# Some results related to the continuity problem*

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**Abstract**

The continuity problem, i.e., the question whether effective maps between effectively given topological spaces are effectively continuous, is reconsidered. In earlier work it was shown that this is always the case, if the effective map also has a witness for noninclusion. The extra condition does not have an obvious topological interpretation. As is shown in the present paper, it appears naturally where in the classical proof that sequentailly continuous maps are continuous the Axiom of Choice is used. The question is therefore whether the witness condition appears in the general continuity theorem only for this reason, i.e., whether effective operators are effectively sequentially continuous. For two large classes of spaces covering all important applications it is shown that this is indeed the case. The general question, however, remains open.

## 1 Introduction

Computations are usually required to end in finite time. Because of this only a *finite* amout of information about the input can be used during a computation. Moreover, an output once written on the output tape cannot be changed anymore: given more information about the input, the machine can only extend what is already written on the output tape (Monotonicity).

These properties not only hold for functions on the natural numbers, but also for the computation of operators on such functions. A natural topology can be defined on such spaces with respect to which computable operators turn out to be (effectively) continuous.

If one restricts one's interest to functions which are computable and can therefore be presented by the programs computing them (or their codings), there is another way of specifying the computability of operators: an operator is *effective* if it is tracked by a computable function on the code.

The continuity problem is the question whether effective operators are the restrictions (to computable inputs) of (effectively) continuous operators. Obviously, both approaches are rather unconnected. Nevertheless for certain important cases positive solutions were presented: In the case of operators on the partial computable functions this is due to Myhill and Shepherdson [14]; in the case of the total computable functions to Kreisel, Lacombe and

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Shoenfield [11]. In the first case the result has been generalised to certain types of directed-complete partial orders with the Scott topology (cf. e.g. Egli and Constable [3]; Sciore and Tang [17]; Weihrauch [28]), in the other to separable metric spaces (Ceitin [2]; Moschovakis [13]). These two types of spaces are quite different, not only topologically: they also offer different algorithmic techniques to use. As follows from an example by Friedberg [8], effective operators are not continuous, in general.

The situation remained unclear for quite a while. In 1984 Spreen and Young [26] showed that for second-countable topological $T_0$ spaces effective maps are effectively continuous if they have a witness for noninclusion. Later Spreen [20] showed that for quite a large class of spaces this property is also necessary. It says that if the image of a basic open set under the operator is not included in a given basic open sets in its co-domain, then one needs be able to effectively produce a witness for this.

The condition seems natural when dealing with continuity. In the present note we will give even more evidence for its canonicity. In classical topology it is well known that for second-countable spaces sequentially continuous maps are continuous. The proof can be transferred into a constructive framework. There is however one step in which the classical proof uses the Axiom of Choice and the effective information needed here is exactly what is provided by the witness for noninclusion condition.

So, the question comes up whether effective operators are effectively sequentially continuous and the extra condtion is only needed for the step from effective sequential continuity to effective continuity. We will show for a large class of spaces that effective operators are effectively continuous. To this end we require the spaces to come equipped with a set of canonical computable sequences which are such that sequences can be *stretched* by wait-and-see strategies and the operator taking convergent sequences to their limits is effective. In addition all basic open sets need be completely enumerable, uniformly in their index. All conditions seem very natural, but as we will see, in particular the combination of wait-and-see strategies with limit computations has a strong impact on the topology.

If we deal with spaces as the total computable functions or the computable real numbers, then a metric is at hand which allows putting stronger conditions on the convergence of sequences. These conditions are important in order to be able to render the limit operator computable, however they are not compatible with wait-and-see strategies. Other algorithmic techniques like decision procedures are at hand instead. Also for spaces of this kind it is shown that effective operators are effectively sequentially continuous. However, we have not been able to present a uniform approach to the question whether effective operators are effectively sequentially continuous as we did in the case of effective continuity. It is even not clear whether this holds in general. A modification of Friedberg's example shows that effective operators are not sequentially continuous in general. But this still leaves open the possibility that they are effecively sequentially continuous as we are dealing with computable sequences only in this case.

As is well known, limits of point sequences in a $T_0$ space are not uniquely determined. In the joint paper [26] we had to make special assumption to handle this problem. Later, in [20] we based our approach on filter convergence to get rid of it. In both cases we had to assume that one can effectively pass from a computable enumeration of the sequence elements and/or a filter base to the points they converge to. The relationship between both conditions will be studied as well.

The paper is organized as follows: Section 2 contains basic definitions. In Section 3 notions and results from the theory of effective spaces are recalled. A new construction of an acceptable numbering is given. Important special cases of such spaces are considered in Section 4. The condition of a numbering having a limit algorithm and the existence of such numberings is dicussed in Section 5. In Section 6 the relationship between effective continuity notions of different strength is investigated, in particular the relationship between effective continuity and effective sequential continuity. Finally, in Section 7, the question of when an effective map is effectively sequentially continuous is examined.

## 2 Basic definitions

In what follows, let $\langle\ ,\ \rangle : \omega^2 \to \omega$ be a computable pairing function with corresponding projections $\pi_1$ and $\pi_2$ such that $\pi_i(\langle a_1, a_2\rangle) = a_i$. We extend the pairing function in the usual way to an $n$-tupel encoding. The projection functions are then denoted by $\pi_i^n$ with $1 \le i \le n$. Let $P^{(n)}$ ($R^{(n)}$) denote the set of all $n$-ary partial (total) computable functions, and let $W_i$ be the domain of the $i$th partial computable function $\varphi_i$ with respect to some Gödel numbering $\varphi$. We let $\varphi_i(a)\downarrow$ mean that the computation of $\varphi_i(a)$ stops, $\varphi_i(a)\downarrow \in C$ that it stops with value in $C$, and $\varphi_i(a)\downarrow_n$ that it stops within $n$ steps. In the opposite cases we write $\varphi_i(a)\uparrow$ and $\varphi_i(a)\uparrow_n$ respectively. Moreover, we write $F\colon X \rightharpoonup Y$ to mean that $F$ is a partial function from set $X$ into set $Y$ with domain $\mathrm{dom}(F)$.

A *(partial) numbering* $\nu$ of a set $S$ is a partial map $\nu\colon \omega \rightharpoonup S$ (onto). The value of $\nu$ at $n \in \mathrm{dom}(\nu)$ is denoted, by $\nu_n$. If $s \in S$ and $n \in \mathrm{dom}(\nu)$ with $\nu_n = s$, then $n$ is said to be an *index* of $s$. Numberings $\nu$ with $\mathrm{dom}(\nu) = \omega$, are called *total.* Note that instead of numbering we also say *indexing.*

**Definition 2.1.** Let $\nu, \kappa$ be numberings of set $S$.

1. $\nu \le \kappa$, read $\nu$ is *reducible* to $\kappa$, if there is some function $g \in P^{(1)}$ with $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(g)$, $g(\mathrm{dom}(\nu)) \subseteq \mathrm{dom}(\kappa)$, and $\nu_m = \kappa_{g(m)}$, for all $m \in \mathrm{dom}(\nu)$.

2. $\nu \equiv \kappa$, read $\nu$ is *equivalent* to $\kappa$, if $\nu \le \kappa$ and $\kappa \le \nu$.

A subset $X$ of $S$ is *completely enumerable*, if there is a computably enumerable set $W_n$ such that $\nu_i \in X$ if and only if $i \in W_n$, for all $i \in \mathrm{dom}(\nu)$. Set $M_n = X$, for any such $n$ and $X$, and let $M_n$ be undefined, otherwise. Then $M$ is a numbering of the class *CE* of completely enumerable subsets of $S$. If $W_n$ is decidable, the $X$ is called *completely decidable.*

$X$ is *enumerable*, if there is a computably enumerable set $A \subseteq \mathrm{dom}(\nu)$ such that $X = \{\, \nu_i \mid i \in A \,\}$. Thus, $X$ is enumerable if we can enumerate a subset of the index set of $X$ which contains at least one index for every element of $X$, whereas $X$ is completely enumerable if we can enumerate all indices of elements of $X$ and perhaps some numbers which are not used as indices by the numbering $\nu$.

**Definition 2.2.** A map $F\colon S \to T$ between sets $S$ and $T$ with numberings $\nu$ and $\kappa$, respectively, is *effective*, if there is a function $f \in P^{(1)}$ such that $f(i)\downarrow \in dom(\kappa)$ and $F(\nu_i) = \kappa_{f(i)}$, for all $i \in \mathrm{dom}(\nu)$. Function $f$ is said to *track* $F$ and any Gödel number of $f$ is called *index* of $F$.

Note that the preimage of a completely enumerable set with respect to an effective map is completely enumerable again.

A sequence $(y_a)_{a\in\omega}$ of elements of $S$ is *computable* if there some function $g \in R^{(1)}$ with $\mathrm{range}(g) \subseteq \mathrm{dom}\,\nu)$ so that $y_a = \nu_{g(a)}$, for all $a \in \omega$. Every Gödel number of $g$ is called *index* of $(y_a)_a$. Let $\omega$ be enumerated by its identity. Then the computable sequences in $S$ are the effective maps from $\omega$ to $S$.

## 3 Effective spaces

Let $\mathcal{T} = (T, \tau)$ be a countable topological $T_0$ space with a countable basis $\mathcal{B}$. As has been demonstrated by the author in a series of papers [18, 19, 20, 21, 22, 23, 24, 25], topological spaces of this kind are well suited for effectivity considerations.

Assume further that $B$ is a total numbering of $\mathcal{B}$. In the applications we have in mind the basic open sets can be described in a finite way. The indexing $B$ is then obtained by an encoding of the finite descriptions. If we want to deal with the points and open sets of space $\mathcal{T}$ in an effective way, the interplay between both should at least be such that we can effectively list the points of each basic open set, uniformly in its index.

