The Gelfand widths of ℓ_p-balls for $0 < p \leq 1$

Tino Ullrich

tino.ullrich@hcm.uni-bonn.de

Hausdorff Center for Mathematics, University of Bonn

Joint work with:
Simon Foucart (Paris), Alain Pajor (Marne-la-Vallée), Holger Rauhut (Bonn)

For a subset K of $\mathbb{R}^N$ (quasi-)normed by $\|\cdot\|_X$, the Gelfand widths are defined as the numbers

$$d^m(K,X) := \inf_{A \in \mathbb{R}^{m \times N}} \sup_{v \in K \cap \ker A} \|v\|_X, \quad , \quad m < N,$$

and represent an important concept in classical and modern approximation and complexity theory. They have found recent interest in the rapidly emerging field of compressive sensing because they give general performance bounds for sparse recovery methods. Since vectors in ℓ_p-balls $K = B_p^N$, $0 < p \leq 1$, can be well-approximated by sparse vectors in $X = \ell_q^N$ if $0 < p < q$, the Gelfand widths of such balls are particularly relevant in this context. In substantial papers from the 1970s and 80s due to Kashin, Gluskin, and Garnaev, upper and lower estimates for the Gelfand widths of ℓ_1-balls are provided. Donoho extends these estimates to the Gelfand widths of ℓ_p-balls with $p < 1$. For $0 < p \leq 1$ and $p < q \leq 2$ it holds

$$\min\left\{1, \frac{\ln(N/m)+1}{m}\right\}^{1/p-1/q} \lesssim d^m(B_p^N, \ell_q^N) \lesssim \min\left\{1, \frac{\ln(N/m)+1}{m}\right\}^{1/p-1/q}. \tag{1}$$

Unfortunately, his proof of the lower bound contains a gap and so the result remained unproved. In this talk we show how to use compressive sensing methods to establish this lower bound in a more intuitive way. Our method is new even for the case $p = 1$. In the remaining time we comment on the proof of the upper bound based on RIP estimates. Our techniques also provide the same sharp asymptotic behavior for the Gelfand widths of weak-ℓ_p-balls.