Information Systems Revisited: The General Continuous Case*

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Abstract

In this paper a new notion of continuous information system is introduced. It is shown that the information systems of this kind generate exactly the continuous domains. The new information systems are of the same logic-oriented style as the information systems first introduced by Scott in 1982: they consist of a set of tokens, a consistency predicate and an entailment relation satisfying a set of natural axioms.

It is shown that continuous information systems are closely related to abstract bases. Indeed, both categories are equivalent. Since it is known that the categories of abstract bases and/or continuous domains are equivalent, it follows that the category of continuous information systems is also equivalent to that of continuous domains.

In applications mostly subclasses of continuous domains are considered. The domains have e.g. to be pointed, algebraic, bounded-complete or FS. Conditions are presented that when fulfilled by an continuous information system force the generated domain to belong to the required subclass. In most cases the requirements are not only sufficient but also necessary.

1 Introduction

In 1982, in his seminal paper [14] Dana Scott introduced information systems and approximable mappings as a logic-oriented approach to denotational semantics of programming languages. Larsen and Winskel [12] then showed that the category of these structures is equivalent to the category of Scott domains with Scott continuous functions and explained how domain equations can be solved exactly (not just up to isomorphism) by using information systems. In the sequel, the new concept was used in many domain-theoretic studies (cf. e.g. [13]). Moreover, similar structures were introduced in order to characterize various other kinds of domains that have turned out important in studies of computation [17, 4, 5, 7, 19, 21, 22].

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Nearly all these formalisms represent only subclasses of algebraic domains. Hoofman [11] and Vickers [16] were the first to present a generalization of information systems to the continuous case. Hoofman's continuous information systems are in the spirit of Scott's information systems, which means they consist of a set of tokens representing bits of information and an entailment relation between finite sets of tokens, but they capture only the bounded-complete continuous domains. Vickers's approach, however, is more general and allows to represent all continuous domains, but is not Scott-style: he considers transitive dense relations. Bedregal [3] contains a Scott-style approach that claims to capture exactly the continuous domains. Unfortunately, this paper contains no proofs but various errors. It seems that the requirements used there are too weak.

In the present paper we introduce a new notion of continuous information systems. The definition uses several conditions that appear already in Scott's and/or Hoofman's approach. Moreover, a new requirement is added. As is shown, these information systems generate exactly the continuous (not necessarily pointed) domains, or more mathematically: the categories of both structures are equivalent.

As is well known, up to isomorphism continuous domains are completely determined by their bases and the restriction of the approximation relation to them. Important domain constructions can be reduced to analogous constructions on the bases. In the same way Scott continuous functions can be recovered from their restriction to the bases of the domains involved. Bases of continuous domains together with the approximation relations are examples of what is called abstract basis. The category of these structures is known to be equivalent to the category of continuous domains.

In this paper it is analyzed how abstract bases and continuous information systems correspond to each other. As we will see, both structures are closely related. Abstract bases form a bridge between continuous information systems and domains. It will be shown that the categories of continuous information systems and/or abstract bases are equivalent as well. The above mentioned equivalence of continuous information systems and continuous domains is a consequence of both results.

The paper is organized as follows: Section 2 contains the necessary definitions and results about domains and abstract bases. In Section 3 continuous information systems are introduced and their relationship with abstract bases is analyzed. By this we obtain that the states of a continuous information system form a continuous domain with respect to set inclusion. The analysis defines the object part of the functors that establish the equivalence between both categories. The morphism part is studied in Section 4. The appropriate morphisms for information systems are approximable mappings. They correspond to approximable relations in a natural way. This leads to the just mentioned equivalence.

As has already been said, a continuous information system generates a—not necessarily pointed—continuous domain. In most computer science application domains are required to be pointed or satisfy even more restrictive requirements. In Section 5 conditions are derived which when fulfilled by an information system force the generated domain to satisfy the additional requirements. In most cases the conditions are not only sufficient but also necessary.

Final remarks can be found in Section 6.

2 Domains and abstract bases

For any set A, we write $X \subseteq_{\text{fin}} A$ to mean that X is finite subset of A.

Let $(D, \sqsubseteq)$ be a poset. D is *pointed* if it contains a least element $\bot$. A subset S of D is called *consistent* if it has an upper bound. S is *directed*, if it is nonempty and every pair of elements in S has an upper bound in S. D is a *directed-complete partial order* (*dcpo*), if every directed subset S of D has a least upper bound $\bigsqcup S$ in D, and D is *bounded-complete* if every consistent subset has a least upper bound in D.

Assume that x, y are elements of a dcpo D. Then x is said to *approximate* y, written $x \ll y$, if for any directed subset S of D the relation $y \sqsubseteq \bigsqcup S$ always implies the existence of some $u \in S$ with $x \sqsubseteq u$. Moreover, x is *compact* if $x \ll x$. A subset B of D is a *basis* of D, if for each $x \in D$ the set $\{ u \in B \mid u \ll x \}$ contains a directed subset with least upper bound x. Note that the set of all compact elements of D is included in every basis of D. A dcpo D is said to be *continuous* (or a *continuous domain*) if it has a basis and it is called *algebraic* (or an *algebraic domain*) if its compact elements form a basis. Standard references for domain theory and its applications are [9, 8, 1, 15, 2, 6].

Definition 2.1. Let D and D' be posets. A function $f\colon D \to D'$ is *Scott continuous* if for any directed subset S of D with a least upper bound,

$$f(\bigsqcup S) = \bigsqcup f(S).$$

The above definition demands that $f(\bigsqcup S)$ is the least upper bound of the set $f(S)$. As a consequence one obtains that f is also monotone. Under the stronger assumption that D and D' are dcpo's one needs not care about the existence of directed least upper bounds. Instead, one requires f to be monotone and preserve directed least upper bounds.

With respect to the pointwise order the set of all Scott continuous functions between two dcpo's D and D' is a dcpo again. Observe that it need not be continuous even if D and D' are.

Definition 2.2. Let D be a pointed dcpo.

1. A Scott continuous function$f\colon D \to D$ is said to be *finitely separated* from the identity id_D on D, if there exists a finite subset M_f of D such that for any $X \in D$ there is $m \in M_f$ with $f(x) \sqsubseteq m \sqsubseteq x$. One speaks of *strong separation* if for each $x \in D$ there are elements $m, m' \in M_f$ with $f(x) \sqsubseteq m \ll m' \sqsubseteq x$.

2. D is called an *FS-domain* if there is a directed collection $(f_i)_{i \in I}$ of Scott continuous functions on D, each finitely separated from id_D, with the identity function as their least upper bound.

FS-domains are continuous: the set of all points $f_i(m)$ with $m \in M_{f_i}$ and $i \in I$ is a basis. Algebraic FS-domains are also called *bifinite* domains.

By [10, Lemma 2], for every f_i, the function $f_i \circ f_i$ is strongly finitely separated from Id_D. Since also $\bigsqcup_{i \in I}(f_i \circ f_i) = \mathrm{id}_D$, we can assume that an FS-domain comes with a strongly finitely separating function family and hence that the separating sets M_{f_i} contain only base elements.

As is well known, if D is a continuous domain D with basis B, then B with the restriction of the approximation relation to it is an abstract basis.

Definition 2.3. An *abstract basis* is given by a set B together with a transitive relation $\prec$ on B, such that the following interpolation law holds for all elements u and finite subsets M of B:

$$M \prec u \Rightarrow (\exists v \in B) M \prec v \prec u.$$

Here $M \prec u$ means that $m \prec u$, for any $m \in M$.

Recall that for a transitive relation $\prec$ on a set B, a subset S of B is called *dense* if for all $u, v \in B$ with $u \prec v$ there is some $w \in S$ such that $u \prec w \prec v$. If B is dense, one also says that the relation $\prec$ is dense. For abstract bases $(B, \prec)$ we therefore have that $\prec$ is a dense transitive relation.

Conversely, if we are given an abstract basis then a continuous domain can be derived from it by considering all round ideals and ordering them with respect to set inclusion.

Definition 2.4. Let $(B, \prec)$ be an abstract basis. A nonempty subset I of B is a *round ideal* if

1. $(\forall u \in B)(\forall v \in I)(u \prec v \Rightarrow u \in I)$,
2. $(\forall u, v \in I)(\exists w \in I)(u \prec w \wedge v \prec w)$.

The set $\mathrm{RI}(B)$ of all round ideals of B ordered by set inclusion is called the *round ideal completion* of B. Let $\mathrm{pi}_B \colon B \to \mathrm{RI}(B)$ be the embedding which maps $u \in B$ to the principal ideal $\downarrow_{\prec} u$.

