Domains with Approximation Structure and Their Canonical Quasi-Metrics

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Abstract

In this note continuous directed-complete partial orders with least element (domains) are enriched by a family of projections, called approximation structure, that assigns to each point a sequence of canonical approximations. The morphisms are continuous maps which commute with the projections. Each approximation structure induces a quasi-ultrametric on the domain, the associated topology of which is finer than the Scott topology of the domain. It coincides with the Scott topology exactly if the subdomains generated by the canonical approximations at the various levels contain only compact elements, which is the case if and only if none of them contains infinite strictly increasing chains and if in each of them the lengths of the chains in any given set of finite strictly increasing chains with the same supremum are bounded. The condition implies that the subdomains as well as the domain must be algebraic. Similarly, the topology coming with the metric that is associated with the quasi-metric is finer than the Lawson topology of the underlying domain and coincides with it if the condition is satisfied.

For categories of domains that are closed under the construction of Cartesian products, function spaces and bilimits of ω-chains, approximation structures on the composed domains can be defined from those coming with the components in a natural way. The quasi-metrics induced by the newly constructed approximation structures are nicely related to those generated by the approximation structures of the components.

1 Introduction

Domains as introduced by Dana Scott [13] and, independently, by Yuri L. Ershov [5] are partial orders intended to model the notion of approximation (and of computation) in a qualitative way. The smaller elements are usually regarded as providing less information than those higher up; they are partial elements from which the total (ideal, fully developed) elements evolve as one goes up in the order.

In general, there is no measure for the speed of approximation, but for a large—and important—class of domains there exists a uniform way of approximation, which is compatible with the elementwise one. In these cases the domains are approximated by increasing chains of subdomains, or more precisely: they can be represented as bilimits of ω-chains of domains with embedding/projections as bonding maps. Examples of such domains are the SFP domains introduced by Plotkin [12] in order to provide a class of domains that is closed under the constructions usually used in programming language semantics, and the domains that are obtained as solutions of recursive domain equations (cf. [16]).

The approximating subdomains in such a chain can be understood as uniform levels of approximation. Every point of the domain has a best approximation in each of the subdomains. With respect to these levels the growth of a continuous function can be measured as well as the amount of information about an input needed to produce an output of a certain level. We call the family of maps projecting points to their best approximations an approximation structure. As is shown, any domain having an approximation structure can be represented as bilimit of an ω-chain of subdomains with embedding/projections as bonding maps.

Approximation structures have been introduced in [17] in the case of stable domains. It was proved that various models for a very strong untyped lambda calculus can be constructed on the basis of such domains. The morphisms in the corresponding category respect the approximation levels.

In this note the more general case of continuous domains is considered. The requirements that link the approximation structure with the continuity structure of the domain are rather weak. Stronger conditions will be discussed in the paper in connection with topological considerations. It is shown that any approximation structure defines a canonical quasi-ultrametric on the domain the induced topology of which is finer than the Scott topology of the domain. It coincides with the Scott topology just if the strong continuity axiom holds, which requires for all point x, y and all levels n that if every level n approximant of x approximates y, then also the best approximation of x at level n approximates y. The conditions seems to be quite natural, but as is shown, it is equivalent to demanding that all elements appearing at the various approximation levels are compact. Consequently, the domain must be algebraic. As is moreover demonstrated, the axiom holds if and only if the subdomains generated by the approximations at the different levels contain only finite strictly increasing sequences and if in each such subdomain the lengths of the sequences in any collection of finite strictly increasing sequences having the same supremum are bounded.

Quasi-metrics on domains have first been considered by M. B. Smyth [14] in order to allow for quantitative statements about approximations in domains. They satisfy essentially the same conditions as a metric, except that they must not be symmetric. As a consequence of this every quasi-metric induces a partial order on its domain. If it is generated by an approximation structure the corresponding partial order matches with the domain order. In this case the distance from a point x to a point y is determined by the maximal level such that all approximations of x of that level approximate y.

With each quasi-metric a metric is associated, which is obtained by symmetrizing the quasi-metric. In our case the distance of two points in this metric is fixed by the maximal level up to which their approximations coincide. The topology generated by the metric is finer than the Lawson topology of the domain and agrees with it in case the strong continuity axiom holds. Moreover, the domain is complete with respect to this metric, if a weaker condition, called the weak continuity axiom, is satisfied. It requires for each level n that every approximant of x of level n also approximates the best approximation of x at that level.

Note that domains with approximation structure are a special case of the projection spaces studied by Ehrig et al. [6, 7, 8, 10] as a generalization of the projective model of process algebra by Bergstra and Klop [3]. Moreover, the class of these domains contains the rank-ordered sets introduced by Bruce and Mitchell [4] in a model construction for type systems that allow subtyping, recursive types and higher-order polymorphism.

Classes of domains used in programming language semantics are usually closed under certain constructions which reflect the constructions used in building up larger programs and data types from smaller ones. Important examples are the construction of Cartesian products, function spaces and bilimits of ω-chains. As will be seen, approximation structures on the composed domains can be constructed in a natural way from those of the components in these cases. The quasi-metrics associated with the approximation structures thus obtained are in a nice relationship with the quasi-metrics coming with the approximation structures on the components. In the case of Cartesian products the distance from a pair of points to another such pair is the maximum of the distances between the points in the first component and those in the second component, respectively. Similar results hold in the other two cases.

The paper is organized as follows. In Section 2 domains with approximation structure and their morphisms are introduced and fundamental properties are given. The canonical quasi-metric of an approximation structure and its associated topology is studied in Section 3. Similarly, in Section 4 the metric generated by the quasi-metric defined in Section 3 and its topology is considered. In Section 5, finally, the above mentioned constructions on the class of domains with approximation structure are discussed and relationships between the quasi-metrics induced by the newly constructed approximation structures and those coming with the components are derived. Final remarks appear in Section 6.

