Sensorics Exam

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Name:	Task:	T1	T2	T3	T4	T5	T6	Sum
Mat.-No.:	Scores:	10	30	10	28	30	12	120
Grade:	Accomplished:							

Task 1: Comprehension Questions

Mark the correct answers clearly.
Every question has one or two correct answers!
For every correctly marked answer you will get one point. If there is one correct answer marked and one incorrect answer marked, you will get no point for that subtask.

a) The capacity of a plate capacitor depends on ...
- ☐ ... the distance between the plates.
- ☐ ... the voltage of the circuit.
- ☐ ... the area of the plates.

b) Assess the following statements regarding temperature measurement methods.
- ☐ PTC Thermometers are faster but less accurate than NTC Thermometers.
- ☐ Thermocouples can only measure temperatures up to 200°C.
- ☐ PTC Resistance Thermometers have almost linear characteristics.

c) The direct measurement of speed can be done by ...
- ☐ ... using the Doppler effect of acoustic or electromagnetic waves.
- ☐ ... using the Combination of 2 cameras and correlation analysis.
- ☐ ... using the piezoelectric effect.

d) If you differentiate a noisy signal with respect to time t ...
- ☐ ... the gain of the transfer function changes.
- ☐ ... the noise is amplified.
- ☐ ... the system becomes unstable.

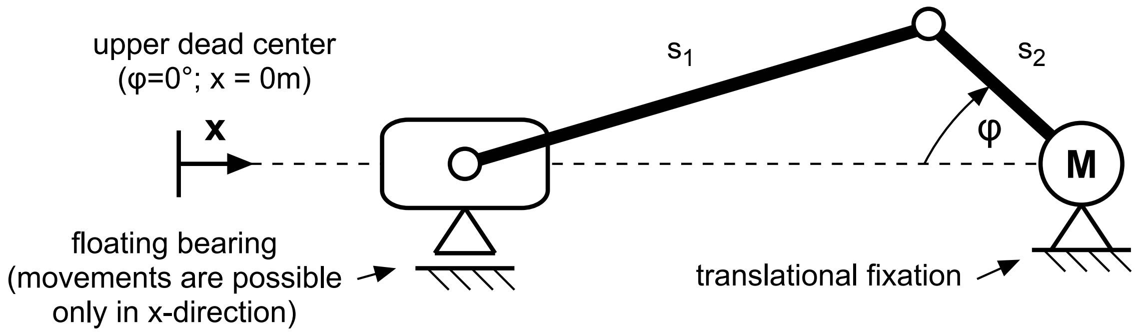
e) Assess the following statements regarding measurement techniques.
- ☐ Voltage meters should have an internal resistance as small as possible.
- ☐ The ideal operational amplifier has an infinite input resistance.
- ☐ The ideal operational amplifier has an infinite output resistance.

f) Assess the following statements regarding digital signals.
- ☐ Increasing the sampling time avoids aliasing.
- ☐ Changing the sampling time does not influence the quantization error.
- ☐ The sampling frequency must be at least the double of the highest significant frequency component of the signal.

g) Assess the following statements regarding frequency measurement.
- ☐ The measurement of the cycle duration is well suited for signals of low frequency.
- ☐ The frequency is calculated by the Shannon theorem using the sampling time.
- ☐ Counting the number of cycles within one time interval is well suited for signals of low frequency.

Task 2: Slider-Crank Mechanism

The picture below shows a slider-crank mechanism, that is driven by an electric motor **M** (on top of the translational fixation). s_1 and s_2 denote the length of the two cranks. The x-position of the slider is measured from the upper dead center.



The position x, the velocity v and the acceleration a of the slider should be measured.

a) Name two possible measurement methods, that can be used to measure the position x of the slider.

b) Name two measurement principles, that can be used to measure speed in general.

c) If a Piezoelectric sensor is applied to the slider, would this type of sensor be suited to measure the slider's acceleration? What drawback do Piezoelectric sensors have (Hint: What about a rotational speed near zero $\dot{\varphi} \approx 0$)? Sketch the position of the sensor into the picture above, where you would apply this sensor.

d) Now it is assumed to have only one sensor measuring the angle φ. The angular velocity as well as the angular acceleration should be determined with the help of derivatives of the measured angle φ. What could be problematic with this approach in the real world? Hint: What happens when the angle measurement is disturbed? How does the frequency of the disturbance influence the speed and acceleration?

e) The position of the slider can be approximated with the following equation:

$$x(\varphi(t)) = s_2 \left(1 - \cos(\varphi(t)) + \frac{\lambda}{4} - \frac{\lambda}{4}\cos(2\varphi(t))\right) ,$$

