# Domains under bitopological glasses

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22nd Summer Conference on Topology and Its Applications,
Castellón 2007

1. A short domain-theory fresh-up

## Definition

- A poset $(D, \sqsubseteq)$ with least element $\perp$ is called domain if every directed set $S \subseteq D$ has a least upper bound $\bigsqcup S$ in $D$.
- $D$ is bounded-complete if every bounded subset has a least upper bound in $D$.
- Let $x, y \in D$. Then $x$ approximates $y$ ($x \ll y$) if for all directed sets $S \subseteq D$,

$$y \sqsubseteq \bigsqcup S \quad \Rightarrow \quad (\exists s \in S) x \sqsubseteq s.$$

- Element $x \in D$ is compact if $x \ll x$.
- $B \subseteq D$ is a basis for $D$ if for every $x \in D$, $B_x = \{ z \in B \mid z \ll x \}$ is directed with least upper bound $x$.

### Remark

*$B$ basis for $D$* $\quad\Rightarrow\quad B \supseteq K_D(= \{\,$*compact elements*$\,\})$

### Definition

- $D$ is continuous if $D$ has a basis.
- $D$ is algebraic if $K_D$ is a basis for $D$.

### Remark

*In a continuous domain $\ll$ has the interpolation property:*

$$M \subseteq_f D \,\&\, M \ll y \quad\Rightarrow\quad (\exists z \in B) M \ll z \ll y.$$

Let $D$ be a continuous domain with basis $B$. For $x \in D$ and $z \in B$ set

- $\uparrow x = \{ y \in D \mid x \sqsubseteq y \}$,
- $\twoheaduparrow z = \{ y \in D \mid z \ll y \}$.

## Definition

- The Scott topology $\sigma$ on $D$ is generated by the sets $\twoheaduparrow z$ with $z \in B$.
- The lower topology $\omega$ on $D$ has all principal filters $\uparrow x$ with $x \in D$ as a subbasis for the closed sets.
- The Lawson topology $\lambda$ on $D$ is the join of both the Scott and the lower topology.

For a topology $\tau$ let $\leq_\tau$ denote its specialization order.

## Remark

- $\leq_\sigma = \sqsubseteq$,
- $\leq_\omega = \sqsubseteq^{-1}$.
- $(D, \lambda)$ *is Hausdorff.*

## 2. A theorem and its appropriate generalization

### Theorem

*A countably based continuous domain with its Lawson topology is completely metrizable.*

In order to get rid of the countability assumption one could try to prove:

### Claim

*A continuous domain with its Lawson topology is completely uniformizable.*

But is this the generalization one is really looking for?

In applications one is mainly interested in the Scott topology. Therefore, a much more informative generalization would be

## Claim

*For every continuous domain $D$ a quasi-uniformity $\mathcal{U}$ can be given such that*

- $\tau_{\mathcal{U}} = \sigma$
- $\tau_{\mathcal{U}^{-1}} = \omega$
- $(D,\mathcal{U}^*)$ *is complete, i.e.* $(D,\mathcal{U})$ *is bicomplete.*

Here,

- $\tau_{\mathcal{U}}$ denotes the topology induced by $\mathcal{U}$,
- $\mathcal{U}^{-1}$ is the converse of $\mathcal{U}$ and
- $\mathcal{U}^*$ is the uniformity generated by $\mathcal{U}$.

The advantage of this generalization is that in the countably based case one would automatically obtain a quasi-metric the topology of which is compatible with the Scott topology. In certain cases one would even obtain a partial metric. This is what one is really looking for in applications.

## 3. The algebraic case

Let $D$ be an algebraic domain. Then $K_D$, the set of its compact elements is a basis of $D$ and the collection of all principal filters $\uparrow z$, for $z \in K_D$, is a base of the Scott topology on $D$.

For $z \in K_D$ set

$$P_z = \{ (x, y) \in K^2 \mid z \sqsubseteq x \Rightarrow z \sqsubseteq y \}.$$

Then $\{ P_z \mid z \in K_D \}$ is a subbasis of a quasi-uniformity $\mathcal{P}$ on $D$.