Proposition 2.5 (cf. [1, 18])**.** *Let $(B, \prec)$ be an abstract basis. Then the following two statements hold:*

1. *The round ideal completion $\mathrm{RI}(B)$ is a continuous domain with basis $\mathrm{pi}_B(B)$.*
2. *$\mathrm{RI}(B)$ is algebraic exactly if the subset of all reflexive elements is dense, i.e., if for all $u, v \in B$ with $u \prec v$ there is some $w \in B$ so that $u \prec w \prec w \prec v$.*

Proposition 2.6. *Let D be a continuous domain with basis B. Then B with the restriction of the approximation relation to B is an abstract basis such that the corresponding round ideal completion is isomorphic to D.*

The isomorphism is given by the two functions $\mathrm{il}_D \colon D \to \mathrm{RI}(B)$ and $\mathrm{sp}_D \colon \mathrm{RI}(B) \to D$ which are inverse to each other:

$$\mathrm{il}_D(x) = \{\, u \in B \mid u \ll x \,\}, \qquad \mathrm{sp}_D(I) = \bigsqcup I.$$

The usual morphisms between abstract bases are approximable relations.

Definition 2.7. A relation R between abstract bases B and C is called *approximable* if the following four conditions hold for all $u, u' \in B$ and $v, v' \in C$ and all finite subsets M of C:

1. $(uRv \wedge v \succ_C v') \Rightarrow uRv'$,
2. $(\forall v'' \in M) uRv'' \Rightarrow (\exists w \in C)(uRw \wedge w \succ_C M)$,
3. $(u' \succ_B u \wedge uRv) \Rightarrow u'Rv$,
4. $uRv \Rightarrow (\exists w \in B)(u \succ_B w \wedge wRv)$.

Obviously, for every abstract basis $(B, \prec)$, the relation $\prec$ is approximable. It is the identity morphism IR_B on B.

Let D and D', respectively, are continuous domains with bases of B and B'. Then for any Scott continuous function $f\colon D \to D'$, its *skeleton*

$$\mathrm{sktn}(f) = \{\, (u, v) \in B \times B' \mid v \ll' f(u) \,\}$$

is an approximable relation between $(B, \ll)$ and $(B', \ll')$. Conversely, if $(B, \prec)$ and $(B', \prec')$ are abstract bases and R is an approximable relation between them, then by setting

$$f_R(I) = \{\, v \in B' \mid (\exists u \in I)(u, v) \in R \,\}$$

one obtains a Scott continuous function $f_R\colon \mathrm{RI}(B) \to \mathrm{RI}(B')$. Moreover,

$$f = \mathrm{sp}_{D'} \circ f_{\mathrm{sktn}(f)} \circ \mathrm{il}_D \qquad \text{and} \qquad R = (\mathrm{pi}_B \times \mathrm{pi}_{B'})^{-1}(\mathrm{sktn}(f_R)).$$

Theorem 2.8. *The category* **AB** *of abstract bases and approximable relations is equivalent to the category* **CDOM** *of continuous domains and Scott continuous functions.*

3 Continuous information systems

In this section we introduce our notion of continuous information system and study the relationship of these structures with continuous domains.

Definition 3.1. Let A be a set, Con a collection of finite subsets of A and $\vdash \subseteq \mathrm{Con} \times A$. Then $(A, \mathrm{Con}, \vdash)$ is a *continuous information system* if the following six conditions hold for all sets $X, Y \in \mathrm{Con}$, elements $a \in A$ and nonempty finite subsets F of A:

1. $\{a\} \in \mathrm{Con}$,
2. $X \vdash a \Rightarrow X \cup \{a\} \in \mathrm{Con}$,
3. $(Y \supseteq X \wedge X \vdash a) \Rightarrow Y \vdash a$,

and, defining $X \vdash Y$ to mean that $X \vdash b$, for all $b \in Y$,

4. $X \vdash Y \wedge Y \vdash a \Rightarrow X \vdash a$,
5. $X \vdash a \Rightarrow (\exists Z \in \mathrm{Con})(X \vdash Z \wedge Z \vdash a)$,
6. $X \vdash F \Rightarrow (\exists Z \in \mathrm{Con})(Z \supseteq F \wedge X \vdash Z)$.

If $(A, \mathrm{Con}, \vdash)$ is a continuous information system then the elements of A are usually called *tokens*, the sets in Con *consistent* and the relation $\vdash$ *entailment relation*. Tokens should be thought of as atomic propositions giving information about data and consistent sets as representing consistent finite conjunctions of such propositions. The entailment relation then tells us which propositions are derivable from what.

As we will see in this section, continuous information systems allow the generation of *all* continuous domains. Hoofman [11] introduced a similar structure, also called continuous information system, that captures exactly the bounded-complete continuous domains. Conditions 3.1(1) and (3-5) are used by him as well[1]. Observe that Condition 3.1(2) need not hold in

[1] Note that [11, Definition 19 (5)] should read as $(Y \in \mathrm{Con}_A \wedge X \subseteq Y \wedge X \vdash_A a) \Rightarrow Y \vdash_A a$.

his setting [11, Example 3.6]. As is easily verified, every continuous information system in our sense satisfying the additional requirement that for all finite subsets Y of A and all $X \in \mathrm{Con}$,

$$(Y \subseteq X \wedge X \in \mathrm{Con}) \Rightarrow Y \in \mathrm{Con}$$

is a continuous information systems in Hoofman's sense.

Note that Conditions 3.1(1), (2) and (4) have already been used by Scott [14].

As we will see next, Conditions 3.1(5) and (6) can be replaced by a single requirement.

Proposition 3.2. *Let A be a nonempty set,* Con *be a collection of finite subsets of A and $\vdash \subseteq \mathrm{Con} \times A$ such that Conditions 3.1(2, 3) hold. Then Requirements 3.1(5, 6) together are equivalent to the following statement:*

$$(\forall X \in \mathrm{Con})(\forall F \subseteq_{\mathrm{fin}} A)[X \vdash F \Rightarrow (\exists Z \in \mathrm{Con})(X \vdash Z \wedge Z \vdash F)].$$

Proof. Assume that Conditions 3.1(5, 6) hold. Furthermore, suppose that $X \vdash F$ and let $a \in F$. Then, by 3.1(5), there is some set $Z_a \in \mathrm{Con}$ with $X \vdash Z_a$ and $Z_a \vdash a$. With 3.1(6) it follows that there is some $Z \in \mathrm{Con}$ with $Z \supseteq \bigcup\{ Z_a \mid a \in F \}$ and $X \vdash Z$. Since $Z_a \vdash a$, we also have that $Z \vdash a$, by 3.1(3). Thus, $X \vdash Z$ and $Z \vdash F$.

For the converse implication we only have to show that Condition 3.1(6) holds. Assume that $X \vdash F$. Then there is some $Z \in \mathrm{Con}$ with $X \vdash Z$ and $Z \vdash F$. Set $Y = Z \cup F$. Then $Y \in \mathrm{Con}$, by 3.1(2), and $X \vdash Y$. □

For $X, Y \in \mathrm{Con}$ let $X < Y$ if $Y \vdash X$. Then $<$ is a transitive dense relation on Con, which shows that $(\mathrm{Con}, <)$ is a continuous information systems in the sense of Vickers [16]. These information systems capture exactly the continuous domains.

When we say that a continuous information system generates a continuous domain, we mean that the data (the *states*) that are uniquely described by certain sets of elementary propositions or tokens form a domain with respect to set inclusion.

Definition 3.3 (cf. [3])**.** Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. A subset x of A is a *state* of $(A, \mathrm{Con}, \vdash)$ if the next three conditions hold:

1. $(\forall F \subseteq_{\mathrm{fin}} x)(\exists Y \in \mathrm{Con})(F \subseteq Y \wedge Y \subseteq x)$,
2. $(\forall X \in \mathrm{Con})(\forall a \in A)(X \subseteq x \wedge X \vdash a \Rightarrow a \in x)$,
3. $(\forall a \in x)(\exists X \in \mathrm{Con})(X \subseteq x \wedge X \vdash a)$.

It follows that states are subsets of tokens that are *finitely consistent* (1) and *closed under entailment* (2). Furthermore, each token in a state is *derivable* (3), i.e., for each token there is a finite subset of the state that entails the token.

With respect to set inclusion the states of A form a partially ordered set which we denote by $|A|$.