**Definition 3.1.** Let $\mathcal{T} = (T, \tau)$ be a countable topological $T_0$ space with countable basis $\mathcal{B}$, and let $x$ and $B$ be numberings of $T$ and $\mathcal{B}$ , respectively, such that $B$ is total. We say that $x$ is *computable* if there is some computably enumerable set $L \subseteq \omega$ such that for all $i \in \mathrm{dom}(x)$ and all $n \in \omega$,

$$\langle i, n \rangle \in L \Longleftrightarrow x_i \in B_n.$$

Thus, $x$ is computable if and only if all basic open sets $B$, are completely enumerable, uniformly in $n$.

As said, in the applications we have in mind basic open sets can be described in a finite way and the indexing $B$ is then obtained by an encoding of the finite descriptions. Moreover, in these cases there is a canonical relation between the (code numbers of the) finite descriptions which is stronger than the usual set inclusion between the described sets. This relation is computable enumerable, which is not true for set inclusion, in general.

**Definition 3.2.** Let $\prec_B$ be a transitive binary relation on $\omega$. We say that:

1. $\prec_B$ is a *strong inclusion*, if for all $m, n \in \omega$, from $m \prec_B n$ it follows that $B_m \subseteq B_n$.

2. $\mathcal{B}$ is a *strong basis*, if $\prec_B$ is a strong inclusion and for all $z \in T$ and $m, n \in \omega$ with $z \in B_m \cap B_n$ there is some $a \in \omega$ such that $z \in B_a$, $a \prec_B m$ and $a \prec_B n$.

In what follows, we always assume that $\prec_B$ is a strong inclusion with respect to which $\mathcal{B}$ is a strong basis.

**Definition 3.3.** Space $\mathcal{T}$ is *effective*, if the property of being a strong basis holds effectively, which means that there exists a function $sb \in P^{(3)}$ such that for $i \in \mathrm{dom}(x)$ and $m$, $n \in \omega$ with $x_i \in B_m \cap B_n$, $sb(i, m, n)\downarrow$, $x_i \in B_{sb(i,m,n)}$, $sb(i, m, n) \prec_B m$, and $sb(i, m, n) \prec_B n$.

As is well known, each point $y$ of a $T_0$ space is uniquely determined by its neighbourhood filter $\mathcal{N}(y)$ and/or a base of it.

**Definition 3.4.** Let $\mathcal{H}$ be a filter. A nonempty subset $\mathcal{F}$ of $\mathcal{H}$ is called *strong base* of $\mathcal{H}$ if the following two conditions hold:

1. For all $m$, $n \in \omega$ with $B_m$, $B_n \in \mathcal{F}$ there is some index $a \in \omega$ such that $B_a \in \mathcal{F}$, $a \prec_B m$, and $a \prec_B n$.

2. For all $m \in \omega$ with $B_m \in \mathcal{H}$ there is some index $a \in \omega$ such that $B_a \in \mathcal{F}$ and $a \prec_B m$.

If $x$ is computable, a strong base of basic open sets can effectively be enumerated for each neighbourhood filter. Here, we are interested in enumerations that proceed in a normed way.

**Definition 3.5.** An enumeration $(B_{f(a)})_{a\in\omega}$ with $f : \omega \to \omega$ is said to be *normed* if $f$ is decreasing with respect to $\prec_B$. If $f$ is computable, it is also called *computable* and any Gödel number of $f$ is said to be an *index* of it.

In case $(B_{f(a)})_a$ enumerates a strong base of the neighbourhood filter of some point, we say it *converges* to that point.

**Lemma 3.6** ([20])**.** *Let $\mathcal{T}$ be effective and $x$ be computable. Then there is a function $q \in R^{(1)}$ such that for each $i \in \mathrm{dom}(x)$, $q(i)$ is an index of a normed computable enumeration of basic open sets converging to $x_i$.*

We not only want be able to generate normed recursive enumerations of basic open sets converging to a given point, but conversely, we need also be able to pass effectively from such enumerations to the point they converge to.

**Definition 3.7.** Let $x$ be a numbering of $T$. We say that:

1. $x$ *allows effective limit passing* if there is a function $pt \in P^{(1)}$ such that, if $m$ is an index of a normed computable enumeration of basic open sets converging to some point $y \in T$, then $pt(m)\downarrow \in \mathrm{dom}(x)$ and $x_{pt(m)} = y$.
2. $x$ is *acceptable* if it allows effective limit passing and is computable.

Any two acceptable numberings of $T$ are $m$-equivalent [20]. We will now give an example of an acceptable numbering that shall be used again later.

**Proposition 3.8.** *Let $\mathcal{T}$ be such that $\prec_B$ is computably enumerable and the neighbourhood filter of each point in $T$ has an enumerable strong base of basic open sets. Then, $T$ has an acceptable numbering.*

*Proof.* If $\{ B_n \mid n \in W_e \}$ is a strong base of the neighbourhood filter of some point $y \in T$, set $\bar{x}_e = y$. Otherwise, let $\bar{x}$ be undefined. Because of the assumption, $\bar{x}$ is a numbering of $T$. Let $L = \{ \langle e, n\rangle \mid n \in W_e \}$. Then $L$ is computably enumerable. Moreover, we have for $i \in \mathrm{dom}(\bar{x})$ that

$$\bar{x}_i \in B_n \iff (\exists m \in W_i) m \prec_B n \iff (\exists m)\langle i, m\rangle \in L \wedge m \prec_B n,$$

which shows that $\bar{x}$ is computable.

Next, let $m$ be an index of a normed computable enumeration of basic open sets converging to some point $y \in T$. Then $\{ B_n \mid n \in \mathrm{range}(\varphi_m) \}$ is a strong base of $\mathcal{N}(y)$. Hence, $y = x_{t(m)}$, where $t \in R^{(1)}$ is such that $W_{t(a)} = \mathrm{range}(\varphi_a)$, showing that $\bar{x}$ also allows effective limit passing. $\square$

For basic open sets $B_n$, let

$$\mathrm{hl}(B_n) = \bigcap \{ B_a \mid n \prec_B a \}.$$

Sometimes, when we need to choose certain elements in $B_n$, we may not be able to find them in $B_n$, but then we want to find them as close to $B_n$ as possible.

Let $X$ be an subset of $T$. A typical situation in many proofs is that we need to show for some basic open set $B_n$ that $B_n \subseteq X$. We would try a proof by contradiction and assume that $B_n \not\subseteq X$. Then we would *choose*, uniformly in $n$ and perhaps some index of $X$, an element $z \in B_n \setminus X$ and derive a contradiction. In a non-effective setting it is not worth talking about such situations. In an effective context, however, we have to effectively find such a *witness* $z$. As we will see later, the situation occurs in particular in continuity proofs. In this case $X$ is the preimage of a basic open set $B'_n$ under a map $F\colon T \to T'$, where $\mathcal{T} = (T', \tau')$ is a further countable $T_0$ space with countable basis $\mathcal{B}'$ and a total indexing $B'$ of $\mathcal{B}'$.

**Definition 3.9.** [[26, 20]] $F$ has *a witness for noninclusion*, if there are functions $s \in P^{(2)}$ and $r \in P^{(3)}$ such that for $i \in \mathrm{dom}(x)$ and $e, n \in \omega$ the following hold:

1. If $F(x_i) \in B'_n$, then $s(i,n)\downarrow \in \mathrm{dom}(M)$ and $x_i \in M_{s(i,n)} \subseteq F^{-1}(B'_n)$.
2. If, in addition, $B_e \not\subseteq F^{-1}(B'_n)$, then also $r(i,e,n)\downarrow \in \mathrm{dom}(x)$ with $x_{r(i,e,n)} \in \mathrm{hl}(B_e) \setminus M_{s(i,n)}$.
3. Moreover, for every index $m$ of a computable normed enumeration of basic open sets converging to $x_i$, if $B_{\varphi_m(e+1)} \not\subseteq F^{-1}(B'_n)$, then the sequence $(y_a)_a$ with $y_a = x_{\varphi_{p(i,m)}(a)}$, for $a \leq e$, and $y_a = x_{r(i,\varphi_m(e+1),n)}$, otherwise, is in SEQ.

# 4 Special cases

In this section we introduce some important standard examples of effecive $T_0$ spaces.

As is well known, $T_0$ spaces come equipped with a canonical order $\leq_\tau$, called *specialization order*: For $y, z \in T$,

$$y \leq_\tau z \iff (\forall n \in \omega)[y \in B_n \Rightarrow z \in B_n].$$

Every open set is upwards closed under the specialization order and continuous maps are monotone with respect to it.

## 4.1 Constructive domains

Let $Q = (Q, \sqsubseteq)$ be a partial order with least element. A nonempty subset $S$ of $Q$ is *directed*, if for all $y_1$, $y_2 \in S$ there is some $u \in S$ with $y_1$, $y_2 \sqsubseteq u$. The *way-below relation* $\ll$ on $Q$ is defined as follows: $y_1 \ll y_2$ if for every directed subset $S$ of $Q$ the least upper bound of which exists in Q, the relation $y_2 \sqsubseteq \bigsqcup S$ implies the existence of an element $u \in S$ with $y_1 \sqsubseteq u$. Note that $\ll$ is transitive. Elements $y \in Q$ with $y \ll y$ are called *compact*.

A subset $Z$ of $Q$ is a *basis* of $Q$, if for any $y \in Q$ the set $Z_y = \{\, z \in Z \mid z \ll y \,\}$ is directed and $y = \bigsqcup Z_y$. A partial order that has a basis is called *continuous*. If all elements of $Z$ are compact, $Q$ is said to be *algebraic* and $Z$ is called *algebraic basis*.

Now, assume that $Q$ is countable and let $x$ be an indexing of $Q$. Then $Q$ is *constructively d-complete*, if each of its enumerable directed subsets has a least upper bound in $Q$. Let $Q$ be constructively d-complete and continuous with basis $Z$. Moreover, let $\beta$ be a total numbering of $Z$. Then $(Q, \sqsubseteq, Z, \beta, x)$ is said to be a *constructive domain*, if the restriction of the way-below relation to $Z$ as well as all sets $Z_y$, for $y \in Q$, are completely enumerable with respect to the indexing $\beta$, and $\beta \leq x$.

