An Intrinsic Characterisation of Effective Topologies

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1. Effective Topological Spaces

An effective topological space is a two-level structure consisting of two indexed sets:

- points
- basic opens,

as well as two enumerable relations between the two levels:

- membership
- convergence.

Let $\mathcal{T} = (T, \tau, \mathcal{B}, x, B)$ be a countable topological T_0 space with countable basis $\mathcal{B}$ such that

- $x\colon \omega \rightharpoonup T$ (onto) is a (partial) enumeration of the points in T,
- $B\colon \omega \to \mathcal{B}$ (onto) is a (total) enumeration of all basic open sets.

We think of the basic open sets as elementary predicates that are easy to encode.

They determine the points: Because of T_0, equality of points is Leibniz identity w.r.t. these predicates.

In general it is difficult to deal with set inclusion in an effective framework. In most cases we can use a stronger relation on the codes of basic open sets instead.

Definition

A transitive relation $\prec_B$ on ω is a *strong inclusion*, if for all $m, n \in \omega$

$$m \prec_B n \Rightarrow B_m \subseteq B_n.$$

Assume further that

- $\mathcal{B}$ is a *strong basis*, i.e., the property of being a base holds with respect to $\prec_B$ instead of $\subseteq$.

A topology is a structure that allows to talk about *convergence* and *limits*. Here, convergence is defined in terms of normed families of filter bases.

Definition

A family $(B_{a_\nu})_{\nu \geq 0}$ of basic open sets is

- *normed* if
$$a_0 \succ_B a_1 \succ_B \cdots .$$
- *recursive* if the sequence $a_0, a_1, \ldots$ is computable. Any of its Gödel numbers is called *index* of $(B_{a_\nu})_{\nu \geq 0}$.

Definition

Let $(B_{a_\nu})_{\nu \geq 0}$ be a normed family of basic open sets and $y \in T$.

$$(B_{a_\nu})_{\nu \geq 0} \longrightarrow y$$

if $\{ B_{a_\nu} \mid \nu \geq 0 \}$ is a strong basis of the neighbourhood filter $\mathcal{N}(y)$ of y.

For $n \geq 0$, indices i of points and indices m of normed recursive families of basic open sets define

$$i \models n \Leftrightarrow x_i \in B_n$$

$$m \rightsquigarrow i \Leftrightarrow m \text{ is an index of } (B_{a_\nu})_{\nu \geq 0} \text{ and } (B_{a_\nu})_{\nu \geq 0} \longrightarrow x_i.$$

Definition

Space $\mathcal{T}$ is *effective* if the relations

$$\prec_B, \quad \models, \quad \rightsquigarrow$$

are enumerable relative to the sets of numbers over which they are defined.

2. Intrinsic Topologies on Indexed Sets

Indexed sets possess a variety of intrinsic topologies defined by their computability structure.

Let S be a countable set and $\nu\colon \omega \rightharpoonup S$ (onto) be a (partial) numbering of S with domain $\mathrm{dom}(\nu)$.

Definition

$X \subseteq S$ is *completely enumerable*, if there is an r.e. set W_n such that for all $i \in \mathrm{dom}(\nu)$,

$$\nu_i \in X \Leftrightarrow i \in W_n.$$

Set

$$M_n = \begin{cases} X & \text{for any such } n \text{ and } X, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then M is a numbering of the class CE of all completely enumerable subsets of S.

Similarly, X is *completely decidable*, if there is an decidable set A such that for all $i \in \mathrm{dom}(\nu)$, $\nu_i \in X$ iff $i \in A$.

Definition

$X \subseteq S$ is *enumerable*, if there is some r.e. set $A \subseteq \mathrm{dom}(\nu)$ such that
$$X = \{ \nu_i \mid i \in A \}.$$

Thus, X is enumerable if we can enumerate a subset of the index set of X which contains at least one index for every element of X, whereas X is completely enumerable if we can enumerate all indices of all elements of X and perhaps some numbers which are not used as indices by numbering ν.

Definition

1. A topology on S is a *Mal'cev topology*, if it has a basis of completely enumerable subsets of S. Any such basis is called *Mal'cev basis*.
2. The topology $\mathcal{E}$ generated by the collection CE of all completely enumerable subsets of S is called the *Ershov topology*.

Obviously, the Ershov topology is the finest Mal'cev topology on S.

