Pre-Coherence Spaces with Approximation Structure: A Model for Intuitionistic Linear Logic Which is Not a Model of Classical Linear Logic

Dieter Spreen
Theoretische Informatik, Fachbereich Mathematik
Universität Siegen, 57068 Siegen, Germany
Email: spreen@informatik.uni-siegen.de

Abstract

By using additional structure inherent in coherence spaces a new model for intuitionistic linear logic is constructed which is not a model for classical linear logic. The new class of spaces contains also the empty space, whence it yields a logical model, not only a type-theoretic one.

1 Introduction

Linear logic was discovered by Girard [7, 8] while constructing his coherence space semantics for second-order lambda calculus, introduced independently by Girard [5, 6] and Reynolds [11] and called System F by Girard. Since its discovery linear logic has found much interest among logicians and computer scientists. It is a resource-sensitive logic that keeps at the same time the constructivity of intuitionistic logic and the symmetries of classical logic. The coherent space interpretation of linear logic can be regarded as its *standard* semantics, since it has been extracted from this semantics.

Coherence spaces are a special kind of Scott domains and can thus be represented as inverse limits of cochains of their finite subspaces. Each such approximating chain of subspaces defines uniform levels of approximation for the elements of the approximated space. In this paper we present an axiomatic description of this phenomenon: pre-coherence spaces with approximation structure. By doing so we relax the notion of a coherence space in such a way that it need not contain a least element. Its existence is ensured by the approximation structure, if the space is not empty. As is shown in the paper, the category of pre-coherence spaces with approximation structure and linear maps yields a model of intuitionistic linear logic, but not of classical linear logic. The reason is that a pre-coherence spaces with approximation structure may be empty. Thus, we obtain not only a model for the type system given by the derivable formulae, but also of the logic.

The model construction parallels more or less Girard's original construction. But one of the central identities in Girard's construction, namely the one which allows the definition of linear implication by linear negation (dualization) and tensor, does not hold in the construction presented here. The reason is that dualization trivializes the approximation structure.

There are also other order-theoretic models of intuitionistic linear logic. Examples are the category of complete partial orders and strict Scott continuous functions as well as the category of dI-domains and linear maps and generalizations thereof (cf. [4]).

The paper is organized as follows. In Section 2 pre-coherence spaces with approximation structure are introduced and the necessary operations for the interpretation of linear logic are defined. The syntax and the deduction rules for linear logic are given in Section 3. Moreover, it is shown that pre-coherence spaces with approximation structure and linear maps are a model for intuitionistic linear logic, but not of classical linear logic. Final remarks appear in Section 4. Here, we show that in the model linear implication cannot be defined by linear negation (dualization) and tensor.

2 Pre-coherence spaces with approximation structure

Let $(D, \sqsubseteq)$ be a (possibly empty) partially ordered set (poset). For a subset S of D, $\downarrow S = \{ x \in D \mid (\exists y \in S) x \sqsubseteq y \}$ is the *lower set* generated by S. The subset S is called *compatible* if it has an upper bound. S is *directed*, if it is nonempty and every pair of elements in S has an upper bound in S. D is *directed-complete* if every directed subset S of D has a least upper bound $\bigsqcup S$ in D, D is *bounded-complete* if every compatible pair $\{x, y\}$ of elements of D has a least upper bound $x \sqcup y$ in D, and D is *binary-complete* if every nonempty subset S of D such that $x \sqcup y$ exists in D, for all $x, y \in S$, has a least upper bound in D. Note that in a bounded- and directed-complete poset any nonempty compatible subset has a least upper bound.

An element x of D is *compact* if for any directed subset S of D the least upper bound of which exists in D the relation $x \sqsubseteq \bigsqcup S$ always implies the existence of an element $u \in S$ with $x \sqsubseteq u$. We write D^0 for the set of all compact elements of D. D is called *finitary* if for all $u \in D^0$ the lower set generated by $\{u\}$ is finite. If for every $y \in D$ the set of all $u \in D^0$ with $u \sqsubseteq y$ is directed and has y as its least upper bound, the poset D is said to be *algebraic*, and *ω-algebraic* if, in addition, D^0 is countable. Bounded- and directed-complete ω-algebraic posets with least element are also called *Scott domains*. Standard references for domain theory and its applications are [10, 9, 1, 13, 3].