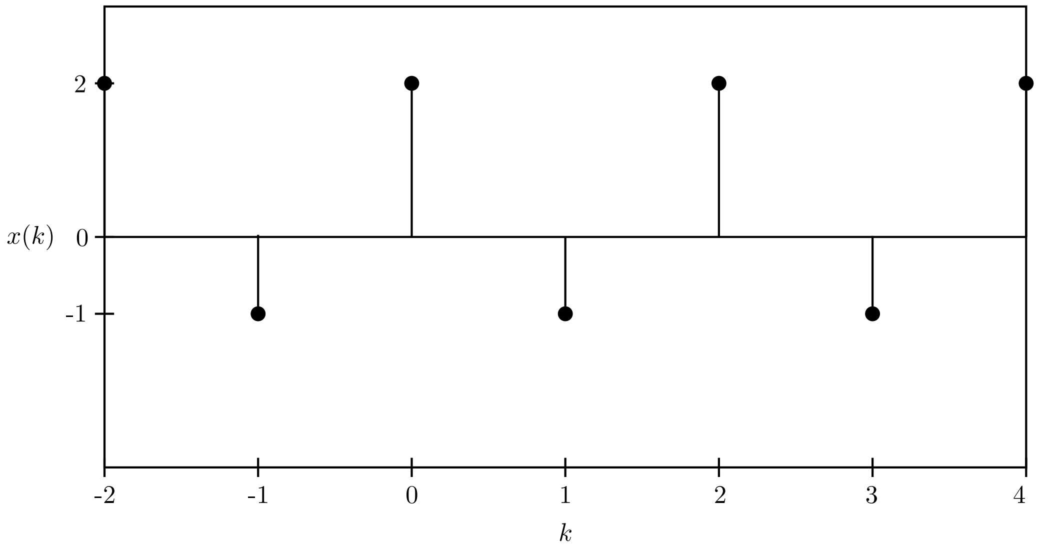
where $\lambda = \frac{s_2}{s_1}$ denotes the fraction between the two crank lengths. Derive the equations to calculate the translational speed and acceleration of the slider.

f) The maximum rotational speed is $n_{max} = 500\text{rpm}$ (rounds per minute). The translational speed maximum is reached at the angle $\varphi = 73.7°$ for $s_2 = 1\text{m}$ and $s_1 = 5\text{m}$. Calculate the maximum translational speed v_{max} for the given values.

g) Determine the sampling frequency f_0 for the angle measurement φ such that the maximum error of the translational position x is less or equal 0.1m at any time. The maximum rotational speed is still $n_{max} = 500\text{rpm}$.

Task 3: Discrete Fourier-Transformation

The following periodic sequence $x(k)$ is given.



a) After how many values N does the series repeat?

b) First derive the equation system, that has to be solved to achieve the discrete Fourier-Transform X(n), and than express the equation system with matrix-vector notation $\underline{X} = \underline{F}\,\underline{x}$.
Use in both cases the abbreviation $W_N = e^{-i\frac{2\pi}{N}}$.
For the equation system use a general signal $x(k)$ and for the matrix-vector notation use a general signal vector $\underline{x}$.

c) Now use the given signal to solve the equation system for $X(n)$, depending on the not yet specified values of W_N.

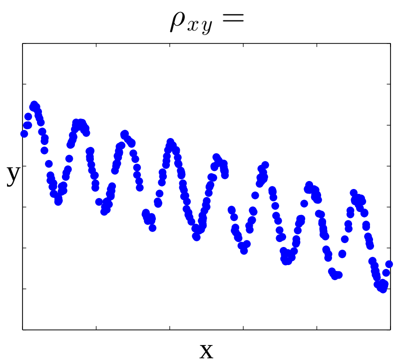
d) Calculate all required powers of W_N and plot them in the complex plane.

e) Use the calculated powers of W_N to calculate the discrete amplitude spectrum $|X(n)|$.

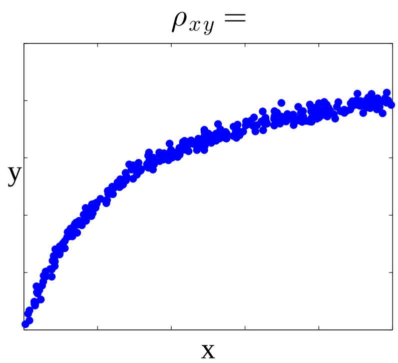
<u>Task 4:</u> Correlation, Probability Densities and Confidence Intervals

All subtasks can be solved independently.

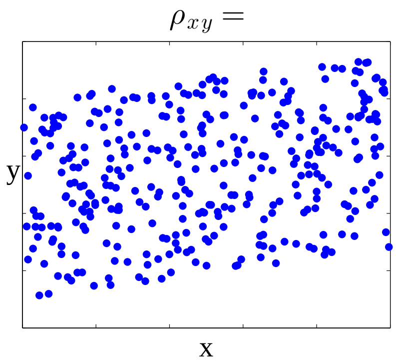
a) Assign the following correlation coefficients to one of the figures below. Every correlation coefficient can only be assigned once. Here are the correlation coefficients: -0.7, 0, 0.2, 0.9.



