Spin Squeezing and entanglement detection through uncertainty relations

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Entanglement criteria from uncertainty relations

- An N-partite separable state is

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)} \quad p_i > 0 \ \sum_i p_i = 1$$

  A non-separable state is *entangled*

- Taking $J_x = \sum_{n=1}^{N} j_x^{(n)}$, $J_y = \sum_{n=1}^{N} j_y^{(n)}$ with

$$(\Delta j_x^{(n)})^2 + (\Delta j_y^{(n)})^2 \geq C_j$$

- We obtain that

$$(\Delta J_x)^2 + (\Delta J_y)^2 < NC_j \quad \Rightarrow \text{entanglement}$$

i.e., $(\Delta J_x)^2 + (\Delta J_y)^2 \geq NC_j$ a necessary condition for separability
Proof. concavity + $\rho = \bigotimes_n \rho^{(n)} \ \Rightarrow \ (\Delta J_k)^2 = \sum_n (\Delta j_k^{(n)})^2$

This method works for $\{A_k = \sum_{n=1}^{N} a_k^{(n)}\}$ with $\{a_1, a_2, \ldots\}$ non-commuting

[H. F. Hofmann and S. Takeuchi, PRA, **68** 032103, (2003); Gühne et al. PRL 99 130504 (2007); Gittsovich et al. PRA 78, 052319 (2008);PRA, 82 032306 (2010); PRA 81, 032333 (2010)] see also [L. Dammeier, R. Schwonnek, R. F. Werner, New J. Phys. **17**, 093046 (2015)]

Entanglement of spin squeezed states

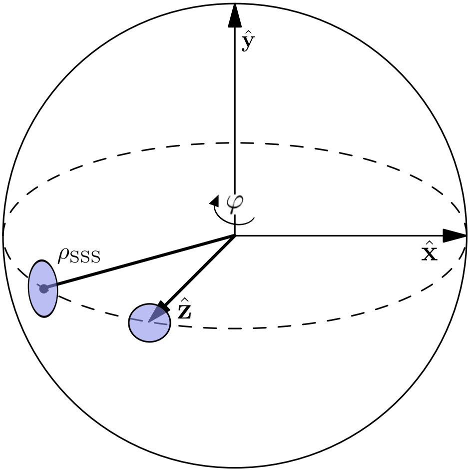
From $(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}|\langle J_z\rangle|^2$ we define a spin-coherent state as

$$(\Delta J_x)^2 = (\Delta J_y)^2 = \frac{1}{2}|\langle J_z\rangle| = \frac{N}{4}$$

and *spin-squeezed states* as

$$|\langle J_z\rangle| \simeq \frac{N}{2}\ ; \qquad (\Delta J_x)^2 < \frac{N}{4}$$

$$\xi_{\rm s}^2 = \frac{N(\Delta J_x)^2}{\langle J_y\rangle^2 + \langle J_z\rangle^2} < 1 \quad \Rightarrow \text{entanglement}$$



They are also very useful for metrology

[A. Sørensen, L.M. Duan, J.I. Cirac, and P. Zoller, Nature **409**, 63 (2001);M. Kitagawa and M. Ueda, Phys. Rev. A **47**, 5138 (1993); D.J. Wineland, J. J. Bollinger, and W. M. Itano, Phys. Rev. A **50**, 67 (1994).]

Generalized spin squeezing

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4}$$

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq \frac{N}{2}$$

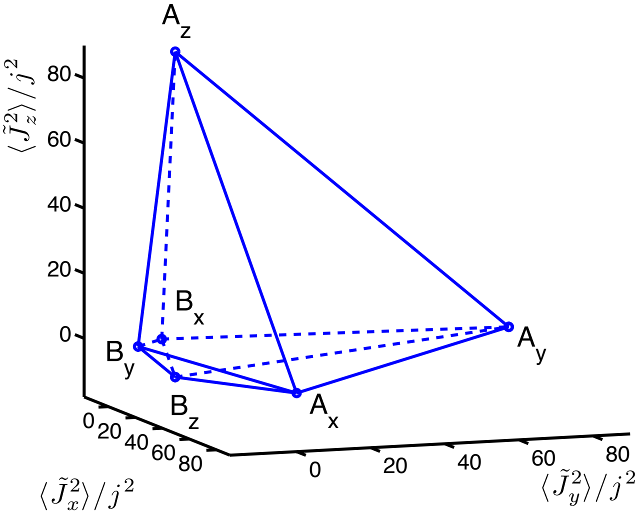
$$(N-1)\left[(\Delta J_x)^2 + (\Delta J_y)^2\right] - \langle J_z^2 \rangle \geq \frac{N(N-2)}{4}$$

$$(N-1)\left[(\Delta J_x)^2\right] - \langle J_y^2 \rangle - \langle J_z^2 \rangle \geq -\frac{N}{2}$$

Violation of one of them implies entanglement.