A numbering $x$ of $Q$ is said to be *admissible*, if the set $\{\, \langle i, j \rangle \mid \beta_i \ll x_j \,\}$ is computably enumerable and there is a function $d \in R^{(1)}$ such that for all indices $i \in \omega$ for which $\beta(W_i)$ is directed, $x_{d(i)}$ is the least upper bound of $\beta(W_i)$. As shown in [28], such numberings always exist. They can even be chosen as total.

Partial orders come with several natural topologies. In the applications we have in mind, one is mainly interested in the *Scott topology* $\sigma$: a subset $X$ of $Q$ is open in $\sigma$, if it is upwards closed with respect to the partial order and intersects each enumerable directed subset of $Q$

of which it contains the least upper bound. In the case of a constructive domain this topology is generated by the sets $B_n = \{ y \in Q \mid \beta_n \ll y \}$ with $n \in \omega$. It follows that $\mathcal{Q} = (Q, \sigma)$ is a countable $T_0$-space with countable basis. Observe that the partial order on $Q$ coincides with the specialization order defined by the Scott topology [9]. Obviously, every admissible numbering is computable.

Define

$$m \prec_B n \Leftrightarrow \beta_n \ll \beta_m.$$

Then $\prec_B$ is a strong inclusion with respect to which the collection of all $B_n$ is a strong basis. Because the restriction of $\ll$ to $Z$ is completely enumerable, $\prec_B$ is computably enumerable. It follows that $\mathcal{Q}$ is effective. Moreover, each admissible indexing allows effective limit passing, i.e., it is acceptable. Conversely, every acceptable numbering of $Q$ is admissible.

Note that the set $P^{(1)}$ of partial computable functions, ordered by $f \sqsubseteq g$, if $g$ extends $f$, is a constructive algebraic domain. The finite functions are its compact elements and each Gödel numbering is admissible.

## 4.2 Constructive $A$- and $f$-spaces

$A$- and $f$-spaces have been introduced by Eršov [4, 5, 6, 7] as a more topologically oriented approach to domain theory. They are not required to be complete. For a set $X$ of a topological space, let $\mathrm{int}(X)$ denote its interior. Set

$$y \ll z \Longleftrightarrow z \in \mathrm{int}(\{ u \in T \mid y \leq_\tau u \}).$$

Now, let $\mathcal{Y} = (Y, \rho)$ be a topological $T_0$-space. $\mathcal{Y}$ is an *$A$-space*, if there is a subset $Y_0$ of $Y$ that satisfies the following three properties:

1. Any two elements of $Y_0$ which are bounded in $Y$ with respect to the specialization order have a least upper bound in $Y_0$.

2. The collection of sets $\mathrm{int}(\{ u \in Y \mid y \leq_\rho u \})$, for $y \in Y_0$, is a basis of topology $\rho$.

3. For any $y \in Y_0$ and $u \in Y$ with $y \ll u$ there is some $z \in Y_0$ such that $y \ll z$ and $z \ll u$.

Any subset $Y_0$ of $Y$ with these properties is called *basic subspace*.

Let $Y$ be countable and $Y_0$ have a total numbering $\beta$. For $m$, $n \in \omega$ let $B_n = \mathrm{int}(\{u \in Y \mid \beta_n \leq_\rho u \})$ and define

$$m \prec_B n \Leftrightarrow \beta_n \ll \beta_m.$$

Then $\prec_B$ is a strong inclusion with respect to which $\{ B_n \mid n \in \omega \}$ is a strong basis. An $A$-space $\mathcal{Y}$ with basic subspace $Y_0$ is *constructive*, if the restriction of $\ll$ to $Y_0$ is completely enumerable, and the neighbourhood filter of each point has an enumerable strong base of basic open sets. As shown in [20], $Y$ has an acceptable numbering $x$ such that $\mathcal{Y}$ is effective.

Let $\mathcal{Y} = (Y, \rho)$ be again an arbitrary topological $T_0$-space. An open set $V$ is an *$f$-set*, if there is a some element $z_V \in V$ such that $V = \{ y \in Y \mid z_V \leq_\rho y \}$. The uniquely determined element $z_V$ is called *$f$-element*. $\mathcal{Y}$ is an *$f$-space*, if the following two conditions hold:

1. If $U$ and $V$ are $f$-sets with nonempty intersection, then $U \cap V$ is also an $f$-set.

2. The collection of all $f$-sets is a basis of topology $\rho$.

An $f$-space is *constructive*, if the set of all $f$-elements has a total numbering $\alpha$ such that the restriction of the specialization order to this set as well as the boundedness of two $f$-elements are completely decidable and there is a function $su \in R^{(2)}$ such that in the case that $\alpha_n$ and $\alpha_m$ are bounded, $\alpha_{su(n,m)}$ is their least upper bound, and if the neighbourhood filter of each point has an enumerable base of $f$-sets. (Similar conditions can be found in [7, 10].)

Obviously, every $f$-space is an $A$-space with basic subspace the set of all $f$-elements. Note that for $y$, $z \in Y$ with $y$ or $z$ being an $f$-element, $y \ll z$ if and only if $y \leq_\tau z$. Thus, also every constructive $f$-space is a constructive $A$-space.

### 4.3 Constructive metric spaces

Whereas domains as well as $A$- and $f$-spaces typicaly do not satisfy $T_2$ separation, in this section we will consider the standard example of an effective Hausdorff space.

Let $\mathcal{M} = (M, \delta)$ be a countable separable metric space and $\beta$ be a total numbering of its dense subset $M_0$. As is well-known, the collection of sets $B_{\langle i,m\rangle} = \{ y \in M \mid \delta(\beta_i, y) < 2^{-m} \}$ ($i$, $m \in \omega$) is a basis of the canonical Hausdorff topology $\Delta$ on $M$.

Define

$$\langle i, m\rangle \prec_B \langle j, n\rangle \Leftrightarrow \delta(\beta_i, \beta_j) + 2^{-m} < 2^{-n}.$$

Using the triangle inequality it is readily verified that $\prec_B$ is a strong inclusion and the collection of all $B_a$ is a strong basis.

**Definition 4.1.** $\mathcal{M}$ is said to be *constructive*, if the sets

$$\{ \langle i, j, a, n\rangle \mid \delta(\beta_i, \beta_j) < a \cdot 2^{-n} \} \quad \text{and} \quad \{ \langle i, j, a, n\rangle \mid \delta(\beta_i, \beta_j) > a \cdot 2^{-n} \}$$

are computably enumerable, and the neighbourhood filter of each point has an enumerable strong base of basic open sets.

Obviously, $\prec_B$ is computably enumerable in this case.

Well-known examples of constructive metric spaces include $\mathbb{R}^n_c$, that is the space of all $n$-tuples of computable real numbers with the Euclidean or the maximum norm; Baire space, that is the set $R^{(1)}$ of all total computable functions with the Baire metric [16]; and the set $\omega$ with the discrete metric. By using an effective version of Weierstraß's approximation theorem [15] and Sturm's theorem [27] it can be shown that $C_c[0, 1]$, the space of all computable functions from $[0, 1]$ to $\mathbb{R}$ with the supremum norm [15], is a constructive metric space too. A proof of this result and further examples can be found in Blanck [1].

## 5 Limit algorithms

In this note we assume each space to come with a rich collection SEQ of canonical computable sequences with the following properties:

1. All sequences that are monotonically increasing with respect to the specialization order are in SEQ.

2. There is a function $p \in R^{(1)}$ such that for each index $m$ of a normed computable enumeration of basic open sets, $p(m)$ is an index of a computable sequence of points in SEQ with $x_{\varphi_{p(m)}(a)} \in \mathrm{hl}(B_{\varphi_m(a)})$.

3. If $(y_a)_a$ is in Seq, then, for every $\bar{a} \in \omega$, $(y'_a)_a$ is in Seq as well, where $y'_a = y_a$, for $a < \bar{a}$, and $y'_a = y_{\bar{a}}$, otherwise.

In the domain case, and similarly for $A$- and $f$-spaces, Seq consists of all computable monotonically increasing sequences. Let $f \in R^{(1)}$ with $\beta = x \circ f$. Then, if $m$ is an index of a normed computable enumeration of basic open sets, $p = f \circ \varphi_m$. In the metric case we let Seq be the set of all computable regular Cauchy sequences, where a Cauchy sequence $(y_a)_a$ is *regular* (or, *fast*), if $\delta(y_m, y_n) < 2^{-m}$, for all $n \geq m$. Instead, one could also take the set of all computable Cauchy sequences with a computable Cauchy criterion (cf. [12]). If $x$ is such that for some $g \in R^{(1)}$, $\beta = x \circ g$, and $m$ is an index of a normed computable enumeration of basic open sets, choose $p = g \circ \pi_1 \circ \varphi_m$.

In Section 3 as well as in other papers we based our approach to the computation of limits on filter convergence. In the earlier paper [26], however, we used point sequence convergence. One of the main reasons for moving to filters was that in $T_0$ spaces the limit of a point sequence in not uniquely determined, in general: if $y$ is a limit point, every $z$ with $z \leq_\tau y$ is a limit point as well. We denote the set of limit points of a sequence $(y_a)_a$ by $\mathrm{Lim}_a\, y_a$.

**Definition 5.1.** A numbering $x$ of $T$ has a *limit algorithm*, if there is a function $\mathrm{li} \in P^{(1)}$ such that the following four conditions hold, for all indices $m, m'$ of convergent sequences in Seq:

1. $\mathrm{li}(m){\downarrow} \in \mathrm{dom}(x)$.
2. $x_{\mathrm{li}(m)} \in \mathrm{Lim}_a\, x_{\varphi_m(a)}$
3. If, for some $\bar{a} \in \omega$, $x_{\varphi_m(a)} = x_{\varphi_m(\bar{a})}$, for all $a \geq \bar{a}$, then $x_{\mathrm{li}(m)} = x_{\varphi_m(\bar{a})}$.
4. If $\mathrm{Lim}_a\, x_{\varphi_m(a)} = \mathrm{Lim}_a\, x_{\varphi_{m'}(a)}$, then $x_{\mathrm{li}(m)} = x_{\mathrm{li}(m')}$.