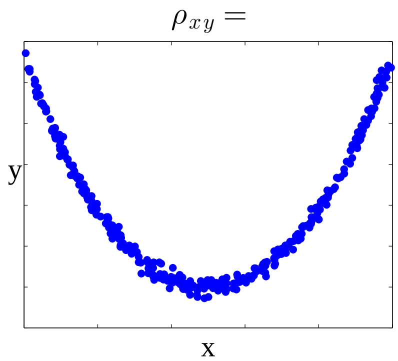
(a)



(b)



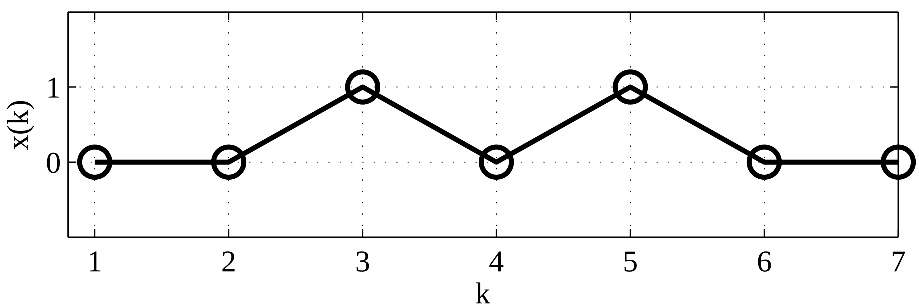
(c)



(d)

b) Calculate the autocorrelation function values for the signal $x(k)$ below. Use the following equation for the calculation of time shifts $\kappa = 0 \ldots 6$:

$$r_{xx}(\kappa) = \frac{1}{N - |\kappa|} \sum_{k=1}^{N-|\kappa|} x(k)x(k+\kappa) \ . \tag{1}$$



State if the estimation of r_{xx} according to equation (1) is biased or unbiased.

c) Probability Densities

1) Sketch **two** normal distributions into **one** coordinate system, that have the following properties:
   - Mean $\mu_1 = 0$, Variance $\sigma_1^2 = 1$.
   - Mean $\mu_2 = 0$, Variance $\sigma_2^2 = 9$.

   Assign each variance to the corresponding (sketched) normal distribution.
2) Sketch the probability density, where all values between 2 and 6 are equally likely (uniform distribution). Take care of the coordinate axis' scaling.

Make sure, that all coordinate axes are labeled.

d) Confidence Intervals

1) The body temperature of a person has been measured five times in a row. The accuracy properties of the measurement instrument are well known. The standard deviation of the disturbance is determined to be $\sigma_T = 0.6$. The following values are obtained:

   $$T[°C] = \begin{bmatrix} 35.8 & 35.9 & 37 & 36.8 & 37 \end{bmatrix}$$

   Calculate the range in which the true temperature will be for an accepted error probability of 0.27%. Therefore use the following table.

Interval	Probability
$\mu_x - 1\sigma_x < x < \mu_x + 1\sigma_x$	68.27%
$\mu_x - 2\sigma_x < x < \mu_x + 2\sigma_x$	95.45%
$\mu_x - 3\sigma_x < x < \mu_x + 3\sigma_x$	99.73%
$\mu_x - 4\sigma_x < x < \mu_x + 4\sigma_x$	99.99%

2) Now it is assumed, that the standard deviation σ_T is not known, but can be estimated from the given values above. Calculate the estimated standard deviation s_T with the help of the five given temperature values.
3) Given s_T from subtask d2), will there anything change regarding the confidence interval calculation for the true temperature value compared with the one from subtask d1)?

Task 5: Time discrete system

The following time continuous impulse response of a first order system is given:

$$g(t) = \frac{K}{T} \cdot e^{-\frac{t}{T}} \, ,$$

with gain $K = 4$ and the time constant $T = 2$ sec.

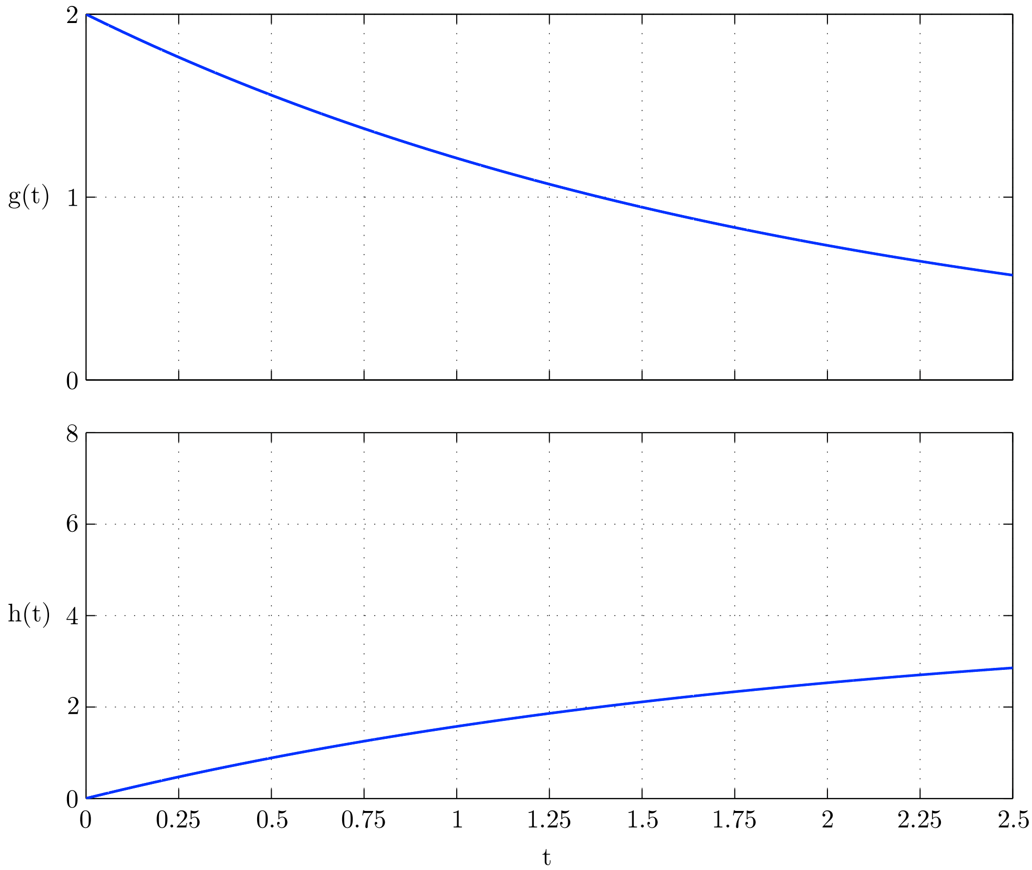
a) Determine the step response $h(t)$ of the given first order system.

