UNIVERSITÄT **SIEGEN**

Representing L-Domains as Information Systems

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Bericht Nr. 08-03

Schriften zur Theoretischen Informatik

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Abstract

A logic-oriented representation of L-domains in the style of Scott's information systems is introduced. The category of these new (L-)information systems with approximable mappings is shown to be equivalent to the category of L-domains with Scott continuous functions. As a consequence it is Cartesian closed. In the paper direct constructions for products and exponents of L-information systems are presented.

1 Introduction

Information systems and approximable mappings have been introduced by Dana Scott [12] as a logic-oriented approach to domain theory. As was shown by Larsen and Winskel [11], the category of bounded-complete algebraic domains with Scott continuous functions is equivalent to the category of information systems with approximable mappings. Information system allow the exact solution of domain equations. As is well known, using domains such solutions are only obtained up to isomorphism.

In the sequel similar structures were introduced to represent various other important classes of domains, like dI-domains, coherence spaces or SFP domains [16, 5, 17, 19, 20], to mention just a few.

Nearly all these formalisms represent only subclasses of algebraic domains. Hoofman [8] and Vickers [15] were the first to present a generalization of information systems to the continuous case. Hoofman's continuous information systems are in the spirit of Scott's information systems. They consist of a set of tokens representing bits of information, a consistency predicate on finite sets of tokens and an entailment relation between consistent sets of tokens, but they capture only the bounded-complete continuous domains. Vicker's approach is more general and allows to represent all continuous domains. However, it is not Scott style: he considers transitive dense relations.

In [13] a Scott-style treatment of the general case was presented. Continuous information systems were introduced and the equivalence of the category of these information systems with approximable mappings with the category of abstract bases and approximable relations was derived. As is well known, the latter category is equivalent to the category of continuous domains and Scott continuous functions [1]. Moreover, the continuous information systems corresponding to several important types of continuous domains were characterized. The case of L-domains, however, was left open.

*This research has been supported by the German Research Foundation (DFG) under grants no. 446 CHV 113/240/0-1, no. 436 RUS 113/850/0-1(R) and no. 445 SUA-113/20/0-1

L-domains were independently introduced by Coquand [3] and Jung [9]. As was shown by Jung [9, 10], they form a maximal Cartesian closed full subcategory of the continuous domains. In the present paper a subclass of continuous information systems, called L-information systems, is defined capturing exactly the L-domains.

It is shown that the category of L-information systems and approximable mappings is equivalent to the category of L-domains and Scott continuous functions. As a consequence many important properties such as the existence of a terminal element, finite products and exponents carry over from L-domains to L-information systems. We will, however, present direct constructions in all these cases.

Scott's original motivation for the introduction of information systems was to provide a more concrete approach to (abstract) domain theory. Therefore, he presented information system analogues of the domain constructions usually needed in giving a denotational programming language semantics.

Especially, the construction of exponents requires special attention in our case. Due to the rather weak conditions which in the L-domain case allow the construction of step functions from single-step functions, it turned out to be not just a straightforward generalization of Scott's construction for algebraic bounded-complete domains.

Note that a logic-oriented approach to L-domains has also been presented by Zhang [18]. However, the representation considered in that paper is motivated by Gentzen-style proof systems and therefore differs from Scott's original approach. Moreover, only algebraic L-domains are captured.

The results of the present research have been presented at the workshops "Logic and Information: From Logic to Constructive Reasoning", Berne, 2007, and "PCC—Proof, Computation, Complexity", Swansea, 2007.

The paper is organized as follows: Section 2 gives a short introduction to domains. In Section 3 continuous information systems are defined and important facts from [13] are recalled. Moreover, a terminal object and finite products are constructed. L-information systems are introduced in Section 4. It is shown that under the relationship between informations systems and domains considered in [13], L-information systems and L-domains correspond to each other. In Section 5 approximable mappings are studied and the equivalence between the category of L-information systems and approximable mappings and the category of L-domains and Scott continuous functions is derived. The function space construction is presented in Section 6 and in Section 7 it is verified that the L-information system thus obtained is indeed the categorical exponent. As a consequence the Cartesian closure of the category of L-information systems follows. Final remarks can be found in Section 8.

2 Domains: basic definitions and results

For any set A, we write $X \subseteq_{\text{fin}} A$ to mean that X is finite subset of A.

Let $(D, \sqsubseteq)$ be a poset. D is *pointed* if it contains a least element $\bot$. For an element $x \in D$, $\downarrow x$ denotes the principal ideal generated by x, i.e., $\downarrow x = \{ y \in D \mid y \sqsubseteq x \}$. A subset S of D is *directed*, if it is nonempty and every pair of elements in S has an upper bound in S. D is a *directed-complete partial order* (*dcpo*), if every directed subset S of D has a least upper bound $\bigsqcup S$ in D.

Assume that x, y are elements of a dcpo D. Then x is said to *approximate* y, written $x \ll y$, if for any directed subset S of D the relation $y \sqsubseteq \bigsqcup S$ always implies the existence of some $u \in S$ with $x \sqsubseteq u$. Moreover, x is *compact* if $x \ll x$. A subset B of D is a *basis* of D, if for each $x \in D$ the set $\{ u \in B \mid u \ll x \}$ contains a directed subset with least upper bound x. Note that the set of all compact elements of D is included in every basis of D. A dcpo D is said to be *continuous* (or a *continuous domain*) if it has a basis and it is called *algebraic* (or an *algebraic domain*) if its compact elements form a basis. Standard references for domain theory and its applications are [7, 6, 1, 14, 2, 4].

Definition 2.1 Let D and D' be posets. A function $f\colon D \to D'$ is *Scott continuous* if it is monotone and for any directed subset S of D with existing least upper bound,

$$\bigsqcup f(S) = f(\bigsqcup S).$$

With respect to the pointwise order the set $[D \to D']$ of all Scott continuous functions between two dcpo's D and D' is a dcpo again. Observe that it need not be continuous even if D and D' are. This is the case, however, if D' is an L-domain [1].

Definition 2.2 A pointed continuous domain D is an *L-domain*, if each pair $x, y \in D$ bounded by $z \in D$ has a least upper bound $x \sqcup^z y$ in $\downarrow z$.

As has been shown by Jung [9, 10], the category **L** of L-domains is one of the two maximal Cartesian closed full subcategories of the category $\mathbf{CONT}_\perp$ of pointed continuous domains and Scott continuous maps. The same holds for the category **aL** of algebraic L-domains with respect to the category $\mathbf{ALG}_\perp$ of pointed algebraic domains. The one-point domain is the terminal object in these categories and the categorical product $D \times E$ of two domains D and E is the Cartesian product of the underlying sets ordered coordinatewise.

Lemma 2.3 ([4]) *Let D be an L-domain and $x_1, x_2 \in D$ with upper bounds $y, z \in D$. Then the following two statements hold:*

1. *If y, z have an upper bound in D, then $x_1 \sqcup^y x_2 = x_1 \sqcup^z x_2$.*
2. *If $x_1 \ll y$ and $x_2 \ll y$, then $x_1 \sqcup^y x_2 \ll y$.*

3 Continuous information systems

As has been shown in [13], continuous domains can be represented in a logic-oriented way by information systems.

Definition 3.1 Let A be a set, Con a collection of finite subsets of A and $\vdash \subseteq \mathrm{Con} \times A$. Then $(A, \mathrm{Con}, \vdash)$ is a *continuous information system* if the following six conditions hold for all sets $X, Y \in \mathrm{Con}$, elements $a \in A$ and nonempty finite subsets F of A:

1. $\{a\} \in \mathrm{Con}$,
2. $X \vdash a \Rightarrow X \cup \{a\} \in \mathrm{Con}$,
3. $(Y \supseteq X \wedge X \vdash a) \Rightarrow Y \vdash a$,
4. $(X \vdash Y \wedge Y \vdash a) \Rightarrow X \vdash a$,
5. $X \vdash a \Rightarrow (\exists Z \in \mathrm{Con})(X \vdash Z \wedge Z \vdash a)$,
6. $X \vdash F \Rightarrow (\exists Z \in \mathrm{Con})(Z \supseteq F \wedge X \vdash Z)$,

where $X \vdash Y$ means that $X \vdash b$, for all $b \in Y$.

Note that Conditions 3.1(5) and (6) can be replaced by a single requirement.

Proposition 3.2 ([13]) *Let A be a nonempty set,* Con *be a collection of finite subsets of A and $\vdash \subseteq \mathrm{Con} \times A$ such that Conditions 3.1(2, 3) hold. Then Requirements 3.1(5, 6) together are equivalent to the following statement:*

$$(\forall X \in \mathrm{Con})(\forall F \subseteq_{\mathrm{fin}} A)[X \vdash F \Rightarrow (\exists Z \in \mathrm{Con})(X \vdash Z \wedge Z \vdash F)].$$

If $(A, \mathrm{Con}, \vdash)$ is a continuous information system then the elements of A are usually called *tokens*, the sets in Con *consistent* and the relation $\vdash$ *entailment relation*. Tokens should be thought of as atomic propositions giving information about data and consistent sets as representing consistent finite conjunctions of such propositions. The entailment relation then tells us which propositions are derivable from what.

In this formalism each element of a data structure corresponds to the set of all atomic statements that can be made about it. Characteristic properties of such collections of tokens are that any finite subcollection is consistent, the collection is closed under deduction, and any of its tokens is derivable from some finite consistent subcollection.

Definition 3.3 Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. A subset x of A is a *state* of $(A, \mathrm{Con}, \vdash)$ if the next three conditions hold:

1. $(\forall F \subseteq_{\mathrm{fin}} x)(\exists Y \in \mathrm{Con})(F \subseteq Y \wedge Y \subseteq x)$,
2. $(\forall X \in \mathrm{Con})(\forall a \in A)(X \subseteq x \wedge X \vdash a \Rightarrow a \in x)$,
3. $(\forall a \in x)(\exists X \in \mathrm{Con})(X \subseteq x \wedge X \vdash a)$.

Similarly to Proposition 3.2 the requirement that $F \subseteq Y$ in 3.3(1) can be replaced by an entailment condition.

Proposition 3.4 ([13]) *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system and x be a subset of A such that Condition 3.3(3) holds. Then Requirement 3.3(1) is equivalent to the following statement:*

$$(\forall F \subseteq_{\mathrm{fin}} x)(\exists Y \in \mathrm{Con})(Y \subseteq x \wedge Y \vdash F).$$

With respect to set inclusion the states of A form a partially ordered set which we denote by $|A|$.

Proposition 3.5 ([13]) *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. Then $\mathcal{D}(A) = (|A|, \subseteq)$ is a continuous domain with basis $\widehat{\mathrm{Con}} = \{\, \widehat{X} \mid X \in \mathrm{Con} \,\}$, where $\widehat{X} = \{\, a \in A \mid X \vdash a \,\}$. Moreover, for $x, y \in |A|$,*

$$x \ll y \Leftrightarrow (\exists V \in \mathrm{Con})(V \subseteq y \wedge V \vdash x).$$

If, conversely, $(D, \sqsubseteq)$ is a continuous domain with basis B, set

$$\mathrm{Con} = \{\, X \subseteq_{\mathrm{fin}} B \mid X \text{ directed with respect to } \sqsubseteq \,\}$$

and define

$$X \vdash u \Leftrightarrow (\exists v \in X) u \ll v,$$

for $X \in \mathrm{Con}$ and $u \in B$. Then the next statement follows as in [13].

Proposition 3.6 *Let D be a continuous domain with basis B. Then $\mathcal{C}(D) = (B, \mathrm{Con}, \vdash)$ is a continuous information system so that the domains $\mathcal{D}(\mathcal{C}(D))$ and D are isomorphic.*

The isomorphism is given by the two functions $\mathrm{ds}_D \in [D \to |\mathcal{C}(D)|]$ and $\mathrm{sp}_D \in [|\mathcal{C}(D)| \to D]$ which are inverse to each other:

$$\mathrm{ds}_D(x) = \{ u \in B \mid u \ll x \}, \quad \mathrm{sp}_D(z) = \bigsqcup z.$$

The next two results characterize those continuous information systems that generate algebraic and/or pointed domains.

Proposition 3.7 *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. Then $|A|$ is algebraic if, and only if, $(A, \mathrm{Con}, \vdash)$ satisfies the following requirement*

$$(\forall X, Y \in \mathrm{Con})[X \vdash Y \Rightarrow (\exists Z \in \mathrm{Con})(X \vdash Z \wedge Z \vdash Z \wedge Z \vdash Y)]. \tag{ALG}$$

Obviously, the extra requirement (ALG) entails Condition 3.1(5). A continuous information system satisfying Condition (ALG) is also called *algebraic information system.*

Proposition 3.8 *Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. Then $|A|$ is pointed if, and only if, $\emptyset \in \mathrm{Con}$ or there is some $a \in A$ with $X \vdash a$, for all $X \in \mathrm{Con}$.*

Definition 3.9 Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system. A *truth element* of A is an element $t \in A$ satisfying the following two conditions:

1. $(\forall X \in \mathrm{Con}) X \vdash t$
2. $(\forall X \in \mathrm{Con})(\forall c \in A)[U \cup \{t\} \vdash c \Rightarrow U \vdash c]$

Next we will show how the one-point domain and the product of continuous domains can be generated from continuous information systems.

Proposition 3.10 *Let $T = (\{\bullet\}, \{\{\bullet\}\}, \{(\{\bullet\}, \bullet)\})$. Then T is an algebraic information system and $|T|$ is the one-point domain.*

Now, let $(A_0, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$ be continuous information systems and pr_0 and pr_1, respectively, be the canonical projections of $A_0 \times A_1$ onto the first and second component. Set $A_\times = A_0 \times A_1$,

$$\mathrm{Con}_\times = \{ X \subseteq_{\mathrm{fin}} A_\times \mid \mathrm{pr}_0(X) \in \mathrm{Con}_0 \wedge \mathrm{pr}_1(X) \in \mathrm{Con}_1 \}$$

and for $X \in \mathrm{Con}_\times$ and $(a_0, a_1) \in A_\times$ define

$$X \vdash_\times (a_0, a_1) \Leftrightarrow \mathrm{pr}_0(X) \vdash_0 a_0 \wedge \mathrm{pr}_1(X) \vdash a_1.$$

Then $(A_\times, \mathrm{Con}_\times, \vdash_\times)$ is a continuous information system, the *product* of $(A_0, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$.

Lemma 3.11 *For $z \in |A_\times|$ and $i = 0, 1$ the following two statements hold:*

1. $\mathrm{pr}_i(z) \in |A_i|$.
2. $z = \mathrm{pr}_0(z) \times \mathrm{pr}_1(z)$.

Proof: (1) Without restriction let $i = 0$ and z be nonempty. We only verify Condition 3.3(2), the other two being obvious. Let $Y_0 \in \mathrm{Con}_0$ be a subset of $\mathrm{pr}_0(z)$ and $a_0 \in A_0$ with $Y_0 \vdash_0 a_0$. Then there is some $X \in \mathrm{Con}_\times$ with $X \subseteq z$. Let a_1 be some element of $\mathrm{pr}_1(z)$. By Condition 3.3(3) there is some $Z \subseteq z$ with $Z \in \mathrm{Con}_\times$ so that $\mathrm{pr}_1(Z) \vdash_1 a_1$. Apply Condition 3.3(1) to obtain some $V \in \mathrm{Con}_\times$ with $X \cup Z \subseteq V \subseteq z$. It follows that $V \vdash_\times (a_0, a_1)$, which implies that $(a_0, a_1) \in z$. Hence $a_0 \in \mathrm{pr}_0(z)$.

(2) follows similarly.

Proposition 3.12 *Let $(A_0, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$ be continuous information systems. Then $(A_\times, \mathrm{Con}_\times, \vdash_\times)$ is a continuous information system as well and the domains $|A_\times|$ and $|A_0| \times |A_1|$ are isomorphic. Moreover:*

1. *If both A_0 and A_1 have a truth element, so does $A_\times$.*

2. *If A_0 and A_1 are both algebraic, then $A_\times$ is algebraic too.*

4 L-information systems

The following definition captures essential properties of local least upper bounds.

Definition 4.1 Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system and $F \subseteq_{\mathrm{fin}} A$. Define $\mathrm{Sup}(F)$ to be the collection of all sets $Z \in \mathrm{Con}$ with $F \subseteq Z$ such that for all $Y \in \mathrm{Con}$ with $F \subseteq Y$, if for some $V \in \mathrm{Con}$, $V \vdash Z \cup Y$, then

1. $Z \cup Y \in \mathrm{Con}$,

2. $(\forall X \in \mathrm{Con})[X \vdash Y \Rightarrow X \vdash Z]$,

3. $(\forall c \in A)[Z \cup Y \vdash c \Rightarrow Y \vdash c]$.

As follows from the correspondence between continuous information systems and continuous domains, finite sets in Con represent existing finite least upper bounds in the corresponding domain. In L-domains least upper bounds of arbitrary finite sets need exist only relative to principal ideals. Thus, different representations of a least upper bound must behave consistently in this case. This is expressed in Condition (1). The second requirement says that an upper bound of an upper bound of some finite set must be an upper bound of the supremum of the set.

Lemma 4.2 1. *If $F \in \mathrm{Con}$ then $F \in \mathrm{Sup}(F)$.*

2. *If A has a truth element t, then $\{t\} \in \mathrm{Sup}(\emptyset)$.*

The sets in $\mathrm{Sup}(F)$ are thought of as representations of local least upper bounds of the finite set represented by F. A least upper bound of finitely many elements of a principal ideal may have several representations, but they should at least be equivalent.

Definition 4.3 Let $(A, \mathrm{Con}, \vdash)$ be a continuous information system and $F \subseteq_{\mathrm{fin}} A$. Two consistent sets Y and Z are *equivalent* with respect to F, written $Y \underset{F}{\sim} Z$, if the following conditions are satisfied:

1. $F \subseteq Y \wedge F \subseteq Z$,

2. $(\forall U \in \mathrm{Con})[Y \cup U \in \mathrm{Con} \Leftrightarrow Z \cup U \in \mathrm{Con}]$,

3. $(\forall X \in \mathrm{Con})[X \vdash Y \Leftrightarrow X \vdash Z]$,

4. $(\forall U \in \mathrm{Con})[Y \cup U, Z \cup U \in \mathrm{Con} \Rightarrow (\forall c \in A)[Y \cup U \vdash c \Leftrightarrow Z \cup U \vdash c]$.

Lemma 4.4 *Let $F \subseteq_{\mathrm{fin}} A$ and $Z, Z' \in \mathrm{Sup}(F)$ such that $V \vdash Z \cup Z'$, for some $V \in \mathrm{Con}$. Then $Z \underset{F}{\sim} Z'$.*

Let us now introduce the notion of an L-information system. The definition is motivated by the second statement in Lemma 2.3.

Definition 4.5 A continuous information system $(A, \mathrm{Con}, \vdash)$ is an *L-information system* if it contains a truth element and the following condition (L) holds for all $X \in \mathrm{Con}$ and all nonempty finite subsets F of A:

$$X \vdash F \Rightarrow (\exists Z \in \mathrm{Sup}(F)) X \vdash Z. \tag{L}$$

Obviously, Condition (L) strengthens Condition 3.1(6). An algebraic information system that has a truth element and satisfies Condition (L) is called *algebraic* L-information system.

Proposition 4.6 *Let D be an L-domain. Then $\mathcal{C}(D)$ is an L-information system.*

Proof: As is easily verified, the smallest element of D is a truth element of $\mathcal{C}(D)$. Because of Proposition 3.6 it remains to show that Condition (L) is satisfied. Let to this end F be a nonempty finite subset of A and $X \in \mathrm{Con}$ with $X \vdash F$.