A *complete prime* (*coprime*) of D is an element d such that for any nonempty compatible subset S of D which has a least upper bound in D the relation $d \sqsubseteq \bigsqcup S$ always implies the existence of an element $x \in S$ with $d \sqsubseteq x$. We denote the set of coprimes of D by D^p. D is called *prime algebraic* if $d = \bigsqcup \{ p \in D^p \mid p \sqsubseteq d \}$, for all $d \in D$.

Definition 2.1 A poset D is a *pre-coherence space* if it is binary-complete, bounded-complete and prime algebraic such that any two different coprimes are incomparable with respect to the partial order.

As is easily verified, pre-coherence spaces are finitary. Moreover, the pre-coherence spaces with least element are exactly the coherence spaces defined by Girard [8]. Hence, they are finitary Scott domains that satisfy Berry's Axiom d which says that for all $x, y, z \in D$ such that $\{y, z\}$ is bounded, $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$. Such domains are called dI-domains [3].

When dealing with dI-domains one usually considers stable functions as morphisms.

Definition 2.2 Let D and E be posets. A function $f \colon D \to E$ is said to be

1. *Scott continuous* if it is monotone and if for any directed subset S of D such that the least upper bound of S exists in D or the least upper bound of $f(S)$ exists in E, one has that also the other least upper bound exists and

$$f(\bigsqcup S) = \bigsqcup f(s).$$

2. *stable* if it is Scott continuous and for all compatible pairs $\{x, y\}$ in D such that the greatest lower bound $x \sqcap y$ of x and y exists in D or the greatest lower bound of $f(x)$ and $f(y)$ exists in E, one has that also the other greatest lower bound exists and

$$f(x \sqcap y) = f(x) \sqcap f(y).$$

Stable functions $f, g\colon D \to E$ are usually ordered by the *stable ordering* $\sqsubseteq_S$, where $f \sqsubseteq_s g$ if for all $x, y \in D$

$$x \sqsubseteq y \Rightarrow f(x) = f(y) \sqcap g(x).$$

It is widely known that every dI-domain D can be represented as the inverse limit of an ω-cochain $(D_i)_{i\in\omega}$ of finite subdomains [2]. Since the subdomains D_i are closed under the operation of taking existing least upper bounds, any element x has a best approximation $\bigsqcup\{\, z \in D_i \mid z \sqsubseteq x \,\}$ in each of them. Abstracting from such properties one obtains the notion of an approximation structure.

Definition 2.3 Let D be a pre-coherence space and for each $i \in \omega$, $[\cdot]_i^D\colon D \to D$. $([\cdot]_i^D)_{i\in\omega}$ is said to be an *approximation structure* on D if the following conditions (1)-(7) hold:

1. $[\cdot]_i^D$ is stable.
2. $\downarrow D_i \subseteq D_i$, where $D_i = \{\, x \in D \mid [x]_i^D = x \,\}$.
3. $D^0 \subseteq \bigcup_i D_i$.
4. $[\cdot]_i^D \circ [\cdot]_j^D = [\cdot]_{\min\{i,j\}}^D$.
5. $[\cdot]_i^D \sqsubseteq_s \mathrm{id}_D$.
6. $\bigsqcup_i [\cdot]_i^D = \mathrm{id}_D$.
7. $(\forall x, y \in D)[x]_0^D = [y]_0^D$.

Approximation structures satisfying conditions (1)-(6) of the above definition have been studied by the author in the case of dI-domains [12].

Note that a pre-coherence space D with approximation structure $([\cdot]_i^D)_{i\in\omega}$ may be empty. In this case the maps $[\cdot]_i^D$ are the empty maps. But if D is not empty, it has a least element, by conditions (5) and (7), which we denote by $\bot_D$. Thus, a pre-coherence space with approximation structure is either empty or a coherence space.

Examples of pre-coherence spaces with approximation structure which we need later are the two-point space $\mathbb{S} = \{\bot, \top\}$ with $\bot \sqsubseteq \top$, $[x]_i^{\mathbb{S}} = x$, for $x \in \mathbb{S}$ and $i > 0$, and $[x]_0^{\mathbb{S}} = \bot$, and the one-point space $\perp\!\!\!\perp = \{\bot\}$ with $[\cdot]_i^{\perp\!\!\!\perp}$ being the identity on $\perp\!\!\!\perp$, for all $i \in \omega$.