b) Determine the transfer function $G_S(z)$ via **step response invariance**. Use the sampling time $T_0 = 0.25$ sec.

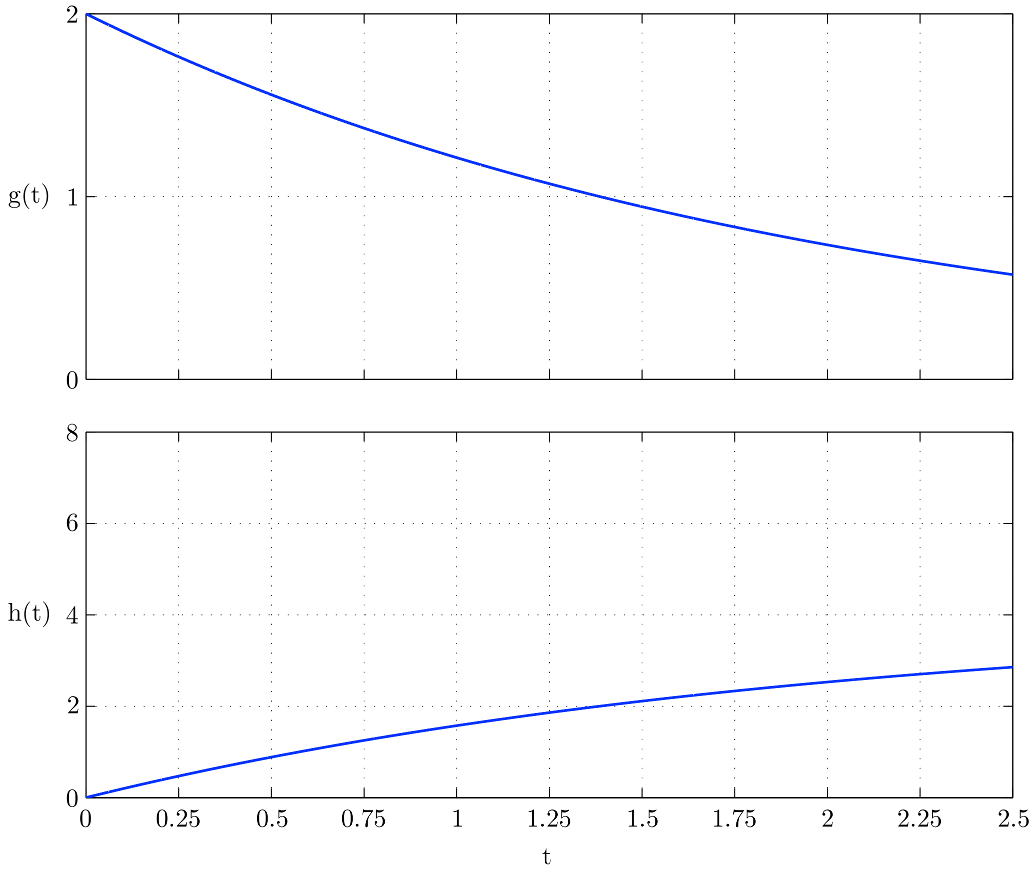
c) Determine the transfer function $G_I(z)$ via **impulse response invariance**. Use the sampling time $T_0 = 0.25$ sec.

d) To compare the two Systems $G_S(z)$ and $G_I(z)$ calculate the gain, the properness and the stability.

e) Plot the first $k = 0, 1, 2, 3$ samples of the impulse and step response for the **System** $G_S(z)$.