2 Domains with approximation structure

Let $(D, \sqsubseteq)$ be a partially ordered set with least element $\bot$. A subset S of D is *directed*, if it is nonempty and every pair of elements in S has an upper bound in S. D is a *directed-complete* partial order (cpo) if every directed subset S of D has a least upper bound $\bigsqcup S$ in D.

For elements x, $y \in D$ one says that x is *way-below* y (or that x *approximates* y), and writes $x \ll y$, if for every directed subset S of D such that $y \sqsubseteq \bigsqcup S$ there is some $u \in S$ with $x \sqsubseteq u$. If $x \ll x$ then x is called *compact*. We denote the set of compact elements of D by D^0. The way-below relation is transitive. The next lemma lists some of its important properties.

Lemma 2.1 *Let D be a cpo and $u, v, x, y \in D$. Then the following two statements hold:*

1. *If $x \ll y$ then $x \sqsubseteq y$.*
2. *If $u \sqsubseteq x \ll y \sqsubseteq v$ then $u \ll v$.*

Definition 2.2 Let D be cpo.

1. A subset Z of D is a *basis* of D if for any $x \in D$ the set Z_x $(= \{ z \in Z \mid z \ll x \})$ contains a directed subset which has x as its least upper bound.
2. D is called *continuous* or a *domain* if it has a basis.
3. D is called *algebraic* or an *algebraic domain* if it has a basis of compact elements. If D^0 is a countable basis, D is called an *ω-algebraic domain*.

Note that D^0 is included in any basis of D. Standard references for domain theory and its applications are [11, 9, 1, 19, 2].

On domains the way-below relation has a further useful property, which is known as the interpolation property.

Lemma 2.3 *Let D be a domain with basis Z. Then for any $x, y \in D$ with $x \ll y$ there exists $u \in Z$ so that $x \ll u \ll y$.*

The next corollary contains a useful consequence.

Corollary 2.4 *Let D be a domain, $x \in D$ and S be a directed subset of D. If $x \ll \bigsqcup S$ then there is some $z \in S$ with $x \ll z$.*

Proof: Assume that $x \ll \bigsqcup S$. By the interpolation property there is some $y \in D$ such that $x \ll y \ll \bigsqcup S$. Hence there is some $z \in S$ so that $y \sqsubseteq z$. With Lemma 2.1(2) it follows that $x \ll z$.

As is well known, on each cpo D there is a canonical topology σ: the *Scott* topology. A subset X is open, if it is upwards closed with respect to the order and intersects each directed subset of D of which it contains the least upper bound. In case that D is continuous with basis Z, this topology is generated by the sets $\twoheaduparrow\{z\}(= \{ y \in D \mid z \ll y \})$ with $z \in Z$.

Definition 2.5 Let D and E be cpo's. A map $f\colon D \to E$ is said to be *continuous* if it is monotone and for any directed subset S of D,

$$f(\bigsqcup S) = \bigsqcup f(S).$$

It is well known that the order-based notion of continuity coincides with the topological one. Denote the collection of all continuous maps from D to E by $[D \to E]$. Endowed with the *pointwise order*, that is, $f \sqsubseteq_p g$ if $f(x) \sqsubseteq g(x)$ for all $x \in D$, it is a cpo again, but it need not be continuous, even if D and E are.

Definition 2.6 An *embedding/projection* (e, p) from a cpo D to a cpo E is a pair of continuous maps $e\colon D \to E$ and $p\colon E \to D$ such that $p \circ e = \mathrm{id}_D$, the identity map on D, and $e \circ p \sqsubseteq_p \mathrm{id}_E$. The map e is called *embedding* and p *projection*.

Embeddings are one-to-one and preserve existing least upper bounds as well as the way-below relation. Projections are onto. Note that $e \circ p$ is an idempotent map on E. Such maps are called *retractions*.

An ω-chain in a category $\mathbf{K}$ is a diagram of the form $\Delta = D_0 \xrightarrow{f_0} D_1 \xrightarrow{f_1} \cdots$. Dually an ω-cochain is a diagram of the form $\Delta = D_0 \xleftarrow{f_0} D_1 \xleftarrow{f_1} \cdots$. If $\mathbf{K^{ep}}$ is a category of cpo's with embedding/projections as morphisms and $(D_i, (e_i, p_i))_{i\in\omega}$ is an ω-chain in $\mathbf{K^{ep}}$, then $(D_i, e_i)_{i\in\omega}$ is an ω-chain in the category $\mathbf{K^{e}}$ and $(D_i, p_i)_{i\in\omega}$ is an ω-cochain in the category $\mathbf{K^{p}}$, where $\mathbf{K^{e}}$ and $\mathbf{K^{p}}$ have the same objects as $\mathbf{K^{ep}}$, but have embeddings and/or projections as morphisms. As is well known, $(D_i, e_i)_{i\in\omega}$ has a colimit which is the limit of $(D_i, p_i)_{i\in\omega}$. It is called the bilimit of $(D_i, (e_i, p_i))_{i\in\omega}$. Moreover, if D is an object of $\mathbf{K^{ep}}$ and $(f_i, g_i)_{i\in\omega}$ a family of embedding/projections such that $(D, (f_i)_{i\in\omega})$ is a cocone for $(D_i, e_i)_{i\in\omega}$ and $(D, (g_i)_{i\in\omega})$ is a cone for $(D_i, p_i)_{i\in\omega}$, then $(D, (f_i)_{i\in\omega})$ is colimiting ($(D, (g_i)_{i\in\omega})$ is limiting) exactly if $\bigsqcup_i f_i \circ g_i = \mathrm{id}_D$ (cf. [1, Lemma 3.3.8]).