Similarly to Proposition 3.2 the requirement that $F \subseteq Y$ in 3.3(1) can be replaced by an entailment condition.

Proposition 3.4. *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system and x be a subset of A such that Condition 3.3(3) holds. Then Requirement 3.3(1) is equivalent to the following statement:*

$$(\forall F \subseteq_{\mathrm{fin}} x)(\exists Y \in \mathrm{Con})(Y \subseteq x \wedge Y \vdash F).$$

Proof. Assume that Condition 3.3(1) holds, let x be a subset of A and F a finite subset of x. Then, by 3.3(3), there is some consistent subset Z_a of x with $Z_a \vdash a$, for every $a \in F$. Let $Z = \bigcup\{ Z_a \mid a \in F \}$. Then Z is a finite subset of x and hence, by 3.3(1), there is some consistent superset Y of Z that is still contained in x. As $Y \supseteq Z_a$, it follows with 3.1(3) that $Y \vdash a$, for every $a \in F$.

For the converse implication, let F be again a finite subset of x. Then there is some consistent subset Y of x with $Y \vdash F$. Set $Z = Y \cup F$. As F is finite, it follows with 3.1(2) and (3) that Z is consistent. Moreover, $F \subseteq Z \subseteq x$. $\square$

As we will see now, there is a close connection between continuous information systems and abstract bases such that states and round ideals correspond to each other.

Let $(B, \prec)$ be an abstract basis and for $u, v \in B$, let $u \preceq v$ if $u \prec v$ or $u = v$. Set

$$\mathrm{Con} = \{ X \subseteq_{\mathrm{fin}} B \mid X \text{ directed with respect to } \preceq \}$$

and for $X \in \mathrm{Con}$ and $a \in B$ define

$$X \vdash a \Leftrightarrow (\exists c \in X) a \prec c$$

Lemma 3.5. $\mathcal{C}(B) = (B, \mathrm{Con}, \vdash)$ *is a continuous information system.*

Proof. As is readily seen, Conditions 3.1(1-5) hold. For the verification of Condition 3.1(6) let $X \in \mathrm{Con}$ and F be a finite nonempty subset of B so that $X \vdash F$. Then there is some $c_a \in X$ with $a \prec c_a$, for every $a \in F$. Since X is directed with respect to $\preceq$, there is moreover some $d \in X$ with $c_a \preceq d$, for all $a \in F$. It follows that $F \prec d$. By the interpolation law there is hence some $e \in B$ with $F \prec e \prec d$. Set $Z = F \cup \{e\}$. Then $Z \in \mathrm{Con}$, $Z \supseteq F$, and $X \vdash Z$. $\square$

Lemma 3.6. $\mathrm{RI}(B) = |\mathcal{C}(B)|$.

Proof. Let $I \in \mathrm{RI}(B)$. We need to show that I is a state of $\mathcal{C}(B)$. Let to this end F be a finite subset of I. If F is empty, set $Y = \{a\}$, for some $a \in I$. Otherwise, assume that $F = \{u_1, \ldots, u_r\}$. Then we can inductively find elements $v_1, \ldots, v_r \in I$ with $v_1 = u_1$ and $v_{i-1}, u_i \prec v_i$, for $i = 2, \ldots, r$. Set $Y = F \cup \{v_2, \ldots, v_r\}$ in this case. Then we have in both cases that Y is a consistent subset of I.

Next assume that X is a subset of I and $u \in B$ with $X \vdash u$. Then $u \prec v$, for some $v \in X$. Thus $v \in I$ and hence also $u \in I$.

Finally, let $u \in I$. Since I is a round ideal, there is some $v \in I$ with $u \prec v$. It follows that $\{v\} \in \mathrm{Con}$, $\{v\} \subseteq I$, and $\{v\} \vdash u$.

Now, let conversely x be a state of $\mathcal{C}(B)$. We have to verify that x is a round ideal. Assume that $v \in x$ and $u \in B$ with $u \prec v$. Then $\{v\} \in \mathrm{Con}$ and $\{v\} \vdash u$. Hence $u \in x$, by 3.3(2).

For Condition 2.4(2) let $u, v \in x$. By 3.3(3) there are finite consistent subsets U and V of x with $U \vdash u$ and $V \vdash v$. With 3.3(1) it follows that there is a consistent subset Z of x which contains both U and V. Hence, $Z \vdash \{u, v\}$, by 3.1(3). Since Z is directed with respect to $\preceq$, we obtain that $u, v \prec z$, for some $z \in Z$. $\square$

By Proposition 2.5(1) it follows that $|\mathcal{C}(B)|$ is a continuous domain.

Next we will show that conversely also every continuous information system $(A, \mathrm{Con}, \vdash)$ generates an abstract basis: For $X, Y \in \mathrm{Con}$ set

$$Y \prec X \Leftrightarrow X \vdash Y.$$

Lemma 3.7. *$\mathcal{B}(A) = (\mathrm{Con}, \prec)$ is an abstract basis.*

Proof. By Condition 3.1(4), the relation $\prec$ is transitive. To see that it also satisfies the interpolation law, let $V_1, \ldots, V_r, X \in \mathrm{Con}$ with $V_i \prec X$, for $i = 1, \ldots, r$. Then we have that $X \vdash V_i$, for each such i, and hence that $X \vdash \bigcup\{\, V_i \mid 1 \leq i \leq r \,\}$. Set $V = \bigcup\{\, V_i \mid 1 \leq i \leq r \,\}$. With Proposition 3.2 it follows that there is some $Z \in \mathrm{Con}$ with $X \vdash Z$ and $Z \vdash V$. In particular we have that $Z \vdash V_i$, for all $1 \leq i \leq r$. Thus, $V_i \prec Z \prec X$, for $i = 1, \ldots, r$. □

Lemma 3.8. *1. For $x \in |A|$, $\{\, X \in \mathrm{Con} \mid X \subseteq x \,\} \in \mathrm{RI}(\mathcal{B}(A))$.*

2. For $I \in \mathrm{RI}(\mathcal{B}(A))$, $\bigcup I \in |A|$.

Proof. (1) As a consequence of 3.3(2) the set $\{\, X \in \mathrm{Con} \mid X \subseteq x \,\}$ is downwards closed with respect to $\prec$. To see that it is also directed, let X and Y be consistent subsets of x. By Proposition 3.4 there is a consistent subset Z of x that contains both X and Y and is such that $Z \vdash X \cup Y$, which means that $X, Y \prec Z$.

(2) We have to verify the conditions in Definition 3.3. Let F be a finite subset of $\bigcup I$. Then, for each $a \in F$ there is some $X_a \in I$ with $a \in X_a$. As I is a round ideal and F is finite, there is moreover some $Y \in I$ so that $X_a \prec Y$, for all $a \in F$. Then $Y \in \mathrm{Con}$ and $Y \vdash X_a$, for every such a. Since $a \in X_a$, this implies that $Y \vdash F$. With Condition 3.1(6) we now obtain that there is some $Z \in \mathrm{Con}$ with $Y \vdash Z$. Moreover, $Z \supseteq F$. Thus, $Z \prec Y$, which means that $Z \in I$. Hence, $Z \subseteq \bigcup I$.

For the second condition let X be a consistent subset of $\bigcup I$ and a be an element of A so that $X \vdash a$. Since X is finite, we can again find some $Y \in I$ with $Y \vdash X$. By 3.1(4) we have that $Y \vdash a$, that is $\{a\} \prec Y$. Since $Y \in I$, it follows that $\{a\} \in I$ as well. Thus, $a \in \bigcup I$.

For the third condition let $a \in \bigcup I$. Then there is some $X_a \in I$ with $a \in X_a$. As I is a round ideal, we can moreover find some $Y \in I$ with $X_a \prec Y$. Thus, $Y \vdash X_a$, which in particular means that $Y \vdash a$. Furthermore, $Y \subseteq \bigcup I$. □

The last lemma allows us to define functions $\mathcal{F}\colon |A| \to \mathrm{RI}(\mathcal{B}(A))$ and $\mathcal{G}\colon \mathrm{RI}(\mathcal{B}(A)) \to |A|$ by setting

$$\mathcal{F}(x) = \{\, X \in \mathrm{Con} \mid X \subseteq x \,\} \qquad \text{and} \qquad \mathcal{G}(I) = \bigcup I.$$

Obviously, they are both monotone.