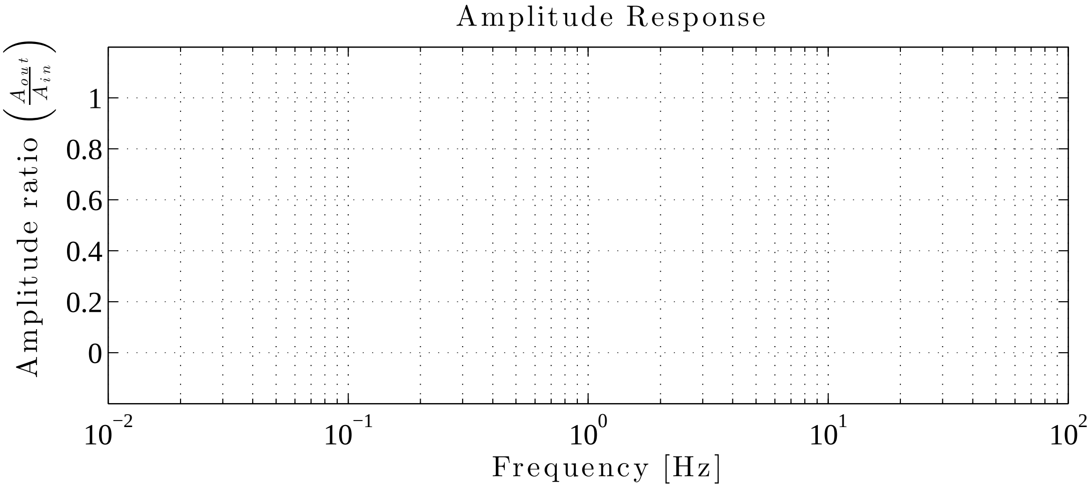
f) Plot the first $k = 0, 1, 2, 3$ samples of the impulse and step response for the **System** $G_I(z)$.


g(t)
h(t)
t
0
1
2
4
6
8
0.25
0.5
0.75
1.25
1.5
1.75
2.25
2.5

Task 6: Filter

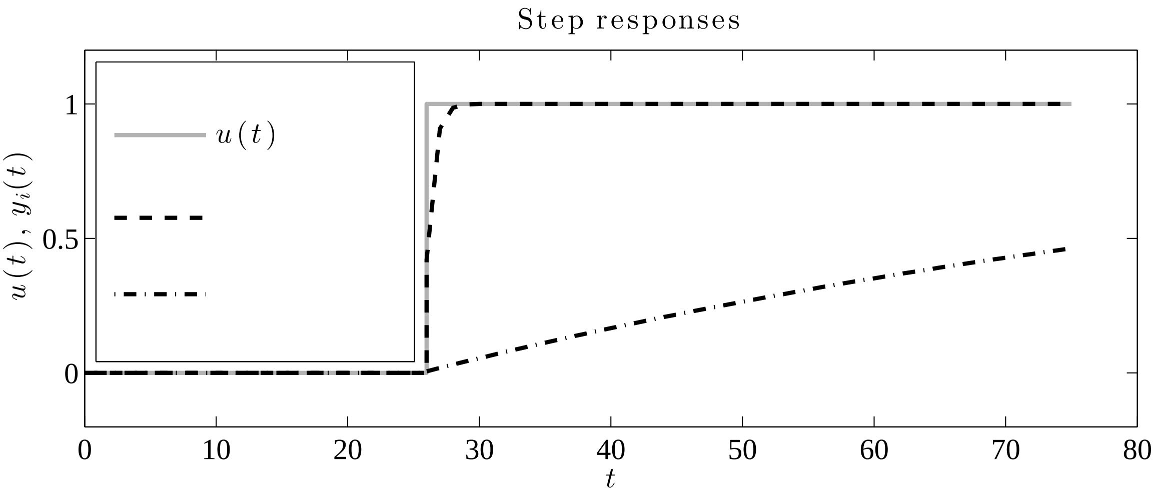
There are two filters, that should be compared. Both filters are low-pass filters, with the transfer functions G_1 and G_2. The limit (cut-off) frequency of filter G_1 is $f_1 = 0.05$ Hz, and the limit (cut-off) frequency of filter G_2 is $f_2 = 5$ Hz.

a) Sketch the **idealized** amplitude responses of both filters G_1 and G_2 in the figure below. Label your sketched amplitude responses with the correct transfer function.

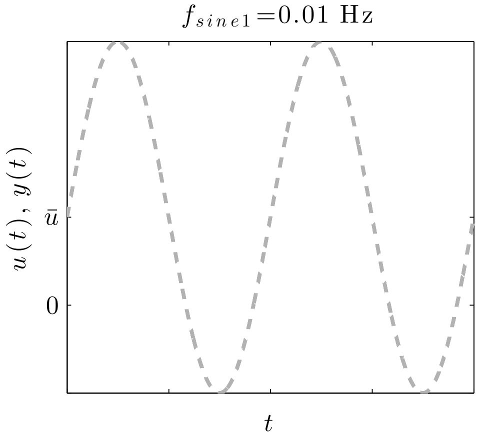


From now on assume that the ideal low-pass filters are approximated by first order systems.

