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Full Algorithmic Consistency in Viscoelasticity – Enabling High Performance Computations

Motivation and Goals

The need for rapid prototyping in industry calls for high-performance algorithms to speed-up engineering simulations. For the design of structures made of metals or polymers, inelastic deformations must be considered in finite element (FE) simulations. Here, time integration methods for inelastic rate equations are most relevant and the questions arise:

- *Which is the best embedding of higher-order Runge-Kutta (RK) methods for inelastic FEM analyses?*
- *Which speed-up can be achieved compared with the (linear) Backward-Euler (BE)?*

Solution. In the FE solution framework of inelasticity the variational form of the balance of momentum is solved on the "global" level for displacements. From the displacements a deformation/strain measure is calculated; they are required as input for the solution of the Initial Value Problem (IVP) of inelastic flow on the "local" quadrature (typically Gauss) point level.

RK methods require total strain values at their *stages*. Compared with the true, nonlinear strain-path the approximation of deformation as constant $\boldsymbol{F}(t) = \boldsymbol{F}_{n+1} = \text{const.}$, $t \in [t_n, t_{n+1}]$ is poor; higher order polynomials (based on data at t_{n+1}, t_n and earlier times) improve the approximation, see Fig. 1.

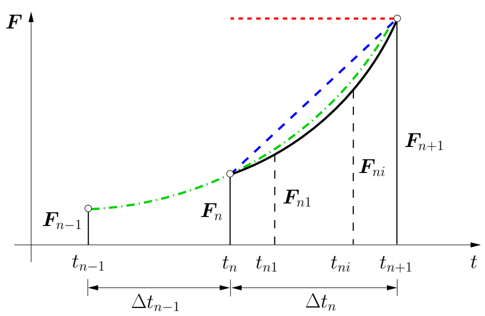


Fig. 1. The true deformation path (solid line) and its polynomial approximations (constant, linear, quadratic).

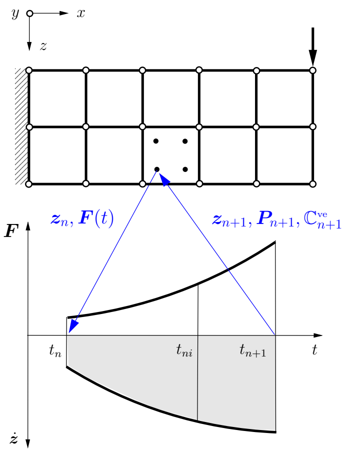
The *necessary* polynomial order follows from a consistency analysis. The differential equation for viscoelastic flow ($\boldsymbol{z}$ represents inelastic strain) typically exhibits the format

$$\dot{\boldsymbol{z}} = \boldsymbol{f}\left(\boldsymbol{F}(t), \boldsymbol{z}(t)\right) . \tag{1}$$

Therefore, for RK methods of the nominal order p and a representation of $\boldsymbol{F}(t)$ by interpolation polynomial of degree $q-1$ (hence of order q) it holds, see [1],

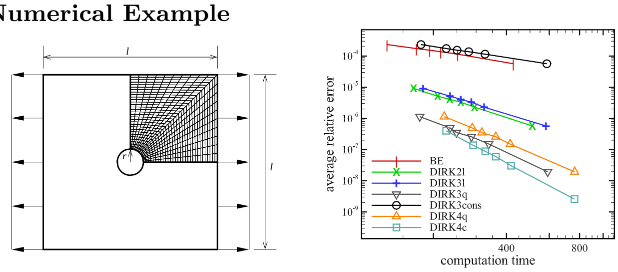
$$\boxed{\boldsymbol{F} = \mathcal{O}(\Delta t^q)\,, \ \boldsymbol{z} = \mathcal{O}(\Delta t^p) \ \rightarrow \ \boldsymbol{z} = \mathcal{O}(\Delta t^{\min\{p,q\}})\,. \quad (2)}$$

Algorithmic solution framework of inelasticity within FEM



- *"global" level:* solution of the BVP by FEM
- data transfer: global ↔ local
  - $\mathbf{z}_n$: internal variable at t_n
  - $\mathbf{F}(t)$: deformation gradient
  - $\mathbf{z}_{n+1}$: internal variable at t_{n+1}
  - $\mathbf{P}_{n+1}$: stress tensor at t_{n+1}
  - $\mathbb{C}^{\text{ve}}_{n+1} = \dfrac{\partial \boldsymbol{P}_{n+1}}{\partial \boldsymbol{F}_{n+1}}$ algo.tangent
- *"local" Gauss-point level:* solution of the IVP

Numerical Example



error	p	q	$e(\boldsymbol{C})$	$e(\boldsymbol{S}^{\text{OV}})$	speed-up
BE	1	1	1.02	1.02	1.0
DIRK2l	2	2	2.00	2.00	**21.6**
DIRK3q	3	3	2.67	2.95	**48.1**
DIRK4c	4	4	3.03	3.64	**48.7**

Conclusions

(A) Full order for $\boldsymbol{z}$ and $\boldsymbol{P}$ is obtained, if and only if $q = p$.

(B) For lower order strain interpolation $q < p$, the order of $\boldsymbol{z}$ and of stress $\boldsymbol{P}$ is reduced to q, order reduction!

(C) Representation of total strain by interpolation based on data at t_{n+1}, t_n, $\ldots$, t_{n+2-p}, $p \geq 2$ is effective.

(D) Drastic speed-up compared with Backward-Euler (BE).

(E) Concept of **algorithmic consistency** in inelasticity must be augmented:

Standard condition: Nagtegaal (1982), Simo & Taylor (1985)
Algorithmic tangent moduli $\mathbb{C}^{\text{ve}}_{n+1} = \partial \boldsymbol{P}_{n+1} / \partial \boldsymbol{F}_{n+1}$
→ quadratic convergence in global solution.

Novel condition for *consistent coupling*: $q = p$
→ full convergence order p in time integration.

References

[1] B. Eidel, *Habilitation Thesis* (2011).
[2] B. Eidel, C. Kuhn, *Int. J. Numer. Meth. Eng.*, **87** (2011).
[3] B. Eidel, F. Tempel, and J. Schröder, *Comput. Mech.*, **52** (2013).

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