Note that

$$P_z^{-1} = \{ (y, x) \in K^2 \mid z \not\sqsubseteq y \Rightarrow z \not\sqsubseteq x \}$$

Therefore, we have for $x, y \in D$ and $z \in K_D$ that

$$P_z[x] = \{\, y \mid (x, y) \in P_z \,\} = \begin{cases} \uparrow z & \text{if } x \in \uparrow z, \\ D & \text{otherwise,} \end{cases}$$

and

$$P_z^{-1}[y] = \begin{cases} D \setminus \uparrow z & \text{if } y \notin \uparrow z, \\ D & \text{otherwise.} \end{cases}$$

Proposition

*Let $D$ be an algebraic domain. Then the following hold:*

1. $\tau_{\mathcal{P}} = \sigma$.
2. $\tau_{\mathcal{P}^{-1}} = \omega$.
3. $\mathcal{P}$ *is the coarsest quasi-uniformity on $D$ compatible with $\sigma$.*
4. $\mathcal{P}$ *is totally bounded.*

## 4. The general (continuous) case

Let $D$ be continuous domain with basis $B$. For $z, z' \in B$ with $z \ll z'$ define

$$K_{z,z'} = \{ (x, y) \in K^2 \mid z' \ll x \Rightarrow z \ll y \}.$$

Then

$$\{ K_{z,z'} \mid z, z' \in B \text{ with } z' \ll z \}$$

is a subbasis of the Künzi-Brümmer quasi-uniformity $\mathcal{K}$ on $D$.

**Lemma**

*If $D$ is algebraic, then $\mathcal{K} = \mathcal{P}$.*

### Proposition

*Let $D$ be a continuous domain. Then the following hold:*

1. $\tau_{\mathcal{K}} = \sigma$.
2. $\tau_{\mathcal{K}^{-1}} = \omega$.
3. $\mathcal{K}$ *is the coarsest quasi-uniformity on $D$ compatible with* $\sigma$.
4. $\mathcal{K}$ *is totally bounded.*

The first three statements are a special case of a more general result of J. Lawson.

## 5. Bicompleteness

### Definition

A domain is coherent if the intersection of two Scott-compact saturated sets is again Scott-compact.

### Theorem

*Let $D$ be a coherent continuous domain. Then $(D, \mathcal{K})$ is bicomplete.*

### Corollary

$(D, \tau_{\mathcal{K}^*})$ *is compact.*

### Remark

- *As we have seen above,* $\tau_{\mathcal{K}^*}$ *coincides with the Lawson topology on D.*
- *It is well known that*

$$(D, \lambda) \text{ is compact} \Leftrightarrow D \text{ is coherent.}$$

*Thus, in the above theorem we cannot dispense with coherence.*

Remember the initial

### Claim

*For every continuous domain $D$ there is a quasi-uniformity $\mathcal{U}$ such that*

- $\tau_{\mathcal{U}} = \sigma$
- $\tau_{\mathcal{U}^{-1}} = \omega$
- $(D, \mathcal{U})$ *is bicomplete.*

We do not know whether this claim is true. If so, we would have

- $\mathcal{K} \subsetneq \mathcal{U}$ and
- $\mathcal{U}$ is not totally bounded.

## 6. The Urysohn construction

Let $D$ be a continuous domain with Basis $B$ and $\mathbb{D}$ be the set of dyadic rationals in the interval $[0, 1]$.

By repeated interpolation we can, for any pair $(z, z') \in B^2$ with $z \ll z'$, construct a family $\langle e_p \rangle_{p \in \mathbb{D}}$ such that

- $e_0 = z', \qquad e_1 = z$
- $e_q \ll e_p$, for all $p, q, \in \mathbb{D}$ with $p < q$.

Define $f_z^{z'} : D \to [0, 1]$ by

$$f_z^{z'}(x) = \inf \{\, p \in \mathbb{D} \mid e_p \ll x \,\}.$$

**Lemma**

- $f_z^{z'}$ *is $\sigma$-upper and $\omega$-lower semicontinuous.*
- $f_z^{z'}(\uparrow z') = \{0\}$.
- $f_z^{z'}(D \setminus \twoheaduparrow z) = \{1\}$.

For $z, z' \in B$ with $z \ll z'$ and $m > 0$ let

$$U_{z,z',m} = \{\, (x,y) \in D^2 \mid f_z^{z'}(y) - f_z^{z'}(x) < 2^{-m} \,\}.$$

## Proposition

*The collection*

$$\{\, U_{z,z'm} \mid m > 0 \text{ and } z, z' \in B \text{ with } z \ll z' \,\}$$

*is a subbasis of a quasi-uniformity $\mathcal{U}$ on $D$ such that*

$$\mathcal{U} = \mathcal{K}.$$

Assume now that $D$ is countably based. Let

$$(z_0, z_0'), (z_1, z_1'), \ldots$$

be an enumeration of all pairs $(z, z') \in B^2$ with $z \ll z'$. Set

$$f_i = f_{z_i}^{z_i'}, \qquad U_{i,m} = U_{z_i, z_i', m}$$

and define for $x, y \in D$

$$\delta(x, y) = \sum_{i=0}^{\infty} 2^{-(i+1)} \max\{0, f_i(y) - f_i(x)\}.$$

## Lemma

- *$\delta$ is a quasi-metric on $D$.*
- $\mathcal{V}_\delta = \mathcal{U}$

Here, $\mathcal{V}_\delta$ is the quasi-uniformity induced by $\delta$.