A typical technique in enumeration is to wait and see, i.e., to repeat what has aleady been enumerated till new information becomes available. This motivated the following condition.

**Definition 5.2.** A sequence $(y_a)_a$ in Seq is said to *allow delaying*, if for all $\bar{a}, m \in \omega$ the sequence $(y'_a)_a$ with $y'_a = y_a$, for $a < \bar{a}$, $y'_a = y_{\bar{a}}$, for $\bar{a} \leq a < \bar{a} + m$, and $y'_a = y_{a-m+1}$, otherwise, is in Seq as well.

We start with a general result.

**Proposition 5.3.** *Let $x$ have a limit algorithm. Moreover, let $X$ be a completely enumerable subset of $T$ and $(y_a)_a$ a sequence in* Seq *that allows delaying. Then for every index $m$ of $(y_a)_a$ and any numbers $\bar{a} \in \omega$, if $y_{\bar{a}} \in X$, also $x_{\mathrm{li}(m)} \in X$.*

*Proof.* Let $W_e$ witness the complete enumerability of $X$ and set $g(\bar{b}) = \mu c \geq \bar{a} : \varphi_e(\mathrm{li}(\bar{b})){\downarrow}_c$. By the recursion theorem there is then some $b \in \omega$ with

$$\varphi_b(a) = \begin{cases} \varphi_m(a) & \text{if } a < \bar{a}, \\ \varphi_m(\bar{a}) & \text{if } g(b){\uparrow}_a \text{ and } a \geq \bar{a}, \\ \varphi_m(\bar{a} + a - g(b) + 1) & \text{if } g(b){\downarrow}_a \text{ and } a \geq \bar{a}. \end{cases}$$

Suppose that $g(b){\uparrow}$. Because of Property 3 of Seq, the sequence $(x_{\varphi_b(a)})_a$ is in Seq in this case. Moreover, it converges to $y_{\bar{a}}$. With Condition 5.1(3) we therefore obtain that

$x_{\mathrm{li}(b)} = y_{\bar{a}}$. By our assumption, $y_{\bar{a}} \in X$, i.e., $\varphi_m(\bar{a}) \in W_e$. It follows that $\mathrm{li}(b) \in W_e$ as well, which means that $g(b)\downarrow$, a contradiction.

So we have that $g(b)\downarrow$. Since $(y_a)_a$ allows delaying, it follows that the just defined sequence with index $b$ is in Seq, also in this case. Moreover, $\mathrm{Lim}_a\, x_{\varphi_b(a)} = \mathrm{Lim}_a\, y_a$ and hence, by Condition 5.1(4), $x_{\mathrm{li}(m)} = x_{\mathrm{li}(b)}$. As a further consequence, $\mathrm{li}(b) \in W_e$, which means that $x_{\mathrm{li}(b)} \in X$. This shows that $x_{\mathrm{li}(m)} \in X$. $\square$

Let $u, z \in T$ with $y \leq_\tau z$. Then the sequence with $y_a = u$, for $a < \bar{a}$ and $y_a = z$, otherwise, for some $\bar{a} \in \omega$, is in Seq, by Condition 3 for Seq, and obviously allows delaying. Thus, if $u \in X$, $z \in X$ as well.

**Corollary 5.4.** *Let $x$ have a limit algorithm. Then each completely enumerable subset of $T$ is upwards closed under the specialization order.*

Next, suppose that $\mathcal{T}' = (T', \tau')$ is a further countable $T_0$ space with countable basis $\mathcal{B}'$ and numberings $x'$ and $B'$ of $T'$ and $\mathcal{B}'$, respectively, such that $B'$ is total. Moreover, recall that the preimage of a completely enumerable set under an effective map is completely enumerable again.

**Corollary 5.5.** *Let $x$ have a limit algorithm and $x'$ be computable. Then every effective map $F\colon T \to T'$ is monotone with respect to the specialization order.*

Now, assume that $X$ is a basic open set.

**Corollary 5.6.** *Let $x$ be computable and have a limit algorithm. Then, for any index $m$ of a convergent sequence in* Seq *that allows delaying, the following two statements hold:*

1. $x_{\varphi_m(a)} \leq_\tau x_{\mathrm{li}(m)}$, *for all $a \in \omega$.*
2. $y \leq_\tau x_{\mathrm{li}(m)}$, *for all $y \in \mathrm{Lim}_a\, x_{\varphi_m(a)}$.*

*Proof.* (1) Let $a, n \in \omega$ with $x_{\varphi_m(a)} \in B_n$. Then it follows with the preceding proposition that $x_{\mathrm{li}(m)} \in B_n$ as well. Thus, $x_{\varphi_m(a)} \leq_\tau x_{\mathrm{li}(m)}$.

(2) Let $y \in \mathrm{Lim}_a\, x_{\varphi_m(a)}$ and $n \in \omega$ with $y \in B_n$. Then there is some $\bar{a} \in \omega$ so that $x_{\varphi_m(a)} \in B_n$, for all $a \geq \bar{a}$. By the preceding proposition it follows that also $x_{\mathrm{li}(m)} \in B_n$, which shows that $y \leq_\tau x_{\mathrm{li}(m)}$. $\square$

In case that the function $\mathrm{li} \in P^{(1)}$ in Definition 5.1 also satisfies the condition in Statement 5.6(2), we say that the limit operator *computes maximal limits.*

Is is well known that if space $\mathcal{T}$ satifies $T_1$ separation its specialization order coincides with the identity relation on $T$. Under the assumptions of the above corollary we therefore obtain that sequences in Seq that allow delaying must be constant. In other words, except in the case of $T_0$ spaces that violate the $T_1$ condition, only trivial sequences in Seq satisfy the assumption.

As we will see next, the property in Corollary 5.6(1) is characteristic for limit algorithms.

**Lemma 5.7.** *Let* $\mathrm{li} \in P^{(1)}$ *such that for all indices $m$ of convergent sequences in* Seq, $\mathrm{li}(m)\downarrow \in \mathrm{dom}(x)$ *with* $x_{\mathrm{li}(m)} \in \mathrm{Lim}_a\, x_{\varphi_m(a)}$ *and* $x_{\varphi_m(a)} \leq_\tau x_{\mathrm{li}(m)}$, *for all $a \in \omega$. Then $x$ has a limit algorithm.*

*Proof.* As in the preceding proof we obtain that $y \leq_\tau x_{\mathrm{li}(m)}$, for all $y \in \mathrm{Lim}_a\, x_{\varphi_m(a)}$. Hence, $x_{\mathrm{li}(m)} = \max_{\leq_\tau} \mathrm{Lim}_a\, x_{\varphi_m(a)}$. If $m'$ is an index of a further converging sequence in SEQ so that $\mathrm{Lim}_a\, x_{\varphi_{m'}(a)} = \mathrm{Lim}_a\, x_{\varphi_m(a)}$, we therefore have that $x_{\mathrm{li}(m')} = \max_{\leq_\tau} \mathrm{Lim}_a\, x_{\varphi_{m'}(a)} = \max_{\leq_\tau} \mathrm{Lim}_a\, x_{\varphi_m(a)} = x_{\mathrm{li}(m)}$.

If there is some $\bar{a} \in \omega$ such that $\varphi_m(a) = \varphi_m(\bar{a})$, for all $a \geq \bar{a}$, then $x_{\varphi_m(\bar{a})} \in \mathrm{Lim}_a\, x_{\varphi_m(a)}$. Thus, $x_{\varphi_m(\bar{a})} \leq_\tau x_{\mathrm{li}(m)}$. To see that also the converse inequality holds, let $n \in \omega$ with $x_{\mathrm{li}(m)} \in B_n$. Since $x_{\mathrm{li}(m)}$ is a limit point of $(x_{\varphi_m(a)})_a$, there is some $\hat{a} \in \omega$ with $x_{\varphi_m(a)} \in B_n$, for all $a \geq \hat{a}$. In particular, we have that $x_{\varphi_m(\bar{a})} \in B_n$, which shows that $x_{\mathrm{li}(m)} \leq_\tau x_{\varphi_m(\bar{a})}$. □

**Proposition 5.8.** *Let $x$ be computable and all sequences in* SEQ *allow delaying. Then $x$ has a limit algorithm if, and only if, there is some function* li $\in P^{(1)}$ *so that for all indices $m$ of convergent sequences in* SEQ*,* $\mathrm{li}(m)\!\downarrow\, \in \mathrm{dom}(x)$ *with* $x_{\mathrm{li}(m)} \in \mathrm{Lim}_a\, x_{\varphi_m(a)}$ *and* $x_{\varphi_m(a)} \leq_\tau x_{\mathrm{li}(m)}$*, for all $a \in \omega$.*

This gives us a hint of how to construct a numbering of $T$ that has a limit algorithm in the case of $T_0$ spaces that do not satisfy $T_1$ separation. As in the case of constructive domains, $A$- and $f$-spaces, we now let SEQ only contain sequences that monotonically increase with respect to the specialization order. Such sequences always allow delaying. Note that the greatest limit point, if it exists, is the least upper bound in this case. Moreover, we assume that $\prec_B$ is computably enumerable and that the neighbourhood filter of each point has an enumerable strong base of basic open sets. As we have seen in Section 4, the just mentioned spaces always satisfy these assumptions. Let $\bar{x}$ be the numbering constructed in Proposition 3.8 and li $\in R^{(1)}$ with

$$W_{\mathrm{li}(m)} = \{\, n \in \omega \mid (\exists a \in \omega) n \in W_{\varphi_m(a)} \,\}.$$

Suppose that $m$ is an index of a monotonically increasing sequence with least upper bound $y$. By what we have seen above, all $B_n$ with $n \in W_{\mathrm{li}(m)}$ contain $y$. On the other hand, if $B_n$ contains $y$ then there is some $a \in \omega$ with $x_{\varphi_m(a)} \in B_n$, as $y$ is a limit point. Thus, $n \in W_{\mathrm{li}(m)}$. It follows that $\{\, B_n \mid n \in W_{\mathrm{li}(m)} \,\}$ is the set of all basic open sets containing $y$ and hence a strong base of $\mathcal{N}(y)$. So, $y = \bar{x}_{\mathrm{li}(m)}$, which shows that $\bar{x}$ has a limit algorithm.

