Quasi-probability representation and quantum joint measurements

Ali Asadian, Siegen University

Maria Laach, August 2017

Outline

- Joint quantum measurements of compatible observables
- **Quasi-probability representations of quantum mechanics**
- **Quasi-PRs of the joint measurements**
- Example: qubit and ~~higher dimension~~
- Concluding remarks

Joint measurements

Joint measurement $\mathcal{M}$

Is a compound measurement of a set of compatible observables $(\text{e.g. } M_1, M_2)$

described by $\{E_{k_1,k_2|\mathcal{M}}\}$, $\sum_{k_1,k_2} E_{k_1,k_2|\mathcal{M}} = I$

Joint measurements

Joint measurement $\mathcal{M}$

Is a compound measurement of a set of compatible observables $(\text{e.g. } M_1, M_2)$

described by $\{E_{k_1,k_2|\mathcal{M}}\}$, $\sum_{k_1,k_2} E_{k_1,k_2|\mathcal{M}} = I$

Marginality condition:

$$\sum_{k_2} E_{k_1,k_2|\mathcal{M}} = E_{k_1|M_1},$$

$$\sum_{k_1} E_{k_1,k_2|\mathcal{M}} = E_{k_2|M_2}$$

Does (an optimal) joint measurements exist? Compatibility of the observables

How to construct them?

Can we answer with a **single recipe**?

Quasi-probability representation

- Quasi-PR of the quantum states and measurements:

$$\mu : \rho \mapsto \mu_\rho(\lambda) \in \mathbb{R} \quad \text{and} \quad \sum_{\lambda \in \Lambda} \mu_\rho(\lambda) = 1$$

So called ontic state

$$\xi : E_{k|M} \mapsto \xi_M(k|\lambda) \in \mathbb{R} \quad \text{and} \quad \sum_k \xi_E(k|\lambda) = 1 \quad , \ \forall \lambda$$

Born's rule (the total law of probability):

$$p_k = \mathrm{Tr}(\rho E_k) = \sum_{\lambda \in \Lambda} \mu_\rho(\lambda) \xi_E(k|\lambda)$$

.. Much like a classical probability

Ferrie & Emerson JPA (2008)

Quasi-probability representation

- Quasi-PR of the quantum states and measurements:

$$\mu : \rho \mapsto \mu_\rho(\lambda) \in \mathbb{R} \quad \text{and} \quad \sum_{\lambda \in \Lambda} \mu_\rho(\lambda) = 1$$

So called ontic state

$$\xi : E_{k|M} \mapsto \xi_M(k|\lambda) \in \mathbb{R} \quad \text{and} \quad \sum_{k} \xi_E(k|\lambda) = 1 \quad , \ \forall \lambda$$

Born's rule (the total law of probability):

$$p_k = \mathrm{Tr}(\rho E_k) = \sum_{\lambda \in \Lambda} \mu_\rho(\lambda) \xi_E(k|\lambda)$$

.. Much like a classical probability

Frame representation(orthogonal basis):

$$\{F_\lambda\} \quad , \quad \sum_{\lambda} F_\lambda = \mathbb{I} \quad \Rightarrow \quad \sum_{\lambda} \mu_\rho(\lambda) = 1 \qquad \mu_\rho(\lambda) = \mathrm{Tr}[\rho F_\lambda]$$

$$\{D_\lambda\} \quad , \quad \mathrm{Tr} D_\lambda = 1 \quad \Rightarrow \quad \sum_{k} \xi_E(k|\lambda) = 1 \qquad \xi_E(k|\lambda) = \mathrm{Tr}[E_k D_\lambda]$$

$$\mathrm{Tr}(F_\lambda D_{\lambda'}) = \delta_{\lambda,\lambda'}$$ **Orthogonal**, and thus form a **complete basis**

Ferrie & Emerson NJP (2008)