The class of algebraic domains contains the large subcollection **SFP** of domains D that can be represented as bilimits of ω-chains $(D_i)_{i\in\omega}$ of finite subdomains [12]. Since the subdomains D_i are closed under the operation of taking existing least upper bounds, any element x of D has a best approximation $\bigsqcup \{ z \in D_i \mid z \sqsubseteq x \}$ in each of them. Abstracting from such properties one obtains the notion of an approximation structure.

Definition 2.7 Let D be a domain with basis Z and for each $i \in \omega$, $[\cdot]_i\colon D \to D$. The family $([\cdot]_i)_{i\in\omega}$ of maps is said to be an *approximation structure* on D if the following five conditions hold for all $i, j \in \omega$:

1. $[\cdot]_i$ is continuous.
2. $Z \subseteq \bigcup_n D_n$, where $D_n = \{ x \in D \mid [x]_n = x \}$.
3. $[\cdot]_i \circ [\cdot]_j = [\cdot]_{\min\{i,j\}}$.

4. $[\cdot]_i \sqsubseteq_p \mathrm{id}_D$.

5. $\bigsqcup_n [\cdot]_n = \mathrm{id}_D$.

In an earlier paper [17] the author introduced approximation structures in the context of dI-domains and showed that various models of an important strong untyped lambda calculus can be constructed on this basis. The morphisms used then respected the approximation structure. In that case stronger requirements were made reflecting the special properties of stable domains.

Note that by conditions (3) and(4), $[x]_i = [[x]_{i+1}]_i \sqsubseteq [x]_{i+1}$. Thus, the maps $[\cdot]_i$ form an increasing chain under the pointwise order. By stipulation (3) we moreover have that D_i is contained in D_{i+1}. Let $\iota_{i,i+1}$ be the corresponding inclusion map and $\iota_i \colon D_i \to D$ be the inclusion of D_i in D. Then it follows with condition (4) that $(\iota_{i,i+1}, [\cdot]_i \restriction D_{i+1})$ and $(\iota_i, [\cdot]_i)$ are embedding/projections.

Let $Z_i = Z \cap D_i$. Then Z_i is contained in $\{ [v]_i \mid v \in Z \}$, i.e., $Z_i \subseteq [Z]_i$. Conversely, if $v \in [Z]_i$, then there is some $u \in Z$ with $v = [u]_i$. Hence, $[v]_i = [[u]_i]_i = [u]_i = v$, i.e., $v \in Z_i$. Thus, we have that $Z_i = [Z]_i$.

Proposition 2.8 *Let D be a domain with basis Z and approximation structure $([\cdot]_i)_{i\in\omega}$. Then the following two statements hold:*

1. *D_i is a subdomain of D with basis Z_i, for every $i \in \omega$.*

2. *D is isomorphic to the bilimit of the ω-chain $(D_i, (\iota_{i,i+1}, [\cdot]_i \restriction D_{i+1}))_{i\in\omega}$.*

The first statement is a special case of the general result that the image of a retraction r on a domain with basis Z is a continuous sub-cpo with basis $r(Z)$ (cf. [1, Lemma 3.1.3 and Theorem 3.1.4]) and the second statement is a consequence of condition 2.7(5). It follows that if D is algebraic then the same is true for the D_i, and conversely.

We have already realized that an approximation structure on a domain D can be obtained from a representation of the domain as a bilimit of an ω-chain of subdomains. Now, we see that also conversely any approximation structure on D determines such a representation.

Lemma 2.9 *Let D be a domain with approximation structure, $x \in D$ and $i \in \omega$. Then*

$$[x]_i = \bigsqcup \{ u \in Z_i \mid u \ll x \}.$$

Proof: Note first that the set of all $u \in Z_i$ with $u \ll x$ is directed and thus has a least upper bound. In order to see this let $u, v \in Z_i$ with $u, v \ll x$. Since Z is a basis there is some $w \in Z_x$ with $u, v \sqsubseteq w$. It follows that $u = [u]_i \sqsubseteq [w]_i \sqsubseteq w \ll x$ and hence that $u \sqsubseteq [w]_i \ll x$. Similarly, we obtain that $v \sqsubseteq [w]_i \ll x$. Because $Z_i = [Z]_i$, we have that $[w]_i \in Z_i$.

Now, let $u \in Z_i$ with $u \ll x$, then $u \sqsubseteq x$ and hence $u = [u]_i \sqsubseteq [x]_i$. For the converse inequality note that Z_i is a basis of D_i, by Proposition 2.8(1), and that the way-below relation on D_i is the restriction of the way-below relation on D to D_i. Moreover, observe that for $u \in Z_i$ with $u \ll [x]_i$ we have that $u \ll x$, as $[x]_i \sqsubseteq x$.

Except condition (2) there are no requirements in Definition 2.7 which combine the approximation structure with the continuity structure of the domain, i.e., there are no conditions relating the maps $[\cdot]_i$ with the way-below relation. By the above lemma we only have for $u \in Z_i$ and $x \in D$ with $u \ll x$ that $u \sqsubseteq [x]_i$, but later on we would need that $u \ll [x]_i$.

Definition 2.10 Let D be a domain with approximation structure. Then we call the condition that for all $i \in \omega$, $u \in Z_i$ and $x \in D$

$$u \ll x \Rightarrow u \ll [x]_i$$

the *weak continuity axiom*.

The weak continuity axiom does obviously hold, if all base elements are compact, i.e., if D is algebraic.

As we go on, we will meet a further (and more powerful) proviso as fallout of our topological considerations, but as we will also see, this condition has a strong impact on the order structure of the domain.