**Proposition 5.9.** *Let $\mathcal{T}$ be a $T_0$ space that does not satisfy $T_1$ separation and is such that $\prec_B$ is computably enumerable and the neighbourhood filter of each point has an enumerable strong base of basic open sets. Then every acceptable numbering of $T$ has a limit algorithm.*

*Proof.* As we have seen, the numbering $\bar{x}$ constructed in Proposition 3.8 is acceptable and has a limit algorithm. Moreover, it has been mentioned that all acceptable numberings of $T$ are $m$-equivalent. Obviously, the property of having a limit algorithm is inherited under $m$-equivalence. □

**Proposition 5.10.** *Let $\mathcal{M}$ be a constructive metric space. Then every acceptable numbering of $M$ has a limit algorithm.*

*Proof.* Again it suffices to show that the numbering $\bar{x}$ constructed in Proposition 3.8 has a limit algorithm. Let $(y_a)_a$ be a computable regular Cauchy sequence that converges to some point $y \in M$. Because of regularity we have that $\delta(y_a, y) \leq 2^{-a}$. Let $a > 1$. Since $M_0$ is dense in $M$, there is some $\beta_i \in M_0$ such that $\delta(\beta_i, y_a) < 2^{-a}$. By the triangular inequation it then follows that $\{\, u \in M \mid \delta(y_a, u) \leq 2^{-a} \,\} \subseteq B_{\langle i, a-1\rangle}$. We need to enumerate a strong base of $\mathcal{N}(y)$. To this end we will enumerate all pairs $\langle i, a-1\rangle$ with $a > 0$ and $y_a \in B_{\langle i,a\rangle}$. Let $m$ be an index of $(y_a)_a$. By definition of $\bar{x}$,

$$x_{\varphi_m(a)} \in B_{\langle i,a\rangle} \Longleftrightarrow \langle i, a\rangle \in W_{\varphi_m(a)}.$$

Therefore, if we let $\mathrm{li} \in R^{(1)}$ such that

$$W_{li(m)} = \{ \langle i, a-1 \rangle \mid a > 0 \wedge \langle i, a \rangle \in W_{\varphi_m(a)} \},$$

all basic open set $B_{\langle i,a-1 \rangle}$ with $\langle i, a-1 \rangle \in W_{\mathrm{li}(m)}$ contain the limit point $y$. It remains to show that they form a strong filter base.

Let $\langle i, a-1 \rangle, \langle j, c-1 \rangle \in W_{\mathrm{li}(m)}$. Then $y \in B_{\langle i,a-1 \rangle} \cap B_{\langle j,c-1 \rangle}$. Since the set of all $B_d$ forms a strong basis of the metric topology, there exist $b, n \in \omega$ such that $y \in B_{\langle b,n \rangle}$ and $\langle b, n \rangle \prec_B \langle i, a-1 \rangle$ as well as $\langle b, n \rangle \prec_B \langle j, c-1 \rangle$. Let $\bar{n} \in \omega$ with $2^{-\bar{n}} < 2^{-n} - \delta(\beta_b, y)$. Moreover, choose $\hat{n} > \bar{n} + 2$ so that $\delta(y_{\hat{n}}, y) \leq 2^{-\hat{n}}$ and $e \in \omega$ with $\delta(\beta_e, y_{\hat{n}}) < 2^{-\hat{n}}$ as well. It then follows that

$$\begin{aligned}\delta(\beta_e, \beta_b) + 2^{-\hat{n}+1} &\leq \delta(\beta_e, y) + \delta(y, \beta_b) + 2^{-\hat{n}+1} \\ &< 2 \cdot 2^{-\hat{n}+1} + \delta(y, \beta_b) < 2^{-\bar{n}} + \delta(y, \beta_b) < 2^{-n},\end{aligned}$$

which means that $\langle e, \hat{n}-1 \rangle \prec_B \langle b, n \rangle$. Thus we have that $\langle e, \hat{n}-1 \rangle \prec_B \langle i, a-1 \rangle$ as well as $\langle e, \hat{n}-1 \rangle \prec_B \langle j, c-1 \rangle$. Moreover, $\langle e, \hat{n}-1 \rangle \in W_{\mathrm{li}(m)}$. □

So far we have seen for two important and large classes of effective spaces that acceptable numberings also have a limit algorithm. We will now, conversely, study when numberings that have a limit algorithm also allow effective limit passing. In doing so, we will meet the witness for noninclusion condition. This time the identity function on $T$ is required to have it. We say in this case that space $\mathcal{T}$ *has a witness for noninclusion.*

**Proposition 5.11.** *Let let $\mathcal{T}$ have a witness for noninclusion, and every sequence in* SEQ *allow delaying. Moreover, let $x$ be computable as well as have a limit algorithm. Then $x$ also allows effective limit passing.*

*Proof.* Let $\mathrm{li} \in P^{(1)}$ witness that $x$ has a limit algorithm, and $p \in R^{(1)}$ be as in Property 2 of SEQ. Moreover, let $s \in P^{(2)}$ and $r \in P^{(3)}$ testify that space $\mathcal{T}$ has a witness for noninclusion.

Now, let $m$ be an index of a normed computable enumeration of basic open sets converging to $y \in T$. We will show that $y = x_{\mathrm{li}(p(m))}$. Then $\mathrm{pt} = \mathrm{li} \circ p$ witnesses that $x$ allows effective limit passing.

Let $B_n \in \mathcal{N}(y)$. Then exists $\bar{a} \in \omega$ with $\varphi_m(\bar{a}) \prec_B n$. Hence $x_{\varphi_{p(m)}(a)} \in B_{\varphi_m(a-1)} \subseteq B_n$, for all $a \geq \bar{a}+1$, from which it follows that $(x_{\varphi_{p(m)}(a)})_a$ converges to $y$. Thus, $\mathrm{li}(p(m))\!\downarrow \in \mathrm{dom}(x)$. Furthermore, with Corollary 5.6(2) we obtain that $y \leq_\tau x_{\mathrm{li}(p(m))}$.

For the converse inequality, assume that $n \in \omega$ with $x_{\mathrm{li}(p(m))} \in B_n$. By Condition 3.9(1) we then have that $x_{\mathrm{li}(p(m))} \in M_{s(\mathrm{li}(p(m)),n)}$ as well. Let $b = s(\mathrm{li}(p(m)), n)$ and set $g(e) = \mu c : \varphi_b(\mathrm{li}(e))\!\downarrow_c$. By the recursion theorem there is some $u \in \omega$ with

$$\varphi_u(a) = \begin{cases} \varphi_{p(m)}(a) & \text{if } g(u)\!\uparrow_a, \\ r(i, \varphi_m(g(u)), n) & \text{otherwise.} \end{cases}$$

Assume that $g(u)\!\uparrow$. Then $x_{\varphi_u(a)} = x_{\varphi_{p(m)}(a)}$, for all $a \in \omega$. Thus, the sequence is in SEQ. In addition, $\mathrm{li}(u)\!\downarrow \in \mathrm{dom}(x)$ and $x_{\mathrm{li}(u)} = x_{\mathrm{li}(p(m))}$, which implies that $\mathrm{li}(u) \in W_b$, a contradiction.

Suppose next that $B_{\varphi_m(g(u))} \not\subseteq B_n$. Then $r(i, \varphi_m(g(u)), n)\!\downarrow \in \mathrm{dom}(x)$. Furthermore,

$$x_{r(i,\varphi_m(g(u)),n)} \in \mathrm{hl}(B_{\varphi_m(g(u))}) \setminus M_{s(\mathrm{li}(p(m)),n)}.$$

By definition, $x_{\varphi_u(a)} = x_{\varphi_{p(m)}(a)}$, for $a < g(u)$, and $x_{\varphi_u(a)} = x_{r(i,\varphi_m(g(u)),n)}$, otherwise. With Condition 3.9(3) it follows that the sequence is in Seq. Moreover, it converges to $x_{r(i,\varphi_m(g(u)),n)}$. Thus, $\mathrm{li}(u){\downarrow} \in \mathrm{dom}(x)$ and, because of Property 5.1(3), $x_{\mathrm{li}(g(u))} = x_{r(i,\varphi_m(g(u)),n)}$. As we have already seen, $\mathrm{li}(u) \in W_b$, i.e., $x_{\mathrm{li}(u)} \in M_{s(\mathrm{li}(p(m)),n)}$. This means that $x_{r(i,\varphi_m(g(u)),n)} \in M_{s(\mathrm{li}(p(m)),n)}$, a contradiction.

Thus, $B_{\varphi_m(g(u))} \subseteq B_n$. As $m$ is an index of a normed enumeration of basic open sets converging to $y$, we have that $y \in B_{\varphi_m(g(u))}$. Therefore, $y \in B_n$ too, which shows that $x_{\mathrm{li}(p(m))} \leq_\tau y$. □

In the above result the assumption that the sequences in Seq allow delaying is only used to show that $y \leq_\tau x_{\mathrm{li}(p(m))}$. The converse inequality, i.e., $x_{\mathrm{li}(p(m))} \leq_\tau y$ was derived without invoking this condition. Now, suppose that $\mathcal{T}$ has the $T_1$ property. Then the specialization order coincides with the identity on $T$. Hence, it follows that $y = x_{\mathrm{li}(p(m))}$. So, we obtain that $x$ allows effective limit passing without using that sequences in Seq allow delaying.

