Representing L-Domains as Information Systems

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1. Introduction

Information systems have been introduced by Dana Scott as a logic-oriented approach to the theory of bounded-complete algebraic domains. In this talk a similar result is presented for continuous L-domains.

2. A short domain-theory fresh-up

Definition

- A poset $(D, \sqsubseteq)$ with least element $\perp$ is called domain if every directed set $S \subseteq D$ has a least upper bound $\bigsqcup S$ in D.
- D is bounded-complete if every bounded subset has a least upper bound in D.
- Let $x, y \in D$. Then x approximates y $(x \ll y)$ if for all directed sets $S \subseteq D$,

$$y \sqsubseteq \bigsqcup S \quad \Rightarrow \quad (\exists s \in S) x \sqsubseteq s.$$

- Element $x \in D$ is compact if $x \ll x$.
- $B \subseteq D$ is a basis for D if for every $x \in D$, $B_x = \{ z \in B \mid z \ll x \}$ is directed with least upper bound x.

Remark

B basis for D $\Rightarrow$ $B \supseteq K_D (= \{\,\textit{compact elements}\,\})$

Definition

- D is continuous if D has a basis.
- D is algebraic if K_D is a basis for D.

Remark

In a continuous domain $\ll$ *has the interpolation property:*

$$M \subseteq_f D \,\&\, M \ll y \quad \Rightarrow \quad (\exists z \in B) M \ll z \ll y.$$

Remark

For the usual domain constructions like sums, products etc. there are corresponding constructions of the canonical bases.

3. Information systems

Definition

$(A, \mathrm{Con}, \vdash)$, where A is a set, $\emptyset \neq \mathrm{Con} \subseteq \wp_f(A)$, and $\vdash \subseteq \mathrm{Con} \times A$, is an information system if

- $a \in A \quad \Rightarrow \quad \{\, a \,\} \in \mathrm{Con}$
- $X \vdash a \quad \Rightarrow \quad X \cup \{\, a \,\} \in \mathrm{Con}$
- $X \vdash Y \,\&\, Y \vdash a \quad \Rightarrow \quad X \vdash a$
- $X \in \mathrm{Con} \;\&\; Y \subseteq X \quad \Rightarrow \quad Y \in \mathrm{Con}$
- $X \in \mathrm{Con} \;\&\; a \in X \quad \Rightarrow \quad X \vdash a.$

Here

$$X \vdash Y \quad \Leftrightarrow \quad (\forall c \in Y) X \vdash c.$$

Definition

$(A, \mathrm{Con}, \vdash)$, where A is a set, $\emptyset \neq \mathrm{Con} \subseteq \wp_f(A)$, and $\vdash \subseteq \mathrm{Con} \times A$, is a continuous information system (c-inf) if

- $a \in A \quad \Rightarrow \quad \{a\} \in \mathrm{Con}$
- $X \vdash a \quad \Rightarrow \quad X \cup \{a\} \in \mathrm{Con}$
- $X \vdash Y \,\&\, Y \vdash a \quad \Rightarrow \quad X \vdash a$
- $\emptyset \in \mathrm{Con}$
- $X, Y \in \mathrm{Con} \,\&\, X \subseteq Y \,\&\, X \vdash a \quad \Rightarrow \quad Y \vdash a$
- $X \vdash a \quad \Rightarrow \quad (\exists Z \in \mathrm{Con}) X \vdash Z \,\&\, Z \vdash a$
- $F \in \wp_f(A) \,\&\, X \vdash F \quad \Rightarrow \quad (\exists Z \in \mathrm{Con}) F \subseteq Z \,\&\, X \vdash Z.$

Definition

A c-inf A is algebraic if

$$X \in \mathrm{Con} \,\&\, X \in a \quad \Rightarrow \quad X \vdash a.$$

Let D be a continuous domain with basis B. Set

- $\mathrm{Con}_D = \{ X \subseteq_f B \mid \bigsqcup X \text{ exists in } D \}$
- $X \vdash_D z \quad \Leftrightarrow \quad z \ll \bigsqcup X.$

Proposition

Let D be a continuous domain with basis B. Then $\mathcal{S}(D, B) = (B, \mathrm{Con}_D, \vdash_D)$ is a c-inf. If D is algebraic with basis K_D, then it its an algebraic c-inf.