Lemma 3.9. *The functions $\mathcal{F}$ and $\mathcal{G}$ are inverse to each other and Scott continuous.*

Proof. Let $x \in |A|$. Obviously, $\bigcup\{\, X \in \mathrm{Con} \mid X \subseteq x \,\} \subseteq x$. For the converse inclusion let $a \in x$. Then $\{a\} \in \mathrm{Con}$. Hence, $\{a\} \in \{\, X \in \mathrm{Con} \mid X \subseteq x \,\}$, which means that $a \in \bigcup\{\, X \in \mathrm{Con} \mid X \subseteq x \,\} \subseteq x$.

Next, let $I \in \mathrm{RI}(\mathcal{B}(A))$. Set $\bar{I} = \bigcup I$. Then $I \subseteq \{\, X \in \mathrm{Con} \mid X \subseteq \bar{I} \,\}$. Conversely, let $X \in \mathrm{Con}$ with $X \subseteq \bar{I}$. As in the proof of Lemma 3.8(2) we can then find some $Y \in \mathrm{Con} \cap I$ with $Y \vdash X$, which means that $X \prec Y$. Since I is a round ideal, it follows that $X \in I$.

To show that both functions are Scott continuous, it is now sufficient to show that $\mathcal{F}$ preserves existing directed least upper bounds. Let to this end $(x_i)_{i \in K}$ a directed family of states. As is readily seen, $\bigcup_{i \in K} x_i$ is a state too, in this case. Moreover, it is the least upper bound of $(x_i)_{i \in K}$. Since similarly the union of a directed family of round ideals is its least upper bound in $\mathrm{RI}(\mathcal{B}(A))$, we have to show that

$$\{\, X \in \mathrm{Con} \mid X \subseteq \textstyle\bigcup_{i \in K} x_i \,\} = \displaystyle\bigcup_{i \in K} \{\, X \in \mathrm{Con} \mid X \subseteq x_i \,\},$$

which is obvious by the directedness of the family $(x_i)_{i\in K}$. $\square$

As we have seen in Proposition 2.5(1), $\mathrm{RI}(\mathcal{B}(A))$ is a continuous domain. It follows that $|A|$ is a continuous domain as well. The consistent subsets of A generate a canonical basis for $|A|$.

Lemma 3.10. *1. For $X \in \mathrm{Con}$, $\widehat{X} = \{\, a \in A \mid X \vdash a \,\}$ is a state of A.*

2. For every $z \in |A|$, the set of all $\widehat{X}$ with $X \in \mathrm{Con}$ so that $X \subseteq z$ is directed and z is its union.

Proof. (1) follows immediately from 3.1(4-6) in this case.

For (2) let $A_z = \bigcup\{\, \widehat{X} \mid X \in \mathrm{Con} \wedge X \subseteq z \,\}$. Because of 3.3(1) A_z is nonempty. Moreover, we have for two consistent subsets X and Y of z that there is a further consistent subset Z of z that includes both of them. With 3.1(3) it follows that $\widehat{X}, \widehat{Y} \subseteq \widehat{Z}$. Thus, A_z is directed. Obviously, $\bigcup A_z \subseteq z$. Let conversely be $a \in z$. Because of 3.3(3) there is some consistent subset X of z such that $X \vdash a$. Then $\widehat{X} \in A_z$ and hence $a \in \bigcup A_z$. $\square$

This result allows to characterize the approximation relation on $|A|$ in terms of the entailment relation. The characterization nicely reflects the intuition that $x \ll y$ if x is covered by some "finite part" of y.

Lemma 3.11. *For $x, y \in |A|$,*

$$x \ll y \Leftrightarrow (\exists V \in \mathrm{Con})(V \subseteq y \wedge V \vdash x).$$

Proof. The "if"-part of the statement is an obvious consequence of the preceding lemma.

For the proof of the converse implication assume that S is a directed collection of states of A such that $y \subseteq \bigcup S$. By the premise there is some finite consistent subset V of y with $x \subseteq \widehat{V}$. It follows that $V \subseteq \bigcup S$. Since V is finite and S directed, there is thus some $s \in S$ so that $V \subseteq s$. As s is a state, we obtain that also $\widehat{V} \subseteq s$ and hence that $x \subseteq s$. Thus $x \ll y$. $\square$

Let us now sum up what we have shown so far.

Theorem 3.12. *1. Let $(B, \prec)$ be an abstract basis. Then the following two statements hold:*

(a) $\mathcal{C}(B)$ is a continuous information system.

(b) $|\mathcal{C}(B)|$ is a continuous domain with $|\mathcal{C}(B)| = \mathrm{RI}(B)$.

2. Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. Then the following three statements hold:

(a) $|A|$ is a continuous domain.

(b) $\mathcal{B}(A)$ is an abstract basis.

(c) The domains $\mathrm{RI}(\mathcal{B}(A))$ and $|A|$ are isomorphic.

By Proposition 2.6, every continuous domain D gives rise to an abstract basis the round ideal completion of which is isomorphic to D. The abstract base consists of a fixed basis B_D of D and the restriction of the approximation relation to B_D.

Corollary 3.13. *Let D be a continuous domain. Then the following two statements hold:*

1. $\mathcal{C}(B_D)$ is a continuous information system.

2. The domains $|\mathcal{C}(B_D)|$ and D are isomorphic.

4 Approximable mappings

In a next step we want to turn the collection of continuous information systems into a category. Similar to abstract bases the appropriate morphisms are not proper maps but relations. They share essential properties with entailment relations.

Definition 4.1. An *approximable mapping* H between continuous information systems $(A, \mathrm{Con}, \vdash)$ and $(A', \mathrm{Con}', \vdash')$, written $H\colon A \Vdash A'$, is a relation between Con and A' satisfying for all $X, X' \in \mathrm{Con}$, $Y \in \mathrm{Con}'$ and $b \in A'$, as well as all nonempty finite subsets F of A' the following five conditions, where XHY means that XHc, for all $c \in Y$:

1. $(XHY \wedge Y \vdash' b) \Rightarrow XHb$,
2. $XHF \Rightarrow (\exists Z \in \mathrm{Con}')(F \subseteq Z \wedge XHZ)$,
3. $(X \vdash X' \wedge X'Hb) \Rightarrow XHb$,
4. $(X \supseteq X' \wedge X'Hb) \Rightarrow X'Hb$,
5. $XHb \Rightarrow (\exists Z \in \mathrm{Con})(\exists Z' \in \mathrm{Con}')(X \vdash Z \wedge ZHZ' \wedge Z' \vdash' b)$.

As has already been mentioned, entailment relations are special approximable morphisms. For $X \in \mathrm{Con}$ and $a \in A$ set $X \,\mathrm{Id}\, a$ if $X \vdash a$. Then $\mathrm{Id}\colon A \Vdash A$ such that for all $H\colon A \Vdash A'$, $H \circ \mathrm{Id}_{A'} = H = \mathrm{Id}_A \circ H$, where for approximable mappings $H\colon A \Vdash A'$ and $G\colon A' \Vdash A''$ their composition $H \circ G\colon A \Vdash A''$ is defined by

$$X(H \circ G)c \Leftrightarrow (\exists Y \in \mathrm{Con}')(XHY \wedge YGc).$$

Let **CINF** be the category of continuous information systems and approximable mappings. There is also a close connection between approximable mappings and approximable relations which turns the connection between continuous information systems and abstract bases studied the last section into an equivalence between **AB** and **CINF**.

Let B, B' be abstract bases and R be an approximable relation between B and B'. Moreover, define the relation $\mathcal{C}(R) \subseteq \mathrm{Con}_{\mathcal{C}(B)} \times B'$ by

$$X\mathcal{C}(R)a \Leftrightarrow (\exists c \in X)cRa.$$

Lemma 4.2. $\mathcal{C}(R)\colon \mathcal{C}(B) \Vdash \mathcal{C}(B')$.

Proof. We only consider Conditions 4.1(2) and 4.1(5), the verification of the remaining requirements being straightforward. Let to this end $X \in \mathrm{Con}_{\mathcal{C}(B)}$, $b \in B'$ and F be a finite subset of B'.

Assume first that $X\mathcal{C}(R)F$. Then we have that for every $a \in F$ there is some $c_a \in X$ with c_aRa. Since X is directed with respect to $\preceq$, it contains a greatest element $\bar{c}$. It follows that $\bar{c}Ra$, for all $a \in F$. With Condition 2.7(2) we thus obtain that there is some element $b \in B'$ so that $b \succ F$ and $\bar{c}Rb$. Set $Z = F \cup \{b\}$. Then Z is directed with respect to $\preceq'$, i.e. $Z \in \mathrm{Con}_{\mathcal{C}(B')}$. Moreover, $\bar{c}RZ$ and hence $X\mathcal{C}(R)Z$.