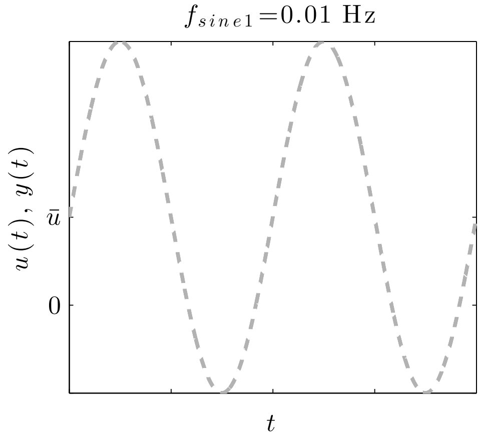
b) Now a unit step $u(t) = \sigma(t)$ is applied to the two filters. The output of filter G_1 should be denoted y_1, the output of filter G_2 should be denoted y_2. In the figure below, both filter responses are shown. Label the two step responses correctly.



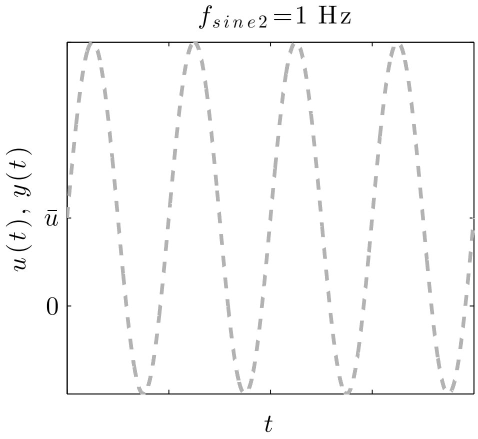
c) Now imagine you would filter sine waves with three different frequencies, $f_{sine1} = 0.01$ Hz, $f_{sine2} = 1$ Hz and $f_{sine3} = 100$ Hz through G_1 and G_2. The sine waves are already shown in the sub-figures (a) to (f) below. Use the left columns to sketch the filtered output $y(t)$ **qualitatively**, when the filter G_1 is used and the right columns for the results of a filtering with G_2. Assume that the responses have already reached the steady state. The value $\bar{u}$ labels the mean of the input signal $u(t)$.



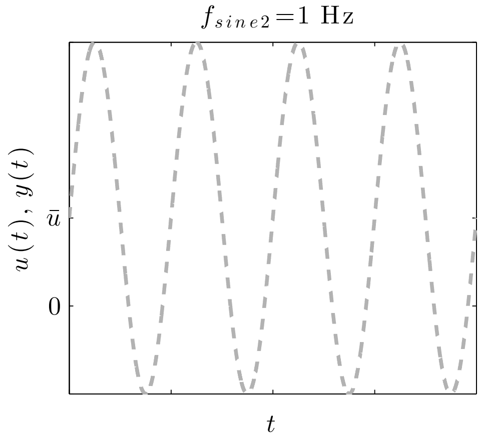
(a) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_1.



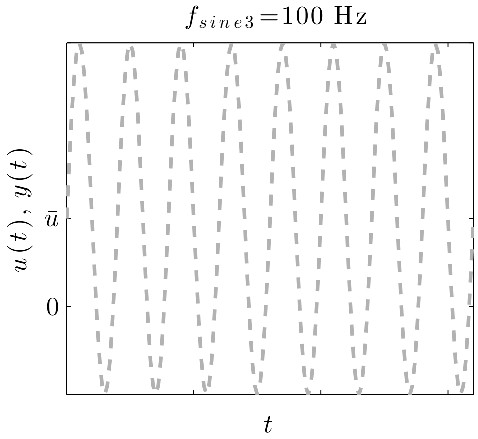
(b) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_2.



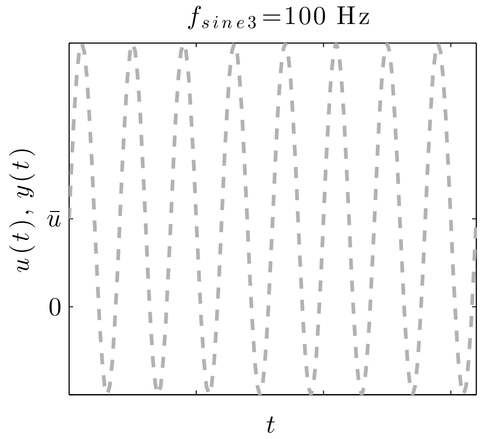
(c) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_1.



(d) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_2.



(e) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_1.



(f) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_2.

Solutions:

Task 1: Comprehension Questions

a) The capacity of a plate capacitor depends on ...

- ■ ... the distance between the plates.
- □ ... the voltage of the circuit.
- ■ ... the area of the plates.

b) Assess the following statements regarding temperature measurement methods.

- □ PTC Thermometers are faster but less accurate than NTC Thermometers.
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c) The direct measurement of speed can be done by ...

- ■ ... using the Doppler effect of acoustic or electromagnetic waves.
- ■ ... using the Combination of 2 cameras and correlation analysis.
- □ ... using the piezoelectric effect.

d) If you differentiate a noisy signal with respect to time t ...

- □ ... the gain of the transfer function changes.
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- □ Voltage meters should have an internal resistance as small as possible.
- ■ The ideal operational amplifier has an infinite input resistance.
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- □ Increasing the sampling time avoids aliasing.
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g) Assess the following statements regarding frequency measurement.

- ■ The measurement of the cycle duration is well suited for signals of low frequency.
- □ The frequency is calculated by the Shannon theorem using the sampling time.
- □ Counting the number of cycles within one time interval is well suited for signals of low frequency.