..continued

Frame representation

$$\mathcal{O} = \sum_\lambda \mathrm{Tr}(\mathcal{O} D_\lambda) F_\lambda \quad , \quad \forall\ \mathcal{O}$$

$$E_k = \sum_\lambda \mathrm{Tr}(E_k D_\lambda) F_\lambda \quad \Rightarrow \quad p_k = \mathrm{Tr}[E_k \rho] = \sum_\lambda \mathrm{Tr}(E_k D_\lambda) \mathrm{Tr}(\rho F_\lambda)$$

$$:= \sum_\lambda \xi_E(k|\lambda) \mu_\rho(\lambda)$$

..continued

Frame representation

$$\mathcal{O} = \sum_\lambda \mathrm{Tr}(\mathcal{O} D_\lambda) F_\lambda \quad , \quad \forall \, \mathcal{O}$$

$$E_k = \sum_\lambda \mathrm{Tr}(E_k D_\lambda) F_\lambda \Rightarrow p_k = \mathrm{Tr}[E_k \rho] = \sum_\lambda \mathrm{Tr}(E_k D_\lambda)\mathrm{Tr}(\rho F_\lambda)$$

$$:= \sum_\lambda \xi_E(k|\lambda)\mu_\rho(\lambda)$$

A general class of a frame: phase-space point operators

Wigner representation (Wootters 1987, Gross 2006, et al)

$$W_\lambda \equiv D_\lambda = d F_\lambda$$

Wigner representation of a single effect:

$$E_k = \frac{1}{d}\sum_\lambda \mathrm{Tr}(E_k W_\lambda) W_\lambda = \frac{1}{d}\sum_\lambda \xi_E(k|\lambda) W_\lambda$$

Qubit example:

Phase-space point operators:

$l, m = 0, 1$

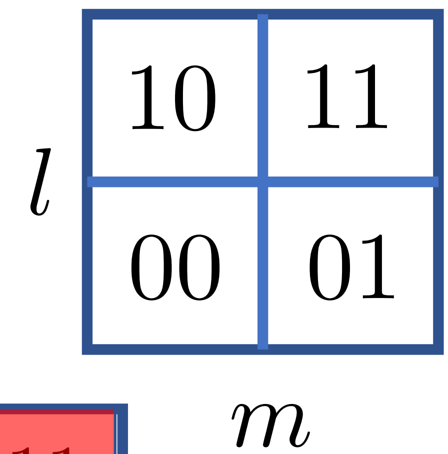
$$W_{lm} = \frac{1}{2}\left[I + (-1)^{l}\sigma_1 + (-1)^{m}\sigma_2 + (-1)^{l+m}\sigma_3\right]$$

$$\sigma_1 = \boldsymbol{r}\cdot\boldsymbol{\sigma} = \sin\theta\cos\varphi\sigma_x + \sin\theta\sin\varphi\sigma_y + \cos\theta\sigma_z$$

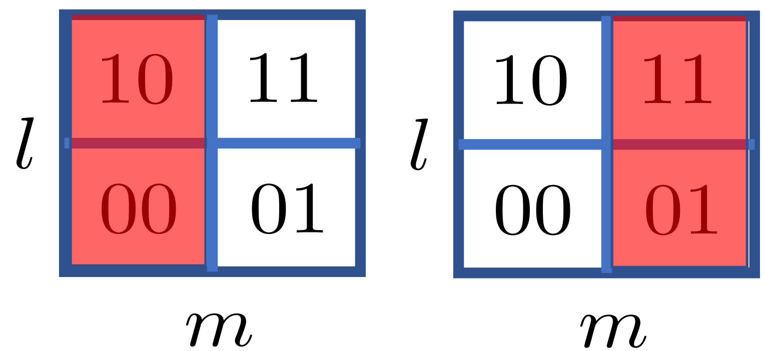
$$\sigma_2 = \boldsymbol{\theta}\cdot\boldsymbol{\sigma} = \cos\theta\cos\varphi\sigma_x + \cos\theta\sin\varphi\sigma_y - \sin\theta\sigma_z$$

$$\sigma_3 = \boldsymbol{\varphi}\cdot\boldsymbol{\sigma} = -\sin\varphi\sigma_x + \cos\varphi\sigma_y$$

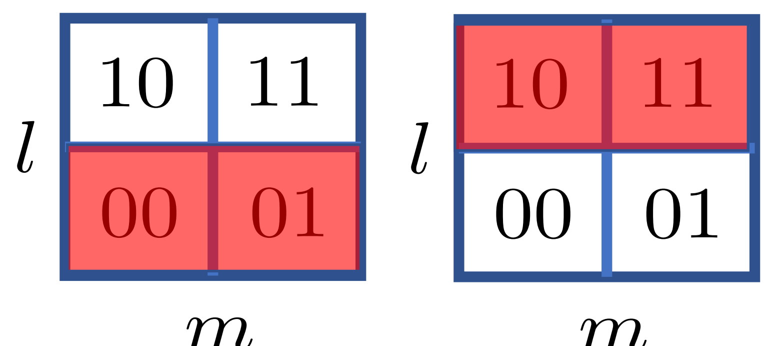
Discrete phase space



$$E_{\pm|X} = \frac{1}{2}(1 \pm \eta_x\sigma_x)$$



$$E_{\pm|Z} = \frac{1}{2}(1 \pm \eta_z\sigma_z)$$



Wootters (1989)