Definition

Let A be a c-inf. A subset $x \subseteq A$ is a state of A if

- $(\forall F \subseteq_f x)(\exists Y \in \mathrm{Con}) F \subseteq Y \subseteq x$
- $X \subseteq x \,\&\, X \vdash a \quad \Rightarrow \quad a \in x$
- $(\forall a \in x)(\exists X \subseteq_f x) X \vdash a$.

Denote the set of states of A by $|A|$. Moreover, for $X \in \mathrm{Con}$ set

$$\overline{X} = \{\, a \in A \mid X \vdash a \,\}.$$

Proposition

Let A be a c-inf. Then $(|A|, \subseteq)$ is a continuous domain with basis $\{\, \overline{X} \mid X \in \mathrm{Con} \,\}$. If A is algebraic, $(|A|, \subseteq)$ is algebraic as well.

Theorem

Let D be a continuous domain with basis B. Then $|\mathcal{S}(D, B)| \cong D$.

4. Scott continuous functions

Definition

A function $f : D \to E$ between domains D and E is Scott continuous if f is monotone and for every directed $S \subseteq D$,

$$\bigsqcup f[S] = f(\bigsqcup S).$$

Remark

If D and E are continuous, a Scott continuous functions is completely determined by its behaviour on the base elements, i.e. by its graph

$$\mathrm{graph}(f) = \{\, (X, z) \in \wp_f(B_D) \times B_E \mid z \ll_E f(\bigsqcup X) \,\}.$$

Definition

Let A and B be c-inf's. A relation $f \subseteq \mathrm{Con}_A \times B$ is an approximable mapping between A and B if

- $\emptyset \neq Y \subseteq_f B \,\&\, XfY \quad \Rightarrow \quad (\exists Z \in \mathrm{Con}_B) Y \subseteq Z \,\&\, XfZ$
- $XfY \,\&\, Y \vdash_B b \quad \Rightarrow \quad Xfb$
- $X \vdash_A X' \,\&\, X'fb \quad \Rightarrow \quad Xfb$
- $X, X' \in \mathrm{Con}_A \,\&\, X \subseteq X' \,\&\, Xfb \quad \Rightarrow \quad X'fb$
- $Xfb \quad \Rightarrow \quad (\exists X' \in \mathrm{Con}_A)(\exists Y \in \mathrm{Con}_B) X \vdash_A X' \,\&\, X'fY \,\&\, Y \vdash_B b.$

Here, $\quad XfY \quad \Leftrightarrow \quad (\forall b \in Y) Xfb.$

Remark

- *$\vdash$ is an approximable mapping on A.*
- *For every Scott continuous $f : D \to E$, $\mathrm{graph}(f)$ is an approximable mapping between $\mathcal{S}(D, B_D)$ and $\mathcal{S}(E, B_E)$.*

Theorem

The categories of continuous (algebraic) domains with Scott continuous functions and continuous (algebraic) information systems with approximable mappings are equivalent.

(Joint work with Luoshan Xu and Xuxin Mao.)

5. L-domains and L-information systems

Definition

A continuous domain D is an L-domain if for every $x \in D$ each finite subset $X \subseteq \downarrow x$ has a least upper bound $\bigsqcup^x X$ in $\downarrow x$.

