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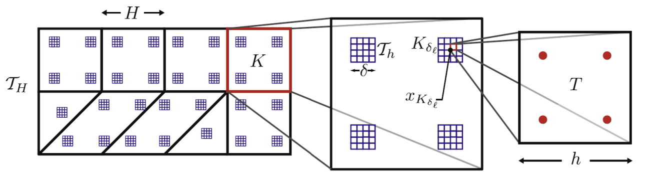
The Heterogeneous Multiscale Finite Element Method (FE-HMM) for the Homogenization of Microheterogeneous Materials

• Motivation

Homogenization theory is concerned with the macroscopic description of a microscopically heterogeneous system. Homogenization is necessary since the impacts of the small scales of such systems at a macroscale are considerable, but the numerical costs of fully resolved microscale information are prohibitive. FE-HMM, developped in [1-3], exhibits a sound mathematical basis that allows for a priori estimates, error controlled mesh-refinement, and optimal micro-macro refinements achieving the full convergence order for minimal costs.

• Main Aims

(i) The development of the FE-HMM for problems of solid mechanics, here for the case of elasticity in a linear setting.
(ii) A parallel solution strategy with favorable scaling on HPC machines.



(a) macro discretization with sampling domains (microproblems) (b) microproblems (c) example of quadrature for a microproblem

Concept of FE-HMM, picture from A. Abdulle, A. Nonnenmacher Comput. Methods Appl. Mech. Engrg. 198 (2009).

• Variational form of the macro FE-HMM problem

Find $\boldsymbol{u}^H \in \mathcal{S}_{\Omega_D}(\Omega, \mathcal{T}_H)$ such that

$$B_H(\boldsymbol{u}^H, \boldsymbol{v}^H) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}^H d\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{v}^H ds \;\forall \boldsymbol{v}^H \in \mathcal{S}_{\partial\Omega_D}(\Omega, \mathcal{T}_H)\,.$$

• The modified macro bilinear form

In the absence of explicit knowledge of $\mathbb{A}^0(\boldsymbol{x})$, we approximate

$$B_H(\boldsymbol{u}^H, \boldsymbol{v}^H) = \sum_{K\in\mathcal{T}_H} \sum_{l=1}^{N_{qp}} \omega_l \left(\mathbb{A}^0\, \varepsilon^h : \delta\varepsilon^h\right)(\boldsymbol{x}_l)$$

$$\approx \sum_{K\in\mathcal{T}_H} \sum_{l=1}^{N_{qp}} \frac{\omega_{K_l}}{|K_{\delta_l}|} \int_{K_{\delta_l}} \mathbb{A}^\varepsilon(\boldsymbol{x})\, \varepsilon(\boldsymbol{u}^h_{K_{\delta_l}}) : \delta\varepsilon(\boldsymbol{v}^h_{K_{\delta_l}})\, d\boldsymbol{x}$$

where $K_{\delta_l} = x_{K_{\delta_l}} + \varepsilon[-1/2, +1/2]^d$ is a sampling domain, ω_{K_l} is the quadrature weight, $|K_{\delta_l}|$ the volume of K_{δ_l}.
The macro element stiffness matrix is obtained according to

$$\boldsymbol{k}_K^{e,mac} = B_H^e \left[N_I^H, N_J^H\right]_{I,J=1}^{N_{node}} = \sum_{l=1}^{N_{qp}} \frac{\omega_{K_l}}{|K_{\delta_l}|} \boldsymbol{T}_{K_l}^T\, \boldsymbol{K}_{K_l}^{mic}\, \boldsymbol{T}_{K_l}$$

where $\boldsymbol{T}_{K_l}$ is a micro-macro stiffness transfer operator containing micro solution vectors $\boldsymbol{d}^{h(I,x_i)}$ as columns:

where $\boldsymbol{T}_{K_l} = [[[\boldsymbol{d}^{h(I,x_i)}]_{i=1,\dots,d}]_{I=1,\dots,N_{node}}]$,
with $\boldsymbol{d}^{h(I,x_i)} = (d^{h(I,x_i)}_{1,x_1}, d^{h(I,x_i)}_{1,x_2}, d^{h(I,x_i)}_{1,x_3}, \dots, d^{h(I,x_i)}_{M_{mic},x_3})^T$.