Lemma 2.11 *Let D be a domain with approximation structure. Then the approximation structure satisfies the weak continuity axiom if and only if for all $i \in \omega$, $u \in Z_i$ and $x \in D$*

$$u \ll x \Rightarrow (\exists v \in Z_i) u \ll v \ll x.$$

Proof: For the "if" part note that D_i is a subdomain of D with basis Z_i, by Proposition 2.8(1), and use the interpolation property. The "only-if" part follows with Lemmas 2.9 and 2.1(2).

Observe that in general we only have that if $u \ll x$ then there is some $j \geq i$ and some $v \in Z_j$ such that $u \ll v \ll x$. To see this use the interpolation property and condition 2.7(2).

The morphisms we want to use in this paper shall not only respect the order structure, but also the approximation structure.

Definition 2.12 Let D and E be domains with approximation structures $([\cdot]_i^D)_{i \in \omega}$ and $([\cdot]_i^E)_{i \in \omega}$, respectively. A continuous function $f \colon D \to E$ is

1. *rank-preserving* if for all $x \in D$ and $i, j \in \omega$ with $j \geq i$,

$$[f(x)]_i^E = [f([x]_j^D)]_i^E.$$

2. a *projection morphism* if for all $i \in \omega$

$$[\cdot]_i^E \circ f = f \circ [\cdot]_i^D.$$

The next lemma follows as in [17].

Lemma 2.13 *A continuous function $f \colon D \to E$ is a projection morphism if and only if it is rank-preserving and for all $i \in \omega$, $f(D_i) \subseteq E_i$.*

In order to produce an output of level i, i.e. in E_i, a rank-preserving map requires only an approximation of the input of at most the same level. A projection morphism in addition preserves levels of approximation.

Note that every domain has a trivial approximation structure with respect to which every continuous function is a projection morphism: set $[\cdot]_i = \mathrm{id}_D$. An easy nontrivial example of a domain with approximation structure is the topped domain $\overline{\omega}$ of the natural numbers, i.e., the ordinal $\overline{\omega} = \omega \cup \{\omega\}$ with its canonical order, which has ω as its set of compact elements. For $k \in \overline{\omega}$ and $i \in \omega$ set

$$[k]_i^{\overline{\omega}} = \min\{i, k\}.$$

Then $([\cdot]_i^{\overline{\omega}})_{i\in\omega}$ is an approximation structure on $\overline{\omega}$.

Let D be a domain with approximation structure and set $[\cdot]_\omega = \mathrm{id}_D$. Then $\lambda(k,x) \in \overline{\omega} \times D.[x]_k$ is a map from $\overline{\omega} \times D$ to D. Note that $\overline{\omega} \times D$ is a domain with approximation structure again. The order and the projection maps are defined componentwise. As in [17] we obtain the following result.

Lemma 2.14 *The map* $[\cdot]_\cdot$ *is a projection morphism.*

3 The canonical quasi-metric

As will be shown in this section, every approximation structure defines a quasi-ultrametric on its underlying domain in a natural way.

Definition 3.1 Let X be a set. A map $d\colon X \times X \to [0,\infty)$ is a *quasi-metric* if the following two conditions hold for all $x,y,z \in X$:

1. $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$
2. $d(x,z) \le d(x,y) + d(y,z)$.

If instead of (2) the stronger condition

2'. $d(x,z) \le \max\{d(x,y), d(y,z)\}$

holds, d is called a *quasi-ultrametric.*

Note that a quasi-metric need not be symmetric. Every quasi-metric d defines a partial order $\le_d$ and a T_0 topology τ_d on X. The partial order is given by

$$x \le_d y \Leftrightarrow d(x,y) = 0$$

and the topology generated by the sets

$$B_n(x) = \{\, z \in X \mid d(x,z) < 2^{-n} \,\}$$

with $x, y \in X$ and $n \in \omega$.

With respect to the induced partial order each quasi-metric is monotone in its first argument and antimonotone in its second argument. Moreover, it is continuous in its topology.

In what follows, let D be a domain with basis Z and approximation structure $([\cdot]_i)_{i\in\omega}$. As is well known

$$x \sqsubseteq y \Leftrightarrow (\forall u \in Z)(u \ll x \Rightarrow u \ll y).$$

For $n \in \omega$ set

$$x \sqsubseteq_n y \Leftrightarrow (\forall u \in Z_n)(u \ll x \Rightarrow u \ll y).$$

Assume that $x \sqsubseteq_n y$. Then it follows with Lemma 2.9 that $[x]_n \sqsubseteq y$. The converse implication is true, if $x \sqsubseteq_n [x]_n$, which holds exactly in case the weak continuity axiom is satisfied.

We will now introduce a distance function on D, where the distance of a point x to a point y is determined by the maximal level such that all approximations of x of that level approximate y. Define

$$d(x,y) = \begin{cases} 0 & \text{if } x \sqsubseteq y, \\ 2^{-\min\{\, i \mid x \not\sqsubseteq_i y \,\}} & \text{otherwise.} \end{cases}$$

It is readily verified that d is a quasi-ultrametric on D. We call it the *canonical quasi-ultrametric of the approximation structure.* As is easily seen, its induced partial order coincides with the domain order.

By definition $[x]_n \in B_n(x)$ just if $x \sqsubseteq_n [x]_n$. Thus, the approximation structure satisfies the weak continuity axiom if and only if $[x]_n \in B_n(x)$, for all $x \in D$ and $n \in \omega$.

Proposition 3.2 *Let D be a domain with an approximation structure that satisfies the weak continuity axiom. Then $\bigcup_i D_i$ is a dense subspace of D with respect to the topology defined by the canonical quasi-ultrametric of the approximation structure.*

Next, we will study the relationship between the topology τ_d defined by d and the Scott topology on D. It is well known that the Scott topology captures essential properties of computations. Therefore, in order to be able to use a quasi-metric on a domain for studying quantitative aspects of computations, its associated topology should coincide with the Scott topology. Note that $y \in B_n(x)$ exactly if $x \sqsubseteq_n y$.