**Corollary 5.12.** *Let $\mathcal{T}$ be a $T_1$ space and have a witness for noninclusion. Moreover, let $x$ be computable as well as have a limit algorithm. Then $x$ also allows effective limit passing.*

# 6 Continuity

By definition, a sequence $(y_a)_a$ converges to a point $y$, if for any $n \in \omega$ with $y \in B_n$ there is some $N_n \in \omega$ with $y_a \in B_n$, for all $a \geq N_n$. Any function that maps $n$ with $y \in B_n$ to such an $N_n$ is called *convergence module*.

**Definition 6.1.** A sequence $(y_a)_a$ of elements of $T$ *converges effectively* to some point $y \in T$, if there is some function $k \in P^{(1)}$ such that for all $n \in \omega$ with $y \in B_n$ it follows that $k(n){\downarrow}$ and $y_a \in B_n$, for all $a \geq k(n)$.

Thus, a sequence converges effectively if it has a computable convergence module.

Let $m$ be an index of a computable normed enumeration of basic open sets converging to $y \in B_n$. Then, for all $a > 0$, $x_{\varphi_{p(m)}(a)} \in B_{\varphi_m(a-1)}$. Assume that $\prec_B$ is computably enumerable and set

$$A = \{ \langle a', m', n' \rangle \mid a' > 0 \wedge \varphi_{m'}(a'-1) \prec_B n' \}.$$

As $\{ B_a \mid a \in \mathrm{range}(\varphi_m) \}$ is a strong basis of $\mathcal{N}(y)$, $A$ is not empty. With respect to some fixed enumeration, let $\langle \bar{a}, \bar{m}, \bar{n} \rangle$ be the first element enumerated in $A$ with $\bar{m} = m$ and $\bar{n} = n$. Set $\varphi_{k(m)}(n) = \bar{a}$. Then $\varphi_{k(m)}(n)$ witnesses that $(x_{\varphi_{p(m)}(a)})_a$ converges effectively to $y$, uniformly in $m$.

**Lemma 6.2.** *Let $\{p(m)\}_m$ be the family of indices of canonical computable point sequences in Seq associated with converging normed computable enumerations of basic open sets. Then each sequence $(x_{\varphi_{p(m)}(a)})_a$ converges effectively, uniformly in $m$.*

We say that $\mathcal{T}$ has a *uniform computable convergence module*, if there is a function $\mathrm{cm} \in R^{(1)}$ such that for all indices $m$ of effectively converging sequences in Seq, $\varphi_{\mathrm{cm}(m)}$ is a corresponding convergence module.

**Definition 6.3.** A map $F\colon T \to T'$ is *effectively sequentially continuous*, if for every sequence $(y_a)_a$ in Seq effectively converging to a point $y \in T$, the sequence $(F(y_a))_a$ effectively converges to $F(y)$, uniformly in the index $m$ of $(y_a)_a$, i.e., there is some function $k \in R^{(1)}$ so that $\lambda n.\, \varphi_{k(m)}(n)$ witnesses the effective convergence of $(F(y_a))_a$.

For topological spaces with countable topological basis it is well known that sequentially continuous maps are continuous, and vice versa. In this section we will study this relationship in the effective context described so far.

**Definition 6.4.** A map $F\colon T \to T'$ is said to be

1. *effectively pointwise continuous*, if there is a function $h \in P^{(2)}$ such that for all $i \in \mathrm{dom}(x)$ and $n \in \omega$ with $F(x_i) \in B'_n$, $h(i,n)\downarrow$, $x_i \in B'_{h(i,n)}$, and $F(B_{h(i,n)}) \subseteq B'_n$;

2. *effectively continuous*, if there is a function $g \in R^{(1)}$ such that for all $n \in \omega$, $F^{-1}(B'_n) = \bigcup \{ B_a \mid a \in W_{g(n)} \}$.

The equivalence between effective continuity and effective pointwise continuity was investigated in [20]. To show that effectively pointwise continuous maps are effectively continuous one had to assume that $\mathcal{T}$ has an enumerable dense subset.

For the study of the remaining relationship between effective sequential continuity and effective pointwise continuity, let us first see how this is usually done, in a non-effective context. Assume to this end that $\mathcal{T} = (T, \tau)$ and $\mathcal{T}' = (T', \tau')$ are $T_0$ space with countable bases $\mathcal{B}$ and $\mathcal{B}'$, respectively, let $F\colon T \to T'$ be a sequentially continuous map, and $y$ a point in $T$.

- First, one uses the countability of $\mathcal{B}$ to construct a sequence of basic open sets $U_0 \supseteq U_1 \supseteq \dots$ that forms a basis of the neighbourhood filter of $y$.
- Next, one assumes that $F$ is not continuous. Hence, there is some basic open set $V$ containing $F(y)$ such that $F(U_a) \nsubseteq V$, for all $a \in \omega$.
- Using the Axiom of Choice, one then selects some point $y_a \in U_a$, for each $a \in \omega$, such that $F(y_a) \notin V$.
- It follows that $(y_a)_a$ converges to $y$, but $(F(y_a))_a$ does not converge to $F(y)$, a contradiction.

Let us now assume that $B$ and $B'$ are total numberings of $\mathcal{B}$ and $\mathcal{B}'$, respectively, and that $T$ and $T'$ are both countable with numberings $x$ and $x'$, respectively. Moreover, suppose that $\mathcal{T}$ is effective and $x$ is computable. Then, by Lemma 3.6, there is some function $q \in R^{(1)}$ such that, for each $i \in \mathrm{dom}(x)$, $q(i)$ is an index of a normed computable enumeration of basic open set converging to $x_i$. In particular, we have that $B_{\varphi_{q(i)}(0)} \supseteq B_{\varphi_{q(i)}(1)} \supseteq \dots$

In the second step, assuming that

$$B_{\varphi_{q(i)}(a)} \nsubseteq F^{-1}(B'_n)$$

we need be able to effectively find a witness $y_a$ for this, uniformly in $n$, $a$ and $i$. It is here where we need $F$ to have a witness for noninclusion. As was shown in [26, 20], effective maps e.g. have a witness for noninclusion, if their domain is a constructive $A$- or $f$-space or a constructive domain, or their codomain is a constructive metric space or, more general, an effective $T_3$ space. However, as follows from an example by Friedberg [8], this is not the case, in general.

**Theorem 6.5.** *Let $\mathcal{T}$ be effective and $x$ be computable. Then every effectively sequentially continuous map $F\colon T \to T'$ that has a witness for noninclusion is effectively pointwise continuous.*

*Proof.* Let $k \in R^{(1)}$ witness the effective sequential continuity of $F$ and $s \in P^{(2)}$ as well as $r \in P^{(3)}$ its having a witness for noninclusion. Moreover, let $q \in R^{(1)}$ be as in Lemma 3.6 and $p \in R^{(1)}$ as in Property 2 of SEQ. By the recursion theorem there is a function $g \in R^{(2)}$ with

$$\varphi_{g(i,n)}(a) = \begin{cases} \varphi_{p(q(i))}(a) & \text{if } a < \varphi_{k(g(i,n))}(n), \\ r(i, \varphi_{q(i)}(\varphi_{k(g(i,n))}(n)), n) & \text{otherwise.} \end{cases}$$

Set $h(i,n) = \varphi_{k(g(i,n))}(n)$ and assume that $h(i,n)\uparrow$, for some $i \in \mathrm{dom}(x)$ and $n \in \omega$ with $F(x_i) \in B'_n$. Then $x_{\varphi_{g(i,n)}(a)} = x_{\varphi_{p(q(i))}(a)}$, for all $a \in \omega$. Thus, $(x_{\varphi_{g(i,n)}(a)})_a$ is a sequence in SEQ effectively converging to $x_i$. It follows that $\varphi_{k(g(i,n))}(n)$ is defined, a contradiction. Thus, $h(i,n)$ is defined, for all $i \in \mathrm{dom}(x)$ and $n \in \omega$ with $F(x_i) \in B'_n$.

Next, assume that $B_{\varphi_{q(i)}(h(i,n))} \not\subseteq F^{-1}(B'_n)$, for some $i \in \mathrm{dom}(x)$ and $n \in \omega$ with $F(x_i) \in B'_n$. Then $r(i, \varphi_{q(i)}(h(i,n)), n)\downarrow \in \mathrm{dom}(x)$ with

$$x_{r(i,\varphi_{q(i)}(h(i,n)),n)} \in \mathrm{hl}(B_{\varphi_{q(i)}(h(i,n))}) \setminus F^{-1}(B'_n).$$

By Condition 3.9(3), $(x_{\varphi_{g(i,n)}(a)})_a$ is in SEQ, effectively converging to $x_{r(i,\varphi_{q(i)}(h(i,n)),n)}$.

Since $h(i,n)$ is a module of the convergence of $(F(x_{\varphi_{g(i,n)}(a)}))_a$ to $F(x_{r(i,\varphi_{q(i)}(h(i,n)),n)})$, it follows that $F(x_{\varphi_{g(i,n)}(h(i,n))}) \in B'_n$, i.e., $F(x_{r(i,\varphi_{q(i)}(h(i,n)),n)}) \in B'_n$, a contradiction. Consequently, the function $h$ witnesses that $F$ is effectively pointwise continuous. □

For the derivation of the converse implication we recall the fact that in $T_0$ spaces which are not Hausdorff limit points are no longer uniquely determined. We say that SEQ *has maximal limits* if $\mathrm{Lim}_a\, y_a$ has a greatest element, for each sequence $(y_a)_a$ in SEQ.