Without restriction, let $F \notin \mathrm{Con}$. Since X is directed, its least upper bound $\bigsqcup X$ exists in D. Moreover, $u \ll \bigsqcup X$, for all $u \in F$. Hence F has a least upper bound z in $\downarrow\bigsqcup X$. Moreover, $z \ll \bigsqcup X$. Set $Z = F \cup \{z\}$. Then $Z \supseteq F$. Moreover, Z is directed. Hence $Z \in \mathrm{Con}$.

Now, let $Y \in \mathrm{Con}$ with $Y \supseteq F$ and assume that $V \vdash Y \cup Z$, for some $V \in \mathrm{Con}$. We have to verify Conditions 4.1(1–3).

(1) Since Y and V, respectively, are directed finite subsets of D, they contain a greatest element, say v and y. Then we have that $y, z \ll v$. Moreover, y and z are upper bounds of F. Let z' be the least upper bound of F in $\downarrow v$. Then $z' \sqsubseteq z$. Hence $z' \ll \bigsqcup X$, which implies that $z' = z$. As a consequence we have that $z \sqsubseteq y$. Thus $Y \cup Z$ is directed, i.e. $Y \cup Z \in \mathrm{Sup}(F)$.

(2) Let $X \in \mathrm{Con}$ with $X \vdash Y$. Then $y \ll x$, for some $x \in X$. As we have just seen, $z \sqsubseteq y$. Thus, $z \ll x$ as well, i.e. $X \vdash Z$.

(3) Let c be a base element of D with $Z \cup Y \vdash c$. Then $c \ll y$, as y is the greatest element of $Z \cup Y$. Consequently $Y \vdash c$.

It follows that $Z \in \mathrm{Sup}(F)$. Since $z \ll \bigsqcup X$ we also have that $X \vdash Z$.

Conversely, we have that every L-information system generates an L-domain.

Proposition 4.7 *Let $(A, \mathrm{Con}, \vdash)$ be an L-information system. Then $\mathcal{D}(A)$ is an L-domain.*

Proof: Because of Propositions 3.5 and 3.8 we only have to show that finite least upper bounds exist in every principal ideal of $\mathcal{D}(A)$.

Let $z_1, z_2, x \in |A|$ with $z_1, z_2 \subseteq x$ and set

$$z = \{\, a \in A \mid (\exists F \subseteq_{\mathrm{fin}} z_1 \cup z_2)(\exists X \in \mathrm{Con})(\exists Z \in \mathrm{Sup}(F))(X \subseteq x \wedge X \vdash Z \wedge Z \vdash a) \,\}.$$

Then $z \subseteq x$. We will show that z is the least upper bound of z_1 and z_2 in $\downarrow x$.

Claim 1 $z_i \subseteq z$, for $i = 1, 2$.

Let $a \in z_i$. Then, by 3.3(3), there is a consistent set $Y \subseteq z_i$ with $Y \vdash a$. Since $z_i \subseteq x$, we obtain that also $Y \subseteq x$. By Proposition 3.4, there is then a consistent set $X \subseteq x$ with $X \vdash Y$. Note that $Y \in \mathrm{Sup}(Y)$. Moreover, $Y \subseteq z_1 \cup z_2$. It follows that $a \in z$.

Claim 2 $z \in |A|$.

We have to verify Conditions 3.3(1–3).

(1) Let $G \subseteq_{\mathrm{fin}} z$. We will show that there is a set $Y \in \mathrm{Con}$ with $G \subseteq Y \subseteq z$.

For any $a \in G$ there are sets $F_a \subseteq_{\mathrm{fin}} z_1 \cup z_2$, $X_a \in \mathrm{Con}$ and $Z_a \in \mathrm{Sup}(F_a)$ so that $X_a \vdash Z_a$, $X_a \subseteq x$ and $Z_a \vdash a$. Set $\widetilde{F} = \bigcup\{ F_a \mid a \in G \}$ and $\widetilde{X} = \{ X_a \mid a \in G \}$. Then $\widetilde{X} \subseteq_{\mathrm{fin}} x$. Hence there is some $\overline{X} \in \mathrm{Con}$ with $\widetilde{X} \subseteq \overline{X} \subseteq x$. It follows that $\overline{X} \vdash Z_a$, for every $a \in G$. Since $Z_a \supseteq F_a$, we thus have that $\overline{X} \vdash \widetilde{F}$. By Condition (L) there is some $\overline{Z} \in \mathrm{Sup}(\widetilde{F})$ with $\overline{X} \vdash \overline{Z}$. Then $\overline{Z} \supseteq \widetilde{F} \supseteq F_a$, for every $a \in G$. Since $\overline{X} \vdash Z_a$ and $Z_a \in \mathrm{Sup}(F_a)$, we obtain for each $c \in A$ with $Z_a \vdash c$ that also $\overline{Z} \vdash c$. Thus $\overline{Z} \vdash G$. With 3.1(6) we have that there is some consistent superset Y of G with $\overline{Z} \vdash Y$.

It remains to show that $Y \subseteq z$. Let $c \in Y$. Then $\overline{Z} \vdash c$. Since $\widetilde{F} \subseteq_{\mathrm{fin}} z_1 \cup z_2$, $\overline{Z} \in \mathrm{Sup}(\widetilde{F})$ and $\overline{X} \in \mathrm{Con}$ such that $\overline{X} \vdash \overline{Z}$ and $\overline{X} \subseteq x$, it follows from the definition of z that $c \in z$.

(2) Let $X \in \mathrm{Con}$ and $a \in A$ with $X \subseteq z$ and $X \vdash a$. We need to show that $a \in z$.

As above we have for any $c \in X$ that there are sets $F_c \subseteq_{\mathrm{fin}} z_1 \cup z_2$, $X_c \in \mathrm{Con}$ und $Z_c \in \mathrm{Sup}(F_c)$ such that $X_c \subseteq x$, $X_c \vdash Z_c$ and $Z_c \vdash c$. Set $\widetilde{F} = \bigcup\{ F_c \mid c \in X \}$ and $\widetilde{X} = \bigcup\{ X_c \mid c \in X \}$. Then $\widetilde{X} \subseteq_{\mathrm{fin}} x$ and hence, by 3.3(1), there is some $\overline{X} \in \mathrm{Con}$ with $\widetilde{X} \subseteq \overline{X} \subseteq x$. Since $\overline{X} \supseteq X_c$, $X_c \vdash Z_c$ and $Z_c \supseteq F_c$, for $c \in X$, we have that $\overline{X} \vdash F_c$, for every $c \in X$, i.e. $\overline{X} \vdash \widetilde{F}$. Apply Condition (L) to obtain a set $\overline{Z} \in \mathrm{Sup}(\widetilde{F})$ with $\overline{X} \vdash \overline{Z}$. Then $\overline{Z} \supseteq \widetilde{F} \supseteq F_c$, for each $c \in X$. Since $\overline{X} \vdash \overline{Z} \cup Z_c$ and $Z_c \in \mathrm{Sup}(F_c)$, we have for all $b \in A$ with $Z_c \vdash b$ that also $\overline{Z} \vdash b$. Remember that $Z_c \vdash c$, for all $c \in X$. Thus $\overline{Z} \vdash X$, from which we obtain that $\overline{Z} \vdash a$, as $X \vdash a$, by our assumption. In addition we have that $\overline{X} \in \mathrm{Con}$, $\widetilde{F} \subseteq_{\mathrm{fin}} z_1 \cup z_2$ and $\overline{Z} \in \mathrm{Sup}(\widetilde{F})$ with $\overline{X} \subseteq x$ and $\overline{X} \vdash \overline{Z}$. It follows that $a \in z$.

(3) Let $a \in z$. We have to show that there is some $Y \in \mathrm{Con}$ with $Y \subseteq z$ and $Y \vdash a$.

Since $a \in z$, there are sets $F \subseteq_{\mathrm{fin}} z_1 \cup z_2$, $X \in \mathrm{Con}$ and $Z \in \mathrm{Sup}(F)$ so that $X \subseteq x$, $X \vdash Z$ and $Z \vdash a$. With Condition 3.1(5) we moreover obtain that there is some $Y \in \mathrm{Con}$ with $Z \vdash Y \vdash a$. It remains to show that $Y \subseteq z$. Note hereto that $Z \vdash b$, for every $b \in Y$, $Z \in \mathrm{Sup}(F)$, $F \subseteq_{\mathrm{fin}} z_1 \cup z_2$, $X \vdash Z$ and $X \subseteq x$.

Claim 3 $(\forall y \in |A|)[z_1, z_2 \subseteq y \subseteq x \Rightarrow z \subseteq y]$.

Let $y \in |A|$ with $z_1, z_2 \subseteq y \subseteq x$ and $a \in z$. Then there are sets $F \subseteq_{\mathrm{fin}} z_1 \cup z_2$, $X \in \mathrm{Con}$ and $Z \in \mathrm{Sup}(F)$ so that $X \subseteq x$, $X \vdash Z$ and $Z \vdash a$. Since $z_1, z_2 \subseteq y$, it follows that $F \subseteq_{\mathrm{fin}} y$. By Proposition 3.4 there is therefore some $Y \in \mathrm{Con}$ with $Y \subseteq y$ and $Y \vdash F$. With Condition 3.1(6) we obtain the existence of some set $V \in \mathrm{Con}$ such that $V \supseteq F$ and $Y \vdash V$. Then $V \subseteq y$, by 3.3(2). Since $y \subseteq x$, we have that $X \cup V \subseteq_{\mathrm{fin}} x$. Thus there is some $\overline{X} \in \mathrm{Con}$ with $\overline{X} \subseteq x$ and $\overline{X} \vdash X \cup V$. It follows that $\overline{X} \vdash Z \cup V$. Remember that $Z \vdash a$ and $Z \in \mathrm{Sup}(F)$. Therefore $V \vdash a$ as well. Because $V \subseteq y$, we obtain that $a \in y$.

Obviously the one-point information system T is an L-information system.

Proposition 4.8 *Let* $(A_0, \mathrm{Con}_0, \vdash_0)$ *and* $(A_1, \mathrm{Con}_1, \vdash_1)$ *be L-information systems. Then* $(A_\times, \mathrm{Con}_\times, \vdash_\times)$ *is also an L-information system.*

Note here that for $F \subseteq_{\mathrm{fin}} A_0 \times A_1$ and $Z_i \in \mathrm{Sup}_i(\mathrm{pr}_i(F))$ $(i = 0, 1)$, $Z_0 \times Z_1 \in \mathrm{Sup}_\times(F)$.

Lemma 4.9 *Let* $(A, \mathrm{Con}, \vdash)$ *be an L-information system and* $U, X_1, \ldots, X_n \in \mathrm{Con}$ *such that* $U \vdash X_i$ *and* $X_i \vdash X_i$*, for* $i = 1, \ldots, n$*. Then* $Y \vdash Y$*, for all* $Y \in \mathrm{Sup}(X_1 \cup \cdots \cup X_n)$ *with* $U \vdash Y$*.*

Proof: Since $Y \supseteq X_1 \cup \cdots \cup X_n$ and $X_i \vdash X_i$, for $i = 1, \ldots, n$, we have that $Y \vdash X_1 \cup \cdots \cup X_n$. Therefore, by Condition (L), there is some $Z \in \mathrm{Sup}(X_1 \cup \cdots \cup X_n)$ with $Y \vdash Z$. Because $U \vdash Y$, it follows that $U \vdash Y \cup Z$. Thus

$$Y \underset{X_1 \cup \cdots \cup X_n}{\sim} Z,$$

by Lemma 4.4. As $Y \vdash Z$, we obtain that $Y \vdash Y$.

5 Approximable mappings

In order to study the relationship between information systems and domains from a category-theoretic point of view, appropriate morphisms between information systems have to be introduced.

Definition 5.1 Let $(A, \mathrm{Con}, \vdash)$ and $(A', \mathrm{Con}', \vdash')$ be continuous information systems. An *approximable mapping* H between A and A', written $H \colon A \Vdash A'$, is a relation between Con and A' satisfying for all $X, X' \in \mathrm{Con}$, $Y \in \mathrm{Con}'$ and $b \in A'$, as well as all nonempty finite subsets F of A' the following six conditions, where Condition (6) is only required, if A and A' have truth elements t and t', respectively:

1. $(XHY \wedge Y \vdash' b) \Rightarrow XHb$,
2. $XHF \Rightarrow (\exists Z \in \mathrm{Con}')(F \subseteq Z \wedge XHZ)$,
3. $(X \vdash X' \wedge X'Hb) \Rightarrow XHb$,
4. $(X \supseteq X' \wedge X'Hb) \Rightarrow X'Hb$,
5. $XHb \Rightarrow (\exists Z \in \mathrm{Con})(\exists Z' \in \mathrm{Con}')(X \vdash Z \wedge ZHZ' \wedge Z' \vdash' b)$,
6. $\{t\}Ht'$,

where XHY means that XHc, for all $c \in Y$.

As follows from Definition 3.1, entailment relations are special approximable morphisms. For $X \in \mathrm{Con}$ and $a \in A$ set $X \,\mathrm{Id}\, a$ if $X \vdash a$. Then $\mathrm{Id} \colon A \Vdash A$ such that for all $H \colon A \Vdash A'$, $H \circ \mathrm{Id}_{A'} = H = \mathrm{Id}_A \circ H$, where for approximable mappings $H \colon A \Vdash A'$ and $G \colon A' \Vdash A''$ their composition $H \circ G \colon A \Vdash A''$ is defined by

$$X(H \circ G)c \Leftrightarrow (\exists Y \in \mathrm{Con}')(XHY \wedge YGc).$$

Let **LINF** be the category of L-information systems and approximable mappings and **aLINF** the full subcategory of algebraic L-information systems.

Proposition 5.2 *The one-point information system* T *is a terminal object in* **aLINF** *and* **LINF**.

Proof: Let $(A, \mathrm{Con}, \vdash)$ be an L-information system and $F = \mathrm{Con} \times \{\bullet\}$. It suffices to show that $F\colon A \Vdash T$. We will only verify Condition 5.1(5), the other ones being obvious.

Let to this end $X \in \mathrm{Con}$ and $t \in A$ be a truth element of A. Then $X \vdash t$. Moreover, $\{t\} \in \mathrm{Con}$, $\{t\}F\{\bullet\}$ and $\{\bullet\} \vdash_T \bullet$.

For two L-information systems $(A_0, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$ define the relations $\mathrm{Pr}_i \subseteq \mathrm{Con}_\times \times A_i$ $(i = 0, 1)$ by

$$X \,\mathrm{Pr}_i\, a_i \Leftrightarrow \mathrm{pr}_i(X) \vdash_i a_i$$

Lemma 5.3 *For $i = 0, 1$, $\mathrm{Pr}_i\colon A_\times \Vdash A_i$*

Proof: Again, we only verify Condition 5.1(5). Let $X \in \mathrm{Con}_\times$ and $a_i \in A_i$ with $X \,\mathrm{Pr}_i\, a_i$. Then $\mathrm{pr}_i(X) \vdash_i a_i$. Hence there are $Z_i, Z_i' \in \mathrm{Con}_i$ so that $\mathrm{pr}_i(X) \vdash_i Z_i \vdash_i Z_i' \vdash_i a_i$. Let t_{1-i} be a truth element of A_{1-i}. Then $\mathrm{pr}_{1-i}(X) \vdash_{1-i} t_{1-i}$. Set $Z = Z_0 \times \{t_1\}$, if $i = 0$, and $Z = \{t_0\} \times Z_1$, otherwise. We obtain that $X \vdash_\times Z \,\mathrm{Pr}_i\, Z_i' \vdash_i a_i$.

Proposition 5.4 *For L-information systems A_0 and A_1, $(A_\times, \mathrm{Pr}_0, \mathrm{Pr}_1)$ is their categorical product.*

Note that for approximating mappings $F_0\colon A \Vdash A_0$ and $F_1\colon A \Vdash A_1$ the mediating morphism $\langle F_0, F_1\rangle\colon A \Vdash A_\times$ is given by

$$X\langle F_0, F_1\rangle(a_0, a_1) \Leftrightarrow \mathrm{pr}_0(X)F_0a_0 \wedge \mathrm{pr}_1(X)F_1a_1.$$

As we have seen above, there is a close connection between L-information systems and L-domains. It can be extended to the corresponding morphisms, i.e. approximable mappings and Scott continuous functions, so that we obtain an equivalence between **LINF** and **L**.

Let D and D' be L-domains with bases B and B', respectively. For $f \in [D \to D']$ define the relation $\mathcal{C}(f) \subseteq \mathrm{Con}_{\mathcal{C}(D)} \times B'$ by

$$X\mathcal{C}(f)a \Leftrightarrow (\exists c \in X)a \ll' f(c).$$

Lemma 5.5 $\mathcal{C}(f)\colon \mathcal{C}(D) \Vdash \mathcal{C}(D')$

Next, let A, A' be L-information systems and $H\colon A \Vdash A'$. For $x \in |A|$ set

$$\mathcal{D}(H)(x) = \{\, b \in A' \mid (\exists X \in \mathrm{Con}_A)(X \subseteq x \wedge XHb) \,\}.$$

Lemma 5.6 $\mathcal{D}(H) \in [\mathcal{D}(A) \to \mathcal{D}(A')]$.

Both results follow as in [13]. It is readily seen that $\mathcal{C}\colon \mathbf{L} \to \mathbf{LINF}$ and $\mathcal{D}\colon \mathbf{LINF} \to \mathbf{L}$ are functors. As will be shown in what follows, they constitute an equivalence between both categories.

For a category $\mathbf{C}$ let $\mathcal{I}_{\mathbf{C}}$ be the identical functor on $\mathbf{C}$. We first show that there is a natural isomorphism $\tau\colon \mathcal{I}_{\mathbf{L}} \to \mathcal{D} \circ \mathcal{C}$. Let to this end D be an L-domain with basis B. Then $\mathcal{D}(\mathcal{C}(D)) = (|B|, \subseteq)$. As we have seen in Section 3, the domains D and $\mathcal{D}(\mathcal{C}(D))$ are isomorphic via the functions $\mathrm{ds}_D \in [D \to |B|]$ and $\mathrm{sp}_D \in [|B| \to D]$. Set $\tau_D = \mathrm{ds}_D$.