$\sum$ 10

Task 2: Slider-Crank Mechanism

a) Name two possible measurement methods, that can be used to measure the position x of the slider.

Here a few possible measurement methods:

1) Inductive measurement method
2) Capacitive measurement method
3) Optical angle measurement and transformation via the given equation
4) Optical displacement measurement of the slider
5) etc.

2

b) Name two measurement principles, that can be used to measure speed in general.

Possibilities for speed measurement:

1) Measure time interval Δt and distance Δx, then calculate the velocity $v = \frac{\Delta x}{\Delta t}$.
2) Measurement of rotational speed ω and conversion into the translational speed with $v = \omega r$.
3) Direct measurement of speed by the use of the Doppler effect (of acoustic or electromagnetic waves).

2

c) If a Piezoelectric sensor is applied to the slider, would this type of sensor be suited to measure the slider's acceleration? What drawback do Piezoelectric sensors have (Hint: What about rotational speed near zero $\dot{\varphi} \approx 0$)?

In this case a piezoelectric sensor is suited for acceleration measurement. Only for a rotational speed of $\dot{\varphi} \approx 0$ the acceleration might not be accurate, because it would be a quasi-stationary measurement. Piezoelectric sensors are only suited for dynamic measurements, because of their transient behavior. (Sensor Position: see sketch...) 3

d) What could be problematic with this approach in the real world?

Derivatives enhance the noise! Especially for the acceleration where the signal is derived two times with respect to time, the signal might become very noisy. It has to be checked if the achieved accuracy is sufficient for the task at hand. 5

e) Derive the equations to calculate the translational speed and acceleration of the slider.

$$\dot{x}(\varphi) = v(\varphi) = s_2\dot{\varphi}\left(\sin(\varphi) + \frac{\lambda}{2}\sin(2\varphi)\right)$$

$$\ddot{x}(\varphi) = a(\varphi) = s_2\dot{\varphi}^2\left(\cos(\varphi) + \lambda\cos(2\varphi)\right) + s_2\ddot{\varphi}\left(\sin(\varphi) + \frac{\lambda}{2}\sin(2\varphi)\right) \;,$$

8

f) Calculate the maximum translational speed v_{max} for the given values.

$$\begin{aligned} v(\varphi) &= s_2\dot{\varphi}\left(\sin(\varphi) + \frac{\lambda}{2}\sin(2\varphi)\right) \\ &= \frac{1\text{m} \cdot 500 \cdot \frac{2\pi}{\text{per round}}}{60\frac{\text{sec}}{\text{min}} \cdot \text{min}}\left(\sin(73.7° \cdot \frac{\pi}{180°}) + \frac{1}{10}\sin(2 \cdot 73.7° \cdot \frac{\pi}{180°})\right) \\ &= 53.08\frac{\text{m}}{\text{sec}} = v_{max} \end{aligned}$$

$\boxed{6}$

g) Determine the sampling frequency f_0 for the angle measurement φ such that...

To guarantee one measured value at least every $\Delta x = 0.1\text{m}$, we determine how long it takes to cover that distance at maximum translational speed v_{max}:

$$\begin{aligned} \Delta t &= \frac{\Delta x}{v_{max}} \\ &= \frac{0.1\text{m}}{53.08\frac{\text{m}}{\text{sec}}} \\ &= 0.0019\text{sec} \ . \end{aligned}$$

With the help of this time period, the sampling frequency becomes:

$$\begin{aligned} f_0 &= \frac{1}{\Delta t} \\ &= 530.76\text{Hz} \ . \end{aligned}$$

$\boxed{4}$

$\boxed{\sum 30}$

Task 3: Discrete Fourier-Transformation

In general: $\mathrm{DFT}\{x(k)\} = X(n) = \sum_{k=0}^{N-1} x(k) \cdot W_N^{nk}$ with $W_N = e^{-i\frac{2\pi}{N}}$.

a) The periodic, time discrete signal is $N = 2$ long. $\boxed{1}$

b) Therefore the equation for the discrete Fourier transformation is:

$$\mathrm{DFT}\{x(k)\} = X(n) = \sum_{k=0}^{1} x(k) \cdot W_2^{nk} \ .$$

The equation system for achieving the Fourier transform $X(n)$ is:

$$\begin{aligned} X(0) &= W_2^0 \cdot x(0) + W_2^0 \cdot x(1) \\ X(1) &= W_2^0 \cdot x(0) + W_2^1 \cdot x(1) \ . \end{aligned}$$ $\boxed{2}$