Proposition 3.3 *Let D be a domain with approximation structure. Then its Scott topology is coarser than the topology defined by the canonical quasi-ultrametric of the approximation structure.*

Proof: Let $u \in Z$ and $x \in D$ with $u \ll x$. By condition 2.7(2) $u \in Z_m$, for some $m \in \omega$. Therefore $B_m(x) \subseteq \Uparrow\{u\}$.

The converse inclusion does not hold in general. Smyth [15] considers the above quasi-metric in the context of ω-algebraic domains D with a rank function on the set of their compact elements, i.e. a map $\mathrm{rk}\colon D^0 \to \omega$. In this case the quasi-metric coincides with the Scott topology, if $\mathrm{rk}^{-1}(\{n\})$ is finite, for every $n \in \omega$. Domains with approximation structure have a canonical rank function on their basis, which is given by $\mathrm{rk}(u) = \min\{\, i \mid u \in D_i \,\}$.

In this section we will characterize when the quasi-metric topology coincides with the Scott topology. Let us first study when a ball $B_n(x)$ is Scott open. Observe to this end that $B_n(x)$ is always upwards closed with respect to the domain order.

Lemma 3.4 *Let $x, y \in D$ and $n \in \omega$ such that $[x]_n \ll y$. Then $y \in B_n(x)$.*

Proof: Let $u \in Z_n$ with $u \ll x$. Then it follows with Lemma 2.9 that $u \sqsubseteq [x]_n \ll y$. Hence $u \ll y$. This shows that $y \in B_n(x)$.

The converse implication in this lemma does not hold in general. As we will see next, it characterizes the Scott openness of $B_n(x)$.

Lemma 3.5 *For $x \in D$ and $n \in \omega$, $B_n(x)$ is Scott open if and only if $[x]_n \ll y$, for all $y \in B_n(x)$.*

Proof: Let S be a directed subset of D. Moreover, for the "if" part, let $y \in B_n(x)$ and assume that $y \sqsubseteq \bigsqcup S$. Then $\bigsqcup S \in B_n(x)$, which implies that there is some $z \in S$ with $z \in B_n(x)$. As we have already seen, this means that $[x]_n \sqsubseteq z$, what was to be shown.

Now, for the "only-if" part, assume that $\bigsqcup S \in B_n(x)$. Then $[x]_n \ll \bigsqcup S$. By Corollary 2.4 there is thus some $z \in S$ so that $[x]_n \ll z$, which, by Lemma 3.4, implies that $z \in B_n(x)$.

Definition 3.6 Let D be a domain with approximation structure. We call the requirement that for $x, y \in D$ and $n \in \omega$

$$x \sqsubseteq_n y \Rightarrow [x]_n \ll y$$

the *strong continuity axiom.*

This axiom and in particular its consequence that $[x]_n \ll x$, for all $x \in D$ and $n \in \omega$, which we call the *special continuity axiom* and which strengthens condition (4) in the definition of an approximation structure, expresses very well our intuition about the approximations $[x]_n$ as being canonical finitary elements that approximate x and as being the best possible approximation of x at level n. In combination with condition (5) the special continuity axiom says that the approximation structure endows the domain with a canonical basis with respect to which condition (2) always holds.

If we choose $x \in D_n$ in the special continuity axiom, then we obtain that $x = [x]_n \ll x$, i.e., x is compact. So, it follows from the special continuity axiom that D has a canonical basis of compact elements. This is a strong consequence, since it means that D must be algebraic. As we will see next, this is not only a consequence, but equivalent to the continuity axiom.

Lemma 3.7 *The approximation structure on D satisfies the strong continuity axiom exactly if $D_n \subseteq D^0$, for all $n \in \omega$.*

Proof: It remains to show the "only-if" part. Let to this end $n \in \omega$ and $x, y \in D$ such that $x \sqsubseteq_n y$. As we have already seen, this implies that $[x]_n \sqsubseteq y$. Hence $[x]_n \ll y$, since $[x]_n$ is compact.

Note that a compact element cannot be the least upper bound of an infinite strictly increasing sequence. For two finite increasing sequences $(y_i)_{i \le N}$, $(z_j)_{j \le M}$ of elements of D we say that they are *compatible* if for every $i \le N$ there is some $j \le M$ with $y_i \sqsubseteq z_j$, and vice versa. Moreover, for any collection of finite sequences its *length set* is the class of lengths of the sequences in the given collection.

Lemma 3.8 *For $n \in \omega$, $D_n \subseteq D^0$ just if D_n contains only strictly increasing sequences of finite length and the length set of any collection of compatible strictly increasing sequences in D_n is bounded.*

Proof: As has already been said, D_n contains no infinite strictly increasing sequences, if $D_n \subseteq D^0$. Moreover, all sequences in a set of compatible strictly increasing sequences in D_n have the same least upper bound, say $x \in D_n$. Then the members of all sequences in this set are contained in the set $\downarrow_n\{x\}$ of all elements $y \in D_n$ with $y \sqsubseteq x$. Since D_n is algebraic by our assumption and its elements are compact, $\downarrow_n\{x\}$ is directed. Therefore, if the set of lengths of the given sequences were not bounded, we could construct an infinite strictly increasing sequence with least upper bound x, which is impossible, as we have seen.

Conversely, if $x \in D_n$ and S is a directed subset of D_n with $x \sqsubseteq \bigsqcup S$, then S contains only strictly increasing sequences of finite length. Moreover, among them there is one of maximal length. Its greatest element, say z, must be the least upper bound of S, otherwise we could construct a longer sequence, since S is directed. It follows that $z \in S$ and $x \sqsubseteq z$. Thus, x is compact.