Bloch vs Frame

Bloch

$$\{I, \sigma_1, \sigma_2, \sigma_3\} \quad \mathrm{Tr}(\sigma_i \sigma_j) = \delta_{ij}$$

State

$$\rho = \sum_{j=0}^{3} \eta_j \sigma_j$$

Effect

$$E = \sum_{j=0}^{3} e_j \sigma_j$$

Born's rule

$$p_k = \mathrm{Tr}(E_k \rho) = \sum_{j=0}^{3} e_j^k \eta_j$$

Frame

$$\{W_{00}, W_{10}, W_{01}, W_{11}\} \quad \mathrm{Tr}(W_\lambda W_{\lambda'}) = 2\delta_{\lambda\lambda'}$$

State

$$\rho = \frac{1}{2} \sum_{l,m=0,1} \mu_\rho(lm) W_{lm}$$

Effect

$$E_k = \frac{1}{2} \sum_{l,m=0,1} \xi_E(k|lm) W_{lm}$$

$$\mu_\rho(lm) = \mathrm{Tr}(\rho W_{lm})/d$$

Born's rule

$$p_k = \mathrm{Tr}(E_k \rho) = \sum_{l,m=0,1} \xi_E(k|lm) \mu_\rho(lm)$$

Quasi-PR of the joint measurements

Ansatz: For a given set of observables $\{M_1, M_2\}$

$$E_{k_1,k_2|\mathcal{M}} = \frac{1}{d}\sum_{\lambda} \mathrm{Tr}(E_{k_1|M_1} W_\lambda)\mathrm{Tr}(E_{k_2|M_2} W_\lambda) W_\lambda$$

$$= \frac{1}{d}\sum_{\lambda} \xi_{M_1}(k_1|\lambda)\xi_{M_2}(k_2|\lambda) W_\lambda \geq 0$$

Factorized conditional quasi-probabilities

Recall:

$$p_{k_1 k_2|\mathcal{M}} = \sum_{\lambda} \xi_{M_1}(k_1|\lambda)\xi_{M_2}(k_2|\lambda)\mu_\rho(\lambda)$$

Marginalizing

$$E_{k_1} = \sum_{k_2} E_{k_1,k_2|\mathcal{M}} = \frac{1}{d}\sum_{\lambda} \mathrm{Tr}(E_{k_1|M_1} W_\lambda)\mathrm{Tr}(\sum_{k_2} E_{k_2|M_2} W_\lambda) W_\lambda$$

$$= \frac{1}{d}\sum_{\lambda} \mathrm{Tr}(E_{k_1|M_1} W_\lambda) W_\lambda$$

Quasi-PR of the joint measurements

Ansatz: For a given set of observables $\{M_1, M_2\}$

$$E_{k_1,k_2|\mathcal{M}} = \frac{1}{d}\sum_{\lambda} \mathrm{Tr}(E_{k_1|M_1}W_\lambda)\mathrm{Tr}(E_{k_2|M_2}W_\lambda)W_\lambda$$

$$= \frac{1}{d}\sum_{\lambda} \xi_{M_1}(k_1|\lambda)\xi_{M_2}(k_2|\lambda)W_\lambda \geq 0$$

Factorized conditional quasi-probabilities

Recall:

$$p_{k_1 k_2|\mathcal{M}} = \sum_{\lambda} \xi_{M_1}(k_1|\lambda)\xi_{M_2}(k_2|\lambda)\mu_\rho(\lambda)$$