• Solution of the micro-problem.

The Lagrange functional $J(\boldsymbol{d}^{h(I,x_i)}, \lambda^{(I,x_i)})$ is minimized under the constraints of macro-micro coupling and micro-periodicity. Solve for each nodal unit displacement state (I, x_i) at a macro finite element node I in x_i-direction for $\boldsymbol{d}^{h(I,x_i)}$ and $\lambda^{(I,x_i)}$

$$\begin{bmatrix} \boldsymbol{K}^{mic}_{K_{\delta_l}} & \boldsymbol{G}^T \\ \boldsymbol{G} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}^{h(I,x_i)} \\ \lambda^{(I,x_i)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{G}\,\overline{\boldsymbol{d}}^{H(I,x_i)} \end{bmatrix}.$$

Matrix $\boldsymbol{G}$ contains the coupling conditions, $\lambda^{(I,x_i)}$: vector of Lagrange-multipliers, $\overline{\boldsymbol{d}}^{H(I,x_i)}$: vector of prescribed micro-displacments from a macro unit displacement state.

• A-priori estimates in the H^1- and the L^2-norm

$$||\boldsymbol{u}^0 - \boldsymbol{u}^H||_{H^1(\Omega)} \leq C\left(H^p + \left(\frac{h}{\varepsilon}\right)^{2q}\right) + e_{mod}\,,$$

$$||\boldsymbol{u}^0 - \boldsymbol{u}^H||_{L^2(\Omega)} \leq C\left(H^{p+1} + \left(\frac{h}{\varepsilon}\right)^{2q}\right) + e_{mod}\,,$$

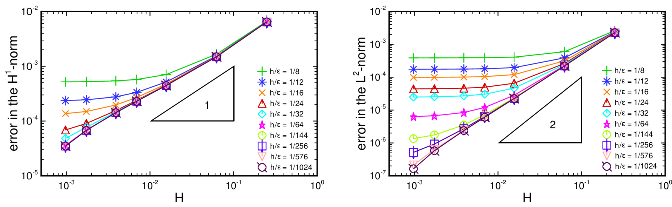
p/q macro/micro shape function polynomial order, $p = q = 1$.

• Benchmark problem for optimal mic-mac refinement

$$\mathbb{A}^\varepsilon(\boldsymbol{x}) = \begin{bmatrix} \mathbb{A}^\varepsilon_{11} & 35 & 0 \\ 35 & \mathbb{A}^\varepsilon_{22} & 0 \\ 0 & 0 & 50 \end{bmatrix}, \quad \begin{aligned} \mathbb{A}^\varepsilon_{11} &= [500/(5 + 3.5 \cdot \sin(2\pi x_1/\varepsilon))] \\ \mathbb{A}^\varepsilon_{22} &= [500/(5 + 3.5 \cdot \cos(2\pi x_1/\varepsilon))] \end{aligned}$$

Set $h/\varepsilon = 1/N_{mic}$, and $H = l/N_{mac}$. N_{mac} (N_{mic}) is the number of macro (micro) nodes per dimension.

Optimal convergence for minimal costs is obtained,
if $N_{mic} = N_{mac}$, i.e. $h/\varepsilon = H$ in the L^2-norm,
if $N_{mic} = \sqrt{N_{mac}}$, i.e. $h/\varepsilon = \sqrt{H}$ in the H^1-norm.



• Conclusions

(i) First FE-HMM formulation for a vector-valued field problem thus opening the door to solid mechanics.
(ii) Numerical results for the convergence order are in excellent agreement with a priori estimates.
(iii) Optimal micro-macro uniform refinement strategies from theory are underpinned by simulations to full extent.

References

[1] W. E and B. Engquist, *Commun. Math. Sci.* 49, 87–132 (2003).
[2] A. Abdulle, *Math. Mod. Meth. Appl. Sci.* 16(4), 615–635 (2006).
[3] A. Abdulle, *Math. Sci. Appl.* 31, 133–181 (2009).
[4] B. Eidel, A. Fischer, *PAMM* (2016), submitted.

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