Next, suppose that $X\mathcal{C}(R)b$. Then there is some $c \in X$ with cRb. By Condition 2.7(2) there is thus some element $\bar{b} \in B'$ such that $cR\bar{b}$ and $b \prec \bar{b}$. Set $Z' = \{\bar{b}\}$. Then $Z' \in \mathrm{Con}_{\mathcal{C}(B')}$ and $Z' \vdash_{\mathcal{C}(B')} b$. With Condition 2.7(4) we moreover obtain that there is some $\bar{c} \in B$ so that $\bar{c}R\bar{b}$ and $c \prec \bar{c}$. Set $Z = \{\bar{c}\}$. Then we have that $Z \in \mathrm{Con}_{\mathcal{C}(B)}$, $X \vdash_{\mathcal{C}(B)} Z$ and $Z\mathcal{C}(R)Z'$. $\square$

Now, let A, A' be continuous information systems and $H\colon A \Vdash A'$. For $X \in \mathcal{B}(A)$ and $Y' \in \mathcal{B}(A')$ let

$$X\mathcal{B}(H)Y' \Leftrightarrow XHY'.$$

Lemma 4.3. *$\mathcal{B}(H)$ is an approximable relation between $\mathcal{B}(A)$ and $\mathcal{B}(A')$.*

Proof. We only verify Conditions 2.7(2) and 2.7(4), the others being straightforward. Let to this end $X \in \mathrm{Con}$, $Y' \in \mathrm{Con}'$, and be $\mathfrak{M}'$ a finite subset of Con'.

Suppose first that for all $X' \in \mathfrak{M}'$, $X\mathcal{B}(H)X'$. Then it follows with Condition 4.1(5) that for each such X' and all $a' \in X'$ there is some $Z'_{X',a'} \in \mathrm{Con}'$ such that $XHZ'_{X',a'}$ and $Z'_{X',a'} \vdash' a'$. By Conditions 4.1(4) and 3.1(3) we therefore have that there is some $Z' \in \mathrm{Con}'$ with $Z' \supseteq \bigcup\{\, Z'_{X',c'} \mid X' \in \mathfrak{M}' \wedge c' \in X' \,\}$, XHZ' and $Z' \vdash' a'$, which means that $X\mathcal{B}(H)Z'$ and $Z' \succ_{\mathcal{B}(A')} \mathfrak{M}$.

Next, assume that $X\mathcal{B}(H)Y'$. Then XHa', for all $a' \in Y'$. With Condition 4.1(5) we obtain that there is some $Z_{a'} \in \mathrm{Con}$ so that $X \vdash Z_{a'}$ and $Z_{a'}Ha'$, for each such a'. By applying Conditions 3.1(6) and 4.1(2) it follows that there is some $Z \in \mathrm{Con}$ with $Z \supseteq \bigcup\{\, Z_{c'} \mid c' \in Y' \,\}$, $X \vdash Z$ and ZHa', for all $a' \in Y'$. Thus, we have that $X \succ_{\mathcal{B}(A)} Z$ and $Z\mathcal{B}(H)Y'$. $\square$

It is readily seen that $\mathcal{C}\colon \mathbf{AB} \to \mathbf{CINF}$ and $\mathcal{B}\colon \mathbf{CINF} \to \mathbf{AB}$ are functors. As will be shown in what follows, they constitute an equivalence between both categories.

For a category $\mathbf{C}$ let $\mathcal{I}_{\mathbf{C}}$ be the identical functor on $\mathbf{C}$. We first show that there is a natural isomorphism $\tau\colon \mathcal{I}_{\mathbf{AB}} \to \mathcal{B} \circ \mathcal{C}$. Let to this end $(B, \prec)$ be an abstract basis. Then $\mathcal{B}(\mathcal{C}(B)) = (\mathrm{Con}_{\mathcal{C}(B)}, \prec_{\mathcal{B}(\mathcal{C}(B))})$ and for $X, Y \in \mathrm{Con}_{\mathcal{C}(B)}$ we have that

$$X \prec_{\mathcal{B}(\mathcal{C}(B))} Y \Leftrightarrow (\forall a \in X)(\exists b \in Y) a \prec b,$$

which means that for $a, b \in B$,

$$\{a\} \prec_{\mathcal{B}(\mathcal{C}(B))} \{b\} \Leftrightarrow a \prec b.$$

Define relations P_B and Q_B, respectively, between B and $\mathcal{B}(\mathcal{C}(B))$ as well as $\mathcal{B}(\mathcal{C}(B))$ and B by

$$P_B = \{\, (b, X) \in B \times \mathrm{Con}_{\mathcal{C}(B)} \mid X \prec_{\mathcal{B}(\mathcal{C}(B))} \{b\} \,\}, \text{and}$$
$$Q_B = \{\, X, b) \in \mathrm{Con}_{\mathcal{C}(B)} \times B \mid \{b\} \prec_{\mathcal{B}(\mathcal{C}(B))} X \,\}.$$

Then P_B and Q_B are approximable relations with $P_B \circ Q_B = \mathrm{RI}_B$ and $Q_B \circ P_B = \mathrm{RI}_{\mathcal{B}(\mathcal{C}(B))}$. Set $\tau_B = P_B$.

Lemma 4.4. *τ is a natural transformation.*

Proof. It remains to show that for abstract bases $(B, \prec), (B', \prec')$ and an approximable relation R between B and B',

$$\tau_B \circ \mathcal{B}(\mathcal{C}(R)) = R \circ \tau_{B'}.$$

Note first that for $X, Y \in \mathrm{Con}_{\mathcal{C}(B)}$,

$$X\mathcal{B}(\mathcal{C}(R))Y \Leftrightarrow (\forall a \in Y)(\exists c \in X) cRa.$$

Now, let $B \in B$ and $Y' \in \mathrm{Con}_{\mathcal{C}(B')}$. Then

$$\begin{aligned} b(\tau_B \circ \mathcal{B}(\mathcal{C}(R)))Y' &\Leftrightarrow (\exists Z \in \mathrm{Con}_{\mathcal{C}(B)}) b\tau_B Z \wedge Z\mathcal{B}(\mathcal{C}(R))Y' \\ &\Leftrightarrow (\exists Z \in \mathrm{Con}_{\mathcal{C}(B)})\{b\} \succ_{\mathcal{B}(\mathcal{C}(B))} Z \wedge Z\mathcal{B}(\mathcal{C}(R))Y' \\ &\Leftrightarrow \{b\}\mathcal{B}(\mathcal{C}(R))Y' \\ &\Leftrightarrow (\exists z' \in B')\{b\}\mathcal{B}(\mathcal{C}(R))\{z'\} \wedge \{z'\} \succ_{\mathcal{B}(\mathcal{C}(B'))} Y' \\ &\Leftrightarrow (\exists z' \in B') bRz' \wedge z'\tau_{B'}Y' \\ &\Leftrightarrow b(R \circ \tau_{B'})Y'. \end{aligned}$$

□

Let us summarize what we have just shown.

Proposition 4.5. *$\tau\colon \mathcal{I}_{\mathbf{AB}} \to \mathcal{B} \circ \mathcal{C}$ is a natural isomorphism.*

Next, we show that there is also a natural isomorphism $\eta\colon \mathcal{I}_{\mathbf{CINF}} \to \mathcal{C} \circ \mathcal{B}$. Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. Then $\mathcal{C}(\mathcal{B}(A)) = (\mathrm{Con}, \mathrm{Con}_{\mathcal{C}(\mathcal{B}(A))}, \vdash_{\mathcal{C}(\mathcal{B}(A))})$, where $\mathrm{Con}_{\mathcal{C}(\mathcal{B}(A))}$ is the collection of all finite subsets of Con that are directed with respect to $\preceq_{\mathcal{B}(A)}$ and

$$\mathfrak{X} \vdash_{\mathcal{C}(\mathcal{B}(A))} X \Leftrightarrow (\exists Z \in \mathfrak{X}) Z \vdash X,$$

for $\mathfrak{X} \in \mathrm{Con}_{\mathcal{C}(\mathcal{B}(A))}$ and $X \in \mathrm{Con}$. Set

$$\begin{aligned} S_A &= \{\, (X, Y) \in \mathrm{Con} \times \mathrm{Con} \mid X \vdash Y \,\}, \text{and} \\ T_A &= \{\, (\mathfrak{X}, a) \in \mathrm{Con}_{\mathcal{C}(\mathcal{B}(A))} \times A \mid \mathfrak{X} \vdash_{\mathcal{C}(\mathcal{B}(A))} \{a\} \,\}. \end{aligned}$$

Then $S_A\colon A \Vdash \mathcal{C}(\mathcal{B}(A))$ and $T_A\colon \mathcal{C}(\mathcal{B}(A)) \Vdash A$.