[G. Tóth, C. Knapp, O. Gühne and H.J. Briegel, PRL **99**, 250405 (2007); PRA **79** 042334 (2009)]

It is a complete set of criteria linear in $(\Delta J_k)^2$



the polytope is filled by separable states in the limit $N \gg j$

A compact form for the complete set

- Let us define the following correlation matrices

$$
\begin{aligned}
C_{kl} &:= \frac{1}{2}\langle J_k J_l + J_l J_k \rangle \\
\Gamma_{kl} &:= C_{kl} - \langle J_k \rangle \langle J_l \rangle \\
Q_{kl} &:= \frac{1}{N} \sum_n \left(\frac{1}{2} \langle j_k^{(n)} j_l^{(n)} + j_l^{(n)} j_k^{(n)} \rangle \right) \\
\mathfrak{X} &:= \Gamma + \tfrac{1}{N-1} C - \tfrac{N^2}{N-1} Q
\end{aligned}
$$

- The complete set becomes

$$
\mathrm{Tr}\,(\Gamma) - \sum_{k=1}^{I} \lambda_k^{\mathrm{pos}}(\mathfrak{X}) - Nj \geq 0 \qquad (1)
$$

(which looks like an improvement of $(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 \geq Nj$)

- Eq. (1) follows *just from the LUR* $(\Delta j_x^{(n)})^2 + (\Delta j_y^{(n)})^2 + (\Delta j_z^{(n)})^2 \geq j$ (*Proof. idea:* $\lambda_k^{\mathrm{pos}}(\mathfrak{X}) = 0$ for product states + concavity)

$SU(d)$-squeezing criteria

- A Local Orthogonal Basis $\{g_k\}_{k=0}^{d^2-1}$ is such that

$$\sum_k (\Delta g_k)^2 \geq d-1$$

- Thus, by considering $G_k \sum_{n=1}^{N} g_k^{(n)}$ we find that

$$\mathrm{Tr}\,(\Gamma) - \sum_{k=1}^{I} \lambda_k^{\mathrm{pos}}(\mathfrak{X}) - N(d-1) \geq 0 \qquad (2)$$

is another set of entanglement criteria (with similar definitions of Γ, C and $\mathfrak{X}$)

“Pseudo”-completeness of the $SU(d)$ inequalities

- We define an N-partite *pseudo-separable* state as

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(N)} \quad p_i > 0 \ \sum_i p_i = 1$$

where $\rho_i^{(n)}$ satisfy $\sum_k (\Delta g_k)^2 \geq d - 1$ but need not be positive

- The $SU(d)$ inequalities define a polytope completely filled by pseudo-separable states (in the limit $N \gg d$)

Definition of “entanglement depth”

A state decomposable as

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(M)} \quad p_i > 0 \ \sum_i p_i = 1$$

with $\rho_i^{(n)}$ k-*particle states* is called ρ k-producible

- if not possible ρ has *depth of entanglement* $(k+1)$

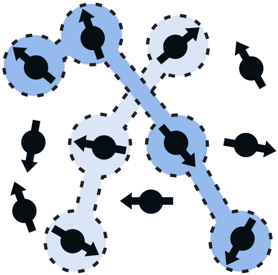


Figure: Entanglement depth of 4

Depth of entanglement of spin-squeezed states

- A necessary condition for k-producibility is (useful for spin squeezed states)

$$(\Delta J_z)^2 \geq \frac{N}{2} F_{\frac{k}{2}} \left(\langle J_x \rangle / \tfrac{N}{2} \right)$$

- Every state that violates it is for sure $k+1$-entangled.
- The function $F_j(x)$ is defined as

$$F_j(X) := \frac{1}{j} \min_{\frac{\langle j_x \rangle_\phi}{j} = X} (\Delta j_z)^2_\phi$$

- We need a convex function $F_j(X)$ because for $\rho = \sum_k p_k \phi_k$ we want

$$(\Delta J_z)^2_\rho \geq \sum_k p_k (\Delta J_z)^2_{\phi_k} \geq \sum_k p_k \frac{N}{2} F_{\frac{k}{2}} \left(\langle J_x \rangle_{\phi_k} / \tfrac{N}{2} \right) \geq$$

$$\frac{N}{2} F_{\frac{k}{2}} \left(\sum_k p_k \langle J_x \rangle_{\phi_k} / \tfrac{N}{2} \right) = \frac{N}{2} F_{\frac{k}{2}} \left(\langle J_x \rangle_\rho / \tfrac{N}{2} \right)$$

[A.S. Sørensen and K. Mølmer, Phys. Rev. Lett. 86, 4431 (2001); Hyllus, Pezze, Smerzi and Tòth, PRA 86(1) 012337 (2012)]

Depth of entanglement of planar squeezed states

- An entanglement depth condition for planar squeezed states

$$(\Delta J_z)^2 + (\Delta J_y)^2 \geq Nj \mathcal{G}_k^{(j)} \left(\langle J_y \rangle / Nj \right),$$