Definition

Let A be a c-inf and $F \subseteq_f A$. Define $\mathrm{Sup}(F)$ to be the collection of all sets $Z \in \mathrm{Con}$ such that

- $F \subseteq Z$
- $(\forall X, Y \in \mathrm{Con})[F \subseteq Y \,\&\, X \vdash Z \cup Y \quad \Rightarrow$
  $Z \cup Y \in \mathrm{Con} \,\&\, (\forall X' \in \mathrm{Con})[X' \vdash Y \Rightarrow X' \vdash Z]]$
- $(\forall Y \in \mathrm{Con})[F \subseteq Y \,\&\, (\exists \hat{X} \in \mathrm{Con})\hat{X} \vdash Y \cup Z \quad \Rightarrow$
  $(\forall c \in A)[Y \cup Z \vdash c \Rightarrow Y \vdash c]]$.

Definition

A c-inf A is an L-information system (L-inf) if the following condition holds:

$$F \subseteq_f A \,\&\, X \vdash F \quad \Rightarrow \quad (\exists Z \in \mathrm{Sup}(F))X \vdash Z.$$

Proposition

- *Let D be an L-domain with basis B. Then $\mathcal{S}(D, B)$ is an L-inf.*
- *Let A be an L-inf. Then $|A|$ is an L-domain.*

Theorem

The categories of (algebraic) L-domains with Scott continuous functions and (algebraic) L-information systems with approximable mappings are equivalent.

6. The function space construction

Definition

Let D and E be continuous domains, $u \in B_D$ and $v \in B_E$. Then the step function $(u \searrow v)\colon D \to E$ is defined by

$$(u \searrow v)(x) = \begin{cases} v & \text{if } u \ll x, \\ \bot_E & \text{otherwise.} \end{cases}$$

Proposition

- *Step functions are Scott continuous.*
- *For Scott continuous $f\colon D \to E$, if $v \ll f(u)$ then $(u \searrow v) \ll f$.*
- $f = \bigsqcup \{ (u \searrow v) \mid v \ll f(u) \}$.

Remark

The set $\{ (u \searrow v) \mid v \ll f(u) \}$ *is not directed, in general.*

Assume that D is bounded-complete and $\{ (u_i, v_i) \in B_D \times B_E \mid i \in I \}$ (I finite) is such that for any $J \subseteq I$,

$$\{ u_i \mid i \in J \} \text{ bounded} \quad \Rightarrow \quad \{ v_i \mid i \in J \} \text{ bounded.}$$

Then $\bigsqcup_{i \in I}(u_i \searrow v_i)$ exists and is Scott continuous. Moreover,

$$\bigsqcup_{i \in I}(u_i \searrow v_i)(x) = \bigsqcup \{ v_i \mid i \in I \,\&\, u_i \ll x \}.$$

In the L-domain case this no longer works. One has to add pairs (u, v), where u runs through all the local suprema of all the sets $\{ u_i \mid i \in J \}$ ($J \subseteq I$).

Definition

Let A be an c-inf, $F \subseteq_f A$ and $Y, Z \in \mathrm{Con}$. Then $Y \sim_F Z$ if

- $F \subseteq Y, Z$
- $(\forall X \in \mathrm{Con})[X \vdash Y \Leftrightarrow X \vdash Z]$
- $(\forall c \in A)[Y \vdash c \Leftrightarrow Z \vdash c]$.

Let A_1 and A_2 be c-infs, $V \subseteq_f \mathrm{Con}_1 \times \mathrm{Con}_2$ and $F_1 \subseteq_f \mathrm{pr}_1(V)$. Set

$$V[F_1] = \bigcup \{ E \in \mathrm{Con}_2 \mid (\exists X \in F_1)(X, E) \in V \}.$$

Definition

Let A_1 and A_2 be c-inf's and $V \subseteq_f \mathrm{Con}_1 \times \mathrm{Con}_2$. Then $W \subseteq \mathrm{Con}_1 \times \mathrm{Con}_2$ is a joinable extension of V if