Proposition 5.7 *$\tau\colon \mathcal{I}_{\mathbf{L}} \to \mathcal{D} \circ \mathcal{C}$ is a natural isomorphism.*

Proof: It remains to show that for L-domains D and D' with bases B and B', respectively, and a function $f \in [D \to D']$,

$$\tau_{D'} \circ f = \mathcal{D}(\mathcal{C}(f)) \circ \tau_D.$$

Let $x \in D$ and remember that $\tau_D(x) = \{\, v \in B \mid v \ll x \,\}$. Then

$$\begin{aligned}
\tau_{D'}(f(x)) &= \{\, u \in B' \mid u \ll' f(x) \,\} \\
&= \{\, u \in B' \mid (\exists v \in B)(v \ll x \wedge u \ll' f(v)) \,\} \\
&= \{\, u \in B' \mid (\exists v \in \tau_D(x)) u \ll' f(v) \,\} \\
&= \{\, u \in B' \mid (\exists X \in \mathrm{Con}_{\mathcal{C}(D)})(X \subseteq \tau_D(x) \wedge (\exists v \in X) u \ll' f(v)) \,\} \\
&= \{\, u \in B' \mid (\exists X \in \mathrm{Con}_{\mathcal{C}(D)})(X \subseteq \tau_D(x) \wedge X \mathcal{C}(f) u) \,\} \\
&= \mathcal{D}(\mathcal{C}(f))(\tau_D(x)).
\end{aligned}$$

Next we show that there is also a natural isomorphism $\eta \colon \mathcal{I}_{\mathbf{LINF}} \to \mathcal{C} \circ \mathcal{D}$. Let $(A, \mathrm{Con}, \vdash)$ be an L-information system. Then $\mathcal{C}(\mathcal{D}(A)) = (\widehat{\mathrm{Con}}, \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))}, \vdash_{\mathcal{C}(\mathcal{D}(A))})$, where $\mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))}$ is the collection of all finite subsets of $\widehat{\mathrm{Con}}$ that are directed with respect to set inclusion, and

$$\mathfrak{X} \vdash_{\mathcal{C}(\mathcal{D}(A))} \widehat{X} \Leftrightarrow (\exists \widehat{Z} \in \mathfrak{X})(\exists U \in \mathrm{Con})(Z \vdash U \wedge U \vdash \widehat{X}),$$

for $\mathfrak{X} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))}$ and $\widehat{X} \in \widehat{\mathrm{Con}}$. Set

$$\begin{aligned}
S_A &= \{\, (X, \widehat{Y}) \in \mathrm{Con} \times \widehat{\mathrm{Con}} \mid \{\widehat{X}\} \vdash_{\mathcal{C}(\mathcal{D}(A))} \widehat{Y} \,\}, \\
T_A &= \{\, (\mathfrak{X}, a) \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))} \times A \mid (\exists \widehat{Z} \in \mathfrak{X}) Z \vdash a \,\}.
\end{aligned}$$

Lemma 5.8 *1.* $S_A \colon A \Vdash \mathcal{C}(\mathcal{D}(A))$.

2. $T_A \colon \mathcal{C}(\mathcal{D}(A)) \Vdash A$.

Proof: (1) Conditions 5.1(1–4, 6) are readily verified. We only verify the remaining one. Let to this end $X \in \mathrm{Con}$ and $\widehat{Y} \in \widehat{\mathrm{Con}}$ with $X S_A \widehat{Y}$. Then $\{\widehat{X}\} \vdash_{\mathcal{C}(\mathcal{D}(A))} \widehat{Y}$. Hence there is some $\mathfrak{Z} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))}$ such that $\{\widehat{X}\} \vdash_{\mathcal{C}(\mathcal{D}(A))} \mathfrak{Z} \vdash_{\mathcal{C}(\mathcal{D}(A))} \widehat{Y}$. The first of the two entailment statements means that $\{\widehat{X}\} \vdash_{\mathcal{C}(\mathcal{D}(A))} \widehat{Z}$, for all $\widehat{Z} \in \mathfrak{Z}$. It follows that there are $U_{\widehat{Z}}, U'_{\widehat{Z}} \in \mathrm{Con}$, for each $\widehat{Z} \in \mathfrak{Z}$, with $X \vdash U_{\widehat{Z}} \vdash U'_{\widehat{Z}} \vdash Z$. As a consequence there is some $V \in \mathrm{Con}$ such that $V \supseteq \bigcup \{\, U_{\widehat{Z}} \mid \widehat{Z} \in \mathfrak{Z} \,\}$, $X \vdash V$ and $V \vdash U'_{\widehat{Z}} \vdash \widehat{Z}$. So we have that there is some $V \in \mathrm{Con}$ and some $\mathfrak{Z} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))}$ with $X \vdash V$, $\{\widehat{V}\} \vdash_{\mathcal{C}(\mathcal{D}(A))} \mathfrak{Z}$ and $\mathfrak{Z} \vdash_{\mathcal{C}(\mathcal{D}(A))} \widehat{Z}$.

(2) Again we only consider Condition 5.1(5). Let $\mathfrak{X} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))}$ and $a \in A$ with $\mathfrak{X} T_A a$. Then there is some $X \in \mathfrak{X}$ with $X \vdash a$. It follows that there are $U, V, Z \in \mathrm{Con}$ so that $X \vdash U \vdash V \vdash Z \vdash a$. Thus $X \vdash U \vdash \widehat{V}$. Set $\mathfrak{Z} = \{\widehat{V}\}$. Then we have that $\mathfrak{X} \vdash_{\mathcal{C}(\mathcal{D}(A))} \mathfrak{Z}$, $\mathfrak{Z} T_A Z$ and $Z \vdash a$.

Lemma 5.9 *1.* $S_A \circ T_A = \mathrm{Id}_A$.

2. $T_A \circ S_A = \mathrm{Id}_{\mathcal{C}(\mathcal{D}(A))}$.

Proof: (1) Let $X \in \mathrm{Con}$ and $a \in A$. Then we have that

$$\begin{aligned} X(S_A \circ T_A)a &\Leftrightarrow (\exists \mathfrak{Y} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))})(X S_A \mathfrak{Y} \wedge \mathfrak{Y} T_A a) \\ &\Leftrightarrow (\exists \mathfrak{Y} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))})(\forall \widehat{Y} \in \mathfrak{Y})(\{\widehat{X}\} \vdash_{\mathcal{C}(\mathcal{D}(A))} \widehat{Y} \wedge (\exists \widehat{Y} \in \mathfrak{Y}) Y \vdash a) \\ &\Leftrightarrow (\exists \mathfrak{Y} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))})(\forall \widehat{Y} \in \mathfrak{Y})(\exists U \in \mathrm{Con})(X \vdash U \wedge U \vdash \widehat{Y} \wedge (\exists \widehat{Y} \in \mathfrak{Y}) Y \vdash a) \\ &\Leftrightarrow X \vdash a. \end{aligned}$$

The left-to-right implication in the last line is obvious. For the converse direction note that from $X \vdash a$ we obtain that $X \vdash U \vdash Z \vdash a$, for some $U, Z \in \mathrm{Con}$. Set $\mathfrak{Y} = \{\widehat{Z}\}$.

(2) Next, let $\mathfrak{X} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))}$ and $\widehat{X} \in \widehat{\mathrm{Con}}$. Then we obtain

$$\begin{aligned} \mathfrak{X}(T_A \circ S_A)\widehat{X} &\Leftrightarrow (\exists Y \in \mathrm{Con})(\mathfrak{X} T_A Y \wedge Y S_A \widehat{X}) \\ &\Leftrightarrow (\exists Y \in \mathrm{Con})(\exists Z \in \mathfrak{X})(Z \vdash Y \wedge \{\widehat{Y}\} \vdash_{\mathcal{C}(\mathcal{D}(A))} \widehat{X}) \\ &\Leftrightarrow (\exists Y \in \mathrm{Con})(\exists Z \in \mathfrak{X})(Z \vdash Y \wedge (\exists U \in \mathrm{Con}) Y \vdash U \wedge U \vdash \widehat{X}) \\ &\Leftrightarrow (\exists Z \in \mathfrak{X})(\exists U \in \mathrm{Con})(Z \vdash U \wedge U \vdash \widehat{X}) \\ &\Leftrightarrow \mathfrak{X} \vdash_{\mathcal{C}(\mathcal{D}(A))} \widehat{X}. \end{aligned}$$

Let $\eta_A = S_A$.

Proposition 5.10 *$\eta\colon \mathcal{I}_{\mathbf{LINF}} \to \mathcal{C} \circ \mathcal{D}$ is a natural isomorphism.*

Proof: We only have to show that for L-information systems $(A, \mathrm{Con}, \vdash)$ and $(A', \mathrm{Con}', \vdash')$ and an approximable mapping $H\colon A \Vdash A'$

$$\eta_A \circ \mathcal{C}(\mathcal{D}(H)) = H \circ \eta_{A'}.$$

Note to this end that for $\mathfrak{X} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))}$ and $X' \in \mathcal{C}(\mathcal{D}(A'))$,

$$\begin{aligned} \mathfrak{X}\mathcal{C}(\mathcal{D}(H))\widehat{X'} &\Leftrightarrow (\exists \widehat{Z} \in \mathfrak{X}) \widehat{X'} \ll_{\mathcal{D}(A')} \mathcal{D}(H)(\widehat{Z}) \\ &\Leftrightarrow (\exists \widehat{Z} \in \mathfrak{X})(\exists V \in \mathrm{Con}')(V \subseteq \mathcal{D}(H)(\widehat{Z}) \wedge V \vdash' \widehat{X'}) \\ &\Leftrightarrow (\exists \widehat{Z} \in \mathfrak{X})(\exists V \in \mathrm{Con}')(\exists U \in \mathrm{Con})(U \subseteq \widehat{Z} \wedge UHV \wedge V \vdash' \widehat{X'}) \\ &\Leftrightarrow (\exists \widehat{Z} \in \mathfrak{X})(\exists V \in \mathrm{Con}')(\exists U \in \mathrm{Con})(Z \vdash U \wedge UHV \wedge V \vdash' \widehat{X'}) \\ &\Leftrightarrow (\exists \widehat{Z} \in \mathfrak{X})(\exists V \in \mathrm{Con}')(ZHV \wedge V \vdash' \widehat{X'}). \end{aligned}$$

Then we have for $X \in \mathrm{Con}$ and $\widehat{Y'} \in \mathcal{C}(\mathcal{D}(A'))$ that

$$\begin{aligned} X(\eta_A \circ \mathcal{C}(\mathcal{D}(H)))\widehat{Y'} &\Leftrightarrow (\exists \mathfrak{V} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))})(X S_A \mathfrak{V} \wedge \mathfrak{V}\mathcal{C}(\mathcal{D}(H))\widehat{Y'}) \\ &\Leftrightarrow (\exists \mathfrak{V} \in \mathrm{Con}_{\mathcal{C}(\mathcal{D}(A))})(\{\widehat{X}\} \vdash_{\mathcal{C}(\mathcal{D}(A))} \mathfrak{V} \wedge \mathfrak{V}\mathcal{C}(\mathcal{D}(H))\widehat{Y'}) \\ &\Leftrightarrow \{\widehat{X}\}\mathcal{C}(\mathcal{D}(H))\widehat{Y'} \\ &\Leftrightarrow (\exists V \in \mathrm{Con}')(XHV \wedge V \vdash' \widehat{Y'}) \\ &\Leftrightarrow (\exists W, V \in \mathrm{Con}')(XHW \wedge W \vdash' V \wedge V \vdash' \widehat{Y'}) \\ &\Leftrightarrow (\exists W \in \mathrm{Con}')(XHW \wedge \{\widehat{W}\} \vdash_{\mathcal{C}(\mathcal{D}(A'))} \widehat{Y'}) \\ &\Leftrightarrow (\exists W \in \mathrm{Con}')(XHW \wedge W S_{A'} \widehat{Y'}) \\ &\Leftrightarrow X(H \circ \eta_{A'})\widehat{Y'}. \end{aligned}$$

Together with Proposition 5.7 we now obtain what we were aiming for in this section.

Theorem 5.11 *The category* **LINF** *of L-information systems and approximable mappings is equivalent to the category* **L** *of L-domains and Scott continuous functions.*

Corollary 5.12 *The category* **aLINF** *of algebraic L-information systems and approximable mappings is equivalent to the category* **aL** *of algebraic L-domains and Scott continuous functions.*

6 The function space construction

As has already been mentioned, the categories **L** and **aL** are Cartesian closed. Because of the equivalence of these categories with **LINF** and **aLINF**, respectively, we know that the latter categories are Cartesian closed as well. However, this means that in concrete cases we have to pass back and forth between information systems and domains in order to construct the exponent of two L-information systems. In this and the next section we show how this can be done within **LINF** and **aLINF**, respectively. To this end we show how the collection of approximable mappings between two L-information systems can be represented as an L-information system again. We start with discussing what will be the tokens of this new information system.

Let $(A_1, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$ be L-information systems. Moreover, for $\mathfrak{V} \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$ and $Y \in \mathrm{Con}_0$ set

$$\mathrm{ET}_0(Y, \mathfrak{V}) = \bigcup \{\, X \in \mathrm{pr}_0(\mathfrak{V}) \mid Y \vdash_0 X \,\},$$
$$\mathrm{ET}_1(Y, \mathfrak{V}) = \bigcup \{\, E \in \mathrm{Con}_1 \mid (\exists X \in \mathrm{Con}_0)((X, E) \in \mathfrak{V} \wedge Y \vdash_0 X) \,\}.$$

Definition 6.1 Let $\mathfrak{V} \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$. $\mathfrak{W} \subseteq \mathrm{Con}_0 \times \mathrm{Con}_1$ is a *joinable extension* of $\mathfrak{V}$, if $\mathfrak{V} \subseteq \mathfrak{W}$ and the following five conditions hold:

1. For all $Y \in \mathrm{Con}_0$ there are $S \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{V}))$ and $T \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}))$ so that $Y \vdash_0 S$ and $(S, T) \in \mathfrak{W}$.

2. For all $(S, T) \in \mathfrak{W}$ either $(S, T) \in \mathfrak{V}$ or there is some $Y \in \mathrm{Con}_0$ such that $S \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{V}))$, $Y \vdash_0 S$, and $T \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}))$.

3. For all $(S, T), (S', T') \in \mathfrak{W} \setminus \mathfrak{V}$ and all $Y \in \mathrm{Con}_0$,
$$(S \underset{\mathrm{ET}_0(Y,\mathfrak{V})}{\sim} S' \wedge T \underset{\mathrm{ET}_1(Y,\mathfrak{V})}{\sim} T') \Rightarrow (S, T) = (S', T').$$

4. For all $(S, T), (S', T') \in \mathfrak{W}$, if there are $Y, Y' \in \mathrm{Con}_0$ such that $S \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{V}))$ and $T \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}))$ as well as $S' \in \mathrm{Sup}_0(\mathrm{ET}_0(Y', \mathfrak{V}))$ and $T' \in \mathrm{Sup}_1(\mathrm{ET}_1(Y', \mathfrak{V}))$, and if $Y'' \vdash_0 S \cup S'$, for some $Y'' \in \mathrm{Con}_0$, then there is some $Z \in \mathrm{Con}_1$ with $Z \vdash_1 T \cup T'$.

Let $\mathrm{JE}(\mathfrak{V})$ be the set of all joinable extensions of $\mathfrak{V}$.

Lemma 6.2 *Let* $\mathfrak{V} \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$, $\mathfrak{W} \in \mathrm{JE}(\mathfrak{V})$ *and* $Y \in \mathrm{Con}_0$. *Then the following two statements hold:*

1. $\{\, (X, E) \in \mathfrak{W} \mid Y \vdash_0 X \,\}$ *is finite.*

2. $\mathrm{ET}_0(Y, \mathfrak{W}) \in \mathrm{Con}_0$ *and* $\mathrm{ET}_1(Y, \mathfrak{W}) \in \mathrm{Con}_1$.

Proof: (1) Let $(X', E'), (X'', E'') \in \mathfrak{W} \setminus \mathfrak{V}$ with $Y \vdash_0 X' \cup X''$. Then there are $Y', Y'' \in \mathrm{Con}_0$ such that $Y' \vdash_0 X'$, $X' \in \mathrm{Sup}_0(\mathrm{ET}_0(Y', \mathfrak{V}))$ as well as $E' \in \mathrm{Sup}_1(\mathrm{ET}_1(Y', \mathfrak{V}))$, and $Y'' \vdash_0 X''$, $X'' \in \mathrm{Sup}_0(\mathrm{ET}_0(Y'', \mathfrak{V}))$ as well as $E'' \in \mathrm{Sup}_1(\mathrm{ET}_1(Y'', \mathfrak{V}))$. Assume that $\{\, \widetilde{X} \in \mathrm{pr}_0(\mathfrak{V}) \mid Y' \vdash_0 \widetilde{X} \,\} = \{\, \widetilde{X} \in \mathrm{pr}_0(\mathfrak{V}) \mid Y'' \vdash_0 \widetilde{X} \,\}$. Then $\mathrm{ET}_0(Y', \mathfrak{V}) = \mathrm{ET}_0(Y', \mathfrak{V})$ ($= G$, say) and $\mathrm{ET}_1(Y', \mathfrak{V}) = \mathrm{ET}_1(Y'', \mathfrak{V})$ ($= H$, say). Since $Y \vdash_0 X' \cup X''$, it follows with Lemma 4.4 that $X' \underset{G}{\sim} X''$. Moreover, because of 6.1(4), there is some $V \in \mathrm{Con}_1$ with $V \vdash_1 E' \cup E''$. Therefore we also have that $E' \underset{H}{\sim} E''$. Thus $(X', E') = (X'', E'')$, by 6.1(3).

This shows that the number of pairs $(X, E) \in \mathfrak{W} \setminus \mathfrak{V}$ with $Y \vdash_0 X$ is bounded by the cardinality of the powerset of $\mathrm{pr}_0(\mathfrak{V})$. Since $\mathfrak{V}$ is finite, the powerset of $\mathrm{pr}_0(\mathfrak{V})$ is finite as well. Consequently, there are only finitely many $(X, E) \in \mathfrak{W}$ with $Y \vdash_0 X$.

(2) By Condition 6.1(1) there is some $(S, T) \in \mathfrak{W}$ such that $Y \vdash_0 S$, $S \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{V}))$, and $T \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}))$. In particular we have that $S \in \mathrm{Con}_0$ and $T \in \mathrm{Con}_1$. As we have seen in (1), there are only finitely many $(X, E) \in \mathfrak{W}$ with $Y \vdash_0 X$, say $(X_0, E_0), \dots, (X_n, E_n)$. Without restriction, let $(X_0, E_0) = (S, T)$ and assume for $i < n$ that $X_0 \cup \cdots \cup X_i \in \mathrm{Con}_0$ as well as $E_0 \cup \cdots \cup E_i \in \mathrm{Con}_1$.

By construction $(X_{i+1}, E_{i+1}) \in \mathfrak{V}$ or there is some $\overline{Y} \in \mathrm{Con}_0$ such that $\overline{Y} \vdash_0 X_{i+1}$, $X_{i+1} \in \mathrm{Sup}_0(\mathrm{ET}_0(\overline{Y}, \mathfrak{V}))$ and $E_{i+1} \in \mathrm{Sup}_1(\mathrm{ET}_1(\overline{Y}, \mathfrak{V}))$. In the first case, $X_{i+1} \subseteq \mathrm{ET}_0(Y, \mathfrak{V}) \subseteq X_0$ and $E_{i+1} \subseteq \mathrm{ET}_1(Y, \mathfrak{V}) \subseteq E_0$.

In the other case, if $(U, V) \in \mathfrak{V}$ with $\overline{Y} \vdash_0 U$, then $U \subseteq X_{i+1}$. Thus, $Y \vdash_0 U$, which implies that $\mathrm{ET}_0(\overline{Y}, \mathfrak{V}) \subseteq \mathrm{ET}_0(Y, \mathfrak{V}) \subseteq X_0 \subseteq X_0 \cup \cdots \cup X_i$ and $\mathrm{ET}_1(\overline{Y}, \mathfrak{V}) \subseteq \mathrm{ET}_1(Y, \mathfrak{V}) \subseteq E_0 \cup \cdots \cup E_i$. Moreover, $Y \vdash_0 X_0 \cup \cdots \cup X_{i+1}$. As we have just seen, for $j = 1, \dots, i+1$, (X_j, E_j) is such that either $X_j \subseteq X_0$ and $E_j \subseteq E_0$, or $(X_j, E_j) \in \mathfrak{W} \setminus \mathfrak{V}$. Therefore, by Condition 6.1(4), there is some $Z \in \mathrm{Con}_1$ so that $Z \vdash_1 E_0 \cup \cdots \cup E_{i+1}$. With 4.1(1) we thus obtain that $X_0 \cup \cdots \cup X_{i+1} \in \mathrm{Con}_0$ and $E_0 \cup \cdots \cup E_{i+1} \in \mathrm{Con}_1$. This concludes the induction. For $i = n$ we obtain that $\mathrm{ET}_0(Y, \mathfrak{W}) \in \mathrm{Con}_0$ and $\mathrm{ET}_1(Y, \mathfrak{W}) \in \mathrm{Con}_1$.