**Proposition 6.6.** *Let $\mathcal{T}$ have a uniform computable convergence module,* SEQ *maximal limits, and $x$ a limit operator that computes maximal limits. Then every effectively pointwise continuous maps $F: T \to T'$ is effectively sequentially continuous.*

*Proof.* Let $cm \in R^{(1)}$ and $h \in P^{(2)}$, respectively, witness that $\mathcal{T}$ has a uniform effective convergence module and $F$ is effectively pointwise continuous. Now, assume that $m$ is an index of a computable sequence in SEQ effectively converging to a point $y \in T$ and $n \in \omega$ so that $F(y) \in B'_n$. Then $(x_{\varphi_m(a)})_a$ also converges effectively to its largest limit point $x_{\mathrm{li}(m)}$. Moreover, $F(x_{\mathrm{li}(m)}) \in B'_n$. It follows that $h(\mathrm{li}(m), n)\downarrow$, $x_{\mathrm{li}(m)} \in B_{h(\mathrm{li}(m),n)}$, and $F(B_{h(\mathrm{li}(m),n)}) \subseteq B'_n$. Hence, $x_{\varphi_m(a)} \in B_{h(\mathrm{li}(m),n)}$ and therefore $F(x_{\varphi_m(a)}) \in B'_n$, for all $a \geq \varphi_{\mathrm{cm}(m)}(h(\mathrm{li}(m), n))$, which shows that $F$ is effectively sequentially continuous. □

# 7 Effective maps

In this section we will investigate when an effective map $F: T \to T'$ is effectively sequentially continuous.

Assume that $(y_a)_a$ is a computable sequence in SEQ converging to $y \in T$. Then $(F(y_a))_a$ is computable as well. We will now show that in certain general cases it converges effectively to $F(y)$, i.e., we will show that in theses cases each effective maps is effectively sequentially continuous.

Let $m$ be an index of $(y_a)_a$ and $n \in \omega$ with $F(y) \in B'_n$. Then, uniformly in $m$ and $n$, we construct a computable sequence $(z_a)_a$ with index $b$:

1. We follow the sequence $(y_a)_a$ as long as the computation of $\mathrm{li}(b)$ has *not* terminated or $F(x_{\mathrm{li}(b)})$ has *not* been found in $B'_n$.

2. If the computation of $\mathrm{li}(b)$ has terminated and $F(x_{\mathrm{li}(b)})$ has been found in $B'_n$, say in step $N_0$, we delay our strategy to follow the sequence $(y_a)_a$ and repeat the element $y_{N_0}$ as long as $F(y_{N_0})$ has *not* been found in $B'_n$.

3. If, in step $N_1$, we have found $F(y_{N_0})$ in $B'_n$, we go to element $y_{N_0+1}$ and repeat it as long as we have *not* found $F(y_{N_0+1})$ in $B'_n$.

4. If, in step $N_2$, we have found $F(y_{N_0+1})$ in $B'_n$, we go to element $y_{N_0+2}$ and repeat it as long as we have *not* found $F(y_{N_0+2})$ in $B'_n$, and so on.

As we will see, all steps $N_0, N_1, N_2 \ldots$ exist and depend computably on $m, n$. Obviously, then $N_0$ is a module for the convergence of $(F(y_a))_a$ to $F(y)$.

**Theorem 7.1.** *Let $x$ have a limit algorithm, $x'$ be computable, and the sequences in* SEQ *allow delaying. Then every effective map $F\colon T \to T'$ is effectively sequentially continuous.*

*Proof.* Since $x'$ is computable, $B'_n$ is completely enumerable, uniformly in $n$. Hence, as $F$ is effective, $F^{-1}(B'_n)$ is completely enumerable as well. Let this be witnessed by $W_{v(n)}$ with $v \in R^{(1)}$. Obviously, $v$ uniformly depends on the index of $F$. Finally, let $\mathrm{li} \in P^{(1)}$ witness that $x$ has a limit algorithm and let $t \in R^{(1)}$ with $\mathrm{range}(t) \subseteq \mathrm{dom}(x)$ so that $(x_{t(a)})_a$ is a sequence in SEQ converging to some $y \in T$.

Set $\bar{g}(\bar{b}, n) = \mu c : \varphi_{v(n)}(\mathrm{li}(\bar{b}))\downarrow_c$ and define $\bar{u} \in P^{(3)}$ by

$$\bar{u}(\bar{b}, n, 0) = 0,$$

$$\bar{u}(\bar{b}, n, a+1) = \begin{cases} \bar{u}(\bar{b}, n, a) + 1 & \text{if } \bar{g}(\bar{b}, n)\uparrow_{a+1}, \text{ or } \bar{g}(\bar{b}, n)\downarrow_{a+1} \\ & \text{and } \varphi_{v(n)}(t(\bar{u}(\bar{b}, n, a)))\downarrow_{a+1}, \\ \bar{u}(\bar{b}, n, a) & \text{if } \bar{g}(\bar{b}, n)\downarrow_{a+1} \text{ and } \varphi_{v(n)}(t(\bar{u}(\bar{b}, n, a)))\uparrow_{a+1}. \end{cases}$$

In addition, let $h \in R^{(2)}$ with

$$\varphi_{h(\bar{b}, n)}(a) = t(\bar{u}(\bar{b}, n, a)).$$

By the recursion theorem there is then a function $b \in R^{(1)}$ with $\varphi_{b(n)} = \varphi_{h(b(n),n)}$. Set $g(n) = \bar{g}(b(n), n)$ and $u(n, a) = \bar{u}(b(n), n, a)$.

Suppose that $g(n)\uparrow$, for some $n \in \omega$ with $F(y) \in B'_n$. Then $u(n, a) = a$ and hence $x_{\varphi_{b(n)}(a)} = x_{t(a)}$. It follows that $(x_{\varphi_{b(n)}(a)})_a$ converges to $y$. Hence, $y \leq_\tau x_{\mathrm{li}(b(n))}$ because of Corollary 5.6. By assumption, $y \in F^{-1}(B'_n)$. With Corollary 5.4 we therefore obtain that $x_{\mathrm{li}(b(n))} \in F^{-1}(B'_n)$ as well, since $F^{-1}(B'_n)$ is completely enumerable. Consequently, $\mathrm{li}(b(n)) \in W_{v(n)}$, i.e., $g(n)$ is defined, a contradiction.

Next, let $k(n, e) = \mu c : \varphi_{v(n)}(t(g(n) + e))\downarrow_c$ and assume that there is some $n, \bar{a} \in \omega$ so that $k(n, \bar{a})\uparrow$. Let $\bar{a}$ be minimal with this property. Then

$$\varphi_{b(n)}(a) = \begin{cases} t(a) & \text{if } a < g(n), \\ t(g(n)) & \text{if } g(n) \leq a < k(n, 0)), \\ t(g(n)+1)) & \text{if } k(n, 0) \leq a < k(n, 1), \\ \quad\vdots & \\ t(g(n) + \bar{a} - 1) & \text{if } k(n, \bar{a} - 2) \leq a < k(n, \bar{a} - 1), \\ t(g(n) + \bar{a}) & \text{if } a \geq k(n, \bar{a} - 1). \end{cases}$$

As $\textsc{Seq}$ is closed under delaying, the sequence $(x_{\varphi_{b(n)}(a)})_a$ is in $\textsc{Seq}$. Moreover, it is a sequence that is eventually constant. With Condition 5.1(3) it therefore follows that $x_{\mathrm{li}(b(n))} = x_{t(g(n)+\bar{a})}$. As we have already seen, $\mathrm{li}(b(n)) \in W_{v(n)}$, i.e., $x_{\mathrm{li}(b(n))} \in F^{-1}(B'_n)$. The latter set is completely enumerable. Therefore, also $t(g(n) + \bar{a}) \in W_{v(n)}$, which means that $k(n, \bar{a})\downarrow$, a contradiction.

This shows that $k(n, e)\downarrow$, for all $n, e \in \omega$. In other words, $g$ is a computable convergence module, i.e., the sequence $(F(x_{t(a)})_a$ converges effectively to $F(y)$. As follows from the construction, $g$ depends computably on the Gödel number of $t$. So, we have that $F$ is sequentially continuous, effectively. $\square$

Note that the construction of the convergence module $g$ not only depends computably on the index of the sequence transformed by $F$, but also on the index of $F$. Moreover, we do not know whether effective maps have a witness for noninclusion under the assumptions of the theorem.

**Corollary 7.2.** *Let $x$ have a limit algorithm and be computable. Moreover, let the sequences in $\textsc{Seq}$ allow delaying. Then $\mathcal{T}$ has a uniform computable convergence module.*

*Proof.* Let $F$ be the identity on $T$ in the above theorem. $\square$

The most restrictive assumption in the above result is that sequences in $\textsc{Seq}$ should allow delaying. In the proof the constructed sequence had to be delayed several times. This is certainly not possible for sequences that have to satisfy strong conditions as the regular Cauchy sequences. In what follows we will derive an analogous result for spaces like constructive metric spaces in which the requirement that sequences in $\textsc{Seq}$ allow delaying is no longer used. As we will see, the construction in the proof is very much the same as the one in the previous proof, only where we had to wait and see whether a certain computation will terminate, we can now use a decision precedure. We will derive the result for the rather general class of effective $T_3$ spaces. For a subset $X$ of a topological space, let $\mathrm{cl}(X)$ denote its closure and $\mathrm{ext}(X)$ ist exterior.

**Definition 7.3.** $\mathcal{T}$ is *effectively* $T_3$, if there is some function $s \in P^{(2)}$ such that $s(i, m)\downarrow$ with
$$x_i \in B_{s(i,m)} \subseteq \mathrm{cl}(B_{s(i,m)}) \subseteq B_m,$$
for all $i \in \mathrm{dom}(x)$ and $m \in \omega$ with $x_i \in B_m$.

We moreover say that numbering $x$ is *co-computable* if $\mathrm{ext}(B_n)$ is completely enumerable, uniformly in $n$. Note that in a constructive metric space $x_i \in \mathrm{ext}(B_{\langle j,n\rangle})$, exactly if there is some basic open set $B_{\langle e,m\rangle}$ containing $x_i$ so that $\delta(\beta_j, \beta_e) > 2^{-m}+2^{-n}$. Hence the numbering constructed in Proposition 3.8 is also co-computable.