Marginalizing

$$E_{k_1} = \sum_{k_2} E_{k_1,k_2|\mathcal{M}} = \frac{1}{d}\sum_{\lambda} \mathrm{Tr}(E_{k_1|M_1}W_\lambda)\mathrm{Tr}(\sum_{k_2} E_{k_2|M_2}W_\lambda)W_\lambda$$

$$= \frac{1}{d}\sum_{\lambda} \mathrm{Tr}(E_{k_1|M_1}W_\lambda)W_\lambda$$

$$\sum_{k_2} E_{k_2|M_2} = I \quad , \quad \mathrm{Tr}(W_\lambda) = 1$$

Joint measurement of n compatible observables:

$$E_{\boldsymbol{k}|\mathcal{M}} = \frac{1}{d}\sum_{\lambda} \xi_{\mathcal{M}}(\boldsymbol{k}|\lambda)W_\lambda$$

$$= \frac{1}{d}\sum_{\lambda}\prod_{j=1}^{n} \xi_{M_j}(k_j|\lambda)W_\lambda$$

$$\mathcal{M} = \{M_1, \cdots, M_n\}$$

$$\boldsymbol{k} = (k_1, \cdots, k_n)$$

Quasi-PR of the joint measurements

Joint measurement of n compatible observables:

$$\mathcal{M} = \{M_1, \cdots, M_n\}$$

$$\boldsymbol{k} = (k_1, \cdots, k_n)$$

Conjecture:

One can always find a suitable frame by which the joint measurement of the n compatible observable can be represented as

$$E_{\boldsymbol{k}|\mathcal{M}} = \frac{1}{d} \sum_{\lambda} \xi_{\mathcal{M}}(\boldsymbol{k}|\lambda) W_{\lambda}$$

$$= \frac{1}{d} \sum_{\lambda} \prod_{j=1}^{n} \xi_{M_j}(k_j|\lambda) W_{\lambda}$$

Compatibility criteria..

Sufficient(for arbitrary number of measurements) : Come from positivity condition

$$\sum_{l=0,1}\Big(\xi_{\mathcal{M}}(\boldsymbol{k}|l,l)-\xi_{\mathcal{M}}(\boldsymbol{k}|l,l\oplus 1)\Big)^2\leq 2\prod_{l=0,1}\Big(\xi_{\mathcal{M}}(\boldsymbol{k}|l,l)+\xi_{\mathcal{M}}(\boldsymbol{k}|l,l\oplus 1)\Big)\;*$$

Probabilistic version

Recall:

$$\xi_{\mathcal{M}}(\boldsymbol{k}|l,m)=\prod_{j=1}^{n}\xi_{M_j}(k_j|l,m)=\prod_{j=1}^{n}\mathrm{Tr}(E_{k_j|M_j}W_{lm})$$

Compatibility criteria..

Sufficient(for arbitrary number of measurements) : Come from positivity condition

$$\sum_{l=0,1}\Big(\xi_{\mathcal{M}}(\boldsymbol{k}|l,l)-\xi_{\mathcal{M}}(\boldsymbol{k}|l,l\oplus 1)\Big)^2 \leq 2\prod_{l=0,1}\Big(\xi_{\mathcal{M}}(\boldsymbol{k}|l,l)+\xi_{\mathcal{M}}(\boldsymbol{k}|l,l\oplus 1)\Big) \ *$$

Probabilistic language

Recall:

$$\xi_{\mathcal{M}}(\boldsymbol{k}|l,m)=\prod_{j=1}^{n}\xi_{M_j}(k_j|l,m)=\prod_{j=1}^{n}\mathrm{Tr}(E_{k_j|M_j}W_{lm})$$

For $n=2$

Necessary (Busch Criterion):

$$|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2|+|\boldsymbol{\eta}_1+\boldsymbol{\eta}_2|\leq 2 \ **$$

Quantum language

Symmetric (unbiased) qubit effects:

$* \Leftrightarrow **$

Necessary and sufficient

** Paul Busch PRD(1986)

Concluding remarks..

Advantages of the quasiprobability approach to quantum compatibility & joint measurability:

Conceptual

- A (quasi-)probabilistic (“classical-like”) description of the compatibility and joint measurability; offers unifying picture

Compatibility **outside** QM and Compatibility **inside** QM

Practical

- General construction of (optimal) joint measurements of multiple measurements using frame representation

Concluding remarks..

Advantages of the quasiprobability approach to quantum compatibility & joint measurability:

Conceptual

- A (quasi-)probabilistic (“classical-like”) description of the compatibility and joint measurability; offers unifying picture

Compatibility **outside** QM and Compatibility **inside** QM

Practical

- General construction of (optimal) joint measurements of multiple measurements using frame representation

Thank you