Lemma 4.6. *1. $S_A \circ T_A = \mathrm{Id}_A$,*

2. $T_A \circ S_A = \mathrm{Id}_{\mathcal{C}(\mathcal{B}(A))}$.

Proof. Note that for $X, Y \in \mathrm{Con}$, $X \vdash Y$ just if $\{X\} \vdash_{\mathcal{C}(\mathcal{B}(A))} Y$. Then (1) is a consequence of the conditions for entailment relations. For (2) let $\mathfrak{X} \in \mathrm{Con}_{\mathcal{C}(\mathcal{B}(A))}$ and $X \in \mathrm{Con}$. Then we have that

$$\begin{aligned} \mathfrak{X}(T_A \circ S_A)X &\Leftrightarrow (\exists Y \in \mathrm{Con})\mathfrak{X} T_A Y \wedge Y S_A X \\ &\Leftrightarrow (\exists Y \in \mathrm{Con})\mathfrak{X} \vdash_{\mathcal{C}(\mathcal{B}(A))} Y \wedge Y \vdash X \\ &\Leftrightarrow (\exists Y \in \mathrm{Con})(\exists Z \in \mathfrak{X}) Z \vdash Y \wedge Y \vdash X \\ &\Leftrightarrow (\exists Z \in \mathfrak{X}) Z \vdash X \\ &\Leftrightarrow \mathfrak{X} \vdash_{\mathcal{C}(\mathcal{B}(A))} X. \end{aligned}$$

□

Set $\eta_A = S_A$.

Lemma 4.7. *η is a natural transformation.*

Proof. We only have to show that for continuous information systems $(A, \mathrm{Con}, \vdash)$ and $A', \mathrm{Con}', \vdash')$ and an approximable mapping $H\colon A \Vdash A'$,

$$\eta_A \circ \mathcal{C}(\mathcal{B}(H)) = H \circ \eta_{A'}.$$

Note to this end that for $\mathfrak{X} \in \mathrm{Con}_{\mathcal{C}(\mathcal{B}(A))}$ and $X' \in \mathcal{C}(\mathcal{B}(A'))$,

$$\mathfrak{X}\mathcal{C}(\mathcal{B}(H))X' \Leftrightarrow (\exists Z \in \mathfrak{X}) Z H X'.$$

Then we have for $X \in \mathcal{C}(\mathcal{B}(A))$ and $Y' \in \mathcal{C}(\mathcal{B}(A'))$ that

$$\begin{aligned}
X(\eta_A \circ \mathcal{C}(\mathcal{B}(H)))Y' &\Leftrightarrow (\exists \mathfrak{Z} \in \mathrm{Con}_{\mathcal{C}(\mathcal{B}(A))}) X S_A \mathfrak{Z} \wedge \mathfrak{Z}\mathcal{C}(\mathcal{B}(H))Y' \\
&\Leftrightarrow (\exists \mathfrak{Z} \in \mathrm{Con}_{\mathcal{C}(\mathcal{B}(A))})(\forall Z \in \mathfrak{Z}) X \vdash Z \wedge (\exists U \in \mathfrak{Z}) Z H Y' \\
&\Leftrightarrow X H Y' \qquad (1) \\
&\Leftrightarrow (\exists V' \in \mathrm{Con}') X H V' \wedge V' \vdash' Y' \\
&\Leftrightarrow X(H \circ \eta_{A'})Y'.
\end{aligned}$$

In line (1) the implication from left to right is a consequence of Condition 4.1(3). For the converse direction use Condition 4.1(5) to obtain some $U \in \mathrm{Con}$ such that $X \vdash U$ and UHY'. Then set $\mathfrak{Z} = \{U\}$. □

Let us summarize again what we have achieved in this step.

Proposition 4.8. *$\eta\colon \mathcal{I}_{\mathbf{CINF}} \to \mathcal{C} \circ \mathcal{B}$ is natural isomorphism.*

Putting Propositions 4.5 and 4.8 together, we obtain what we were aiming for in this section.

Theorem 4.9. *The category* **AB** *of abstract bases and approximable relations is equivalent to the category* **CINF** *of continuous information systems and approximable mappings.*

Corollary 4.10. *The category* **CINF** *of continuous information systems and approximable mappings is equivalent to the category* **CDOM** *of continuous domains and Scott continuous functions.*

5 Special cases

In this section we are going to study for some important kinds of domains how they can be represented as continuous information systems. Algebraic domains are certainly one of the most frequently used kinds.

Proposition 5.1. *Let A be a set,* Con *a collection of finite subsets of A and $\vdash \subseteq \mathrm{Con} \times A$. Then $(A, \mathrm{Con}, \vdash)$ is a continuous information system with $|A|$ being algebraic if, and only if, $(A, \mathrm{Con}, \vdash)$ satisfies Conditions 3.1(1-4, 6) and in addition the following requirement*

$$(\forall X, Y \in \mathrm{Con})[X \vdash Y \Rightarrow (\exists Z \in \mathrm{Con})(X \vdash Z \wedge Z \vdash Z \wedge Z \vdash Y)]. \qquad \text{(ALG)}$$

Proof. We have seen in Section 3 that if A is a continuous information system, then $\mathcal{B}(A)$ is an abstract basis. Moreover, $\mathrm{RI}(\mathcal{B}(A))$ and $|A|$ are isomorphic domains. So, if $|A|$ is algebraic the same holds for $\mathrm{RI}(\mathcal{B}(A))$. Condition (ALG) is therefore a consequence of Proposition 2.4(2).

For the converse implication note that Condition 3.1(5) follows from the extra requirement (ALG) with Condition 3.1(1). Therefore, A is a continuous information system. Moreover, $\mathrm{RI}(\mathcal{B}(A))$ is algebraic because of Proposition 2.4(2). □

For a continuous information system Condition (ALG) is obviously satisfied if

$$(\forall X \in \mathrm{Con})(\forall a \in X) X \vdash a.$$

This requirement has been used by Scott [14].

In most computer science applications of domain theory one requires the domains to be pointed. Therefore, we are interested in when $|A|$ has a least and, similarly, when it has a greatest element.

Proposition 5.2. *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. Then $|A|$ is pointed if, and only if, $\emptyset \in \mathrm{Con}$ or there is some $a \in A$ with $X \vdash a$, for all $X \in \mathrm{Con}$.*

Proof. Let us first assume that $|A|$ has a least state $\perp$. If $\perp$ is empty, then by choosing $F = \emptyset$ in 3.3(1) we have that there is some $X \in \mathrm{Con}$ with $\emptyset \subseteq X \subseteq \emptyset$, which means that $\emptyset \in \mathrm{Con}$.

Suppose now that $\perp$ is not empty, say $a \in \perp$. Then a is contained in every state of A, in particular in the states $\widehat{X}$. Thus $X \vdash a$, for every $X \in \mathrm{Con}$.

For the converse implication assume first that $\emptyset \in \mathrm{Con}$. Then $\widehat{\emptyset}$ is a state of A. If $c \in \widehat{\emptyset}$, i.e., if $\emptyset \vdash c$, then $X \vdash c$, for any $X \in \mathrm{Con}$, by 3.1(3). With 3.3(1, 2) it follows that c is contained in every state of A, i.e. $\widehat{\emptyset}$ is its least state.

If, on the other hand, A has some element a with $X \vdash a$, for every $X \in \mathrm{Con}$, then we obtain again that a is contained in every state of $|A|$, which implies that $\widehat{\{a\}}$ is its least state. $\square$

Let $A^{\vdash} = \{\, a \in A \mid (\exists X \in \mathrm{Con}) X \vdash a \,\}$ be the *derivable kernel* of A. Obviously, $A^{\vdash} = \bigcup |A|$. Moreover, $|A|$ has a greatest element exactly if $\bigcup |A|$ is a state of A.

Lemma 5.3. *$\bigcup |A|$ is a state* $\Leftrightarrow (\forall F \subseteq_{\mathrm{fin}} A^{\vdash})(\exists X \in \mathrm{Con})(F \subseteq X \wedge X \subseteq A^{\vdash})$.