For $x \in D$ let the *rank* of x, written $\mathrm{rk}(x)$, be the smallest number i with $x \in D_i$, if there is such an i, and let it be ω, otherwise. Note that for every coprime p, $\mathrm{rk}(p) > 0$, since $\bot_D$ is not a coprime.

Definition 2.4 Let D and E be pre-coherence spaces with approximation structure. A stable function $f\colon D \to E$ is *linear* if $f \circ [\cdot]_0^D = [\cdot]_0^E \circ f$ and for all compatible pairs $\{x, y\}$ of elements of D,

$$f(x \sqcup y) = f(x) \sqcup f(y).$$

Lemma 2.5 *Let D be a pre-coherence space with approximation structure. Then every map $[\cdot]_i^D$ is linear.*

Proof: The first requirement follows from condition 2.3(4). For the verification of the second one let $\{x, y\}$ be a compatible pair of elements of D. By the monotonicity of $[\cdot]_i^D$ we have that $[x]_i^D \sqcup [y]_i^D \sqsubseteq [x \sqcup y]_i^D$. Now, let $p \in D^p$ with $p \sqsubseteq [x \sqcup y]_i^D$. Then $p \in D_i$ by condition 2.3(2). Moreover, $p \sqsubseteq x \sqcup y$, by condition 2.3(5). Hence $p \sqsubseteq x$ or $p \sqsubseteq y$, since p is a coprime. Thus $p = [p]_i^D \sqsubseteq [x]_i^D$ or $p = [p]_i^D \sqsubseteq [y]_i^D$, which implies that $p \sqsubseteq [x]_i^D] \sqcup [y]_i^D$. Since D is prime algebraic it follows that $[x \sqcup y]_i^D \sqsubseteq [x]_i^D \sqcup [y]_i^D$.

For stable maps $f: D \to E$ between pre-coherence spaces one has that for each $p \in E^p$ there is a least element $u \in D^0$ with $p \sqsubseteq f(u)$ (see e.g. [4] where similar spaces are considered). This is used for an alternative description of stable functions.

Definition 2.6 Let D and E be pre-coherence spaces and $f: D \to E$ be stable. The set

$$\mathrm{Tr}(f) = \{ (u, p) \in D^0 \times E^p \mid u \text{ is minimal with } p \sqsubseteq f(u) \}$$

is called the *trace* of f.

As is easily verified, $\mathrm{Tr}([\cdot]_i^D) = \{ (p, p) \mid p \in D^p \wedge \mathrm{rk}(p) \leq i \}$, for all $i \in \omega$.

The trace of a stable function has certain characteristic properties such that each set of pairs of compact elements and coprimes with these properties is the trace of a stable function. Let pr_1 and pr_2, respectively, denote the projection on the first and the second component.

Lemma 2.7 *Let D and E be pre-coherence spaces and $f: D \to E$ be stable. Then $\mathrm{Tr}(f)$ satisfies the following properties:*

1. *For every $x \in D$ there is some $u \in \mathrm{pr}_1(\mathrm{Tr}(f))$ with $u \sqsubseteq x$.*
2. *For every finite subset X of $\mathrm{Tr}(f)$, if $\mathrm{pr}_1(x)$ is compatible, so is $\mathrm{pr}_2(x)$.*
3. *For all $(u, p), (v, p) \in \mathrm{Tr}(f)$, if $\{u, v\}$ is compatible then $u = v$.*
4. *For all $q \in E^p$ with $q \sqsubseteq p$, for some $(u, p) \in \mathrm{Tr}(f)$, there is a $v \sqsubseteq u$ such that $(v, q) \in \mathrm{Tr}(f)$.*

Lemma 2.8

1. *From $\mathrm{Tr}(f)$ one can compute f via the following formula:*

$$f(x) = \bigsqcup \{ p \mid (\exists u \sqsubseteq x)(u, p) \in \mathrm{Tr}(f) \}.$$

2. *Each set of pairs of compact elements and coprimes satisfying conditions (1)-(4) of Lemma 2.7 is the trace of a stable function defined by the formula above.*

Because of these results one usually identifies a stable function with its trace. Note in addition that for two stable functions f, g between pre-coherence spaces

$$f \sqsubseteq_s g \Leftrightarrow \mathrm{Tr}(f) \subseteq \mathrm{Tr}(g).$$

There is a nice way of seeing whether a stable function is linear or not by looking at its trace.

