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Strain tensors and strain measures in nonlinear elasticity

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1 Strain tensors

The concept of *strain* is of fundamental importance in continuum mechanics. In linearized elasticity, one assumes that the Cauchy stress tensor σ is a linear function of the symmetric infinitesimal strain tensor

$$\varepsilon = \operatorname{sym}\nabla u = \operatorname{sym}(\nabla\varphi - \mathbb{1}) = \operatorname{sym}(F - \mathbb{1}),$$

where $\varphi\colon \Omega \to \mathbb{R}^n$ is the deformation of an elastic body with a given reference configuration $\Omega \subset \mathbb{R}^n$, $\varphi(x) = x + u(x)$ with the displacement u, $F = \nabla\varphi$ is the deformation gradient, $\operatorname{sym}\nabla u = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetric part of the displacement gradient ∇u and $\mathbb{1}$ is the identity tensor. In geometrically nonlinear elasticity models, it is no longer necessary to postulate a linear connection between some stress and some strain. However, nonlinear strain tensors are often used in order to simplify the stress response function, and many constitutive laws are expressed in terms of linear relations between certain strains and stresses [2, 3, 5]: according to Truesdell and Noll [18, p. 347], "*Various authors [...] have suggested that we should select the strain [tensor] afresh for each material in order to get a simple form of constitutive equation. [...] Every invertible stress relation $T = f(B)$ for an isotropic elastic material is linear, trivially, in an appropriately defined, particular strain [tensor $f(B)$].*"

There are many different definitions of what exactly the term "strain" encompasses: while Truesdell and Toupin [19, p. 268] consider "*any uniquely invertible isotropic second order tensor function of [the right Cauchy-Green deformation tensor $C = F^TF$]*" to be a strain tensor, it is commonly assumed [9, p. 230] (cf. [10, 11, 4, 14]) that a (material or Lagrangian) strain takes the form of a *primary matrix function* of the right Biot-stretch tensor $U = \sqrt{F^TF}$ of the deformation gradient $F \in \mathrm{GL}^+(n)$, i.e. an isotropic tensor function $E\colon \mathrm{Sym}^+(n) \to \mathrm{Sym}(n)$ from the set of positive definite tensors to the set of symmetric tensors of the form

$$E(U) = \sum_{i=1}^{n} \mathrm{e}(\lambda_i)\cdot e_i\otimes e_i \quad \text{for} \quad U = \sum_{i=1}^{n} \lambda_i\cdot e_i\otimes e_i \quad (1)$$

with a strictly monotone *scale function* $\mathrm{e}\colon (0,\infty) \to \mathbb{R}$, where $\otimes$ denotes the tensor product, λ_i are the eigenvalues and e_i are the eigenvectors of U. Similarly, a *spatial* or *Eulerian* strain tensor $\widetilde{E}(V)$ depends on the left Biot-stretch tensor $V = \sqrt{FF^T}$ (cf. [6]).

The general idea underlying these definitions is clear: strain is a measure of deformation (i.e. the change in form and size) of a body with respect to a chosen (arbitrary) reference configuration. Furthermore, the strain of the deformation gradient $F \in \mathrm{GL}^+(n)$ should correspond only to the *non-rotational* part of F. In particular, the strain must vanish if and only if F is a pure rotation, i.e. if and only if $F \in \mathrm{SO}(n)$, where $\mathrm{SO}(n) = \{Q \in \mathrm{GL}(n) \,|\, Q^TQ = \mathbb{1}, \det Q = 1\}$ denotes the special orthogonal group. This ensures that the only strain-free deformations are rigid body movements [12].

According to a more general definition [13], (material) **strain tensor** is an injective isotropic tensor function $U \mapsto E(U)$ of the right Biot-stretch tensor U mapping $\mathrm{Sym}^+(n)$ to $\mathrm{Sym}(n)$ with

$$E(Q^TU\,Q) = Q^TE(U)\,Q \quad \text{for all } Q \in \mathrm{O}(n)$$
$$\text{and} \quad E(U) = 0 \iff U = \mathbb{1},$$

where $\mathrm{O}(n) = \{Q \in \mathrm{GL}(n) \,|\, Q^TQ = \mathbb{1}\}$ is the orthogonal group. In particular, these conditions ensure that $E(U) = 0$ if and only if $\mathbb{1} = U = \sqrt{F^TF}$, i.e. if and only if $F \in \mathrm{SO}(n)$.