- $V \subseteq W$
- $(\forall Y \in \mathrm{Con}_1)$
  $(\exists S \in \mathrm{Sup}(\bigcup \{\, X \in \mathrm{pr}_1(V) \mid Y \vdash_1 X \,\}))$
  $(\exists T \in \mathrm{Sup}(V[\{\, X \in \mathrm{pr}_1(V) \mid Y \vdash_1 X \,\}]))$
  $Y \vdash_1 S \,\&\, (S, T) \in W$
- $(\forall (S, T) \in W)(S, T) \in V \vee (\exists Y \in \mathrm{Con}_1)$
  $S \in \mathrm{Sup}(\bigcup \{\, X \in \mathrm{pr}_1(V) \mid Y \vdash_1 X \,\}) \,\&\, Y \vdash_1 S \,\&$
  $T \in \mathrm{Sup}(V[\{\, X \in \mathrm{pr}_1(V) \mid Y \vdash_1 X \,\}])$

- $(\forall (S,T),(S',T') \in W \setminus V)[(\exists Y \in \mathrm{Con}_1)$
  $S \sim_{\bigcup \{ X \in \mathrm{pr}_1(V) \mid Y \vdash_1 X \}} S' \,\&$
  $T \sim_{V[\{ X \in \mathrm{pr}_1(V) \mid Y \vdash_1 X \}} T' \;\Rightarrow\; (S,T) = (S',T')]$

- $(\forall (S,T),(S',T') \in W)[(\exists Y, Y' \in \mathrm{Con}_1)$
  $S \in \mathrm{Sup}(\bigcup \{ X \in \mathrm{pr}_1(V) \mid Y \vdash_1 X \}) \,\&$
  $T \in \mathit{Sup}(V[\{ X \in \mathrm{pr}_1(V) \mid Y \vdash_1 X \}]) \,\&$
  $S' \in \mathrm{Sup}(\bigcup \{ X' \in \mathrm{pr}_1(V) \mid Y' \vdash_1 X' \}) \,\&$
  $T' \in \mathit{Sup}(V[\{ X' \in \mathrm{pr}_1(V) \mid Y' \vdash_1 X' \}]) \,\&$
  $(\exists \hat{Y} \in \mathrm{Con}_1) \hat{Y} \vdash_1 S \cup S' \;\Rightarrow\; (\exists \hat{Z} \in \mathrm{Con}_2) \hat{Z} \vdash_2 T \cup T']$.

Let $\mathrm{JE}(V)$ be the collection of all joinable extensions of V.

Definition

Let A_1 and A_2 be c-inf's. Set

- $A_1 \to A_2 =$
  $\{\, W \subseteq \mathsf{Con}_1 \times \mathsf{Con}_2 \mid (\exists V \subseteq_f \mathsf{Con}_1 \times \mathsf{Con}_2) W \in \mathsf{JE}(V) \,\}$
- $\mathsf{Con}_\to = \{\, \mathcal{W} \subseteq_f A_1 \to A_2 \mid (\forall Y \in \mathsf{Con}_1) \bigcup \{\, E \in \mathsf{Con}_2 \mid (\exists X \in \mathsf{Con}_1)(X, E) \in \bigcup \mathcal{W} \,\&\, Y \vdash_1 X \,\} \in \mathsf{Con}_2 \,\}$
- $\mathcal{W} \vdash_\to V \quad \Leftrightarrow \quad (\forall (Y, Z) \in V)$
  $\bigcup \{\, E \in \mathsf{Con}_2 \mid (\exists X \in \mathsf{Con}_1)(X, E) \in \bigcup \mathcal{W} \,\&\, Y \vdash_1 X \,\} \vdash_2 Z.$

Theorem

Let A_1 and A_2 be L-inf's. Then

- $A_1 \to A_2$ *is an L-inf.*
- $|A_1 \to A_2|$ *is the L-domain of all approximable mappings between* A_1 *and* A_2.

Remark

$|A_1 \to A_2| \cong [|A_1| \to |A_2|]$