Lemma 2.9 *Let D and E be pre-coherence spaces with approximation structure and $f\colon D \to E$ be stable. Then f is linear if and only if* $\mathrm{pr}_1(\mathrm{Tr}(f)) \subseteq D^p$.

Theorem 2.10 *Let D and E be pre-coherence spaces with approximation structure. Then the set $[D \to_\ell E]$ of all linear functions from D into E is a pre-coherence space. Its coprimes are the linear functions that have singletons as trace.*

For $h \in [D \to_\ell E]$ and $i \in \omega$ define

$$[h]_i^\ell(x) = [h(x)]_i^E \qquad (x \in D).$$

Then $[h]_i^\ell$ is linear again. The trace of $[h]_i^\ell$ can easily be computed from the trace of h.

Lemma 2.11 *Let D and E be pre-coherence spaces with approximation structure, $h \in [D \to_\ell E]$ and $i \in \omega$. Then*

$$\mathrm{Tr}([h]_i^\ell) = \{\, (p,q) \in \mathrm{Tr}(h) \mid \mathrm{rk}(q) \leq i \,\}.$$

Theorem 2.12 *Let D and E be pre-coherence spaces with approximation structure. Then $([\cdot]_i^\ell)_{i\in\omega}$ is an approximation structure on $[D \to_\ell E]$.*

For pre-coherence spaces D and E with approximation structure set

$$D \otimes E = (D \setminus [\cdot]_0^D(D)) \times (E \setminus [\cdot]_0^E(E)) \cup ([\cdot]_0^D(D) \times [\cdot]_0^E(E)),$$

define

$$\begin{aligned} x \sqsubseteq y \Leftrightarrow & [x, y \in (D \setminus [\cdot]_0^D(D)) \times (E \setminus [\cdot]_0^E(E)) \wedge \mathrm{pr}_1(x) \sqsubseteq_D \mathrm{pr}_1(y) \wedge \mathrm{pr}_2(x) \sqsubseteq_E \mathrm{pr}_2(y)] \\ & \vee x \in ([\cdot]_0^D(D) \times [\cdot]_0^E(E)), \end{aligned}$$

and let $[x]_i^\otimes = ([\mathrm{pr}_1(x)]_i^D, [\mathrm{pr}_2(x)]_i^E)$, for $x \in (D \setminus [\cdot]_0^D(D)) \times (E \setminus [\cdot]_0^E(E))$ and $i > 0$. In the remaining cases let $[x]_i^\otimes$ be the unique element of $([\cdot]_0^D(D) \times [\cdot]_0^E(E))$, if this set is not empty, and undefined, otherwise.

Moreover, let $D \times E$ be the Cartesian product of D and E endowed with the component-wise partial order and the mappings $[\cdot]_i^\times$ $(i \in \omega)$ defined by

$$[(x,y)]_i^\times = ([x]_i^D, [y]_i^E).$$

Next, set

$$D \oplus E = [\{0\} \times (D \setminus [\cdot]_0^D(D))] \cup [\{1\} \times (E \setminus [\cdot]_0^E(E))] \cup ([\cdot]_0^D(D) \times [\cdot]_0^E(E)),$$

define

$$\begin{aligned} x \sqsubseteq y \Leftrightarrow & [\mathrm{pr}_1(x) = \mathrm{pr}_1(y) = 0 \wedge \mathrm{pr}_2(x) \sqsubseteq_D \mathrm{pr}_2(y)] \\ & \vee [\mathrm{pr}_1(x) = \mathrm{pr}_1(y) = 1 \wedge \mathrm{pr}_2(x) \sqsubseteq_E \mathrm{pr}_2(y)] \\ & \vee x \in ([\cdot]_0^D(D) \times [\cdot]_0^E(E)), \end{aligned}$$

and for $i > 0$ set $[x]_i^\oplus = [\mathrm{pr}_2(x)]_i^D$, if $x \in \{0\} \times (D \setminus [\cdot]_0^D(D))$, and $[x]_i^\oplus = [\mathrm{pr}_2(x)]_i^E$, if $x \in \{1\} \times (E \setminus [\cdot]_0^E(E))$. For $i > 0$ and $x \in ([\cdot]_0^D(D) \times [\cdot]_0^E(E))$, and for $i = 0$ and $x \in D \oplus E$ let $[x]_i^\oplus$ be the unique element in $([\cdot]_0^D(D) \times [\cdot]_0^E(E))$, if this set is not empty, and let it be undefined, otherwise.