We assume that every Mal'cev basis is indexed by a restriction of indexing M.

In certain cases one needs to be able to completely enumerate not only each basic open set, but also its complement.

Theorem (Rice)

Let $\mathcal{T}$ be effective and connected. Then a subset of T is completely decidable if, and only if, it is empty or the whole space.

It follows that, in general, we cannot expect that the whole complement of a completely enumerable set is completely enumerable as well.

Note that CE is a distributive lattice with respect to union and intersection. For $U \in CE$, let

$$U^* = \mathit{pseudocomplement}(U),$$

i.e. the greatest completely enumerable subset of $S \setminus U$, if it exists.

Definition

U is *regular*, if U^* and U^{**} both exist and $U = U^{**}$.

Set

$$R_{\langle m,n\rangle} = \begin{cases} M_m & \text{if } m, n \in \mathrm{dom}(M) \text{ and } M_m^* = M_n \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then R is a numbering of the class REG of all regular subsets of S.

Definition

1. We say that a topology is a *bi-Mal'cev topology*, if it has a basis of regular sets. Any such basis is called *bi-Mal'cev basis*.
2. Let $\mathcal{R}$ be the topology generated by the collection REG of all regular subsets of S.

We assume that every bi-Mal'cev basis is indexed by a restriction of indexing R.

Obviously, $\mathcal{R}$ is the finest bi-Mal'cev topology on S. Moreover, all of its basic open sets are regular open.

Definition

1. An open set X is *regular open*, if $X = \mathrm{int}(\mathrm{cl}(X))$.
2. A topological space is *semi-regular* if it has a basis of regular open sets.

Definition

Let η be a topology on S with basis $\mathcal{A}$.

1. An open set $X \in \eta$ is *weakly decidable* if both X and $\mathrm{ext}(X)$ are completely enumerable.
2. We say that η is *complemented* if all of its basic open sets are weakly decidable.

Let $\mathcal{C}$ be a class of sets $X \in \eta$ such that

- X is regular open
- $X \in \mathcal{C} \Rightarrow \mathrm{ext}(X) \in \mathcal{C}$.

Then $\mathcal{C}$ generates a topology $\eta_{\mathcal{C}} \subseteq \eta$ such that for all $X \in \mathcal{C}$:

- X is regular open in $\eta_{\mathcal{C}}$.
- $\mathrm{ext}_{\eta_{\mathcal{C}}}(X) = \mathrm{ext}_{\eta}(X)$.

Set

$$\mathcal{C} = \{\, X \in \eta \mid X \text{ regular open and weakly decidable.} \,\}$$

Then $\eta^* = \eta_{\mathcal{C}}$ is the finest complemented semi-regular topology generated by η-open sets.

We call η^* the complemented semi-regular topology *associated* with η.

Note that every regular subset of S is regular open and weakly decidable with respect to the Ershov topology on S, and these are the only such sets.

Proposition

$\mathcal{R}$ is the complemented semi-regular topology associated with the Ershov topology on S, i.e.

$$\mathcal{R} = \mathcal{E}^*.$$

3. Comparing Topologies Effectively

In a second countable space $\mathcal{T} = (T, \tau, \mathcal{B}, B)$ every open set is a countable union of basic open sets. This gives rise to the following notion of being effectively open.

Definition

$O \in \tau$ *Lacombe* set, if there is an r.e. set $A \subseteq \omega$ such that

$$O = \bigcup \{ B_a \mid a \in A \}.$$

Any r.e. index i of A is called *Lacombe index* of O. We write $O = L_i^{\tau}$.

Definition

Let η be a further topology on T with countable basis $\mathcal{C}$ and indexing C of the basis.

1. η is *effectively coarser* than τ (written $\eta \subseteq_e \tau$), if there is a computable total function f such that

$$C_n = L^{\tau}_{f(n)}$$

.

2. η and τ are *effectively equivalent* (written $\eta =_e \tau$), if both $\eta \subseteq_e \tau$ and $\tau \subseteq_e \eta$.

A condition that forces a topology η on T to be effectively coarser than the given topology τ:

Let η be a topology on T with basis $\mathcal{C}$ and numbering C of $\mathcal{C}$. Moreover, assume that

$$x_i \in C_m.$$

We want to find a basic open set B_n with

$$x_i \in B_n \subseteq C_m.$$

As follows from work of Beeson, we have to give a proof by contradiction.