Lemma 6.3 *Let* t_0 *and* t_1*, respectively, be truth elements of* A_0 *and* A_1*, and* $(X, Z) \in \mathrm{Con}_0 \times \mathrm{Con}_1$ *such that* $Z' \vdash_1 Z$*, for some* $Z' \in \mathrm{Con}_1$*. Then* $\{(X, Z)\} \in \mathrm{JE}(\{(X, Z)\})$*, if* $Y \vdash_0 X$*, for all* $Y \in \mathrm{Con}_0$*. Otherwise,* $\{(X, Z), (\{t_0\}, \{t_1\})\} \in \mathrm{JE}(\{(X, Z)\})$*.*

Lemma 6.4 *Let* $\mathfrak{U}' \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$ *and* $\overline{\mathfrak{U}}$ *be a family of pairs* $(X_Y, Z_Y) \in \mathrm{Con}_0 \times \mathrm{Con}_1$*, for* $Y \in \mathrm{Con}_0$*, with the following properties:*

1. $Y \vdash_0 X_Y$,

2. $X_Y \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{U}'))$,

3. $Z_Y \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{U}'))$,

4. $(\forall Y, Y', \overline{Y} \in \mathrm{Con}_0)(\exists \overline{Z} \in \mathrm{Con}_1)(\overline{Y} \vdash_0 X_Y \cup X_{Y'} \Rightarrow \overline{Z} \vdash_1 Z_Y \cup Z_{Y'})$.

Then there is some $\mathfrak{U} \in \mathrm{JE}(\mathfrak{U}')$ *such that*

1. $\mathfrak{U} \subseteq \mathfrak{U}' \cup \overline{\mathfrak{U}}$,

2. $(\forall (X, Z) \in \overline{\mathfrak{U}})(\exists (S, T) \in \mathfrak{U})(\exists Y \in \mathrm{Con}_0)(X \underset{\mathrm{ET}_0(Y, \mathfrak{U}')}{\sim} S \wedge Z \underset{\mathrm{ET}_1(Y, \mathfrak{U}')}{\sim} T)$.

Proof: For every $Y \in \mathrm{Con}_0$, let $\{ (S_i, T_i) \mid i \in I_Y \}$ be a system of representatives of the set $\{ (S,T) \in \overline{\mathfrak{U}} \mid S \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{U}')) \wedge T \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{U}')) \}$ with respect to the equivalence relation

$$\underset{\mathrm{ET}_0(Y,\mathfrak{U}')}{\sim} \times \underset{\mathrm{ET}_1(Y,\mathfrak{U}')}{\sim} .$$

Set $\mathfrak{U} = \mathfrak{U}' \cup \{ (S_i, T_i) \mid (\exists Y \in \mathrm{Con}_0) i \in I_Y \}$. We will show that $\mathfrak{U} \in \mathrm{JE}(\mathfrak{U}')$.

By construction, $\mathfrak{U}' \subseteq \mathfrak{U}$. The remaining conditions in Definition 6.1 are satisfied as well: For (1) let $Y \in \mathrm{Con}_0$. Then there is some $(X_Y, Z_Y) \in \overline{\mathfrak{U}}$ with the required properties. It follows that the corresponding representative in $\mathfrak{U}$ has the same properties. Conditions (2) and (3) hold by construction and (4) is a consequence of Assumption (4).

Now define

$$A_0 \to A_1 = \{ \mathfrak{W} \subseteq \mathrm{Con}_0 \times \mathrm{Con}_1 \mid (\exists \mathfrak{V} \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1) \mathfrak{W} \in \mathrm{JE}(\mathfrak{V}) \},$$

$$\mathrm{Con}_\to = \{ \mathcal{W} \subseteq_{\mathrm{fin}} A_0 \to A_1 \mid (\forall Y \in \mathrm{Con}_1)\, \mathrm{ET}_1(Y, \bigcup \mathcal{W}) \in \mathrm{Con}_1 \},$$

$$\mathcal{W} \vdash_\to \mathfrak{V} \Leftrightarrow (\forall (Y,Z) \in \mathfrak{V})\, \mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_1 Z.$$

Lemma 6.5 *Let $\mathcal{W} \in \mathrm{Con}_\to$ and $\mathfrak{U}, \mathfrak{V} \in A_0 \to A_1$ such that $\mathfrak{W} \vdash_\to \mathfrak{V}$ and $\mathfrak{U} \subseteq \mathfrak{V}$. Then $\mathcal{W} \vdash_\to \mathfrak{U}$.*

Proposition 6.6 *Let $(A_0, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$ be L-information systems. Then $(A_0 \to A_1, \mathrm{Con}_\to, \vdash_\to)$ is an L-information systems as well.*

Proof: We first have to verify Conditions 3.1(1–5).

Claim 1 $(A_0 \to A_1, \mathrm{Con}_\to, \vdash_\to)$ satisfies Condition 3.1(1).

Let $\mathfrak{W} \in A_0 \to A_1$ and $Y \in \mathrm{Con}_0$. Then $\mathrm{ET}_1(Y, \mathfrak{W}) \in \mathrm{Con}_1$, by Lemma 6.2(2), which means that $\{\mathfrak{W}\} \in \mathrm{Con}_\to$.

Claim 2 $(A_0 \to A_1, \mathrm{Con}_\to, \vdash_\to)$ satisfies Condition 3.1(2).

Assume that $\mathcal{W} \vdash_\to \mathfrak{V}$. Then we have for all $(Y,Z) \in \mathfrak{V}$ that

$$\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_1 Z. \tag{1}$$

We have to show that for all $Y \in \mathrm{Con}_0$, $\mathrm{ET}_1(Y, \bigcup(\mathcal{W} \cup \{\mathfrak{V}\})) \in \mathrm{Con}_1$. Let to this end $Y \in \mathrm{Con}_0$. Then

$$\mathrm{ET}_1(Y, \bigcup(\mathcal{W} \cup \{\mathfrak{V}\})) = \mathrm{ET}_1(Y, \bigcup \mathcal{W}) \cup \mathrm{ET}_1(Y, \mathfrak{V}).$$

Note that $\mathrm{ET}_1(Y, \mathfrak{V})$ is finite, by Lemma 6.2. Since $\mathcal{W} \in \mathrm{Con}_\to$, we have that $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \in \mathrm{Con}_1$. With (1) and Conditions 3.1(2, 3) it follows that also $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \cup Z \in \mathrm{Con}_1$.

Claim 3 $(A_0 \to A_1, \mathrm{Con}_\to, \vdash_\to)$ satisfies Condition 3.1(3).

Let $\mathcal{U}, \mathcal{W} \in \mathrm{Con}_\to$ and $\mathfrak{V} \in A_0 \to A_1$ such that $\mathcal{U} \supseteq \mathcal{W}$ and $\mathcal{W} \vdash_\to \mathfrak{V}$. We have to show that also $\mathcal{U} \vdash_\to \mathfrak{V}$. Let for this $(Y,Z) \in \mathfrak{V}$. Then $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_\to Z$. Since $\mathcal{U} \supseteq \mathcal{W}$, we have that $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \subseteq \mathrm{ET}_1(Y, \bigcup \mathcal{U})$. Thus $\mathrm{ET}_1(Y, \bigcup \mathcal{U}) \vdash_1 Z$.

Claim 4 $(A_0 \to A_1, \mathrm{Con}_\to, \vdash_\to)$ satisfies Condition 3.1(4).

Let $\mathcal{U}, \mathcal{W} \in \mathrm{Con}_{\rightarrow}$ and $\mathfrak{V} \in A_0 \to A_1$ and assume that $\mathcal{U} \vdash_{\rightarrow} \mathcal{W} \vdash_{\rightarrow} \mathfrak{V}$. We need to show that $\mathcal{U} \vdash_{\rightarrow} \mathfrak{V}$. Let $(Y, Z) \in \mathfrak{V}$. Then $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_{\rightarrow} Z$. For $\mathfrak{W} \in \mathcal{W}$ and $(X, E) \in \mathfrak{W}$ we moreover have that $\mathrm{ET}(X, \bigcup \mathcal{U}) \vdash_1 E$. Since

$$\bigcup \{ \mathrm{ET}_1(X, \bigcup \mathcal{U}) \mid (\exists \mathfrak{W} \in \mathcal{W})(X \in \mathrm{pr}_1(\mathfrak{W}) \wedge Y \vdash_0 X) \} \subseteq \mathrm{ET}_1(Y, \bigcup \mathcal{U}),$$

it follows that $\mathrm{ET}_1(Y, \bigcup \mathcal{U}) \vdash_1 \mathrm{ET}_1(Y, \bigcup \mathcal{W})$ and hence that $\mathrm{ET}_1(Y, \bigcup \mathcal{U}) \vdash_1 Z$.

Claim 5 $(A_0 \to A_1, \mathrm{Con}_{\rightarrow}, \vdash_{\rightarrow})$ satisfies Condition 3.1(5).

Let $\mathcal{W} \in \mathrm{Con}_{\rightarrow}$ and $\mathfrak{V} \in A_0 \to A_1$ with $\mathcal{W} \vdash_{\rightarrow} \mathfrak{V}$. We have to construct some $\mathcal{U} \in \mathrm{Con}_{\rightarrow}$ so that $\mathcal{W} \vdash_{\rightarrow} \mathcal{U} \vdash_{\rightarrow} \mathfrak{V}$. Let again $(Y, Z) \in \mathfrak{V}$. Then $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_2 Z$. Because of Condition 3.1(5) there is thus some $N_a \in \mathrm{Con}_1$, for each $a \in Z$, with $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_2 N_a \vdash_2 a$.

In the same way we have for each $\mathfrak{W} \in \mathcal{W}$, each $X \in \mathrm{pr}_0(\mathfrak{W})$ with $Y \vdash_0 X$, and each $d \in X$ that there is some $M_d^{Y,X} \in \mathrm{Con}_0$ so that $Y \vdash_0 M_d^{Y,X} \vdash_0 d$. Because of Lemma 6.2, $\mathrm{ET}_0(Y, \mathfrak{W})$ is finite. Since also $\mathcal{W}$ is finite, it follows that $\mathrm{ET}_0(Y, \bigcup \mathcal{W})$ is finite as well. By Proposition 3.2 there is thus some $M^Y \in \mathrm{Con}_0$ such that

$$M^Y \supseteq \bigcup \{ M_d^{Y,X} \mid d \in \mathrm{ET}_0(Y, \bigcup \mathcal{W}) \} \quad \text{and} \quad Y \vdash_0 M^Y \vdash_0 \mathrm{ET}_0(Y, \bigcup \mathcal{W}).$$

Moreover, for every $(Y, Z) \in \mathfrak{V}$ there is some $N^{Y,Z} \in \mathrm{Con}_1$ so that

$$N^{Y,Z} \supseteq \bigcup \{ N_a \mid a \in Z \} \quad \text{and} \quad \mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_1 N^{Y,Z} \vdash_1 Z.$$

Let $\mathfrak{V}' \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$ with $\mathfrak{V} \in \mathrm{JE}(\mathfrak{V}')$ and define $\mathfrak{U}' = \{ (M^Y, N^{Y,Z}) \mid (Y, Z) \in \mathfrak{V}' \}$. Then $\mathfrak{U}' \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$. Now, let $\overline{Y} \in \mathrm{Con}_0$ and $(M^Y, N^{Y,Z}) \in \mathfrak{U}'$ with $\overline{Y} \vdash_0 M^Y$. Then we have for every $X \in \mathrm{pr}_0(\bigcup \mathcal{W})$ with $Y \vdash_0 X$ that $M^Y \vdash_0 X$. Hence $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \subseteq \mathrm{ET}_1(M^Y, \bigcup \mathcal{W})$. Therefore,

$$\mathrm{ET}_1(M^Y, \bigcup \mathcal{W}) \vdash_1 N^{Y,Z}, \tag{2}$$

which implies that $\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W}) \vdash_1 N^{Y,Z}$. By Condition (L) it follows that there are sets $S_{\overline{Y}} \in \mathrm{Sup}_0(\mathrm{ET}_0(\overline{Y}, \mathfrak{U}'))$ with $\overline{Y} \vdash_0 S_{\overline{Y}}$, and $T_{\overline{Y}} \in \mathrm{Sup}_1(\mathrm{ET}_1(\overline{Y}, \mathfrak{U}'))$ with

$$\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W}) \vdash_1 T_{\overline{Y}}. \tag{3}$$

Set $\overline{\mathfrak{U}} = \{ (S_{\overline{Y}}, T_{\overline{Y}}) \mid \overline{Y} \in \mathrm{Con}_0 \}$. Then the requirements in Lemma 6.4 are satisfied. For Condition (4) let $Y, Y' \in \mathrm{Con}_0$ and $(S_Y, T_Y), (S_{Y'}, T_{Y'}) \in \overline{\mathfrak{U}}$ such that $\overline{Y} \vdash_0 S_Y \cup S_{Y'}$, for some $\overline{Y} \in \mathrm{Con}_0$. Then $\mathrm{ET}_0(Y, \mathfrak{U}') \cup \mathrm{ET}_0(Y', \mathfrak{U}') \subseteq S_Y \cup S_{Y'}$, which implies that for all $X \in \mathrm{pr}_0(\mathfrak{U}')$ with $Y \vdash_0 X$ or $Y' \vdash_0 X$ also $\overline{Y} \vdash_0 X$. Since by (3), $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_1 T_Y$ as well as $\mathrm{ET}_1(Y', \bigcup \mathcal{W}) \vdash_1 T_{Y'}$, we obtain that $\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W}) \vdash_1 T_Y \cup T_{Y'}$.

Let $\mathfrak{U} \in \mathrm{JE}(\mathfrak{U}')$ be constructed from $\mathfrak{U}'$ and $\overline{\mathfrak{U}}$ as in Lemma 6.4. Then $\mathfrak{U} \in A_0 \to A_1$. It remains to show that $\mathcal{W} \vdash_{\rightarrow} \{\mathfrak{U}\} \vdash_{\rightarrow} \mathfrak{V}$.

Let to this end $(Y, Z) \in \mathfrak{V}$. If $(Y, Z) \in \mathfrak{V}'$ then we obtain by the above construction that there is some $(M^Y, N^{Y,Z}) \in \mathfrak{U}'$ such that $N^{Y,Z} \vdash_1 Z$ as well as $Y \vdash_0 M^Y$. It follows that also $\mathrm{ET}_1(Y, \mathfrak{U}) \vdash_1 Z$.

If $(Y, Z) \in \mathfrak{V} \setminus \mathfrak{V}'$, there is some $\overline{Y} \in \mathrm{Con}_0$ so that $\overline{Y} \vdash_0 Y$, $Y \in \mathrm{Sup}_0(\mathrm{ET}_0(\overline{Y}, \mathfrak{V}'))$ and $Z \in \mathrm{Sup}_1(\mathrm{ET}_1(\overline{Y}, \mathfrak{V}'))$. As we have just seen, for every $(X, E) \in \mathfrak{V}'$ there is some $(M^X, N^{X,E}) \in \mathfrak{U}'$ with $N^{X,E} \vdash_1 E$ and $X \vdash_0 M^X$. Assume that $\overline{Y} \vdash_0 X$. Since $X \subseteq Y$ in this case, it follows that also $Y \vdash_0 M^X$. Since $\mathfrak{U} \in \mathrm{JE}(\mathfrak{U}')$, there is some $(S_Y, T_Y) \in \overline{\mathfrak{U}}$ such that $Y \vdash_0 S_Y$, $S_Y \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{U}'))$ and $T_Y \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{U}'))$. Then $N^{X,E} \subseteq T_Y$ and hence $T_Y \vdash_1 E$, which shows that $T_Y \vdash_1 \mathrm{ET}_1(\overline{Y}, \mathfrak{U}')$. Because of Condition (L) there is thus some $Z' \in \mathrm{Sup}_1(\mathrm{ET}_1(\overline{Y}, \mathfrak{V}'))$ with $T_Y \vdash_1 Z'$.

By the construction of $\overline{\mathfrak{U}}$ and (3), $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_1 T_Y$. Since $\mathcal{W} \vdash_\rightarrow \mathfrak{V}$, we moreover have that also $\mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_1 Z$. With Lemma 4.4 it follows that

$$Z' \underset{\mathrm{ET}_1(\overline{Y}, \mathfrak{V}')}{\sim} Z.$$

Thus also $T_Y \vdash_1 Z$. Consequently we have that $\mathrm{ET}(Y, \mathfrak{U}) \vdash_1 Z$.

This proves that $\{\mathfrak{U}\} \vdash_\rightarrow \mathfrak{V}$. It remains to show that $\mathcal{W} \vdash_\rightarrow \mathfrak{U}$ as well. Let to this end $(S, T) \in \mathfrak{U}$. If $(S, T) \in \mathfrak{U}'$, there is some $(Y, Z) \in \mathfrak{V}'$ with $S = M^Y$ and $T = N^{Y,Z}$. By (2) we have that $\mathrm{ET}_1(M^Y, \bigcup \mathcal{W}) \vdash_1 N^{Y,Z}$.

If $(S, T) \in \mathfrak{U}$, there is some $\overline{Y} \in \mathrm{Con}_0$ such that $S = S_{\overline{Y}}$ with $\overline{Y} \vdash_0 S_{\overline{Y}}$ and $S_{\overline{Y}} \in \mathrm{Sup}_0(\mathrm{ET}_0(\overline{Y}, \mathfrak{U}'))$, as well as $T = T_{\overline{Y}}$ with $T_{\overline{Y}} \in \mathrm{Sup}_1(\mathrm{ET}_1(\overline{Y}, \mathfrak{U}'))$. Because of (2) we have for $(P, Q) \in \mathfrak{U}'$ with $\overline{Y} \vdash_0 P$ that $\mathrm{ET}_1(P, \bigcup \mathcal{W}) \vdash_1 Q$. Since for all such P, $P \subseteq S_{\overline{Y}}$, we also have that $\mathrm{ET}_1(S_{\overline{Y}}, \bigcup \mathcal{W}) \vdash_1 \mathrm{ET}_1(\overline{Y}, \mathfrak{U}')$. By Condition (L) there is then some $R \in \mathrm{Sup}_1(\mathrm{ET}_1(\overline{Y}, \mathfrak{U}'))$ with $\mathrm{ET}_1(S_{\overline{Y}}, \bigcup \mathcal{W}) \vdash_1 R$. Because $\overline{Y} \vdash_0 S_{\overline{Y}}$, we obtain that $\mathrm{ET}_1(S_{\overline{Y}}, \bigcup \mathcal{W}) \subseteq \mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W})$ and hence that $\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W}) \vdash_1 R$. As a consequence of Lemma 4.4 and (3),

$$T_{\overline{Y}} \underset{(\mathrm{ET}_1(\overline{Y}, \mathfrak{U}')}{\sim} R.$$

It follows that also $\mathrm{ET}_1(S_{\overline{Y}}, \bigcup \mathcal{W}) \vdash_1 T_{\overline{Y}}$. This shows that $\mathcal{W} \vdash_\rightarrow \mathfrak{U}$.

Claim 6 $(A_0 \rightarrow A_1, \mathrm{Con}_\rightarrow, \vdash_\rightarrow)$ has a truth element.