Finally, let $\wp(D)$ be the set of all nonempty bounded subsets of D^0, ordered by set inclusion. For $i \in \omega$ and $X \in \wp(D)$ set

$$[X]_i^{\wp(D)} = \{\, [x]_i^D \mid x \in X \,\}.$$

Then $[X]_i^{\wp(D)}$ is bounded, by condition 2.3(5).

Theorem 2.13 *Let D and E be pre-coherence spaces with approximation structure. Then $D \otimes E$, $D \times E$, $D \oplus E$ and $\wp(D)$ are pre-coherence spaces with approximation structure as well.*

3 Propositional linear logic

We use capital Greek letters $\Gamma, \Delta, \ldots$ for finite sequences of formulae and capital Latin letters $A, B, \ldots$ for single formulae. Formulae are built up as usual from propositional variables $\alpha_0, \alpha_1, \ldots$ and constants $\bot, \top, \mathbf{0}, \mathbf{1}$ with the help of the binary operators $\sqcap, \sqcup, \star, \multimap$ and the unary operator $!$. Note here that we follow the notation of Troelstra [14], from where we have also taken the natural deduction system in Table 1. Moreover, observe that if Γ is the sequence $A_1, \ldots, A_n$ then $!\Gamma$ denotes the sequence $!A_1, \ldots, !A_n$.

Table 1: Natural deduction system for propositional linear logic

$$(\mathrm{Ax})\ A \vdash A \qquad (\top\mathrm{I})\ \Gamma \vdash \top \qquad (\bot\mathrm{E})\ \Gamma, \bot \vdash A$$

$$(\star\mathrm{I})\ \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \star B} \qquad (\star\mathrm{E})\ \frac{\Gamma \vdash A \star B \quad \Delta, A, B \vdash C}{\Gamma, \Delta \vdash C}$$

$$(\sqcap\mathrm{I})\ \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \sqcap B} \qquad (\sqcap\mathrm{E}_i)\ \frac{\Gamma \vdash A_0 \sqcap A_1}{\Gamma \vdash A_i}\ (i \in \{0,1\})$$

$$(\sqcup\mathrm{I}_i)\ \frac{\Gamma \vdash A_i}{\Gamma \vdash A_0 \sqcup A_1}\ (i \in \{0,1\}) \qquad (\sqcup\mathrm{E})\ \frac{\Delta \vdash A \sqcup B \quad A, \Gamma \vdash C \quad B, \Gamma \vdash C}{\Gamma, \Delta \vdash C}$$

$$(\multimap\mathrm{I})\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad (\multimap\mathrm{E})\ \frac{\Gamma \vdash A \multimap B \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}$$

$$(\mathbf{1}\mathrm{I})\ \vdash \mathbf{1} \qquad (\mathbf{1}\mathrm{E})\ \frac{\Gamma \vdash \mathbf{1} \quad \Delta \vdash A}{\Gamma, \Delta \vdash A}$$

$$(\mathbf{0}\text{-rule})\ \frac{\Gamma, A \multimap \mathbf{0} \vdash \mathbf{0}}{\Gamma \vdash \mathbf{0}}\ (\text{CLL only})$$

$$(!\mathrm{I})\ \frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \qquad (!\mathrm{E}^w)\ \frac{\Gamma \vdash !B \quad \Delta \vdash A}{\Gamma, \Delta \vdash A}$$

$$(!\mathrm{E})\ \frac{\Gamma \vdash !B \quad \Delta, B \vdash A}{\Gamma, \Delta \vdash A} \qquad (!\mathrm{E}^c)\ \frac{\Gamma \vdash !B \quad \Delta, !B, !B \vdash A}{\Gamma, \Delta \vdash A}$$

A sequent $\Gamma \vdash A$ is derivable in classical propositional linear logic, if it has a derivation using the axioms and rules given in Table 1. If it has a derivation in which the **0**-rule is not used, it is derivable in intuitionistic propositional linear logic. We denote this by $\Gamma \vdash_I A$.