Note that the right hand side is Condition 3.3(1) for the set $\bigcup |A|$. As is readily verified, Conditions 3.3(2, 3) hold in this case.

Proposition 5.4. *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. Then $|A|$ has a greatest element if, and only if,*

$$(\forall F \subseteq_{\mathrm{fin}} A^{\vdash})(\exists X \in \mathrm{Con})(F \subseteq X \wedge X \subseteq A^{\vdash}). \tag{TOP}$$

Observe that $A^{\vdash}$ may be a proper subset of A, i.e., a continuous information system may contain tokens that are never entailed. Because of Condition 3.3(3) they are of no use in domain construction. The following construction shows that they can easily be eliminated.

Definition 5.5. A continuous information system $(A, \mathrm{Con}, \vdash)$ is called *reduced* if for every $a \in A$ there is some $X \in \mathrm{Con}$ with $X \vdash a$.

Let $(A_{\mathrm{red}}, \mathrm{Con}_{\mathrm{red}}, \vdash_{\mathrm{red}})$ be defined by

$$\begin{aligned} A_{\mathrm{red}} &= \{\, a \in A \mid (\exists a \in A) X \vdash a \,\}, \\ \mathrm{Con}_{\mathrm{red}} &= \{\, X \in \mathrm{Con} \mid X \subseteq A_{\mathrm{red}} \,\}, \text{ and} \\ X \vdash_{\mathrm{red}} a &\Leftrightarrow X \vdash a, \end{aligned}$$

where $X \in \mathrm{Con}_{\mathrm{red}}$ and $a \in A_{\mathrm{red}}$. The next result is now readily verified.

Lemma 5.6. *$(A_{\mathrm{red}}, \mathrm{Con}_{\mathrm{red}}, \vdash_{\mathrm{red}})$ is a reduced continuous information system with $|A_{\mathrm{red}}| = |A|$.*

A further type of domains that is often used in computer science are the bounded-complete ones. As follows from the definition, bounded-complete continuous domains are always pointed.

Proposition 5.7. *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system that satisfies the additional requirement that*

$$(\forall Y \subseteq_{\mathrm{fin}} A)(\forall X \in \mathrm{Con})[X \vdash Y \Rightarrow Y \in \mathrm{Con}]. \tag{WBC}$$

Then $|A|$ is bounded-complete.

Proof. Since $|A|$ is directed-complete, it suffices to show that any two states which are bounded from above have a least upper bound. Note first that because of Proposition 3.4 and Condition (WBC) any finite subset of a state is consistent. Now, let x, x' and y be states with $x, x' \subseteq y$. We want to show that

$$z = \{\, a \in A \mid (\exists X \in \mathrm{Con})(X \subseteq x \cup x' \wedge X \vdash a) \,\}$$

is also a state and that it is the least upper bound of x and x'.

Let $a \in z$. Then there is some consistent subset X of $x \cup x'$ with $X \vdash a$. Since $X \subseteq y$, it follows that $a \in y$. Thus $z \subseteq y$. As a consequence, any finite subset of z is consistent. Hence, z satisfies Condition 3.3(1). With Condition 3.3(3) we obtain that $x, x' \subseteq z$. Therefore, by its definition, z satisfies this requirement as well.

For Condition 3.3(2), let $Z \subseteq z$ and $c \in A$ with $Z \vdash c$. Then there is some consistent $X_a \subseteq x \cup x'$, for each $a \in Z$, with $X_a \vdash a$. Let X be the union of these sets X_a. As $X \subseteq y$, X is consistent. Moreover, $X \subseteq x \cup x'$ and $X \vdash a$. Hence $a \in z$.

Finally, let y' be any other state with $x, x' \subseteq y'$. Then $x \cup x' \subseteq y'$. With Condition 3.3(2) it follows that also $z \subseteq y'$. $\square$

It would be interesting to find a condition that is also necessary, as we did in the other cases. Note that Condition (WBC) implies Condition 3.1(6).

In his definition of continuous information system Hoofman [11] uses the above Condition (WBC) as well as the requirement that for all finite subsets Y of A and all $X \in \mathrm{Con}$,

$$(Y \subseteq X \wedge X \in \mathrm{Con}) \Rightarrow Y \in \mathrm{Con}\,. \tag{SBC}$$

He does not use Condition 3.1(2). Observe that (WBC) follows from (SBC) if 3.1(2) is assumed.

A continuous domain which is a complete lattice is called a *continuous lattice*. If the domain is even algebraic, one calls it an *algebraic domain*. Obviously, a poset is a continuous lattice just if it is a bounded-complete continuous domain with greatest element, correspondingly in the algebraic case.

Proposition 5.8. *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system.*

1. *If A satisfies the additional requirements (TOP) and (WBC). Then $|A|$ is a continuous lattice.*

2. *If A satisfies the additional requirements (ALG), (TOP) and (WBC). Then $|A|$ is an algebraic lattice.*

One of the earliest and most important applications of domain theory was in programming language semantics. Since any advanced algorithmic language allows the formation of function spaces as data type, only Cartesian closed full subcategories of **CDOM** are of interest in this case. Moreover, the domains have to be pointed. As has been shown by Jung [10], there are just two such maximal subcategories, generated by FS-domains and L-domains, respectively. We will consider only the first class here. The representation of L-domains by continuous information systems will be the subject of a separate paper.

Proposition 5.9. *Let A be a set and* Con *a countable collection of finite subsets of A including the empty set. Then there is a relation $\vdash \subseteq \mathrm{Con} \times A$ such that $(A, \mathrm{Con}, \vdash)$ is a continuous information system with $|A|$ being an FS-domain if, and only if, there exists a directed family $(\Vdash_i)_{i \in I}$ of relations between* Con *and A as well as a family $(\mathfrak{M}_i)_{i \in I}$ of finite subsets of* Con *so that the following eight conditions hold for all indices $i \in I$, sets $X, Y \in \mathrm{Con}$, elements $a \in A$ and nonempty finite subsets F of A:*

1. $\{a\} \in \mathrm{Con}$,
2. $(\exists j \in I) X \Vdash_j a \Rightarrow X \cup \{a\} \in \mathrm{Con}$,
3. $[X \Vdash_i Y \wedge (\exists j \in I) Y \Vdash_j a] \Rightarrow X \Vdash_i a$,
4. $X \Vdash_i F \Rightarrow (\exists Z \in \mathrm{Con})[F \subseteq Z \wedge X \Vdash_i Z]$,
5. $[(\exists j \in I) X \Vdash_j Y \wedge Y \Vdash_i a] \Rightarrow X \Vdash_i a$,
6. $[X \supseteq Y \wedge Y \Vdash_i a] \Rightarrow X \Vdash_i a$,
7. $X \Vdash_i a \Rightarrow (\exists Z, Z' \in \mathrm{Con})(\exists j, j' \in I)[X \Vdash_j Z \wedge Z \Vdash_i Z' \wedge Z' \Vdash_{j'} a]$,
8. $(\exists Z \in \mathfrak{M}_i)[X \vdash Z \wedge (\forall b \in A)(X \Vdash_i b \Rightarrow (\exists j \in I) Z \Vdash_j b)]$.

Proof. Assume that there is a relation $\vdash \subseteq \mathrm{Con} \times A$ such that $(A, \mathrm{Con}, \vdash)$ is a continuous information systems and $|A|$ is an FS-domain. Then there is a directed family $(f_i)_{i \in I}$ on $|A|$ and a family $(M_i)_{i \in I}$ of finite nonempty subsets of $|A|$ such that the identity on $|A|$ is the least upper bound of the functions f_i and each M_i witnesses that f_i is finitely separated from $\mathrm{id}_{|A|}$. Note that for $X \in \mathrm{Con}$, $\widehat{X} = \bigcup_{i \in I} f_i(\widehat{X})$. Thus, for $a \in A$,

$$X \vdash a \Leftrightarrow a \in \widehat{X} \Leftrightarrow (\exists j \in I) a \in f_j(\widehat{X}).$$

For $i \in I$, define $\Vdash_i \subseteq \mathrm{Con} \times A$ by

$$X \Vdash_i a \Leftrightarrow a \in f_i(\widehat{X}).$$

Conditions (1) to (7) are now readily verified. For (7) note that $\widehat{X} = \bigcup \{ \widehat{Z} \mid Z \in \mathrm{Con} \wedge Z \subseteq \widehat{X} \}$ and similarly $f_i(\widehat{Z}) = \bigcup \{ \widehat{Z'} \mid Z' \in \mathrm{Con} \wedge Z' \subseteq f_i(\widehat{Z}) \}$.