- where

$$G_k^{(j)}(X) := \frac{1}{kj} \min_{\substack{\phi \in (\mathbb{C}^d)^{\otimes k} \\ \frac{1}{kj}\langle L_y \rangle_\phi = X}} \left[(\Delta L_y)^2_\phi + (\Delta L_z)^2_\phi \right],$$

- and then we take the convex hull $\mathcal{G}_k^{(j)}$

[GV, G Colangelo, F Martin-Ciurana, M W. Mitchell, R J. Sewell, G Tòth, arXiv:1705.09090]

General method from Legendre transform

- In general, we can find criteria of the form

$$(\Delta A)^2 \geq \mathcal{B}_k^{(d)}\left(\langle W\rangle\right),$$

where $A = \sum_{n=0}^{N} a^{(n)}$ and $W = \sum_{n=0}^{N} w^{(n)}$ are collective observables and

$$B_k^{(d)}(X) := \min_{\substack{\phi\in(\mathbb{C}^d)^{\otimes k} \\ \langle W^{(\mathrm{k-part})}\rangle_\phi = X}} \left[(\Delta A^{(\mathrm{k-part})})^2_\phi\right],$$

- For convexity we use Legendre transforms

$$\mathcal{L}[(\Delta A^{(\mathrm{k-part})})^2_\phi](W^{(\mathrm{k-part})}) := \inf_\phi [(\Delta A^{(\mathrm{k-part})})^2_\phi - \langle W^{(\mathrm{k-part})}\rangle_\phi],$$

$$\mathcal{B}_k^{(d)}(X) := \sup_\lambda \{\lambda X - \tfrac{N}{k}\mathcal{L}[(\Delta A^{(\mathrm{k-part})})^2_\phi](\lambda W^{(\mathrm{k-part})})\},$$

- This is basically a ground state problem over

$$H = (A^{(\mathrm{k-part})} - s)^2 - \lambda W^{(\mathrm{k-part})},$$

[O Marty, M Cramer, GV, G Tòth, M B. Plenio, arXiv:1708.06986] see also [O. Gühne, M. Reimpell, R. F. Werner, PRL **98**, 110502 (2007)]

Depth of entanglement of Dicke states

- A criterion useful for Dicke states is

$$(\Delta J_z)^2 \geq \frac{N}{2} F_{\frac{k}{2}} \left(\frac{\sqrt{\langle J_x^2 + J_y^2 \rangle - \frac{N}{2}\left(\frac{k}{2}+1\right)}}{\frac{N}{2}} \right)$$

- Every state that violates it is for sure $k+1$-entangled.

- The function $F_j(x)$ is the same as for Sørensen-Mølmer's criterion.

[B. Lücke, J. Peise, GV, J. Arlt, L. Santos, G. Tóth and C. Klempt, PRL **112**, 155304 (2014); GV, I. Apellaniz, M. Kleinmann, B. Lücke, C. Klempt and G. Tóth, New J. Phys. **19** (2017)]

Conclusions

Summary

1. We have studied complete sets of entanglement criteria coming from Local Uncertainty Relations and similar to generalized spin squeezing
2. We have derived a general method for detecting the depth of entanglement with collective variances

WORK IN PROGRESS

- Take different operators $\{A_k = \sum_{n=1}^N a_k^{(n)}\}$
- The $a_k^{(n)}$ don't need to be equal to each other for all n

- Example: they might differ by a phase

$$J_k(q) = \sum_{n=1}^{N} e^{iqn} j_k^{(n)}$$

(Those are not Hermitian, but we define $(\Delta J_k(q))^2 = \langle J_k(q)^\dagger J_k(q)\rangle - \langle J_k(q)^\dagger\rangle\langle J_k(q)\rangle$ and a similar set of criteria follows)

- Questions: Which states are detected? How are they related to the A_k? When do the inequalities define polytopes?
- Question-2: How do these criteria relate to the original criteria coming from LURs?

WORK IN PROGRESS/2: Phase space operators

What about considering operators in phase-spaces?

- For example, the position/momentum operators (Q, P)

$$Q = \frac{1}{\sqrt{d}} \sum_{n=0}^{d-1} n|n\rangle\langle n|$$

$$F = \frac{1}{\sqrt{d}} \sum_{nm} \omega^{nm}|n\rangle\langle m|,$$

$$P := FQF^{\dagger} = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} k|\hat{k}\rangle\langle\hat{k}|$$

  of a particle with d-levels $\{|n\rangle\}_{n=0}^{d-1}$ (here $\omega = \exp(i \cdot 2\pi/d)$)

- Or the displacements $D(r, s) = X^r Z^s$ with $X = \omega^{-P}$ and $Z = \omega^Q$

References

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- O Marty, M Cramer, GV, G Tòth, M B. Plenio, arXiv:1708.06986
- + in preparation

THANK YOU FOR YOUR ATTENTION!