)}} f = -kf$$
$$f'(a) = f'(b) = 0$$

Eigenvalue counting function

Definition

Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet Form on $L_2(\mu)$ with a discrete non-negative spectrum. Then

$$N_{(\mathcal{E},\mathcal{F})}(x) = \#\{k \leq x\colon \ k \text{ is eigenvalue of } (\mathcal{E},\mathcal{F})\}, \quad x \geq 0,$$

counted according to multiplicity.

Theorem[1]

Let $(\mathcal{F},\mathcal{E})$ and $(\mathcal{F}',\mathcal{E}')$ Dirichlet forms on $L_2(\mu)$ with $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{E}' = \mathcal{E}|_{\mathcal{F}'\times\mathcal{F}'}$. Then

$$N_{(\mathcal{E}',\mathcal{F}')}(x) \leq N_{(\mathcal{E},\mathcal{F})}(x), \qquad \text{for all } x \geq 0.$$

[1] Kigami, Lapidus: *Weyl's problem for the spectral distribution of laplacians on P.C.F. self-similar fractals*, Commun. Math. Phys. 158 (1993)

Scaling property of the energy form

Proposition

Let $f, g \in \mathcal{F}^{(\xi)}$. Then

$$\mathcal{E}(f,g) = \sum_{i=1}^{N_{\xi_1}} \frac{1}{r_i^{(\xi_1)}} \mathcal{E}(f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)}) + \sum_{i=1}^{N_{\xi_1}-1} f'\left(S_i^{(\xi_1)}(b)\right) g'\left(S_i^{(\xi_1)}(b)\right) \left[S_{i+1}^{(\xi_1)}(a) - S_i^{(\xi_1)}(b)\right].$$

Scaling property of the inner product

Proposition

Let $f, g \in L_2(\mu^{(\xi)})$*. Then*

$$\langle f, g \rangle_{L_2(\mu^{(\xi)})} = \sum_{i=1}^{N_{\xi_1}} m_i^{(\xi_1)} \langle f \circ S_i^{(\xi_1)}, g \circ S_i^{(\xi_1)} \rangle_{L_2(\mu^{(\theta\xi)})}.$$

Scaling property of the eigenvalue counting function

Put

$$m_w^{(\xi)} = m_{w_1}^{(\xi_1)} \cdots m_{w_n}^{(\xi_n)}, \qquad w = (w_1, \dots, w_n) \in W_n^{(\xi)}$$

and write $N^{(\xi)} := N_{(\mathcal{E}, \mathcal{F}^{(\xi)})}$.

Proposition

For all $x \geq 0$ and $n \in \mathbb{N}$,

$$N^{(\xi)}(x) = \sum_{w \in W_n^{(\xi)}} N^{(\theta^n \xi)}(r_w^{(\xi)} m_w^{(\xi)} x).$$

For Dirichlet boundary conditions we make an analogous argumentation. Then, using $N_D^{(\xi)}(x) \leq N^{(\xi)}(x)$, we determine the asymptotical behaviour of N and N_D.

Asymptotic behaviour of the eigenvalue counting function

Let $\gamma_n^{(\xi)}$ be defined by

$$\sum_{w \in W_n^{(\xi)}} \left(r_w^{(\xi)} m_w^{(\xi)} \right)^{\gamma_n^{(\xi)}} = 1, \qquad n \in \mathbb{N}.$$

Then, if the limit exists,

$$\gamma^{(\xi)} := \lim_{n \to \infty} \gamma_n^{(\xi)}.$$

Conjecture

$$\frac{N^{(\xi)}(t)}{t^\alpha} \to \infty, \quad \text{if } \alpha < \gamma$$
$$\frac{N^{(\xi)}(t)}{t^\alpha} \to 0, \quad \text{if } \alpha > \gamma$$

Asymptotic behaviour of the eigenvalue counting function

For every $k \in \{1, \ldots, K\}$ and $n \in \mathbb{N}$ put

$$h_k^{(\xi)}(n) = \frac{1}{n} \#\{i \leq n \colon \xi_i = k\}$$

and assume that

$$h_k^{(\xi)}(n) \to p_k^{(\xi)}, \quad (n \to \infty)$$

for some p_k.

Conjecture

Under the above assumption $\gamma^{(\xi)} = \lim_{n\to\infty} \gamma_n^{(\xi)}$ exists, and if ξ, η are environment sequences with $p_k^{(\xi)} = p_k^{(\eta)}$, then $\gamma^{(\xi)} = \gamma^{(\eta)}$.

Random environment sequences

Let $\xi = (\xi_i)_{i=1}^{\infty}$, where ξ_i are i.i.d. random variables with $\mathbb{P}(\xi_i = k) = p_k$ for $k \in \{1, \ldots, K\}$. Then, $h_k(n) \to p_k$ a.s. and we can apply the above results.

Special case

Assume that $r_i^{(k)} = r^{(k)}$ and $m_i^{(k)} = m^{(k)}$ for all k and i. Then

$$\gamma = -\frac{\mathbb{E} \log N_{\xi_1}}{\mathbb{E} \log(r^{(\xi_1)} m^{(\xi_1)})}.$$