Let t_0 and t_1, respectively, be truth elements of A_0 and A_1. Define $\mathfrak{t} = \{(\{t_0\}, \{t_1\})\}$. As is readily verified, $\mathfrak{t} \in A_0 \rightarrow A_1$ and $\mathcal{W} \vdash_\rightarrow \mathfrak{t}$, for all $\mathcal{W} \in \mathrm{Con}_\rightarrow$. Moreover, we have for $\mathcal{W} \in \mathrm{Con}_\rightarrow$ and $\mathfrak{V} \in A_0 \rightarrow A_1$ with $\mathcal{W} \cup \{\mathfrak{t}\} \vdash_\rightarrow \mathfrak{V}$ that $\mathcal{W} \vdash_\rightarrow \mathfrak{V}$. Note here that for $(Y, Z) \in \mathfrak{V}$, $\mathrm{ET}_1(Y, \bigcup(\mathcal{W} \cup \{\mathfrak{t}\})) = \mathrm{ET}_1(Y, \bigcup \mathcal{W}) \cup \{t_1\}$. Thus, $\mathfrak{t}$ is a truth element of $A_0 \rightarrow A_1$.

Claim 7 $(A_0 \rightarrow A_1, \mathrm{Con}_\rightarrow, \vdash_\rightarrow)$ satisfies Condition (L).

Let $\mathcal{W} \in \mathrm{Con}_\rightarrow$ and $\mathcal{F} \subseteq_{\mathrm{fin}} A_0 \rightarrow A_1$ such that $\mathcal{W} \vdash_\rightarrow \mathfrak{F}$. We need to construct a $\mathcal{Z} \in \mathrm{Sup}_\rightarrow(\mathfrak{F})$ with $\mathcal{W} \vdash_\rightarrow \mathcal{Z}$. Without restriction we assume that $\mathcal{F} \notin \mathrm{Con}_\rightarrow$.

We have for all $\mathfrak{V} \in \mathcal{F}$ and all $(P, Q) \in \mathfrak{V}$ that

$$\mathrm{ET}_1(P, \bigcup \mathcal{W}) \vdash_1 Q. \tag{4}$$

For every $\mathfrak{V} \in \mathcal{F}$ fix some $\mathfrak{V}' \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$ with $\mathfrak{V} \in \mathrm{JE}(\mathfrak{V}')$ and let $\mathcal{F}'$ be the collection of these $\mathfrak{V}'$. Then $\bigcup \mathrm{pr}_0(\bigcup \mathcal{F}')$ and $\bigcup \mathrm{pr}_1(\bigcup \mathcal{F}')$ are finite.

Let $\overline{Y} \in \mathrm{Con}_0$. According to the definition of $\mathrm{Con}_\rightarrow$, $\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W}) \in \mathrm{Con}_1$. As a consequence of (4) we obtain that for all $(P, Q) \in \bigcup \mathcal{F}'$ with $\overline{Y} \vdash_0 P$, $\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W}) \vdash_1 Q$. By Condition (L) there are therefore $S_{\overline{Y}} \in \mathrm{Sup}_0(\mathrm{ET}_0(\overline{Y}, \bigcup \mathcal{F}'))$ with $\overline{Y} \vdash_0 S_{\overline{Y}}$, and $T_{\overline{Y}} \in \mathrm{Sup}_1(\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{F}'))$ so that

$$\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W}) \vdash_1 T_{\overline{Y}}. \tag{5}$$

Construct the set $\mathfrak{Z}_\mathcal{F}$ by applying Lemma 6.4 to $\bigcup \mathcal{F}'$ and $\{\, (S_{\overline{Y}}, T_{\overline{Y}}) \mid \overline{Y} \in \mathrm{Con}_0 \,\}$. Then $\mathfrak{Z}_\mathcal{F} \in \mathrm{JE}(\bigcup \mathcal{F}')$, which implies that $\mathfrak{Z}_\mathcal{F} \in A_0 \rightarrow A_1$. We show that $\mathcal{W} \vdash_\rightarrow \mathfrak{Z}_\mathcal{F}$.

Let to this end $(P, Q) \in \mathfrak{Z}_{\mathcal{F}}$. In case $(P, Q) \in \bigcup \mathcal{F}'$, (4) is what needs to be shown. Otherwise, there is some $\overline{Y} \in \mathrm{Con}_0$ so that $(P, Q) = (S_{\overline{Y}}, T_{\overline{Y}})$, where $S_{\overline{Y}} \in \mathrm{Sup}_0(\mathrm{ET}_0(\overline{Y}, \bigcup \mathcal{F}'))$ with $\overline{Y} \vdash_0 S_{\overline{Y}}$, and $T_{\overline{Y}} \in \mathrm{Sup}_1(\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{F}'))$.

By (4) we have for $(P', Q') \in \bigcup \mathcal{F}'$ with $\overline{Y} \vdash_0 P'$ that $\mathrm{ET}_1(P', \bigcup \mathcal{W}) \vdash_1 Q'$. Since all these sets P' are contained in $S_{\overline{Y}}$, it follows that $\mathrm{ET}_1(S_{\overline{Y}}, \bigcup \mathcal{W}) \vdash_1 \mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{F}')$. Because of Condition 3.1(6) there is now some $R \in \mathrm{Con}_1$ such that $R \supseteq \mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{F}')$ and

$$\mathrm{ET}_1(S_{\overline{Y}}, \bigcup \mathcal{W}) \vdash_1 R.$$

Since $\overline{Y} \vdash_0 S_{\overline{Y}}$, we have that $\mathrm{ET}_1(S_{\overline{Y}}, \bigcup \mathcal{W}) \subseteq \mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W})$. Hence $\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W}) \vdash_1 R$. With Lemma 4.4 and (5) we therefore obtain that also $\mathrm{ET}_1(S_{\overline{Y}}, \bigcup \mathcal{W}) \vdash_1 T_{\overline{Y}}$.

This shows that $\mathcal{W} \vdash_{\rightarrow} \mathfrak{Z}_{\mathcal{F}}$. Set $\mathcal{Z} = \mathcal{F} \cup \{\mathfrak{Z}_{\mathcal{F}}\}$. As $\mathcal{W} \vdash_{\rightarrow} \mathcal{F}$ by assumption, it follows that $\mathcal{W} \vdash_0 \mathcal{Z}$.

It remains to show that $\mathcal{Z} \in \mathrm{Sup}_{\rightarrow}(\mathcal{F})$. First, we prove that $\mathcal{Z} \in \mathrm{Con}_{\rightarrow}$. Let $\mathcal{F} = \{\mathfrak{V}_1, \ldots, \mathfrak{V}_n\}$. We show by induction on i that $\{\mathfrak{Z}_{\mathcal{F}}, \mathfrak{V}_1, \ldots, \mathfrak{V}_i\} \in \mathrm{Con}_{\rightarrow}$.

Since $\mathfrak{Z}_{\mathcal{F}} \in A_0 \to A_1$, Claim 1 tells us that $\{\mathfrak{Z}_{\mathcal{F}}\} \in \mathrm{Con}_{\rightarrow}$. Assume that $\{\mathfrak{Z}_{\mathcal{F}}, \mathfrak{V}_1, \ldots, \mathfrak{V}_i\} \in \mathrm{Con}_{\rightarrow}$, for $i < n$. We need to show that then $\{\mathfrak{Z}_{\mathcal{F}}, \mathfrak{V}_1, \ldots, \mathfrak{V}_{i+1}\} \in \mathrm{Con}_{\rightarrow}$ as well.

Let to this end $\overline{Y} \in \mathrm{Con}_0$. We have to prove that $\mathrm{ET}_1(\overline{Y}, \mathfrak{Z}_{\mathcal{F}} \cup \mathfrak{V}_1 \cup \cdots \cup \mathfrak{V}_{i+1}) \in \mathrm{Con}_1$ and we know that $\mathrm{ET}_1(\overline{Y}, \mathfrak{Z}_{\mathcal{F}} \cup \mathfrak{V}_1 \cup \cdots \cup \mathfrak{V}_i) \in \mathrm{Con}_1$.

Assume that $\mathfrak{V}_{i+1} \in \mathrm{JE}(\mathfrak{V}'_{i+1})$ with $\mathfrak{V}'_{i+1} \in \mathcal{F}'$. By Lemma 6.2(1) there are only finitely many $(X, E) \in \mathfrak{V}_{i+1}$ so that $\overline{Y} \vdash_0 X$, say $(X_1, E_1), \ldots, (X_{m_{i+1}}, E_{m_{i+1}})$. We will show that

$$\mathrm{ET}_1(\overline{Y}, \mathfrak{Z}_{\mathcal{F}} \cup \mathfrak{V}_1 \cup \cdots \cup \mathfrak{V}_i \cup \{(X_1, E_1), \ldots, (X_j, E_j)\}) \in \mathrm{Con}_1 \tag{6}$$

by induction on j. Let to this end $j < m_{i+1}$ and suppose that (6) holds.

If $(X_{j+1}, E_{j+1}) \in \mathfrak{V}'_{i+1}$, then $(X_{j+1}, E_{j+1}) \in \mathfrak{Z}_{\mathcal{F}}$. Otherwise, if $(X_{j+1}, E_{j+1}) \in \mathfrak{V}_{i+1} \setminus \mathfrak{V}'_{i+1}$, then there is some $Y \in \mathrm{Con}_0$ with $Y \vdash_0 X_{j+1}$ such that $X_{j+1} \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{V}'_{i+1}))$ and $E_{j+1} \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}'_{i+1}))$. Since $\mathrm{ET}_0(Y, \mathfrak{V}'_{i+1}) \subseteq X_{j+1}$ and $\overline{Y} \vdash_0 X_{j+1}$, we obtain for $X' \in \mathrm{pr}_0(\mathfrak{V}'_{i+1})$ with $Y \vdash_0 X'$ that also $\overline{Y} \vdash_0 X'$. As moreover $\mathfrak{V}'_{i+1} \subseteq \mathfrak{Z}_{\mathcal{F}}$, we have that

$$\mathrm{ET}_1(Y, \mathfrak{V}'_{i+1}) \subseteq \mathrm{ET}_1(\overline{Y}, \mathfrak{Z}_{\mathcal{F}} \cup \mathfrak{V}_1 \cup \cdots \cup \mathfrak{V}_i \cup \{(X_1, E_1), \ldots, (X_j, E_j)\}).$$

By assumption, $\mathcal{W} \vdash_{\rightarrow} \mathcal{F}$. In addition, we have shown that $\mathcal{W} \vdash_{\rightarrow} \mathfrak{Z}_{\mathcal{F}}$. Because $\mathfrak{V}_1, \ldots, \mathfrak{V}_{i+1} \in \mathcal{F}$, it therefore follows for $(\overline{X}, \overline{E}) \in \mathfrak{Z}_{\mathcal{F}} \cup \mathfrak{V}_1 \cup \cdots \cup \mathfrak{V}_{i+1}$ that

$$\mathrm{ET}_1(\overline{X}, \bigcup \mathcal{W}) \vdash_1 \overline{E}.$$

Hence $\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{W}) \vdash_1 E_{j+1} \cup \mathrm{ET}_1(\overline{Y}, \mathfrak{Z}_{\mathcal{F}} \cup \mathfrak{V}_1 \cup \cdots \cup \mathfrak{V}_i \cup \{(X_1, E_1), \ldots, (X_j, E_j)\})$. With Condition 4.1(1) we thus obtain that

$$\mathrm{ET}_1(\overline{Y}, \mathfrak{Z}_{\mathcal{F}} \cup \mathfrak{V}_1 \cup \cdots \cup \mathfrak{V}_i \cup \{(X_1, E_1), \ldots, (X_{j+1}, E_{j+1})\}) \in \mathrm{Con}_1 .$$

For $j = m_{i+1}$ it follows that $\mathrm{ET}_1(\overline{Y}, \mathfrak{Z}_{\mathcal{F}} \cup \mathfrak{V}_1 \cup \cdots \cup \mathfrak{V}_{i+1}) \in \mathrm{Con}_1$, which concludes our main induction. Consequently, $\mathrm{ET}_1(\overline{Y}, \bigcup \mathcal{Z}) \in \mathrm{Con}_1$, for all $\overline{Y} \in \mathrm{Con}_0$, i.e. $\mathcal{Z} \in \mathrm{Con}_{\rightarrow}$.

It remains to verify Conditions 4.1(1–3). Let to this end $\mathcal{V}, \mathcal{Y} \in \mathrm{Con}_{\rightarrow}$ so that $\mathcal{Y} \supseteq \mathcal{F}$ and $\mathcal{V} \vdash_{\rightarrow} \mathcal{Y} \cup \mathcal{Z}$.

Note that $\bigcup \mathcal{F}' \subseteq \bigcup \mathcal{F} \subseteq \bigcup \mathcal{Y}$. Therefore $(X, E) \in \bigcup (\mathcal{Y} \cup \mathcal{Z})$, exactly if $(X, E) \in \bigcup \mathcal{Y}$ or $(X, E) \in \mathfrak{Z}_{\mathcal{F}} \setminus \bigcup \mathcal{F}'$. By construction we have for any $(X, E) \in \mathfrak{Z}_{\mathcal{F}} \setminus \bigcup \mathcal{F}'$ that there is some $Y^{(X,E)} \in \mathrm{Con}_0$ such that $Y^{(X,E)} \vdash_0 X$, $X \in \mathrm{Sup}_0(\mathrm{ET}_0(Y^{(X,E)}, \bigcup \mathcal{F}'))$ and $E \in \mathrm{Sup}_1(\mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{F}'))$.

(1) Let $U \in \mathrm{Con}_0$. We need to show that $\mathrm{ET}_1(U, \bigcup(\mathcal{Y} \cup \mathcal{Z})) \in \mathrm{Con}_1$. By Lemma 6.2(1) there are only finitely many $(X, E) \in \mathfrak{Z}_{\mathcal{F}} \setminus \bigcup \mathcal{F}'$ with $U \vdash_0 X$, say $(X_1, E_1), \ldots, (X_n, E_n)$. We will prove by induction on $i = 0, \ldots, n$ that

$$\mathrm{ET}_1(U, \bigcup \mathcal{Y}) \cup E_1 \cup \cdots \cup E_i \in \mathrm{Con}_1 .$$

Observe that $\mathrm{ET}_1(U, \bigcup \mathcal{Y}) \in \mathrm{Con}_1$, by the definition of $\mathrm{Con}_\rightarrow$ Now, let $i < n$ and assume that $\mathrm{ET}_1(U, \bigcup \mathcal{Y}) \cup E_1 \cup \cdots \cup E_i \in \mathrm{Con}_1$. Since $\mathcal{V} \vdash_\rightarrow \mathcal{Y} \cup \mathcal{Z}$, we have for $(P, Q) \in \bigcup(\mathcal{Y} \cup \mathcal{Z})$ that $\mathrm{ET}_1(P, \bigcup \mathcal{V}) \vdash_1 Q$. For those $(P, Q) \in \bigcup(\mathcal{Y} \cup \mathcal{Z})$ with $U \vdash_0 P$ we thus obtain that $\mathrm{ET}_1(U, \bigcup \mathcal{V}) \vdash_1 Q$. It follows that $\mathrm{ET}_1(U, \bigcup \mathcal{V}) \vdash_1 \mathrm{ET}_1(U, \bigcup \mathcal{Y}) \cup E_1 \cup \cdots \cup E_{i+1}$. Note that $\mathrm{ET}_1(Y^{(X_{i+1}, E_{i+1})}, \bigcup \mathcal{F}') \subseteq \mathrm{ET}_1(U, \bigcup \mathcal{Y}))$, as $\bigcup \mathcal{F}' \subseteq \bigcup \mathcal{Y}$, $\mathrm{ET}_0(Y^{(X_{i+1}, E_{i+1})}, \bigcup \mathcal{F}') \subseteq X_{i+1}$ and $U \vdash_0 X_{i+1}$. Because of Condition 4.1(1), we therefore have that $\mathrm{ET}_1(U, \bigcup \mathcal{Y}) \cup E_1 \cup \cdots \cup E_{i+1} \in \mathrm{Con}_1$.

(2) Let $\mathcal{U} \in \mathrm{Con}_\rightarrow$ with $\mathcal{U} \vdash_\rightarrow \mathcal{Y}$. We have to demonstrate that $\mathcal{U} \vdash_\rightarrow \mathcal{Z}$. Obviously, it suffices to prove that $\mathcal{U} \vdash_\rightarrow \mathfrak{Z}_{\mathcal{F}} \setminus \bigcup \mathcal{F}'$, i.e., we must show that for any $(X, E) \in \mathfrak{Z}_{\mathcal{F}} \setminus \bigcup \mathcal{F}'$, $\mathrm{ET}_1(X, \bigcup \mathcal{U}) \vdash_1 E$.

Let $(X, E) \in \mathfrak{Z}_{\mathcal{F}} \setminus \bigcup \mathcal{F}'$. Since $\mathcal{U} \vdash_\rightarrow \mathcal{Y}$, we have for all $(P, Q) \in \bigcup \mathcal{Y}$ that $\mathrm{ET}_1(P, \bigcup \mathcal{U}) \vdash_1 Q$. This is true in particular for those $(P, Q) \in \bigcup \mathcal{Y}$ with $Y^{(X,E)} \vdash_0 P$. Thus

$$\mathrm{ET}_1(Y^{(X,E)}, \mathcal{U}) \vdash_1 \mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{Y}).$$

Similarly, by considering only $(P, Q) \in \bigcup \mathcal{F}'$ with $Y^{(X,E)} \vdash_0 P$ and using that

$$\mathrm{ET}_0(Y^{(X,E)}, \bigcup \mathcal{F}') \subseteq X,$$

we obtain that $\mathrm{ET}_1(X, \bigcup \mathcal{U}) \vdash_1 \mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{F}')$. As the information system A_1 satisfies Condition (L), there is thus some $T \in \mathrm{Con}_1$ such that $T \in \mathrm{Sup}_1(\mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{F}'))$ and

$$\mathrm{ET}_1(X, \bigcup \mathcal{U}) \vdash_1 T. \tag{7}$$

Remember that $Y^{(X,E)} \vdash_0 X$. Then we also have that $\mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{U}) \vdash_1 T$. Hence,

$$\mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{U}) \vdash_1 \mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{Y}) \cup T. \tag{8}$$

By our general assumption, $\mathcal{V} \vdash_\rightarrow \mathcal{Y} \cup \mathcal{Z}$. In particular $\mathcal{V} \vdash_\rightarrow \mathfrak{Z}_{\mathcal{F}}$ and $\mathcal{V} \vdash_\rightarrow \mathcal{Y}$. It follows that $\mathrm{ET}_1(X, \bigcup \mathcal{V}) \vdash_1 E$ and consequently

$$\mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{V}) \vdash_1 E. \tag{9}$$

Moreover, we obtain as above that $\mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{V}) \vdash_1 \mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{Y})$. Note that $\mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{Y}) \supseteq \mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{F}')$. With (8) and Condition 4.1(2) it now follows that $\mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{V}) \vdash_1 T$. Because of (9), Lemma 4.4 implies that

$$T \underset{\mathrm{ET}_1(Y^{(X,E)}, \bigcup \mathcal{F}')}{\sim} E.$$

Therefore, as a consequence of (7), we have that $\mathrm{ET}_1(X, \bigcup \mathcal{U}) \vdash_1 E$.