Among the most common examples of material strain tensors used in nonlinear elasticity is the *Seth-Hill family* [15]

$$E_r(U) = \begin{cases} \frac{1}{2r}(U^{2r} - \mathbb{1}) & : r \in \mathbb{R}\setminus\{0\} \\ \log U & : r = 0 \end{cases} \quad (2)$$

of material strain tensors, which includes the *Biot strain tensor* $E_{1/2}(U) = U - \mathbb{1}$, the *Green-Lagrangian strain tensor* $E_1(U) = \frac{1}{2}(U^2 - \mathbb{1})$, the *(material) Almansi strain tensor* [1] $E_{-1}(U) = \frac{1}{2}(\mathbb{1} - C^{-1})$ and, of course, the *(material)* ***Hencky strain tensor***

$$E_0(U) = \log U = \log(\sqrt{F^TF}). \quad (3)$$

Here, $\log\colon \mathrm{Sym}^+(n) \to \mathrm{Sym}(n)$ is the *principal matrix logarithm* on the set $\mathrm{Sym}^+(n)$ of positive definite symmetric matrices. The Hencky (or logarithmic) strain tensor has often been considered the *natural* or *true* strain in nonlinear elasticity [17, 16, 7, 8].

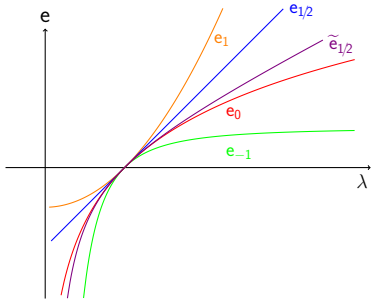

Figure 1: Scale functions $\mathrm{e}_r, \widetilde{\mathrm{e}}_r$ associated with the strain tensors E_r and $\widetilde{E}_r = \frac{1}{2}(E_r - E_{-r})$.

All strain tensors, by the definition employed here, can be seen as *equivalent*: since the mapping $U \mapsto E(U)$ is injective, for every pair E, E' of strain tensors there exists a mapping $\psi\colon \mathrm{Sym}(n) \to \mathrm{Sym}(n)$ such that $E'(U) = \psi(E(U))$ for all $U \in \mathrm{Sym}^+(n)$. Therefore, every constitutive law of elasticity can – in principle – be expressed in terms of any strain tensor and no strain tensor can be inherently superior to any other strain tensor: as Truesdell and Toupin [19, p. 268] state, "*... any [tensor] sufficient to determine the directions of the principal axes of strain and the magnitude of the principal stretches may be employed and is fully general*". Truesdell and Noll [18, p. 348] also argue that there "*is no basis in experiment or logic for supposing nature prefers one strain [tensor] to another*".

2 Strain measures

In contrast to *strain* or *strain tensor*, we use the term **strain measure** to refer to a nonnegative real-valued function $\omega\colon \mathrm{GL}^+(n) \to [0,\infty)$ depending on the deformation gradient which vanishes if and only if F is a pure rotation, i.e. $\omega(F) = 0$ if and only if $F \in \mathrm{SO}(n)$. A simple example of a strain measure in the above sense is the mapping $F \mapsto \|E(\sqrt{F^TF})\|_*$ of F to an orthogonally invariant norm of any strain tensor E.

Note carefully that, in contrast to strain tensors, strain measures cannot simply be used interchangeably: for two different strain measures (as defined above) ω_1, ω_2, there is generally no function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega_2(F) = f(\omega_1(F))$ for all $F \in \mathrm{GL}^+(n)$. Compared to "full" strain tensors, this can be interpreted as an unavoidable **loss of information** for strain measures (which are only scalar quantities).

Since, by our definition, a strain measure attains zero if and only if $F \in \mathrm{SO}(n)$, a simple geometric approach is to consider a **distance function** on the group $\mathrm{GL}^+(n)$ of admissible deformation gradients, i.e. a function $\mathrm{dist}\colon \mathrm{GL}^+(n) \times \mathrm{GL}^+(n) \to [0,\infty)$ with $\mathrm{dist}(A,B) = \mathrm{dist}(B,A)$ which satisfies the triangle inequality and vanishes if and only if its arguments are identical. Such a distance function induces a "natural" strain measure on $\mathrm{GL}^+(n)$ by means of the distance to the special orthogonal group $\mathrm{SO}(n)$:

$$\omega(F) := \mathrm{dist}(F, \mathrm{SO}(n)) := \inf_{Q\in\mathrm{SO}(n)} \mathrm{dist}(F,Q). \quad (4)$$

In this way, the search for an appropriate strain measure reduces to the task of finding a **natural, intrinsic distance function** on $\mathrm{GL}^+(n)$.

3 References

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