It remains to define $\mathfrak{M}_i$ and verify requirement (8). As we has been remarked in Section 2, we may assume that each set M_i contains only base elements, which means in this case that the elements of the M_i are of the form $\widehat{X}$, for some $X \in \mathrm{Con}$. Thus, for every $i \in I$, there are finite subsets $\mathfrak{M}_i$ of Con with $M_i = \{ \widehat{X} \mid X \in \mathfrak{M}_i \}$. Condition (8) is now obvious.

For the converse implication set $\vdash = \bigcup_{i \in I} \Vdash_i$. Then $(A, \mathrm{Con}, \vdash)$ is obviously a continuous information system. Moreover, for $i \in I$ and $x \in |A|$, define

$$f_i(x) = \{ a \in A \mid (\exists X \in \mathrm{Con}) X \subseteq x \wedge X \Vdash_i a \}.$$

As is easily seen, f_i is a Scott continuous function on $|A|$, for each $i \in I$, and the family of all these functions is directed with the identity on $|A|$ as its least upper bound. Finally, for $i \in I$, set $M_i = \{\, \widehat{X} \mid X \in \mathfrak{M}_i \,\}$.

Now, let $i \in I$ and $x \in |A|$. Since Con is countable, the set of all consistent subsets of x is countable as well. Because of Condition 3.3(1) we can therefore construct an increasing sequence $(X_\nu)_{\nu\in\omega}$ of consistent subsets of x such that for every consistent subset Y of x there is some $\nu \in \omega$ with $Y \subseteq X_\nu$. By Assumption (8) there is some $Z_\nu \in \mathfrak{M}_i$, for each such ν, such that $X_\nu \vdash Z_\nu$ and

$$f_i(\widehat{X_\nu}) = \{\, b \in A \mid X_\nu \Vdash_i b \,\} \subseteq \widehat{Z_\nu}.$$

Because $\mathfrak{M}_i$ is finite, there must then be some $Z' \in \mathfrak{M}_i$ such that for infinitely many indices ν, $Z_\nu = Z'$. It follows that for all $\nu \in \omega$, $f_i(\widehat{X_\nu}) \subseteq Z'$ and for some $\nu' \in \omega$, $X_{\nu'} \vdash Z'$. Thus we have that

$$f_i(x) = \bigcup \{\, f_i(\widehat{Y}) \mid Y \in \mathrm{Con} \wedge Y \subseteq x \,\} \subseteq \widehat{Z'} \subseteq \widehat{X_{\nu'}} \subseteq x,$$

which shows that f_i is finitely separated from the identity on $|A|$. □

The next result presents a sufficient condition for a continuous information system to generate a bifinite domains. The countability assumption on Con is not used in this case.

Proposition 5.10. *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system that satisfies Condition (ALG) as well as the requirement that $\emptyset \in \mathrm{Con}$ or there is some $a \in A$ with $X \vdash a$, for all $X \in \mathrm{Con}$. Moreover, for each finite subset F of A let there be a finite family $\mathfrak{F}$ of consistent sets that contains all consistent subsets of F and is such that for all subfamilies $\mathfrak{G}$ of $\mathfrak{F}$ and all $Z \in \mathrm{Con}$,*

$$(\forall X \in \mathfrak{G}) Z \vdash X \Rightarrow (\exists Y \in \mathfrak{F})[Z \vdash Y \wedge (\forall X \in \mathfrak{G}) Y \vdash X].$$

Then $|A|$ is a bifinite domain.

Proof. By Propositions 5.1 and 5.2 we have that $|A|$ is a pointed algebraic domain. Because of [6, Proposition II-2.20] it remains to show that there is a directed family of Scott continuous functions that all have finite range and the identity on $|A|$ as their least upper bound.

Let

$$\mathfrak{I} = \{\, F \subseteq_{\mathrm{fin}} A \mid (\exists X \in \mathrm{Con}) X \subseteq F \,\}.$$

Then $\mathfrak{I}$ is directed. Now, for $F \in \mathfrak{I}$ and $x \in |A|$, define

$$\mathfrak{G}_{F,x} = \{\, X \in \mathfrak{F} \mid X \subseteq x \,\}$$

and set $\widehat{\mathfrak{G}_{F,x}} = \{\, \widehat{X} \mid X \in \mathfrak{G}_{F,x} \,\}$. We show that $\widehat{\mathfrak{G}_{F,x}}$ has a greatest element.

Let $G = \bigcup \mathfrak{G}_{F,x}$. Then G is a finite subset of x. By Proposition 3.4, there is hence some consistent subset Z of x with $Z \vdash G$. It follows that $Z \vdash X$, for all $X \in \mathfrak{G}_{F,x}$. Thus, there is some $Y \in \mathfrak{F}$ such that $Z \vdash Y$ and $Y \vdash X$, for all $X \in \mathfrak{G}_{F,x}$. Since Z is included in x, the same holds for Y. Moreover, $Y \in \mathfrak{G}_{F,x}$ and $\widehat{X} \subseteq \widehat{Y}$, for all $X \in \mathfrak{G}_{F,x}$. Thus $\widehat{Y}$ is the largest element of $\widehat{\mathfrak{G}_{F,x}}$.

Set $g_F(x) = \max \widehat{\mathfrak{G}_{F,x}}$. Then range$(g_F)$ is a subset of the set of all $\widehat{X}$ with $X \in \mathfrak{F}$ and hence finite. We show next that g_F is Scott continuous.

Obviously, g_F is monotone. Let S be a directed subset of $|A|$ and assume that $g_F(\bigcup S) = \widehat{Y}$ with $Y \in \mathfrak{G}_{F,\bigcup S}$. Then $Y \subseteq \bigcup S$. Since S is directed, there is some state $s \in S$ so that $Y \subseteq s$. It follows that $g_F(s) \supseteq \widehat{Y} = g_F(\bigcup S)$, which implies that g_F is Scott continuous.

Note that for $F_1, F_2 \in \mathfrak{I}$, $g_{F_1}, g_{F_2} \sqsubseteq g_{F_1 \cup F_2 \cup \bigcup(\mathfrak{F}_1 \cup \mathfrak{F}_2)}$. Thus, the family $(g_F)_{F \in \mathfrak{I}}$ is directed. It remains to show that $(\bigsqcup_{F \in \mathfrak{I}} g_F)(x) = x$, for all $x \in |A|$.

By the definition of the functions g_F we have that $\bigcup \{ g_F(x) \mid F \in \mathfrak{I} \} \subseteq x$. For the converse inclusion choose $X \in \mathrm{Con}$ with $X \subseteq x$. Then $X \in \mathfrak{I}$. Moreover, $X \in \mathfrak{X}$, where $\mathfrak{X}$ is the family of consistent sets that exists for X by our assumption. It follows that $X \in \mathfrak{G}_{X,x}$ and hence that $\widehat{X} \subseteq g_X(x)$. Thus,

$$x = \bigcup \{ \widehat{Z} \mid Z \in \mathrm{Con} \wedge Z \subseteq x \} \subseteq \bigcup \{ g_Z(x) \mid Z \in \mathrm{Con} \wedge Z \subseteq x \} \subseteq \bigcup \{ g_F(x) \mid F \in \mathfrak{I} \}.$$

□

Obviously, the results given so far allow the introduction of special categories of information system such that equivalence results as in Section 4 hold with respect to the corresponding domain categories. We leave this to the interested reader.

6 Concluding remarks

In this paper a new notion of continuous information system is introduced. As is shown the information systems of this type generate exactly the continuous domains. This result is obtained by analyzing the relationship between information systems and abstract bases. It is shown that both categories are equivalent.

Abstract bases have turned out to be an important tool in domain theory. The main (and up to isomorphism, only) examples for abstract bases are the bases of continuous domains together with the restriction of the approximation relation to the basis. As is well known, up to isomorphism continuous domains are completely determined by their bases. The same is true for Scott continuous functions and their behaviour on the bases of its domain and codomain. This basic relationship led to another well known equivalence, that between the categories of abstract bases and continuous domains.

As a consequence of both equivalence results the equivalence between the categories of continuous information systems and of continuous domains is obtained.

For most applications continuous domains are far too general. A great variety of important subclasses has been studied in the literature. For certain rather large subclasses conditions have been derived in the paper which when fulfilled by a continuous information system force the generated domain to belong to the class in question. It would be interesting to find similar requirements for other subclasses.

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