So, we have to suppose for all n that

$$x_i \in B_n \Rightarrow B_n \not\subseteq C_m.$$

Since we are working in an effective framework, we need to effectively find a point

$$y \in B_n \setminus C_m,$$

realizing that $B_n \not\subseteq C_m$.

As turns out it is sufficient:

- First to find a perhaps smaller completely enumerable set

$$x_i \in M' \subseteq C_m.$$

- Then, for every larger set $B_{n'}$ with $n' \succ_B n$ to find a witness

$$y \in B_{n'} \setminus M'.$$

We say that τ has a *noninclusion realizer* with respect to η, if such M' and y can effectively be found.

Note that M' is completely enumerable, but $y \notin M'$. So, in general there is no way to find y.

4. The Characterization

Definition

$\mathcal{T}$ is *recursively separable* if it has an enumerable dense subset.

Proposition

Let $\mathcal{T}$ be effective as well as recursively separable and η be Mal'cev topology on T. If τ has a noninclusion realizer with respect to η, then $\eta \subseteq_e \tau$.

Remember that by our assumptions every basic open set B_n is completely enumerable. Thus, τ is a Mal'cev topology.

Theorem

Let $\mathcal{T}$ be effective as well as recursively separable and let τ have a noninclusion realizer with respect to itself. Then τ is the effectively finest Mal'cev topology on T relative to which τ has a noninclusion realizer.

5. Special Cases

We will now study important special cases and see when such realizers exist.

- Domain-like spaces

An essential property of continuous domains (dcpo's), just as of Ershov's A- and f-spaces, is that their canonical topology has a basis with every basic open set being an upper set generated by a point which is not necessarily included in B_n but in $\mathrm{hl}(B_n)$ where

$$\mathrm{hl}(B_n) = \bigcap \{ B_m \mid n \prec_B m \}.$$

Definition

$\mathcal{T}$ is *effectively pointed*, if there is a computable function pd such that for all n with $B_n \neq \emptyset$,

1. $x_{\mathrm{pd}(n)} \in \mathrm{hl}(B_n)$,
2. $x_{\mathrm{pd}(n)} \leq_\tau z$, for all $z \in B_n$.

Here $\leq_\tau$ is the *specialization order* associated with τ:

$$y \leq_\tau z \Leftrightarrow \mathcal{N}(y) \subseteq \mathcal{N}(z).$$

Lemma

Let $\mathcal{T}$ be effective and effectively pointed. Then $\mathcal{T}$ is recursively separable with dense base $\{ x_a \mid a \in \mathrm{range}(\mathrm{pd}) \}$.

Lemma

Let $\mathcal{T}$ be effective. Then each completely enumerable subset of T is upwards closed with respect to the specialization order.

Proposition

Let $\mathcal{T}$ be effective and effectively pointed. Then τ has a noninclusion realizer with respect to any Mal'cev topology on T.

Theorem

Let $\mathcal{T}$ be effective and effectively pointed. Then

$$\tau =_e \mathcal{E},$$

i.e., τ is effectively equivalent to the Ershov topology on T.

- Semi-regular spaces

Proposition

Let $\mathcal{T}$ be effective and recursively separable. Then τ has a noninclusion realizer relative to every bi-Mal'cev topology on T.

Lemma

Let $\mathcal{T}$ be effective and recursively separable. Then the following statements hold:

1. *For every weakly decidable open set O,*

$$\mathrm{ext}_{\tau}(O) = O^{*}.$$

2. *For every weakly decidable basic open set B_n,*

$$B_n \text{ regular } \Leftrightarrow B_n \text{ regular open.}$$

3. *If, in addition, $\mathrm{ext}_{\tau}(B_n)$ is completely enumerable, uniformly in n,*

$$\tau \text{ bi-Mal'cev } \Leftrightarrow \tau \text{ semi-regular.}$$

Theorem

Let $\mathcal{T}$ be effective, recursively separable and semi-regular. Moreover, assume that $\mathrm{ext}_\tau(B_n)$ *is completely enumerable, uniformly in n. Then*

$$\tau =_e \mathcal{R},$$

i.e., τ is effectively equivalent to the complemented semi-regular topology associated with the Ershov topology on T.