**Theorem 7.4.** *Let $\mathcal{T}'$ be effectively $T_3$, $x'$ be computable as well as co-computable, and $x$ have a limit algorithm. Then every effective map $F\colon T \to T'$ is effectively sequentially continuous.*

*Proof.* Let $s \in P^{(2)}$ and $f \in P^{(1)}$, respectively, witness that $\mathcal{T}$ is effectively $T_3$ and $F$ is effective. Since $x'$ is computable, it follows as in the proof of Theorem 7.1 that $F^{-1}(B'_n)$ is completely enumerable, uniformly in $n$. Let this be witnessed by $W_{v(n)}$ with $v \in R^{(1)}$.

As $x'$ is also co-computable, it follows in a similar way that there is a function $w \in R^{(2)}$ so that $W_{w(i,n)}$ witnesses that $F^{-1}(\mathrm{ext}(B'_{s(f(i),n)}))$ is completely enumerable, uniformly in $i, n$. Again $v, w$ uniformly depend on the Gödel number of $f$. Let, finally, $\mathrm{li} \in P^{(1)}$ witness that $x$ has limit algorithm and let $t \in R^{(1)}$ with $\mathrm{range}(t) \subseteq \mathrm{dom}(x)$ so that $(x_{t(a)})_a$ is a sequence in Seq converging to some point $y \in T$.

In the construction of the sequence with index $b$ we are going to describe, for certain sequence elements $x_{t(a)}$ we will search whether we find $F(x_{t(a)}) \in B'_n$ or $F(x_{t(a)}) \in \mathrm{ext}(B'_{s(f(\mathrm{li}(b)),n)})$. Possibly both is the case, then we give preference to what we find first. Let

$$\bar{k}_1(\bar{b}, n, a) = \mu c : \varphi_{v(n)}(t(\bar{u}(\bar{b}, n, a)))\downarrow_c, \quad \bar{k}_2(\bar{b}, n, a) = \mu c : \varphi_{w(\mathrm{li}(\bar{b}),n)}(t(\bar{u}(\bar{b}, n, a)))\downarrow_c,$$

and

$$\bar{k}(\bar{b}, n, a) = \begin{cases} \bar{k}_1(\bar{b}, n, a) & \text{if } \bar{k}_1(\bar{b}, n, a) \text{ terminates in at most the same number} \\ & \text{of steps as } \bar{k}_2(\bar{b}, n, a), \\ \bar{k}_2(\bar{b}, n, a) & \text{if } \bar{k}_2(\bar{b}, n, a) \text{ terminates in less steps than } \bar{k}_1(\bar{b}, n, a), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Moreover, set $\bar{g}(\bar{b}, n) = \mu c : \varphi_{v(n)}(\mathrm{li}(\bar{b}))\downarrow_c$ and define $\bar{u} \in P^{(3)}$ by

$$\bar{u}(\bar{b}, n, 0) = 0,$$

$$\bar{u}(\bar{b}, n, a+1) = \begin{cases} \bar{u}(\bar{b}, n, a) + 1 & \text{if } \bar{g}(\bar{b}, n)\uparrow_{a+1}, \text{ or } \bar{g}(\bar{b}, n)\downarrow_{a+1}, \bar{k}(\bar{b}, n, a)\downarrow, \\ & \bar{k}_1(\bar{b}, n, a)\downarrow \text{ and } \bar{k}(\bar{b}, n, a) = \bar{k}_1(\bar{b}, n, a), \\ \bar{u}(\bar{b}, n, a) & \text{if } \bar{g}(\bar{b}, n)\downarrow_{a+1}, \bar{k}(\bar{b}, n, a)\downarrow, \bar{k}_2(\bar{b}, n, a)\downarrow \\ & \text{and } \bar{k}(\bar{b}, n, a) = \bar{k}_2(\bar{b}, n, a), \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Finally, let $h \in R^{(2)}$ with

$$\varphi_{h(\bar{b},n)}(a) = t(\bar{u}(\bar{b}, n, a)).$$

By the recursion theorem there is then some function $b \in R^{(1)}$ with $\varphi_{b(n)} = \varphi_{h(b(n),n)}$. Set

$$k_1(n, a) = \bar{k}_1(b(n), n, a), \quad k_2(n, a) = \bar{k}_2(b(n), n, a), \quad k(n, a) = \bar{k}(b(n), n, a),$$
$$g(n) = \bar{g}(b(n), n) \quad \text{and} \quad u(n, a) = \bar{u}(b(n), n, a).$$

Then it follows as in the proof of Theorem 7.1 that $g(n)$ is defined for all $n \in \omega$ with $F(y) \in B'_n$. Thus, $F(x_{\mathrm{li}(b(n))}) \in B'_n$, i.e., $x'_{f(\mathrm{li}(b(n)))} \in B'_n$, from which we obtain with the effective $T_3$ property that $x'_{f(\mathrm{li}(b(n)))} \in B_{s(f(\mathrm{li}(b(n))),n)}$. In addition, we have for $a \geq g(n)$ that either $F(x_{t(a)}) \in B'_n$, or $F(x_{t(a)}) \in T' \setminus B'_n$, in which case $F(x_{t(a)}) \in \mathrm{ext}(B_{s(f(\mathrm{li}(b(n))),n)})$. Therefore, at least one of $k_1(n, a)$ and $k_2(n, a)$ must be defined, i.e., $k(n, a)$ is always defined.

Assume that there is some $n, \bar{a} \in \omega$ with $\bar{a} \geq g(n)$ such that $F(x_{t(\bar{a})}) \notin B'_n$. Let $\bar{a}$ be minimal with this property. Then $k_1(n, \bar{a})$ is not defined. As a consequence we obtain with what has just been said that $k(n, \bar{a}) = k_2(n, \bar{a})$. Moreover,

$$\varphi_{b(n)}(a) = \begin{cases} t(a) & \text{if } a < \bar{a}, \\ t(\bar{a}) & \text{otherwise.} \end{cases}$$

With Property 3 of Seq it follows that $(x_{\varphi_{b(n)}(a)})_a$ is in Seq. Furthermore, the sequence is eventually constant. Because of Condition 5.1(3) we therefore have that $x_{\mathrm{li}(b(n))} = x_{t(\bar{a})}$. As we have already seen, $\mathrm{li}(b(n)) \in W_{v(n)}$, i.e., $x_{\mathrm{li}(b(n))} \in F^{-1}(B'_n)$. The latter set is completely enumerable. Therefore, also $t(\bar{a}) \in W_{v(n)}$, which means that $F(x_{t(\bar{a})}) \in B'_n$, a contradiction.

This shows that for all $n \in \omega$ with $F(y) \in B'_n$, we have that $F(x_{t(a)}) \in B'_n$ as well, for all $a \geq g(n)$, i.e., the sequence $(F(x_{t(a)}))_a$ converges effectively to $F(y)$. In other words, $F$ is effectively sequentially continuous. $\square$

The two theorems in this section cover a large variety of cases. Unfortunately, however, we were not able to pursue our programme in full generality, i.e., to derive a theorem stating the effective sequential continuity of effective operators that would include all interesting cases, as we did in the continuous case. However, the present situation is also more complicated. In the continuous case we have one decision to make whether to follow a given sequence or to deviate. Now, we have to deal with infinitely many such decisions and the strategies how to make the decisions were quite different in the cases we considered. It is even not clear to us, whether effective operators are effectively sequentially continuous in general, or whether an additional condition is needed.

**Proposition 7.5.** *There is a constructive metric space $\mathcal{M}$, a constructive domain $\mathcal{Q}$, and a map $F\colon M \to Q$ which is effective, but not sequentially continuous.*

*Proof.* The following construction is a modification of an example given by Friedberg [8]. Let $\mathcal{M}$ be Baire space and $\mathcal{Q}$ Sierpinski space $\{\perp, 0\}$ with $\perp \sqsubseteq 0$, $\beta_0 = \perp$, and $\beta_{n+1} = 0$. Moreover, set

$$h(i) = \begin{cases} 1 & \text{if } [(\forall a \leq i)\varphi_i(a) = 0] \vee (\exists c)[\varphi_i(c) \neq 0 \wedge (\forall a < c)\varphi_i(a) = 0 \wedge \\ & (\exists j < c)(\forall b \leq c)\varphi_i(b) = \varphi_j(b)], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then $h \in P^{(1)}$. As it is readily verified, for all $\varphi_i$, $\varphi_j \in R^{(1)}$ with $\varphi_i = \varphi_j$ one has that $h(i) = h(j)$. Let $x$ be an admissible indexing of $Q$. Then there is a function $d \in R^{(1)}$ such that for all $i \in \omega$ for which $\beta(W_i)$ is directed, $x_{d(i)}$ is the least upper bound of $\beta(W_i)$. Let $q \in R^{(1)}$ with $W_{q(i)} = \{0, h(i)\}$, and set $t = d \circ q$. We define the effective mapping $F\colon R^{(1)} \to Q$ by $F(\varphi_i) = x_{t(i)}$. Then $F(\varphi_i) = 0$, if the first condition in the definition of $h$ holds; otherwise, $F(\varphi_i) = \perp$.

Now, for $m \in \omega$, let $k_m = \max\{\, \varphi_i(m+1) + 1 \mid i \leq m \wedge \varphi_i \in R^{(1)} \,\}$ and define

$$g_m(a) = \begin{cases} 0 & \text{if } a \neq m+1, \\ k_m & \text{otherwise.} \end{cases}$$

Then $g_m \in R^{(1)}$, for every $m \in \omega$. Moreover, $(g_m)_m$ is a regular Cauchy sequence that converges to $\lambda n.0$. Since for any Gödel number $j$ of $g_m$ we have that $j > m$ and as $g_m(m+1) \neq 0$, it follows from the definition of $F$ that $F(g_m) = \perp$, for all $m \in \omega$. On the other hand, $F(\lambda n.0) = 0$. Thus, $F$ cannot be sequentially continuous. $\square$

However, this leads still open the question whether effective operators are effectively sequentially continuous. To decide this question negatively, one would need a computable sequence $(g_m)_m$ in the construction.

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