(3) Let $\mathfrak{W} \in A_0 \rightarrow A_1$ such that $\mathcal{Y} \cup \mathcal{Z} \vdash_\rightarrow \mathfrak{W}$. We need to prove that also $\mathcal{Y} \vdash_\rightarrow \mathfrak{W}$, i.e. we must show for $(X, E) \in \mathfrak{W}$ that $\mathrm{ET}_1(X, \bigcup \mathcal{Y}) \vdash_1 E$.

By our assumptions we have for $(X, E) \in \mathfrak{W}$ that

$$\mathrm{ET}_1(X, \bigcup(\mathcal{Y} \cup \mathcal{Z})) \vdash_1 E \tag{10}$$

and for $(P,Q) \in \bigcup(\mathcal{Y} \cup \mathcal{Z})$ that $\mathrm{ET}_1(P, \bigcup \mathcal{V}) \vdash_1 Q$. In particular we have for $(P,Q) \in \bigcup(\mathcal{Y} \cup \mathcal{Z})$ with $X \vdash_0 P$ that

$$\mathrm{ET}_1(X, \bigcup \mathcal{V}) \vdash_1 Q. \tag{11}$$

Let $(X, E) \in \mathfrak{W}$ and remember that $(P,Q) \in \bigcup(\mathcal{Y} \cup \mathcal{Z})$, just if $(P,Q) \in \bigcup \mathcal{Y}$ or $(P,Q) \in \mathfrak{Z}_{\mathcal{F}} \setminus \bigcup \mathcal{F}'$. By Lemma 6.2(1) we know that there are only finitely many $(P,Q) \in \mathfrak{Z}_{\mathcal{F}} \setminus \bigcup \mathcal{F}'$ with $X \vdash_0 P$, say $(P_1, Q_1), \ldots, (P_n, Q_n)$. For $i = 1, \ldots, n$, set

$$\widetilde{Y}_i = \mathrm{ET}_1(X, \bigcup \mathcal{Y}) \cup Q_1 \cup \cdots \cup Q_{n-i}.$$

By induction on i we show that $\widetilde{Y}_i \vdash_1 E$.

Because of (10) the statement holds for $i = 0$. Assume that it holds for $i < n$. By definition, $\widetilde{Y}_i = \widetilde{Y}_{i+1} \cup Q_{n-i}$. Note that $\mathrm{ET}_0(Y^{(P_{n-i}, Q_{n-i})}, \bigcup \mathcal{F}') \subseteq P_{n-i}$ and $X \vdash_0 P_{n-i}$. Therefore,

$$\mathrm{ET}_1(Y^{(P_{n-i}, Q_{n-i})}, \bigcup \mathcal{F}') \subseteq \mathrm{ET}_1(X, \bigcup \mathcal{F}') \subseteq \widetilde{Y}_{i+1}.$$

Since $Q_{n-i} \in \mathrm{Sup}_1(\mathrm{ET}_1(Y^{(P_{n-i}, Q_{n-i})}, \bigcup \mathcal{F}'))$ it follows with Condition 4.1(3) and the induction hypothesis that $\widetilde{Y}_{i+1} \vdash_1 E$.

Proposition 6.7 *Let $(A_0, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$ be algebraic L-information systems. Then $(A_0 \to A_1, \mathrm{Con}_\to, \vdash_\to)$ is algebraic as well.*

Proof: Because of Lemma 4.9 it suffices to show for $\mathcal{W} \in \mathrm{Con}_\to$ and $\mathfrak{V} \in A_0 \to A_1$ that there is some $\mathcal{U} \in \mathrm{Con}_\to$ with $\mathcal{W} \vdash_\to \mathcal{U} \vdash_\to \mathcal{U} \vdash_\to \mathfrak{V}$. Let $\mathfrak{V}' \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$ with $\mathfrak{V} \in \mathrm{JE}(\mathfrak{V}')$ and $(Y, Z) \in \mathfrak{V}'$. As shown in the proof of Claim 5 of the preceding proposition, there are $M^Y \in \mathrm{Con}_0$ and $N^{Y,Z} \in \mathrm{Con}_1$ such that

$$Y \vdash_0 M^Y \vdash \mathrm{ET}_0(Y, \bigcup \mathcal{W}) \quad \text{and} \quad \mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_1 N^{Y,Z} \vdash_1 Z.$$

By Condition (ALG) there are $\widetilde{M}^Y \in \mathrm{Con}_0$ and $\widetilde{N}^{Y,Z} \in \mathrm{Con}_1$ with

$$M^Y \vdash_0 \widetilde{M}^Y \vdash_0 \widetilde{M}^Y \vdash_0 \mathrm{ET}_0(Y, \bigcup \mathcal{W}) \quad \text{and} \quad \mathrm{ET}_1(Y, \bigcup \mathcal{W}) \vdash_1 \widetilde{N}^{Y,Z} \vdash_1 \widetilde{N}^{Y,Z} \vdash_1 N^{Y,Z}.$$

Construct $\mathfrak{U} \in A_0 \to A_1$ as in the proof of Claim 5 by using $(\widetilde{M}^Y, \widetilde{N}^{Y,Z})$ instead of $(M^Y, N^{Y,Z})$. Then it follows as above that $\mathcal{W} \vdash_\to \{\mathfrak{U}\} \vdash_\to \mathfrak{V}$. It remains to prove that $\{\mathfrak{U}\} \vdash_\to \mathfrak{U}$.

Let $(S, T) \in \mathfrak{U}$. Then $S \vdash_0 S$ and $T \vdash_1 T$, by Lemma 4.9. Hence $T \subseteq \mathrm{ET}_1(S, \mathfrak{U})$ which means that $\mathrm{ET}_1(S, \mathfrak{U}) \vdash_1 T$, as was to be shown.

Let $f \in |A_0 \to A_1|$. Then $f \subseteq A_0 \to A_1$, i.e., f is a subset of the powerset of $\mathrm{Con}_0 \times \mathrm{Con}_1$. We will now show that the states of $A_0 \to A_1$ correspond to the approximable mappings between A_0 and A_1 in a one-to-one way.

Lemma 6.8 *For $f \in |A_0 \to A_1|$, let $\mathrm{AM}(f) = \bigcup f$. Then $\mathrm{AM}(f) \colon A_0 \Vdash A_1$.*

Proof: In order to show that $\mathrm{AM}(f)$ is an approximable mapping we need to verify Conditions 5.1(1–5).

(1) Let $X \in \mathrm{Con}_0$, $Y \in \mathrm{Con}_1$ and $b \in A_1$ with $(X, Y) \in \mathrm{AM}(f)$ and $Y \vdash_1 b$. We must show that $(X, \{b\}) \in \mathrm{AM}(f)$.

Since $(X,Y) \in \mathrm{AM}(f)$, there is some $\mathfrak{V} \in f$ with $(X,Y) \in \mathfrak{V}$. Moreover, because of 3.3(3), there is some $\mathcal{W} \in \mathrm{Con}_{\rightarrow}$ with $\mathcal{W} \subseteq f$ and $\mathcal{W} \vdash_{\rightarrow} \mathfrak{V}$. Therefore, $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 Y$, from which it follows that also $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 b$. Apply the construction in Lemma 6.3 to $(X, \{b\})$. Then we have for the resulting set $\mathfrak{U}$ that $\mathcal{W} \vdash_{\rightarrow} \mathfrak{U}$. With Condition 3.3(2) we obtain that $\mathfrak{U} \in f$, i.e., $(X, \{b\}) \in \mathrm{AM}(f)$.

(2) Let $X \in \mathrm{Con}_0$ and F be a finite subset of A_1 with $(X, \{b\}) \in \mathrm{AM}(f)$, for all $b \in F$. We need to construct a $Z \in \mathrm{Con}_1$ so that $F \subseteq Z$ and $(X, Z) \in \mathrm{AM}(f)$.

As $(X, \{b\}) \in \mathrm{AM}(f)$, for all $b \in F$, there is some $\mathfrak{V}_b \in f$ with $(X, \{b\}) \in \mathfrak{V}_b$, for each such b. By Proposition 3.4 there is moreover some $\mathcal{W} \in \mathrm{Con}_{\rightarrow}$ such that $\mathcal{W} \subseteq f$ and $\mathcal{W} \vdash_{\rightarrow} \mathfrak{V}_b$, for all $b \in F$. Hence, we have for all such b that $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_2 b$. It follows that $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_2 F$. Because of Condition 3.1(6) there is thus some $Z \in \mathrm{Con}_1$ with $F \subseteq Z$ and $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_2 Z$. Now, apply the construction in Lemma 6.3 to (X, Z). We obtain a set $\mathfrak{U} \in A_0 \rightarrow A_1$ with $\mathcal{W} \vdash_{\rightarrow} \mathfrak{U}$. Since $\mathcal{W} \subseteq f$, it follows that $\mathfrak{U} \in f$ as well. Consequently, $(X, Z) \in \mathrm{AM}(f)$.

(3) Let $X, X' \in \mathrm{Con}_0$ and $b \in A_1$ such that $X \vdash_0 X'$ and $(X', \{b\}) \in \mathrm{AM}(f)$. We need to show that $(X, \{b\}) \in \mathrm{AM}(f)$.

As $(X', \{b\}) \in \mathrm{AM}(f)$, there is some $\mathfrak{V} \in f$ with $(X', \{b\}) \in \mathfrak{V}$. Moreover, there is some $\mathcal{W} \in \mathrm{Con}_{\rightarrow}$ so that $\mathcal{W} \subseteq f$ and $\mathcal{W} \vdash_{\rightarrow} \mathfrak{V}$. It follows that $\mathrm{ET}_1(X', \bigcup \mathcal{W}) \vdash_2 b$. Since $X \vdash_0 X'$, we have that $\mathrm{ET}_1(X', \bigcup \mathcal{W}) \subseteq \mathrm{ET}_1(X, \bigcup \mathcal{W})$. Thus, $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 b$. As in the case of Condition 5.1(1), we obtain that $(X, \{b\}) \in \mathrm{AM}(f)$.

(4) follows similarly.

(5) Let $X \in \mathrm{Con}_0$ and $b \in A_1$ with $(X, \{b\}) \in \mathrm{AM}(f)$. We have to show that there are $X' \in \mathrm{Con}_0$ and $Y \in \mathrm{Con}_1$ such that $X \vdash_0 X'$, $(X', Y) \in \mathrm{AM}(f)$ and $Y \vdash_1 b$.

Let $\mathfrak{V} \in f$ with $(X, \{b\}) \in \mathfrak{V}$ and $\mathcal{W} \in \mathrm{Con}_{\rightarrow}$ with $\mathcal{W} \subseteq f$ and $\mathcal{W} \vdash_{\rightarrow} \mathfrak{V}$. Then $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 b$. Because of Condition 3.1(5) there is some $Y \in \mathrm{Con}_1$ so that $Y \vdash_1 b$ and $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 Y$. As a consequence of Lemma 6.2(1) we have that there are only finitely many pairs $(P, Q) \in \bigcup \mathcal{W}$ with $X \vdash_0 P$, say $(P_1, Q_1), \ldots, (P_n, Q_n)$. Then $X \vdash_0 P_1 \cup \cdots \cup P_n$. With Proposition 3.2 it follows for some $X' \in \mathrm{Con}_0$ that $X \vdash_0 X'$ and $X' \vdash_0 P_1 \cup \cdots \cup P_n$. Then $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \subseteq \mathrm{ET}_1(X', \bigcup \mathcal{W})$ and therefore $\mathrm{ET}_1(X', \bigcup \mathcal{W}) \vdash_1 Y$. As above it follows that $(X', Y) \in \mathrm{AM}(f)$.

(6) We have seen in the proof of Claim 6 of Proposition 6.6 that $\mathfrak{t} = \{(\{t_0\}, \{t_1\})\}$ is a truth element of $A_0 \rightarrow A_1$. By Condition 3.3(2) a truth element is contained in any state. Thus, $\mathfrak{t} \in f$, i.e. $(\{t_0\}, \{t_1\}) \in \mathrm{AM}(f)$.

Lemma 6.9 *For* $G \colon A_0 \Vdash A_1$ *let* $\mathrm{ST}(G) = \{ \mathfrak{V} \in A_0 \rightarrow A_1 \mid \mathfrak{V} \subseteq G \}$. *Then* $\mathrm{ST}(G) \in |A_0 \rightarrow A_1|$.

Proof: In order to verify that $\mathrm{ST}(G)$ is a state of $A_0 \rightarrow A_1$, we need to check Conditions 3.3(1–3).

(1) Let $\mathcal{F}$ be a nonempty finite subset of $\mathrm{ST}(G)$, say $\mathcal{F} = \{\mathfrak{V}_1, \ldots, \mathfrak{V}_m\}$. We have to show that there is some $\mathcal{Z} \in \mathrm{Con}_{\rightarrow}$ with $\mathcal{F} \subseteq \mathcal{Z} \subseteq \mathrm{ST}(G)$.

For $i = 1, \ldots, m$, let $\mathfrak{V}'_i \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$ such that $\mathfrak{V}_i \in \mathrm{JE}(\mathfrak{V}'_i)$. Set $\mathcal{F}' = \{\mathfrak{V}'_1, \ldots, \mathfrak{V}'_m\}$. Then $\bigcup \mathcal{F}' \subseteq G$.

Now, let $Y \in \mathrm{Con}_0$. By Condition (L) there is some $S_Y \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \bigcup \mathcal{F}'))$ with $Y \vdash_0 S_Y$. It follows for $(P, Q) \in \mathcal{F}'$ that $P \subseteq S_Y$. Since, in addition, PGQ, we have that $S_Y G Q$. Thus, $S_Y G\, \mathrm{ET}_1(Y, \mathcal{F}')$. Note that all $\mathfrak{V}_i$ and hence $\bigcup \mathcal{F}'$ are finite. Consequently, there is some $Z \in \mathrm{Con}_1$ such that $Z \vdash_1 \mathrm{ET}_1(Y, \bigcup \mathcal{F}')$ and $S_Y G Z$. With Condition (L) we obtain some $T_Y \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \bigcup \mathcal{F}'))$ with $Z \vdash_1 T_Y$. Thus $S_Y G T_Y$. By applying Lemma 6.4 to $\bigcup \mathcal{F}'$ and $\{ (S_Y, T_Y) \mid Y \in \mathrm{Con}_0 \}$ we obtain some set $\mathfrak{Z}_{\mathcal{F}} \in \mathrm{JE}(\bigcup \mathcal{F}')$. Then

$\mathfrak{Z}_{\mathcal{F}} \in A_0 \to A_1$. Set $\mathcal{Z} = \mathcal{F} \cup \{\mathfrak{Z}_{\mathcal{F})}\}$. Then it follows as in the proof of Claim 7 of Proposition 6.6 that $\mathcal{Z} \in \mathrm{Con}_{\to}$. By construction, $\mathfrak{Z}_{\mathcal{F}} \subseteq G$. Hence $\mathcal{F} \subseteq \mathcal{Z} \subseteq \mathrm{ST}(G)$.

(2) Let $\mathcal{W} \in \mathrm{Con}_{\to}$ and $\mathfrak{V} \in A_0 \to A_1$ with $\mathcal{W} \subseteq \mathrm{ST}(G)$ and $\mathcal{W} \vdash_{\to} \mathfrak{V}$. We need to show that $\mathfrak{V} \in \mathrm{ST}(G)$.

Let $(X, Y) \in \mathfrak{V}$. Then $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 Y$. Moreover, let $(P_1, Q_1), \ldots, (P_n, Q_n)$ be the finitely many pairs $(P, Q) \in \bigcup \mathcal{W}$ with $X \vdash_0 P$. Then $\mathrm{ET}_1(X, \bigcup \mathcal{W}) = Q_1 \cup \cdots \cup Q_n$. Since $\mathcal{W} \subseteq \mathrm{ST}(G)$, we have that $P_i G Q_i$, for $i = 1, \ldots, n$. Because $X \vdash_0 P_i$, for these i, it follows that $X G \, \mathrm{ET}_1(X, \bigcup \mathcal{W})$ and hence that XGY. This shows that $\mathfrak{V} \subseteq G$, i.e. $\mathfrak{V} \in \mathrm{ST}(G)$.

(3) Let $\mathfrak{V} \in \mathrm{ST}(G)$. We have to show that there is some $\mathcal{W} \in \mathrm{Con}_{\to}$ such that $\mathcal{W} \subseteq \mathrm{ST}(G)$ and $\mathcal{W} \vdash_{\to} \mathfrak{V}$.

Note that $\mathfrak{V} \subseteq G$ and let $\mathfrak{V} \in \mathrm{JE}(\mathfrak{V}')$ with $\mathfrak{V}' = \{(P_1, Q_1), \ldots, (P_n, Q_n)\}$. Then there are $\overline{P}_i \in \mathrm{Con}_0$ and $\overline{Q}_i \in \mathrm{Con}_1$, for $i = 1, \ldots, n$, such that $P_i \vdash_0 \overline{P}_i$, $\overline{P}_i G \overline{Q}_i$ and $Q_i \vdash_1 \overline{Q}_i$. Set $\mathfrak{U}' = \{(\overline{P}_1, \overline{Q}_1), \ldots, (\overline{P}_n, \overline{Q}_n)\}$

Now, let $(X, Y) \in \mathfrak{V} \setminus \mathfrak{V}'$. Then there is some $U \in \mathrm{Con}_0$ such that $U \vdash_0 X$, $X \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{V}'))$ and $Y \in \mathrm{Sup}_1(\mathrm{ET}_1(U, \mathfrak{V}'))$. It follows that $X \vdash_0 \mathrm{ET}_0(U, \mathfrak{U}')$. Therefore, by Condition (L), there is some $S_U \in \mathrm{Sup}_0(\mathrm{ET}_0(U, \mathfrak{U}'))$ with $X \vdash_0 S_U$. Then $S_U G \, \mathrm{ET}_1(U, \mathfrak{U}')$. Thus, there is some $V \in \mathrm{Con}_1$ so that $S_U G V$ and $V \vdash_1 \mathrm{ET}_1(U, \mathfrak{U}')$. Applying Condition (L) again we obtain some $T_U \in \mathrm{Sup}_1(\mathrm{ET}_1(U, \mathfrak{U}'))$ with $V \vdash_1 T_U$.

Since $T_U \vdash_1 \mathrm{ET}_1(U, \mathfrak{V}')$, it follows with Condition (L) that there is some $E' \in \mathrm{Sup}_1(\mathrm{ET}_1(U, \mathfrak{V}'))$ with $T_U \vdash_1 E'$. Thus $XG(E \cup E')$. Hence, there is some $Z \in \mathrm{Con}_1$ with XGZ and $Z \vdash_1 (E \cup E')$. With Lemma 4.4 it follows that

$$E \underset{\mathrm{ET}_1(U, \mathfrak{V}')}{\sim} E',$$

from which we obtain that $T_U \vdash_1 E$ as well.

Construct $\mathfrak{U}$ by applying Lemma 6.4 to $\mathfrak{U}'$ and $\{\, (S_U, T_U) \mid U \in \mathrm{Con}_0 \,\}$. Then $\mathfrak{U} \in \mathrm{JE}(\mathfrak{U}')$, which means that $\mathfrak{U} \in A_0 \to A_1$ and $\{\mathfrak{U}\} \in \mathrm{Con}_{\to}$. It remains to show that $\{\mathfrak{U}\} \vdash_{\to} \mathfrak{V}$.

If $(P, Q) \in \mathfrak{V}'$ then we have for some $(\overline{P}, \overline{Q}) \in \mathfrak{U}'$ that $P \vdash_0 \overline{P}$ and $\overline{Q} \vdash_1 Q$. Thus $\mathrm{ET}_1(P, \mathfrak{U}) \vdash_1 Q$. If, on the other hand, $(X, Y) \in \mathfrak{V} \setminus \mathfrak{V}'$, we have for some $(S_U, T_U) \in \mathfrak{U}$ that $X \vdash_0 S_U$ and $T_U \vdash_1 E$. Therefore $\mathrm{ET}_1(X, \mathfrak{U}) \vdash_1 E$ again. This shows that $\{\mathfrak{U}\} \vdash_{\to} \mathfrak{V}$.