## Definition

A quasi-metric $d$ on a continuous domain $D$ is weakly weighted if there is measurement $|\cdot|\colon D \to [0,1]^{\mathrm{op}}$ such that for $x, y \in D$,

$$x \sqsubseteq y \Rightarrow |y| + d(y,x) \leq |x|.$$

## Conjecture (Smyth)

*For any countably based continuous domain $D$ there is a weakly weighted quasi-metric $d$ with measurement $|\cdot|$ such that*

- $|x| = 0 \Leftrightarrow x$ *is constructively maximal,*
- $d$ *induces the Scott topology,*
- $d^*$ *induces the Lawson topology.*

Here, $d^*$ is the metric associated with $d$.

### Definition

Let $D$ be a domain. We say $x, y \in D$ lie apart from each other and write $x \sharp y$, if both can be separated by disjoint Scott open sets.

Let $\langle e^i_p \rangle_{p \in \mathbb{D}}$ be the family of interpolating basic elements constructed with respect to the pair $(z_i, z'_i)$. Set

$$|x| = 1 - \sum_{i=0}^{\infty} 2^{-(i+1)}[\sup \{\, p \in \mathbb{D} \mid x \sharp e^i_p \,\} + (1 - f_i(x))].$$

Then $|\cdot|$ is a measurement so that

$$|x| = 0 \Leftrightarrow x \text{ is constructively maximal.}$$

Moreover, it turns $\delta$ into a weakly weighted quasi-metric.

### Theorem

*Let $D$ be a countably based continuous domain. Then $\delta$ is a weakly weighted quasi-metric with measurement $|\cdot|$ such that*

- $|x| = 0 \Leftrightarrow x$ *is constructively maximal.*
- $\delta$ *induces the Scott topology,*
- $\delta^*$ *induces the Lawson topology.*

## 7. The Smyth partial metric

### Definition

A partial metric on a set $X$ is a map $p\colon X \times X \to [0,\infty)$ satisfying

1. $p(x,y) = p(y,x)$,
2. $[p(x,y) = p(x,x) = p(y,y)] \Rightarrow x = y$,
3. $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$,
4. $p(x,x) \leq p(x,y)$.

It is well known that the notions $T_0$ weighted quasi-metric and partial metric are equivalent, via the assignment

$$p(x,y) = w(x) + d(x,y)$$

and its inverse

$$w(x) = p(x,x), d(x,y) = p(x,y) - p(x,x).$$

The topology induced by a partial metric is the one induced by the associated quasi-metric.

Let $D$ be a countably based continuous domain. For $x, y \in D$ set

$$\rho(x,y) = 1 - \sum_{i=0}^{\infty} 2^{-(i+1)} [\sup \{ p \mid e^i_p \sharp x, y \} + \sup \{ 1 - p \mid e^i_p \ll x, y \}].$$

## Theorem (Smyth)

*The distance function $\rho$ is a partial metric which induces the Scott topology.*

Let $D$ be a continuous domain with (not necessarily countable) basis $B$. For $z, z' \in B$ with $z \ll z'$ set

$$S_{z,z'} = \{\, (x, y) \in D^2 \mid [z' \ll x \Rightarrow z \ll y] \wedge [z \sharp x \Rightarrow z' \sharp y] \,\}.$$

Then the collection of all such relations $S_{z,z'}$ is a subbasis of a quasi-uniformity $\mathcal{S}$ on $D$.

**Proposition**

$\mathcal{K} \subseteq \mathcal{S}$.

**Corollary**

$\sigma \subseteq \tau_{\mathcal{S}}, \qquad \omega \subseteq \tau_{\mathcal{S}^{-1}}.$

### Proposition

$\tau_{\mathcal{S}} = \sigma$.

### Remark

*I do not know whether*

- $\tau_{\mathcal{S}^{-1}} = \omega$,
- $(D, \mathcal{S})$ *is bicomplete?*

### Proposition

*If $B$ is countable then*

$$\mathcal{S} = \mathcal{V}_\rho$$

*where $\mathcal{V}_\rho$ is the quasi-uniformity induced by the Smyth partial metric.*