In the interpretation of propositional linear logic which we give below a pre-coherence space with approximation structure is assigned to each formula by induction on the structure of the formula. Let to this end η be an assignment of pre-coherence spaces with approximation structure to the propositional variables. The meaning $[\![A]\!]_\eta$ of a formula A with respect to the assignment η is then defined as follows.

$$
\begin{aligned}
&[\![\alpha]\!]_\eta = \eta(\alpha),\\
&[\![\bot]\!]_\eta = [\![\top]\!]_\eta = \perp\!\!\!\perp,\\
&[\![\mathbf{0}]\!]_\eta = [\![\mathbf{1}]\!]_\eta = \mathbb{S},\\
&[\![A \sqcap B]\!]_\eta = [\![A]\!]_\eta \times [\![B]\!]_\eta,\\
&[\![A \sqcup B]\!]_\eta = [\![A]\!]_\eta \oplus [\![B]\!]_\eta,\\
&[\![A \star B]\!]_\eta = [\![A]\!]_\eta \otimes [\![B]\!]_\eta,\\
&[\![A \multimap B]\!]_\eta = [[\![A]\!]_\eta \to_\ell [\![B]\!]_\eta],\\
&[\![!A]\!]_\eta = \wp([\![A]\!]_\eta).
\end{aligned}
$$

A sequence $\Gamma = A_1, \ldots, A_n$ is interpreted by $[\![\Gamma]\!]_\eta = [\![A_1]\!]_\eta \otimes \cdots \otimes [\![A_n]\!]_\eta$, if $n \geq 1$, and $[\![\Gamma]\!]_\eta = \perp\!\!\!\perp$, otherwise, and a sequent $\Gamma \vdash A$ is interpreted by the collection of all linear maps from $[\![\Gamma]\!]_\eta$ to $[\![A]\!]_\eta$, which we denote by $[\![\Gamma \vdash A]\!]_\eta$. A sequent $\Gamma \vdash A$ is *valid* in the pre-coherence space with approximation structure model, written $\mathcal{PCA} \models \Gamma \vdash A$, if $[\![\Gamma \vdash A]\!]_\eta$ is not empty, for every assignment η of pre-coherence spaces with approximation structure to propositional variables.

Theorem 3.1 *Let Γ be a finite sequence of formulae of propositional linear logic and A be a formula of propositional linear logic such that $\Gamma \vdash_I A$. Then $\mathcal{PCA} \models \Gamma \vdash A$.*

The proof of the theorem is straightforward and will be skipped here. Note that if $[\![\Gamma]\!]_\eta$ is empty, then $[\![\Gamma \vdash A]\!]_\eta$ contains exactly the empty map, i.e., $[\![\Gamma \vdash A]\!]_\eta$ is the one-point space. If both $[\![\Gamma]\!]_\eta$ and $[\![\Gamma \vdash A]\!]_\eta$ are not empty, then the same must be true for $[\![A]\!]_\eta$.

Theorem 3.2 *Let Γ be the empty sequence and η be an assignment of pre-coherence spaces with approximation structure to propositional variables such that $\eta(\alpha_0)$ is the empty space. Then $[\![\Gamma, \alpha_0 \multimap \mathbf{0} \vdash \mathbf{0}]\!]_\eta$ is not empty, but $[\![\Gamma \vdash \alpha_0]\!]_\eta$ is empty.*

Proof: Note that $[\![\alpha_0 \multimap \mathbf{0}]\!]_\eta$ consists exactly of the empty map. Thus, $[\![\Gamma]\!]_\eta$, $[\![\Gamma, \alpha_0 \multimap \mathbf{0}]\!]_\eta$ as well as $[\![\mathbf{0}]\!]_\eta$ are the one-point space. It follows that $[\![\Gamma, \alpha_0 \multimap \mathbf{0} \vdash \mathbf{0}]\!]_\eta$ is not empty. But since $[\![\alpha_0]\!]_\eta$ is empty, $[\![\Gamma \vdash \alpha_0]\!]_\eta$ must be empty too.

Thus, the $\mathbf{0}$-rule is not sound in the pre-coherence space with approximation structure semantics. It follows that intuitionistic propositional linear logic is sound with respect to this semantics, but classical propositional linear logic is not.

4 Final remarks

In this paper we presented a model of intuitionistic linear logic, which is not a model of classical linear logic. The model is obtained by making use of some additional structure of coherence spaces, which are used by Girard in his construction of a model for classical linear logic. The additional structure is given by the fact that a coherence space can be represented as the inverse limit of an ω-cochain of finite subspaces. This cochain allows the introduction of uniform levels of approximation for the elements of the coherence space. One advantage of using this additional structure is that one can define spaces which are either coherence spaces or empty so that the model is not only a type-theoretic one, but also a model of the logic. On the other hand it destroys one of the central identities in Girard's construction.