If $D_n \subseteq D^0$, for all $n \in \omega$, it follows from Proposition 2.8 that D is a bilimit of an ω-chain of algebraic domains and is hence itself algebraic.

Summing up what we have gained so far, we obtain the following result.

Theorem 3.9 *Let D be a domain with approximation structure. Then the following four statements are equivalent:*

1. *The Scott topology on D coincides with the topology defined by the canonical quasi-ultrametric of the approximation structure.*
2. *The approximation structure satisfies the strong continuity axiom.*
3. *D_n contains only compact elements, for all $n \in \omega$.*
4. *For all $n \in \omega$, D_n contains only strictly increasing sequences of finite length and the length set of any collection of compatible strictly increasing sequences in D_n is bounded.*

Moreover, if any of these conditions holds, then D must be algebraic.

In the case of domains with approximation structure the condition presented in this characterization improves Smyth's stipulation on the inverse images of the rank function mentioned earlier in this section.

If the Scott topology on D coincides with the topology of the canonical quasi-ultrametric, that is, if all approximations $[x]_n$ are compact, the distance from a point x to a point y is determined by the maximal level up to which the best approximation of x is majorized by y. In that case we have for $x, y \in D$ with $x \not\sqsubseteq y$ that

$$d(x, y) = 2^{-\min\{\, i \mid [x]_i \not\sqsubseteq y \,\}}.$$

4 The associated metric

As is well known, a metric $\bar{d}$ is associated with every quasi-metric d. It is obtained by symmetrizing the quasi-metric:

$$\bar{d}(x, y) = \max\{d(x, y), d(y, x)\}.$$

Let D be a domain with approximation structure and d the canonical quasi-ultrametric associated with it. Then $\bar{d}$ is an ultrametric.

For $x, y \in D$ and $n \in \omega$ define

$$x =_n y \Leftrightarrow (\forall u \in Z_n)(u \ll x \Leftrightarrow u \ll y).$$

Note that $x =_n y$ implies that $[x]_n = [y]_n$. The converse holds if the approximation structure satisfies the weak continuity axiom.

Lemma 4.1 *Let $x, y \in D$ such that $x \neq y$. Then*

$$\bar{d}(x, y) = 2^{-\min\{\, i \mid x \neq_i y \,\}}.$$

Let $\mu_{\bar{d}}$ be the topology associated with $\bar{d}$. It is generated by the sets

$$C_n(x) = \{\, y \in D \mid \bar{d}(x, y) < 2^{-n} \,\}$$

with $x \in D$ and $n \in \omega$.

Note that by Lemma 4.1, $y \in C_n(x)$ just if $x =_n y$. Moreover, $C_n(x) \subseteq B_n(x)$, which implies that $\tau_d \subseteq \mu_{\bar{d}}$.

If the weak continuity axiom holds, every rank-preserving map is nonexpansive with respect to this metric and hence continuous with respect to the associated topology. Because of Lemma 2.13 the same is true for the projection morphisms.

Proposition 4.2 *Let D be a domain with an approximation structure that satisfies the weak continuity axiom. Then D is complete with respect to $\bar{d}$ and $\bigcup_i D_i$ is a dense subspace.*

The proof is as in [17]. Remember that by the weak continuity axiom, $x =_n y$ exactly if $[x]_n = [y]_n$.

In the preceding section we compared the quasi-metric topology induced by the approximation structure with the canonical topology of the underlying domain. There is also a T_2 topology generated by the domain order, the Lawson topology. We will now compare this topology with the metric topology defined by $\bar{d}$.

For $x \in D$ let $\uparrow\{x\} = \{ y \in D \mid x \sqsubseteq y \}$. Then the Lawson topology λ is generated by the subsets of D which are either Scott open or of the form $D \backslash \uparrow\{x\}$, for some $x \in D$.

Lemma 4.3 *For $x \in D$ and $n \in \omega$*

$$C_n(x) = B_n(x) \cap \bigcup \{ D \backslash \uparrow\{y\} \mid y \not\sqsubseteq_n x \}.$$

Proof: Let $z \in C_n(x)$. Then $z \in B_n(x)$ and $x =_n z$. If there were some $y \in D$ with $y \sqsubseteq z$, but $y \not\sqsubseteq_n x$, it followed that $y \sqsubseteq_n z$ and hence that $y \sqsubseteq_n x$, which is impossible by the choice of y. Thus $z \in B_n(x) \cap \bigcup \{ D \backslash \uparrow\{y\} \mid y \not\sqsubseteq_n x \}$.

If, conversely, $z \in B_n(x) \cap \bigcup \{ D \backslash \uparrow\{y\} \mid y \not\sqsubseteq_n x \}$, we have that $x =_n z$, since otherwise we had that $z \in \uparrow\{y\}$, for some $y \in D$ with $y \not\sqsubseteq_n x$, namely $y = z$. Therefore $z \in C_n(x)$.

It follows that $\mu_{\bar{d}} \subseteq \lambda$, if $\tau_d \subseteq \sigma$.

Proposition 4.4 *Let D be a domain with approximation structure. Then the Lawson topology of the domain is coarser than the topology defined by the ultrametric associated with the canonical quasi-ultrametric of the approximation structure.*

Proof: Let X be one of the generating subsets of D generating the Lawson topology. Because $\sigma \subseteq \tau_d$, by Proposition 3.3, and $\tau_d \subseteq \mu_{\bar{d}}$, it suffices to consider the case that $X = D \backslash \uparrow\{y\}$, for some $y \in D$. Without restriction let $y \neq \bot$. Moreover, let $x \in X$ and m be the smallest i with $y \not\sqsubseteq_i x$. We will show that $C_m(x) \subseteq X$. Let to this end $z \in C_m(x)$. Then $z =_m x$ and $y \not\sqsubseteq z$, since otherwise we had $y \sqsubseteq_m z$ and thus $y \sqsubseteq_m x$, in contradiction to our choice of m. Therefore $z \in X$.