Lemma 6.10 *1.* $\mathrm{ST}(\mathrm{AM}(f)) = f$*, for all* $f \in |A_0 \to A_1|$*.*

2. $\mathrm{AM}(\mathrm{ST}(G)) = G$*, for all* $G \colon A_0 \Vdash A_1$*.*

Proof: (1) Obviously $f \subseteq \{\, \mathfrak{V} \in A_0 \to A_1 \mid \mathfrak{V} \subseteq \mathrm{AM}(f) \,\}$. Conversely, if $\mathfrak{V} \in \mathrm{AM}(f)$, then there is some $\mathfrak{U}^{(X,Y)} \in f$ with $(X, Y) \in \mathfrak{U}^{(X,Y)}$, for every $(X, Y) \in \mathfrak{V}$. Moreover, there is some $\mathfrak{V}' \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$ with $\mathfrak{V} \in \mathrm{JE}(\mathfrak{V}')$. Since $\mathfrak{V}'$ is finite, we obtain some $\mathcal{W} \in f$ such that $\mathcal{W} \vdash_{\to} \mathfrak{U}^{(P,Q)}$, for all $(P, Q) \in \mathfrak{V}'$. Thus, $\mathrm{ET}_1(P, \bigcup \mathcal{W}) \vdash_1 Q$, for each such pair (P, Q).

Now, let $(X, Y) \in \mathfrak{V} \setminus \mathfrak{V}'$. Then there exists $U \in \mathrm{Con}_0$ so that $U \vdash_0 X$, $X \in \mathrm{Sup}_0(\mathrm{ET}_0(U, \mathfrak{V}'))$ and $Y \in \mathrm{Sup}_1(\mathrm{ET}_1(U, \mathfrak{V}'))$. It follows that $\mathrm{ET}_0(U, \mathfrak{V}') \subseteq X$. Thus, $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 \mathrm{ET}_1(U, \mathfrak{V}')$. By Condition (L) there is hence some $Z \in \mathrm{Sup}_1(\mathrm{ET}_1(U, \mathfrak{V}'))$ with $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 Z$. In addition, as $\mathcal{W} \cup \{\mathfrak{U}^{(X,Y)}\} \subseteq_{\mathrm{fin}} f$, there is some $\mathcal{V} \in \mathrm{Con}_{\to}$ with $\mathcal{V} \subseteq f$ and $\mathcal{V} \vdash_{\to} \mathcal{W} \cup \{\mathfrak{U}^{(X,Y)}\}$. It follows that $\mathrm{ET}_1(X, \bigcup \mathcal{V}) \vdash_1 Y \cup Z$. Consequently,

$$Y \underset{\mathrm{ET}_1(U, \mathfrak{V}')}{\sim} Z,$$

by Lemma 4.4, which implies that $\mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 Y$. This shows that $\mathcal{W} \vdash_{\to} \mathfrak{V}$. Therefore, $\mathfrak{V} \in f$.

(2) By definition, $\mathrm{AM}(\mathrm{ST}(G)) \subseteq G$. Conversely, let $(X, Y) \in G$. Note that there is some $Z \in \mathrm{Con}_1$ with $Z \vdash_1 Y$ by Condition 5.1(5). Thus, Lemma 6.3 can be applied again and we obtain that $(X, Y) \in \mathrm{AM}(\mathrm{ST}(G))$.

Proposition 6.11 *Let $(A_0, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$ be L-information systems. Then the states of $A_0 \to A_1$ and the approximable mappings between A_0 and A_1 correspond to each other in a one-to-one way:*

1. *$\{ \bigcup f \mid f \in |A_0 \to A_1| \}$ is the set of approximable mappings from A_0 to A_1.*
2. *$|A_0 \to A_1|$ is the collection of all sets $\{ \mathfrak{V} \in A_0 \to A_1 \mid \mathfrak{V} \subseteq G \}$, where G is an approximable mapping between A_0 and A_1.*

In Section 5 we have already studied how approximable mappings between two information systems A_0 and A_1 correspond to Scott continuous functions from $|A_0|$ to $|A_1|$, and vice versa. As we will see now, this correspondence establishes an isomorphism between the domains $|A_0 \to A_1|$ and $[|A_0| \to |A_1|]$.

Let $f \in |A_0 \to A_1|$. Then $\mathrm{AM}(f) \colon A_0 \Vdash A_1$ and hence $\mathcal{D}(\mathrm{AM}(f)) \in [|A_0| \to |A_1|]$. Set

$$\mathrm{fct}(f) = \mathcal{D}(\mathrm{AM}(f)).$$

Then $\mathrm{fct} \in [|A_0 \to A_1| \to [|A_0| \to |A_1|]]$. Note that for $x \in |A_0|$,

$$\begin{aligned}
\mathrm{fct}(f)(x) &= \{ b \in A_1 \mid (\exists X \in \mathrm{Con}_0) X \subseteq x \wedge (X, \{b\}) \in \mathrm{AM}(f) \} \\
&= \{ b \in A_1 \mid (\exists X \in \mathrm{Con}_0)(\exists \mathfrak{V} \in A_0 \to A_1) X \subseteq x \wedge \mathfrak{V} \in f \wedge (X, \{b\}) \in \mathfrak{V} \} \\
&= \{ b \in A_1 \mid (\exists X \in \mathrm{Con}_0)(\exists \mathfrak{V} \in A_0 \to A_1)(\exists \mathcal{W} \in \mathrm{Con}_{\to}) X \subseteq x \wedge \mathcal{W} \subseteq f \\
&\qquad\qquad \wedge \mathcal{W} \vdash_{\to} \mathfrak{V} \wedge (X, \{b\}) \in \mathfrak{V} \} \\
&= \{ b \in A_1 \mid (\exists X \in \mathrm{Con}_0)(\exists \mathcal{W} \in \mathrm{Con}_{\to}) X \subseteq x \wedge \mathcal{W} \subseteq f \wedge \mathrm{ET}_1(X, \bigcup \mathcal{W}) \vdash_1 b \},
\end{aligned}$$

where the equality in the last line follows with Lemma 6.3.

Conversely, let $g \in [|A_0| \to |A_1|]$. Then $\mathcal{C}(g) \colon \mathcal{C}(|A_0|) \Vdash \mathcal{C}(|A_1|)$ and $S_{A_0} \circ \mathcal{C}(g) \circ T_{A_1} \colon A_0 \to A_1$. Set

$$\mathrm{st}(g) = \{ \mathfrak{V} \in A_0 \to A_1 \mid \mathfrak{V} \subseteq S_{A_0} \circ \mathcal{C}(g) \circ T_{A_1} \}.$$

By the preceding proposition, $\mathrm{st}(g) \in |A_0 \to A_1|$. As is readily verified, st is Scott continuous.

We will show now that the functions fct and st are inverse to each other.

Lemma 6.12 $\mathrm{st} \circ \mathrm{fst} = \mathrm{id}_{|A_0 \to A_1|}$.

Proof: Let $f \in |A_0 \to A_1|$. As we have seen in the proof of Proposition 5.10, $S_{A_0} \circ \mathcal{C}(\mathcal{D}(\mathrm{AM}(f))) = \mathrm{AM}(f) \circ S_{A_1}$. Moreover, S_{A_1} and T_{A_1} are inverse to each other. With Lemma 6.10(1) we therefore obtain that

$$\begin{aligned}
\mathrm{st}(\mathrm{fct}(f)) &= \{ \mathfrak{V} \in A_0 \to A_1 \mid \mathfrak{V} \subseteq S_{A_0} \circ \mathcal{C}(\mathcal{D}(\mathrm{AM}(f)) \circ T_{A_1} \} \\
&= \{ \mathfrak{V} \in A_0 \to A_1 \mid \mathfrak{V} \subseteq \mathrm{AM}(f) \} \\
&= f.
\end{aligned}$$

Lemma 6.13 $\mathrm{fst} \circ \mathrm{st} = \mathrm{id}_{[|A_0| \to |A_1|]}$.

Proof: Let $g \in [|A_0| \to |A_1|]$ and note that $\mathcal{D}(S_{A_0}) = \tau_{|A_0|}$ and $\mathcal{D}(T_{A_1}) = \mathrm{sp}_{|A_1|}$, where the function $\mathrm{sp}_{|A_1|}$ defined in Section 3 is the inverse of $\tau_{|A_1|}$. As we have seen in the proof of Proposition 5.7, $\tau_{|A_1|} \circ g = \mathcal{D}(\mathcal{C}(g)) \circ \tau_{|A_0|}$. With Lemma 6.10(2) we hence obtain for $x \in |A_0|$ that

$$\begin{aligned}
\mathrm{fct}(\mathrm{st}(g))(x) &= \mathcal{D}(\bigcup \{\, \mathfrak{V} \in A_0 \to A_1 \mid \mathfrak{V} \subseteq S_{A_0} \circ \mathcal{C}(g) \circ T_{A_1} \,\})(x) \\
&= \mathcal{D}(S_{A_0} \circ \mathcal{C}(g) \circ T_{A_1})(x) \\
&= \mathcal{D}(T_{A_1})(\mathcal{D}(\mathcal{C}(g))(\mathcal{D}(S_{A_0})(x))) \\
&= \mathrm{sp}_{|A_1|}(\mathcal{D}(\mathcal{C}(g))(\tau_{|A_0|}(x))) \\
&= \mathrm{sp}_{|A_1|}(\tau_{|A_1|}(g(x))) \\
&= g(x).
\end{aligned}$$

Proposition 6.14 *Let $(A_0, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$ be L-information systems. Then the domains $|A_0 \to A_1|$ and $[|A_0| \to |A_1|]$ are isomorphic.*

7 Cartesian closure

We will show now that for two L-information systems $(A_0, \mathrm{Con}_0, \vdash_0)$ and $(A_1, \mathrm{Con}_1, \vdash_1)$, $A_0 \to A_1$ is the exponent of A_0 and A_1 in the category **LINF**.

For $Z \in \mathrm{Con}_{(A_0 \to A_1) \times A_0}$ and $b \in \mathrm{Con}_1$ let

$$Z \,\mathrm{EV}\, b \Leftrightarrow \mathrm{ET}_1(\mathrm{pr}_{A_0}(Z), \bigcup \mathrm{pr}_{A_0 \to A_1}(Z)) \vdash_1 b.$$

Lemma 7.1 $\mathrm{EV}\colon (A_0 \to A_1) \times A_0 \Vdash A_1$.

Proof: We only verify Conditions 5.1(3, 5), the others being obvious.

(3) Let $X, X' \in \mathrm{Con}_{(A_0 \to A_1) \times A_0}$ and $b \in A_1$ so that $X \vdash_{(A_0 \to A_1) \times A_0} X'$and $X' \,\mathrm{EV}\, b$. We need to prove that $X \,\mathrm{EV}\, b$.

We have that $\mathrm{pr}_{A_0}(X) \vdash_0 \mathrm{pr}_{A_0}(X')$ and $\mathrm{ET}_1(\mathrm{pr}_{A_0}(X'), \bigcup \mathrm{pr}_{A_0 \to A_1}(X')) \vdash_1 b$. It follows that also $\mathrm{ET}_1(\mathrm{pr}_{A_0}(X), \bigcup \mathrm{pr}_{A_0 \to A_1}(X')) \vdash_1 b$. Moreover, we have that $\mathrm{pr}_{A_0 \to A_1}(X) \vdash_{A_0 \to A_1} \mathrm{pr}_{A_0 \to A_1}(X')$. Thus $\mathrm{ET}_1(P, \bigcup \mathrm{pr}_{A_0 \to A_1}(X)) \vdash_1 Q$, for all $(P, Q) \in \bigcup \mathrm{pr}_{A_0 \to A_1}(X')$ with $\mathrm{pr}_{A_0}(X) \vdash_0 P$. As a consequence,

$$\mathrm{ET}_1(\mathrm{pr}_{A_0}(X), \bigcup \mathrm{pr}_{A_0 \to A_1}(X)) \vdash_1 \mathrm{ET}_1(\mathrm{pr}_{A_0}(X), \bigcup \mathrm{pr}_{A_0 \to A_1}(X')).$$

Hence $\mathrm{ET}_1(\mathrm{pr}_{A_0}(X), \bigcup \mathrm{pr}_{A_0 \to A_1}(X)) \vdash_1 b$, i.e. $X \,\mathrm{EV}\, b$.

(5) Let $X \in \mathrm{Con}_{(A_0 \to A_1) \times A_0}$ and $b \in A_1$ with $X \,\mathrm{EV}\, b$. We have to show that there are $Z \in \mathrm{Con}_{(A_0 \to A_1) \times A_0}$ and $U \in \mathrm{Con}_1$ such that $X \vdash_{(A_0 \to A_1) \times A_0} Z$, $Z \,\mathrm{EV}\, U$ and $U \vdash_1 b$.

By assumption, $\mathrm{ET}_1(\mathrm{pr}_{A_0}(X), \bigcup \mathrm{pr}_{A_0 \to A_1}(X))] \vdash_1 b$. According to Lemma 6.2(1) there only finitely many $(P, Q) \in \bigcup \mathrm{pr}_{A_0 \to A_1}(X)$ with $\mathrm{pr}_{A_0}(X) \vdash_0 P$, say $(P_1, Q_1), \ldots, (P_n, Q_n)$. Therefore, because of Condition 3.1(5), there is some $Y \in \mathrm{Con}_0$ such that $\mathrm{pr}_{A_0}(X) \vdash_0 Y \vdash_0 P_1 \cup \cdots \cup P_n$. Thus $\mathrm{ET}_1(Y, \bigcup \mathrm{pr}_{A_0 \to A_1}(X)) = \mathrm{ET}_1(\mathrm{pr}_{A_0}(X), \bigcup \mathrm{pr}_{A_0 \to A_1}(X))$, which implies that $\mathrm{ET}_1(Y, \bigcup \mathrm{pr}_{A_0 \to A_1}(X)) \vdash_1 b$. This shows that $\mathrm{pr}_{A_0 \to A_1}(X) \vdash_{A_0 \to A_1} \mathfrak{U}$, where $\mathfrak{U} \in \mathrm{JE}(\{(Y, \{b\})\})$ as in Lemma 6.3. It follows that $\mathrm{pr}_{A_0 \to A_1}(X) \vdash_{A_0 \to A_1} \mathcal{V} \vdash_{A_0 \to A_1} \mathfrak{U}$, for some $\mathcal{V} \in \mathrm{Con}_{A_0 \to A_1}$. Hence $\mathrm{ET}_1(Y, \bigcup \mathcal{V}) \vdash_1 b$. Using Condition 3.1(5) again we obtain some $U \in \mathrm{Con}_1$ so that $\mathrm{ET}_1(Y, \bigcup \mathcal{V}) \vdash_1 U \vdash_1 b$. Set $Z = Y \times \mathcal{V}$. Then we have that $X \vdash_{(A_0 \to A_1) \times A_0} Z$, $Z \,\mathrm{EV}\, U$ and $U \vdash_1 b$.

Let $(A_2, \mathrm{Con}_2, \vdash_2)$ be a further L-information system. For $H\colon A_2 \times A_0 \Vdash A_1$, $V \in \mathrm{Con}_2$ and $\mathfrak{V} \in A_0 \to A_1$ define

$$V \Lambda(H) \mathfrak{V} \Leftrightarrow (\forall (X, E) \in \mathfrak{V})(V \times X) H E.$$

Lemma 7.2 $\Lambda(H)\colon A_2 \Vdash A_0 \to A_1$.

Proof: Again we have to verify Conditions 5.1(1–6).

(1) Let $V \in \mathrm{Con}_2$, $\mathcal{W} \in \mathrm{Con}_{A_0 \to A_1}$ and $\mathfrak{V}$ so that $V \Lambda(H) \mathcal{W}$ and $\mathcal{W} \vdash_{A_0 \to A_1} \mathfrak{V}$. Moreover, let $(P, Q) \in \mathfrak{V}$. We have to show that $(V \times P) H Q$.

Since $\mathcal{W} \vdash_{A_0 \to A_1} \mathfrak{V}$, $\mathrm{ET}_1(P, \bigcup \mathcal{W}) \vdash_1 Q$. Moreover, we have that $(V \times X) H E$, for all $(X, E) \in \bigcup \mathcal{W}$ with $P \vdash_0 X$. Thus, $(V \times \mathrm{ET}_0(P, \bigcup \mathcal{W})) H \,\mathrm{ET}_1(P, \bigcup \mathcal{W})$. As $P \vdash_0 \mathrm{ET}_0(P, \bigcup \mathcal{W})$, we obtain that $(V \times P) H Q$.

(2) Let $V \in \mathrm{Con}_2$ and $\mathcal{F} \subseteq_{\mathrm{fin}} A_0 \to A_1$ with $V \Lambda(H) \mathcal{F}$. We have to construct some $\mathcal{Z} \in \mathrm{Con}_{A_0 \to A_1}$ such that $\mathcal{Z} \supseteq \mathcal{F}$ and $V \Lambda(H) \mathcal{Z}$. Without restriction we assume that $\mathcal{F} \notin \mathrm{Con}_{A_0 \to A_1}$.

By our assumption we have for $(P, Q) \in \bigcup \mathcal{F}$ that $(V \times P) H Q$. Now, for $\mathfrak{V} \in \mathcal{F}$, fix some $\mathfrak{V}' \subseteq_{\mathrm{fin}} A_0 \times A_1$ with $\mathfrak{V} \in \mathrm{JE}(\mathfrak{V}')$ and let $\mathcal{F}'$ be the collection of these $\mathfrak{V}'$. Then $\bigcup \mathrm{pr}_{A_0}(\bigcup \mathcal{F}')$ and $\bigcup \mathrm{pr}_{A_1}(\bigcup \mathcal{F}')$ are finite.

Let $Y \in \mathrm{Con}_0$. Then $Y \vdash_0 \mathrm{ET}_0(Y, \bigcup \mathcal{F}')$. By Condition (L) there is thus some $X_Y \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \bigcup \mathcal{F}'))$ with $Y \vdash_0 X_Y$. It follows that $(V \times X_Y) H \,\mathrm{ET}_1(Y, \bigcup \mathcal{F}')$. Hence there is some $U \in \mathrm{Con}_1$ with $(V \times X_Y) H U$ and $U \vdash_1 \mathrm{ET}_1(Y, \bigcup \mathcal{F}')$. Using Condition (L) again we obtain some $E_Y \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \bigcup \mathcal{F}'))$ with $U \vdash_1 E_Y$.

Now construct the set $\mathfrak{Z}_{\mathcal{F}}$ by applying Lemma 6.4 to $\bigcup \mathcal{F}'$ and $\{ (X_Y, E_Y) \mid Y \in \mathrm{Con}_0 \}$. Then $\mathfrak{Z}_{\mathcal{F}} \in \mathrm{JE}(\bigcup \mathcal{F}')$, which implies that $\mathfrak{Z}_{\mathcal{F}} \in A_0 \to A_1$. By construction $(V \times X) H E$, for all $(X, E) \in \mathfrak{Z}_{\mathcal{F}}$. Set $\mathcal{Z} = \mathcal{F} \cup \{\mathfrak{Z}_{\mathcal{F}}\}$. Then $V \Lambda(H) \mathcal{Z}$. As in the proof of Proposition 6.6, Claim 7 it follows that $\mathcal{Z} \in \mathrm{Con}_{A_0 \to A_1}$.