For a pre-coherence space with approximation structure D let $D^\perp = [D \to_\ell \mathbb{S}]$. $D^\perp$ is called the *dual* of D. Then one has with Lemma 2.11 that for every $i > 0$ and every $f \in D^\perp$ with nonempty trace,

$$\begin{aligned} \operatorname{Tr} f &= \{\, (p, q) \in D^p \times \mathbb{S}^p \mid q \sqsubseteq f(p) \,\} \\ &= \{\, (p, \top) \mid p \in D^p \wedge \top = f(p) \,\} \\ &= \operatorname{Tr}([f]_i^\ell), \end{aligned}$$

since $\operatorname{rk}(\top) = 1$. Thus $[\cdot]_i^\ell = [\cdot]_1^\ell$, for all $i > 0$, which means that all coprimes of $D^\perp$ have the same rank.

In Girard's coherence space model one has that the linear function space $[D \to_\ell E]$ is isomorphic to $(D \otimes E^\perp)^\perp$, which is no longer true in the case of coherence spaces with approximation structure, since $(D \otimes E^\perp)^\perp$ has always a trivial approximation structure, which need not be the case for $[D \to_\ell E]$.

References

[1] Abramsky, S. and Jung, A.: Domain theory. In: Abramsky, S., Gabbay, D. M., and Maibaum, T. S. E. (eds.): *Handbook of Logic in Computer Science, Vol. 3, Semantic Structures*. Oxford University Press, Oxford (1994) 1–168

[2] Amadio, R. : Bifinite domains: stable case. In: Pitt, D. H. *et al.* (eds.): Category Theory and Computer Science, 4th Biannual Summer Conf., Paris, France, September 3–6, 1991. Lecture Notes in Computer Science, Vol. 530, Springer, Berlin (1991) 16–33

[3] Amadio, R. M. and Curien, P.-L.: *Domains and Lambda-Calculi*. Cambridge University Press, Cambridge (1998)

[4] Braüner, T.: A model of intuitionistic affine logic from stable domain theory. In: Abiteboul, S. and Shamir, E. (eds.): Automata, Languages and Programming, 21st Intern. Colloq., Jerusalem, Israel, 1994. Lecture Notes in Computer Science, Vol. 820, Springer, Berlin (1994) 340–351

[5] Girard, J.-Y.: Une extension de l'interprétation fonctionelle de Gödel à l'analyse et son application à l'élimination des coupures dans l'analyse et la théorie des types. In: Fenstad, J. F. (ed.): Proceedings of the Second Scandinavian Logic Symposium. North-Holland, Amsterdam (1971) 63–92

[6] Girard, J.-Y.: Interprétation fonctionelle et élimination des coupures de l'arithmétique d'ordre supérieure. These d'Etat. Université Paris VII (1972)

[7] Girard, J.-Y.: Linear logic. *Theoret. Comp. Sci.* 50 (1987) 1–102

[8] Girard, J.-Y., Lafont, Y., and Taylor, P.: *Proofs and Types*. Cambridge University Press, Cambridge (1990)

[9] Gunter, C. A.: *Semantics of Programming Languages*. MIT Press, Cambridge, MA (1992)

[10] Gunter, C. A. and Scott, D. S.: Semantic domains. In: van Leeuwen, J. (ed.): *Handbook of Theoretical Computer Science, Vol. B, Formal Models and Semantics*. Elsevier, Amsterdam (1990) 633–674

[11] Reynolds, J.C.: Towards a Theory of Type Structures. In: Robinet, B. (ed.): Colloque sur la programmation. Lecture Notes in Computer Science, Vol. 19, Springer-Verlag, Berlin (1974) 408–425

[12] Spreen, D.: On functions preserving levels of approximation: A refined model construction for various lambda calculi. *Theoret. Comp. Sci.* 212 (1999) 261–303

[13] Stoltenberg-Hansen, V., Lindström, I., and Griffor, E. R.: *Mathematical Theory of Domains.* Cambridge University Press, Cambridge (1994)

[14] Troelstra, A. S., *Lectures on Linear Logic.* CSLI, Stanford, CA (1992)