The above results show that $\mu_{\bar{d}} = \lambda$, if $\tau_d = \sigma$. The coincidence of these two topologies has been characterized in Theorem 3.9.

Theorem 4.5 *Let D be an algebraic domain with an approximation structure that satisfies the strong continuity axiom. Then the topology defined by the ultrametric associated with the canonical quasi-ultrametric of the approximation structure coincides with the Lawson topology of the domain.*

5 Some constructions

Classes of domains used in programming language semantics are usually closed under certain constructions which reflect the constructions used in building up larger programs and data types from smaller ones. Important examples are the construction of Cartesian products, function spaces and bilimits of ω-chains. The last construction is used in solving recursive domain equations [16].

In this section we will show how in these cases approximation structures on the newly constructed domains can be defined in terms of those on the components in a natural way. Moreover, it will be studied how the canonical quasi-metrics of the approximation structures thus obtained can be computed from the canonical quasi-metrics coming with the components.

We start with the construction of Cartesian products. Let to this end D and E be domains with bases Z^D and Z^E as well as approximation structures $([\cdot]_i^D)_{i\in\omega}$ and $([\cdot]_i^E)_{i\in\omega}$, respectively.

The *Cartesian product* $D \times E$ of D and E is the usual product of sets, augmented with the coordinatewise order. It is a domain again with basis $Z^D \times Z^E$. Note that the way-below relation between two pairs holds, exactly if it holds componentwise. For $(x, y) \in D \times E$ and $n \in \omega$ set

$$[(x, y)]_n^\times = ([x]_n^D, [y]_n^E).$$

Proposition 5.1 *$([\cdot]_i^\times)_{i\in\omega}$ is an approximation structure on $D \times E$.*

As is easily verified,

$$(x, y) \sqsubseteq_n (x', y') \Leftrightarrow x \sqsubseteq_n x' \wedge y \sqsubseteq_n y',$$

for $x, x' \in D$, $y, y' \in E$ and $n \in \omega$.

Now, let d^D, D^E and $d^\times$, respectively, be the canonical quasi-metrics of the approximation structures on D, E and $D \times E$. Moreover, let $(x, y), (x', y') \in D \times E$. Then $d^\times((x, y), (x', y')) = 0$, exactly if $d^D(x, x') = 0$ and $d^E(y, y') = 0$. Moreover, in case that $d^\times((x, y), (x', y')) \neq 0$, its value is determined by the minimal i such that $(x, y) \not\sqsubseteq_i (x', y')$. Let m be the smallest such i. Then $x \not\sqsubseteq_m x'$ or $y \not\sqsubseteq_m y'$. Since $(x, y) \sqsubseteq_j (x', y')$ and hence $x \sqsubseteq_j x'$ as well as $y \sqsubseteq_j y'$, for all $j < m$, it follows that $d^D(x, x'), d^E(y, y') \leq 2^{-m}$ and in at least one case the value is 2^{-m}.

Proposition 5.2 *For any $(x, y), (x', y') \in D \times E$*

$$d^\times((x, y), (x', y')) = \max\{d^D(x, x'), d^E(y, y')\}.$$

Let us next consider the function space construction. As is well known, in general, the function space $[D \to E]$ of two domains D and E is not a domain again. This is the case, however, if E is an L-domain, i.e., a domain in which every subset S bounded by an element z has a least upper bound $\bigsqcup^z S$ in the principal ideal $\downarrow\{z\}$ $(= \{\, y \in E \mid y \sqsubseteq z \,\})$. Then the function space has a basis of step functions.

Definition 5.3 Let D and E, respectively, be a domain and an L-domain.

1. For basic elements $u \in D$ and $v \in E$ the *single-step function* $(u \searrow v)\colon D \to E$ is defined by

$$(u \searrow v)(x) = \begin{cases} v & \text{if } u \ll x, \\ \bot_E & \text{otherwise.} \end{cases}$$

2. A *step function* is the relative join of a bounded finite collection of single-step functions.

Note that a finite family $(u_i \searrow v_i)$, $i = 1, \ldots, n$, of single-step functions is bounded by a continuous function f and has a least upper bound in $\downarrow\{f\}$, exactly if the set $\{ v_i \mid u_i \ll x \}$ is bounded by $f(x)$ for each $x \in D$. As follows from the next lemma [18], for domains D, E the collection of all step functions is a basis of $[D \to E]$.

Lemma 5.4 *Let D and E, respectively, be a domain and an L-domain with bases Z^D and Z^E. Moreover, let $f \in [D \to E]$, and for some finite index set I, let $u_i \in Z^D$ and $v_i \in Z^E$, for $i \in I$. Then the following two statements hold:*

1. $f = \bigsqcup \{ (u \searrow v) \mid v \ll f(u) \}$.
2. $\bigsqcup^f \{ (u_i \searrow v_i) \mid i \in I \} \ll f \;\Leftrightarrow\; (\forall i \in I) v_i \ll f(u_i)$.

Now, assume that D and E have approximation structures $([\cdot]_i^D)_{i\in\omega}$ and $([\cdot]_i^E)_{i\in\omega}$, respectively. For $f \in [D \to E]$, $x \in D$ and $i \in \omega$ set

$$[f]_i^{\to}(x) = [f(x)]_i^E.$$

Since $[\cdot]_i^E$ is continuous, $[f]_i^{\to} \in [D \to E]$.

Proposition 5.5 $([\cdot]_i^{\to})_{i\in\omega}$ *is an approximation structure on $[D \to E]$.*

The verification of the requirements in Definition 2.7 is straightforward. One takes advantage of the fact that the least upper bound of a directed function set can be computed argumentwise. The same is true for the next result, which is a consequence of Lemma 5.4.