(3) Let $V, V' \in \mathrm{Con}_2$ and $\mathfrak{V} \in \mathrm{Con}_{A_0 \to A_1}$ so that $V \vdash_2 V'$ and $V' \Lambda(H) \mathfrak{V}$. We have to verify that for all $(X, E) \in \mathfrak{V}$, $(V \times X) H E$.

Let $(X, E) \in \mathfrak{V}$. Then there is some $U \in \mathrm{Con}_{A_2 \times A_0}$ with $(V' \times X) \vdash_{A_2 \times A_0} U$ and $U H E$. It follows that $V \vdash_2 V' \vdash_2 \mathrm{pr}_{A_2}(U)$ and $X \vdash_0 \mathrm{pr}_{A_0}(U)$. Therefore $V \vdash_2 \mathrm{pr}_{A_2}(U)$ and hence $(V \times X) \vdash_{A_2 \times A_0} U$, which implies that $(V \times X) H E$.

(4) is obvious, as is (6).

(5) Let $V \in \mathrm{Con}_2$ and $\mathfrak{V} \in A_0 \to A_1$ with $V \Lambda(H) \mathfrak{V}$. Then

$$(V \times X) H E, \tag{12}$$

for all $(X, E) \in \mathfrak{V}$. We have to construct $Z \in \mathrm{Con}_2$ and $\mathcal{Z} \in \mathrm{Con}_{A_0 \to A_1}$ so that $V \vdash_2 Z$, $Z \Lambda(H) \mathcal{Z}$ and $\mathcal{Z} \vdash_{A_0 \to A_1} \mathfrak{V}$.

Let $\mathfrak{V}' \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$ with $\mathfrak{V} \in \mathrm{JE}(\mathfrak{V}')$. Then we have for all $(P, Q) \in \mathfrak{V}'$ that $(V \times P) H Q$. Hence there are $M^{(P,Q)} \in \mathrm{Con}_{A_2 \times A_0}$ and $N^{(P,Q)} \in \mathrm{Con}_1$ so that $(V \times P) \vdash_{A_2 \times A_0} M^{(P,Q)}$, $M^{(P,Q)} H N^{(P,Q)}$ and $N^{(P,Q)} \vdash_1 Q$. It follows that $V \vdash_2 \mathrm{pr}_{A_2}(M^{(P,Q)})$, for all $(P, Q) \in \mathfrak{V}'$. Thus, there is some $Z \in \mathrm{Con}_2$ so that $Z \supseteq \bigcup \{ \mathrm{pr}_{A_2}(M^{(P,Q)}) \mid (P, Q) \in \mathfrak{V}' \}$ and $V \vdash_2 Z$. Moreover, $Z \times \mathrm{pr}_{A_0}(M^{(P,Q)}) H N^{(P,Q)}$, for all $(P, Q) \in \mathfrak{V}'$.

Now, let $(X, E) \in \mathfrak{V} \setminus \mathfrak{V}'$. Then there is some $Y \in \mathrm{Con}_0$ such that $Y \vdash_0 X$, $X \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{V}'))$ and $E \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}'))$. For $(P, Q) \in \mathfrak{V}'$ with $Y \vdash_0 P$ we have that $X \vdash_0 \mathrm{pr}_{A_0}(M^{(P,Q)})$, as $P \subseteq X$. Thus, $X \vdash_0 \bigcup \{ \mathrm{pr}_{A_0}(M^{(P,Q)}) \mid (P, Q) \in \mathfrak{V}' \wedge Y \vdash_0 P \}$. By Condition (L) there is hence some $M^Y \in \mathrm{Sup}_0(\bigcup \{ \mathrm{pr}_{A_0}(M^{(P,Q)}) \mid (P, Q) \in \mathfrak{V}' \wedge Y \vdash_0 P \})$ with $X \vdash_0 M^Y$. It follows that $(Z \times M^Y) H \bigcup \{ N^{(P,Q)} \mid (P, Q) \in \mathfrak{V}' \wedge Y \vdash_0 P \}$. Consequently, there is some $T^Y \in \mathrm{Con}_1$ with $(Z \times M^Y) H T^Y$ and $T^Y \supseteq \bigcup \{ N^{(P,Q)} \mid (P, Q) \in$

$\mathfrak{V}' \wedge Y \vdash_0 P\}$. Applying Condition (L) again we obtain some $N^Y \in \mathrm{Sup}_1(\bigcup\{N^{(P,Q)} \mid (P,Q) \in \mathfrak{V}' \wedge Y \vdash_0 P\})$ and $T^Y \vdash_1 N^Y$. Furthermore, since $N^Y \vdash_1 \mathrm{ET}_1(Y, \mathfrak{V}')$, there is some $E' \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}'))$ with $N^Y \vdash_1 E'$. Because of (12) we have that $(V \times X)H(N^Y \cup E)$. Thus there is some $U \in \mathrm{Con}_1$ with $(V \times X)HU$ and $U \vdash_1 (N^Y \cup E)$. Hence $U \vdash_1 (E \cup E')$. With Lemma 4.4 we obtain that also $N^Y \vdash_1 E$.

Construct $\mathfrak{U} \in \mathrm{JE}(\mathfrak{U}')$ with Lemma 6.4 by starting from $\mathfrak{U}' = \{(\mathrm{pr}_{A_0}(M^{(P,Q)}), N^{(P,Q)}) \mid (P,Q) \in \mathfrak{V}'\}$ and $\{(M^Y, N^Y) \mid Y \in \mathrm{Con}_0\}$. Then we have that

$$V \times P \vdash_{A_2 \times A_0} Z \times \mathrm{pr}_0(M^{(P,Q)}), \quad Z \times \mathrm{pr}_{A_0}(M^{(P,Q)}) H N^{(P,Q)} \quad \text{and} \quad N^{(P,Q)} \vdash_1 Q,$$

for $(P,Q) \in \mathfrak{V}'$ and

$$V \times X \vdash_{A_2 \times A_0} Z \times M^Y, \quad Z \times M^Y H N^Y \quad \text{and} \quad N^Y \vdash_1 E,$$

for $(X,E) \in \mathfrak{V} \setminus \mathfrak{V}'$ and $Y \in \mathrm{Con}_0$ such that $Y \vdash_0 X$, $X \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{V}'))$ and $E \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}'))$. Since $N^Y \subseteq \mathrm{ET}_1(X, \mathfrak{U})$, we also have $\mathrm{ET}_1(X, \mathfrak{U}) \vdash_1 E$. It follows that $V \vdash_1 Z$, $Z\Lambda(H)\mathfrak{U}$ and $\{\mathfrak{U}\} \vdash_{A_0 \to A_1} \mathfrak{V}$.

Lemma 7.3 $(\Lambda(H) \times \mathrm{Id}_{A_0}) \circ \mathrm{EV} = H$, *for all* $H\colon C \times A_0 \Vdash A_1$.

Proof: Let $U \in \mathrm{Con}_{A_2 \times A_0}$ and $b \in A_1$ with $U((\Lambda(H) \times \mathrm{Id}_{A_0}) \circ \mathrm{EV})b$. Then there is some $U' \in \mathrm{Con}_{A_2 \times A_0}$ such that $U \vdash_{A_2 \times A_0} U'$ and $U'((\Lambda(H) \times \mathrm{Id}_{A_0}) \circ \mathrm{EV})b$. It follows that for some $Z \in \mathrm{Con}_{(A_0 \to A_1) \times A_0}$, $U'(\Lambda(H) \times \mathrm{Id}_{A_0})Z$ and $Z\,\mathrm{EV}\,b$. Thus, $\mathrm{pr}_{A_2}(U')\Lambda(H)\,\mathrm{pr}_{A_0 \to A_1}(Z)$, $\mathrm{pr}_{A_0}(U') \vdash_0 \mathrm{pr}_{A_0}(Z)$ and $\mathrm{ET}_1(\mathrm{pr}_{A_0}(Z), \bigcup \mathrm{pr}_{A_0 \to A_1}(Z)) \vdash_1 b$. By definition of $\Lambda(H)$ we have that $(\mathrm{pr}_{A_2}(U') \times X)HE$, for all $(X,E) \in \bigcup \mathrm{pr}_{A_0 \to A_1}(Z)$. Hence

$$(\mathrm{pr}_{A_2}(U') \times \mathrm{ET}_0(\mathrm{pr}_{A_0}(Z), \bigcup \mathrm{pr}_{A_0 \to A_1}(Z)))H\,\mathrm{ET}_1(\mathrm{pr}_{A_0}(Z), \bigcup \mathrm{pr}_{A_0 \to A_1}(Z)).$$

Since $\mathrm{pr}_{A_0}(U') \vdash_0 \mathrm{pr}_{A_0}(Z) \vdash_0 \mathrm{ET}_0(\mathrm{pr}_{A_0}(Z), \bigcup \mathrm{pr}_{A_0 \to A_1}(Z))$, it follows that $(\mathrm{pr}_{A_2}(U') \times \mathrm{pr}_{A_0}(U'))Hb$ and thus that UHb.

Conversely, if UHb, there are $U', U'' \in \mathrm{Con}_{A_2 \times A_0}$ and $V \in \mathrm{Con}_1$ such that $U \vdash_{A_2 \times A_0} U' \vdash_{A_2 \times A_0} U''$, $U''Hb$ and $V \vdash_1 b$. Let $\mathfrak{U} \in \mathrm{JE}(\{(\mathrm{pr}_{A_0}(U''), V)\})$ according to Lemma 6.3. Then $\mathfrak{U} \in A_0 \to A_1$ and $\mathrm{pr}_{A_2}(U'')\Lambda(H)\mathfrak{U}$. Thus $U(\Lambda H \times \mathrm{Id}_{A_0})Z$, for $Z = \{\mathfrak{U}\} \times \mathrm{pr}_{A_0}(U')$. Note that $\mathrm{ET}_1(\mathrm{pr}_{A_0}(U'), \mathfrak{U}) \supseteq V$. Therefore, we have that $U(\Lambda H \times \mathrm{Id}_{A_0})Z$ and $Z\,\mathrm{EV}\,b$, i.e. $U((\Lambda H \times \mathrm{Id}_{A_0}) \circ \mathrm{EV})b$.

Lemma 7.4 $\Lambda((G \times \mathrm{Id}_{A_0}) \circ \mathrm{EV}) = G$, *for all* $G\colon A_2 \Vdash A_0 \to A_1$.

Proof: Let $V \in \mathrm{Con}_2$ and $\mathfrak{V} \in A_0 \to A_1$ with $V\Lambda((G \times \mathrm{Id}_{A_0}) \circ \mathrm{EV})\mathfrak{V}$. Then $(V \times X)((G \times \mathrm{Id}_{A_0}) \circ \mathrm{EV})E$, for each $(X,E) \in \mathfrak{V}$. It follows that there is some $Z^{(X,E)} \in \mathrm{Con}_{(A_0 \to A_1) \times A_0}$, for each such (X,E), so that $(V \times X)(G \times \mathrm{Id}_{A_0})Z^{(X,E)}$ and $Z^{(X,E)}\,\mathrm{EV}\,E$, which means that $VG\,\mathrm{pr}_{A_0 \to A_1}(Z^{(X,E)})$, $X \vdash_0 \mathrm{pr}_{A_0}(Z^{(X,E)})$ and $\mathrm{ET}_1(\mathrm{pr}_{A_0}(Z^{(X,E)}), \bigcup \mathrm{pr}_{A_0 \to A_1}(Z^{(X,E)})) \vdash_1 E$. Thus,

$$\mathrm{ET}_1(X, \bigcup \mathrm{pr}_{A_0 \to A_1}(Z^{(X,E)})) \vdash_1 E, \tag{13}$$

for all $(X,E) \in \mathfrak{V}$.

Now, let $\mathfrak{V}' \subseteq_{\mathrm{fin}} \mathrm{Con}_0 \times \mathrm{Con}_1$ with $\mathfrak{V} \in \mathrm{JE}(\mathfrak{V}')$. Then there is some $\mathcal{U} \in \mathrm{Con}_{A_0 \to A_1}$ with $VG\mathcal{U}$ and $\mathcal{U} \supseteq \mathrm{pr}_{A_0 \to A_1}(Z^{(P,Q)})$, for all $(P,Q) \in \mathfrak{V}'$. Moreover, by (13), we have for all such (P,Q) that

$$\mathrm{ET}_1(P, \bigcup \mathcal{U}) \vdash_1 Q. \tag{14}$$

Next, let $(X, E) \in \mathfrak{V} \setminus \mathfrak{V}'$. Then there is some $Y \in \mathrm{Con}_0$ so that $Y \vdash_0 X$, $X \in \mathrm{Sup}_0(\mathrm{ET}_0(Y, \mathfrak{V}'))$ and $E \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}'))$. With (14) it follows that $\mathrm{ET}_1(X, \bigcup \mathcal{U}) \vdash_1 \mathrm{ET}_1(Y, \mathfrak{V}')$. Because of Condition (L) we obtain some $T \in \mathrm{Sup}_1(\mathrm{ET}_1(Y, \mathfrak{V}'))$ with

$$\mathrm{ET}_1(X, \bigcup \mathcal{U}) \vdash_1 T.$$

As $VG(\mathcal{U} \cup \mathrm{pr}_{A_0 \to A_1}(Z^{(X,E)}))$, there is some $\mathcal{U}' \in \mathrm{Con}_{A_0 \to A_1}$ with $\mathcal{U}' \supseteq \mathcal{U} \cup \mathrm{pr}_{A_0 \to A_1}(Z^{(X,E)})$ and $VG\mathcal{U}'$. It follows that $\mathrm{ET}_1(X, \bigcup \mathcal{U}') \vdash_1 T \cup E$. By Lemma 4.4 we therefore have that

$$E \underset{\mathrm{ET}_1(Y, \mathfrak{V}')}{\sim} T.$$

As a consequence, $\mathrm{ET}_1(X, \bigcup \mathcal{U}) \vdash_1 E$.

This shows that $\mathcal{U} \vdash_{A_0 \to A_1} \mathfrak{V}$. Since moreover $VG\mathcal{U}$, we obtain that $VG\mathfrak{V}$.

Conversely, if $VG\mathfrak{V}$, then there is some $\mathcal{Z} \in \mathrm{Con}_{A_0 \to A_1}$ with $VG\mathcal{Z}$ and $\mathcal{Z} \vdash_{A_0 \to A_1} \mathfrak{V}$. It follows that for all $(X, E) \in \mathfrak{V}$, $\mathrm{ET}_1(X, \bigcup \mathcal{Z}) \vdash_1 E$. Since $X \vdash_0 \mathrm{ET}_0(X, \bigcup \mathcal{Z})$ and the latter set is finite by Lemma 6.2(1), we obtain some $X' \in \mathrm{Con}_0$ with $X \vdash_0 X' \vdash_0 \mathrm{ET}_0(X, \bigcup \mathcal{Z})$. Then $\mathrm{ET}_0(X', \bigcup \mathcal{Z}) = \mathrm{ET}_0(X, \bigcup \mathcal{Z})$. Therefore, $\mathrm{ET}_1(X', \bigcup \mathcal{Z}) \vdash_1 E$. Set $Z = \mathcal{Z} \times X'$. Then $Z \in \mathrm{Con}_{(A_0 \to A_1) \times A_0}$, $(V \times X)(G \times \mathrm{Id}_{A_0})Z$ and $Z \,\mathrm{EV}\, E$. Thus, $V \Lambda((G \times \mathrm{Id}_{A_0}) \circ \mathrm{EV})\mathfrak{V}$.

Proposition 7.5 *Let A_0 and A_1 be L-information systems. Then $(A_0 \to A_1, \mathrm{EV})$ is their exponent in* **LINF***.*

With Propostion 6.7 we moreover have that if A_0 and A_1 are both algebraic, then $(A_0 \to A_1, \mathrm{EV})$ is their exponent in **aLINF**.

As we have already seen, **LINF** as well as **aLINF** contain a terminal object. Moreover, we have shown how to construct the categorical product of information systems.

Theorem 7.6 *The category* **LINF** *of L-information systems and approximable mappings as well as its full subcategory* **aLINF** *of algebraic L-information systems are Cartesian closed.*

A well-known result of Jung [9, 10] states that the categories **L** and **aL** are maximal among the Cartesian closed full subcategories of $\mathbf{CONT}_\perp$ and $\mathbf{ALG}_\perp$, respectively. As was shown in [13], $\mathbf{CONT}_\perp$ is equivalent to the category $\mathbf{CINF_t}$ of continuous information systems with truth element and $\mathbf{ALG}_\perp$ is equivalent to the category $\mathbf{AINF_t}$ of algebraic information systems with truth element. Note here that the functors involved in establishing the equivalence result in the present paper are restrictions of those used in [13].

Theorem 7.7 *The categories* **LINF** *and* **aLINF***, respectively, are maximal among the Cartesian closed full subcategory of $\mathbf{CINF_t}$ and $\mathbf{AINF_t}$.*

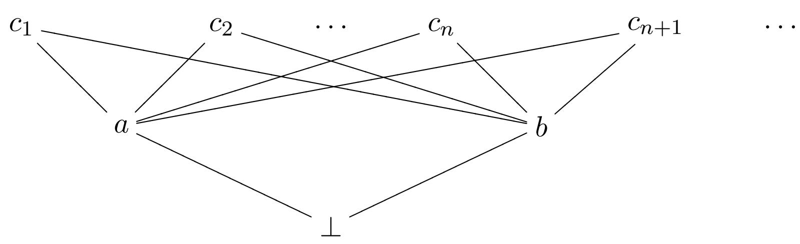
8 Concluding remarks

In this paper the information system representation of L-domains was studied. Continuous information systems were introduced in [13], the equivalence of their category with the category of continuous domains was shown and the information systems of many important classes of domains were derived. The L-domain case, however, was left open.

In information systems base elements of domains are represented by finite consistent sets of tokens. The main problem in finding an information system description of L-domains was the characterization of those consistent sets that represent local suprema, i.e., the least upper bounds a finite set can have with respect to different principal ideals. As was demonstrated, the category of L-information systems with approximating mappings is equivalent to the category of L-domains with Scott continuous functions, which shows that the right notion of L-information system was found. By this result many important properties like the existence of exponents transfer from L-domains to L-information systems. In concrete cases, however, one has to go back and forth between information systems and domains in order to construct the exponent of two information systems. Instead, we presented a direct construction within L-information systems in this paper.

In the case of information systems for bounded-complete domains [12, 8] tokens in $A_0 \to A_1$ correspond to single-step functions $(u \searrow v)$ and finite consistent sets of tokens correspond to the least upper bounds of finite bounded sets of such functions, i.e. step functions. In the case of L-domains such sets do not define a function, in general. Consider e.g. the domain D defined by the diagram in Figure 1.

Figure 1: An infinite L-domain D



Then $(a \searrow b)$ as well as $(b \searrow a)$ are single-step functions in $[D \to D]$, but $(a \searrow b) \sqcup (b \searrow a)$ is not defined. What e.g. should be the value for c_i? Unfortunately, this problem can not be solved by adding finitely many single-step functions, e.g. $(c_i \searrow c_{i+1})$, as is the case for SFP domains. For every c_i one has to add what should be the corresponding function value.

This information can no longer be described by a finite set in Con. Therefore we have taken the descriptions of such functions as elements of the information system $A_0 \to A_1$. The finiteness of the starting objects which is characteristic for information systems is hidden in the fact that these objects now correspond to functions which are generated from a finite set of single-step functions by adding pairs of least upper bounds.

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