Lemma 5.6 *Let $f, g \in [D \to E]$ and $n \in \omega$. Then*

$$f \sqsubseteq_n g \Leftrightarrow (\forall x \in D) f(x) \sqsubseteq_n g(x).$$

For the proof of the "if" part note that for $x \in D$ and $v \in Z_n^E$ with $v \ll f(x)$ there is some $u \in Z^D$ such that $u \ll x$ and $v \ll f(u)$. Moreover, a step function $\bigsqcup^f \{ (u_i \searrow v_i) \mid i \in I \}$ is in $[D \to E]_n$, just if $v_i \in Z_n^E$, for all $i \in I$.

Let $d^{\to}$ and d^E, respectively, be the canonical quasi-metrics of the just defined approximation structure on $[D \to E]$ and that on E.

Proposition 5.7 *For $f, g \in [D \to E]$*

$$d^{\to}(f, g) = \max \{ d^E(f(x), g(x)) \mid x \in D \}.$$

Because of the above lemma this result follows by the same kind of reasoning as in the product case.

Finally, let $\mathbf{DOMA^{ep}}$ be the category the objects of which are domains with approximation structure and the morphisms are embedding/projections which are both projection morphisms. Moreover, let $\Delta = \big(D^i, (e_i, p_i)\big)_{i\in\omega}$ be an ω-chain in this category and set

$$D^\infty = \{ t \in \textstyle\prod_n D_n \mid (\forall n \in \omega) t(n) = p_n(t(n+1)) \}.$$

Endowed with the componentwise defined order D^∞ is a domain (cf. [1, Theorem 3.3.11]).

Let $e_{nm} = e_{m-1} \circ \cdots \circ e_n$ for $n < m$, $e_{nn} = \mathrm{id}_{D^n}$, and $p_{mn} = p_m \circ \cdots \circ p_{n-1}$, for $m < n$. The canonical embedding $\mathrm{in}_n \in [D^n \to D^\infty]$ is given by

$$\mathrm{in}_n(x)(m) = \begin{cases} e_{nm}(x) & \text{if } n \le m, \\ p_{mn}(x) & \text{otherwise.} \end{cases}$$

If D^i comes with basis Z^i, for each $i \in \omega$, then $\bigcup_n \mathrm{in}_n(Z^n)$ is a basis of D^∞.

For $i \in \omega$ let $([\cdot]_j^{D^i})_{j\in\omega}$ be the given approximation structure on D^i. We will now define an approximation structure on D^∞. Let to this end $t \in D^\infty$ and $i, n \in \omega$. Set

$$[t]_i^\infty(n) = [t(n)]_i^{D^n}.$$

Since the projections p_i are projection morphisms, $[t]_i^\infty \in D^\infty$.

Proposition 5.8 *$([\cdot]_i^\infty)_{i\in\omega}$ is an approximation structure on D^∞.*

As is easily verified, D^∞ is the bilimit of the ω-chain Δ in $\mathbf{DOMA^{ep}}$.

Unfortunately, the way-below relation on D^∞ cannot be reduced to the way-below relation on the components as in the case of finite products. Therefore, we restrict ourselves to the case that the D_i are algebraic. Then D^∞ is algebraic as well. We have already seen that under this assumption, $x \sqsubseteq_n y$ exactly if $[x]_n \sqsubseteq y$. Since ordering and approximation structure on D^∞ are defined componentwise, we can proceed as in the other two cases to derive a relationship between the canonical quasi-metrics d^∞ and d^i, respectively, of the approximation structures on D^∞ and the D^i.

Proposition 5.9 *Let D_i be algebraic, for all $i \in \omega$, and $s, t \in D^\infty$. Then*

$$d^\infty(s,t) = \max\{\, d^i(s(i), t(i)) \mid i \in \omega \,\}.$$

6 Conclusion

Several approaches can be found in the literature to enrich domains used in the foundations of computing by weak metrical notions so as to provide a structure for the study of quantitative aspects of computing. In this note domains have been enriched by approximation structures which allow the definition of uniform levels of approximation. Most domains used in program semantics come equipped with this additional structure. In general, the ties between an approximation structure and the continuity structure of the domain are rather weak. Stronger requirements, called the weak and/or strong continuity axiom, have been discussed, which have deep links to topology.

As was shown, each approximation structure defines a quasi-ultrametric in a canonical way. The partial order induced by the quasi-metric agrees with the domain order. Its topology is finer than the Scott topology of the domain, in general. Both topologies coincide exactly if the strong continuity axiom holds, which is the case if and only if the subdomains defining the uniform approximation levels contain only compact elements. This is a strong requirement which forces the subdomains and the domain to be algebraic.

For a special class of algebraic domains a quasi-metric has been introduced by Smyth in the same way. The just mentioned requirement for the equivalence of the two topologies improves his condition.

By symmetrizing a quasi-metric one obtains a metric. In the case of the canonical quasi-metric of an approximation structure the distance of two points in this metric is determined by the maximal level up to which the approximations of the two points agree. The domain is complete in this metric, if the weak continuity axiom is satisfied. Moreover, the metrical topology is finer than the Lawson topology of the domain. Both topologies coincide if the strong continuity axiom holds.

Classes of domains used in studying the meaning of programs have to be closed under constructions reflecting similar constructions on the syntactic level. For three important constructions it has been shown that approximation structures on the newly constructed domains can be defined from those on the components in a natural way. Moreover, the distance of points in the new domains with respect to the canonical quasi-metric of these approximation structures can easily be computed from distances of points in the component domains.

Acknowledgement

This paper has been written while the author was visiting the Department of Mathematics and Statistics of the University of Canterbury, Christchurch, New Zealand. He wants to thank the Department for providing such excellent working conditions.

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