The same in matrix-vector notation:

$$\underbrace{\begin{bmatrix} X(0) \\ X(1) \end{bmatrix}}_{\underline{X}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix}}_{\underline{F}} \cdot \underbrace{\begin{bmatrix} x(0) \\ x(1) \end{bmatrix}}_{\underline{x}} \ . \qquad (2)$$ $\boxed{1}$

c) The given signal with periodic time $N = 2$ is:

$$\underline{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \ .$$

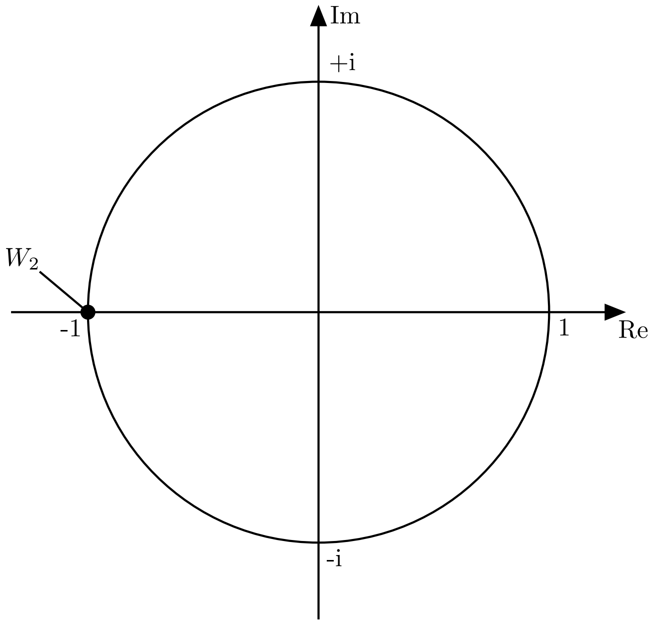
Used in equation 2 it leads to $\underline{X}$:

$$\underline{X} = \begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 - W_2 \end{bmatrix} \ .$$ $\boxed{1}$

d) The only power of W_N to calculate is W_2:

$$W_2 = e^{-i\frac{2\pi}{2}1} = e^{-i\pi} = \cos(-\pi) + i \cdot \sin(-\pi) = -1 \; .$$ 1

Plot of them in the complex plane.



3

e) The discrete amplitude spectrum $|\underline{X}|$ is:

$$|\underline{X}| = \left| \begin{bmatrix} 2-1 \\ 2-(-1) \end{bmatrix} \right| = \left| \begin{bmatrix} 1 \\ 2+1 \end{bmatrix} \right| = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \; .$$ 1

$\sum 10$

Task 4: Correlation, Probability Densities and Confidence Intervals

a) Assign the following correlation coefficients to one of the figures below. Every correlation coefficient can only be assigned once.

(a): -0.7
(b): 0.9
(c): 0.2
(d): 0 4

b) Calculate the autocorrelation function values for the signal $x(k)$.

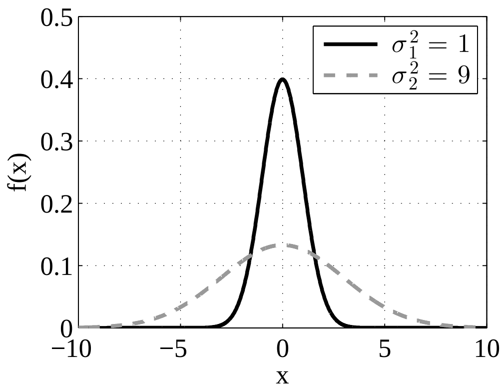
The autocorrelation function values differ from zero only for two time shits: For $\kappa = 0$ and $\kappa = 2$.

$$r_{xx}(\kappa = 0) = \frac{2}{7}$$
$$r_{xx}(\kappa = 2) = \frac{1}{5} \ .$$

The estimation with the given equation is biased. 7

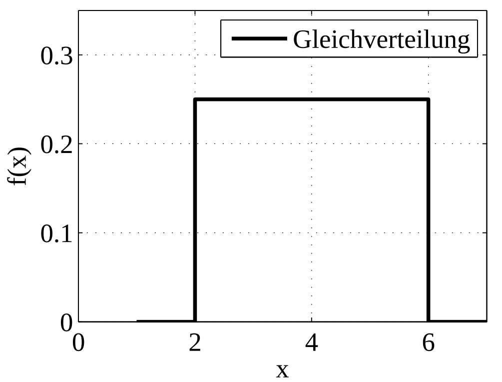
c) Probability Densities 1

1) Normal distributions



4

2) Uniform distribution



2

d) Confidence Intervals

1) Calculate the range in which the true temperature will be for an accepted error probability of 0.27%.
Sample mean: $\bar{T} = \frac{1}{5}\sum_{i=1}^{5} T(i) = \frac{182.5}{5} = 36.5°C$.
Standard deviation of the sample mean: $\sigma_{\bar{T}} = \frac{\sigma_T}{\sqrt{5}} = \frac{0.6}{2.24} = 0.27$.
From the accepted error probability it follows a requested confidence level of 99.73%. This corresponds to $c = 3$ for the range calculation. The range is:

$$\bar{T} - c \cdot \sigma_{\bar{T}} < T < \bar{T} + c \cdot \sigma_{\bar{T}}$$
$$35.70 < T < 37.31$$

3

2) Estimate the instrument's standard deviation of the temperature disturbance with the help of the measured values from above.

$$s_T = \sqrt{\frac{1}{4}\sum_{i=1}^{5}[T(i) - \bar{T}]^2}$$
$$= 0.6$$

4

3) Given s_T from subtask d2), will there anything change regarding the confidence interval calculation for the true temperature value compared with the one from subtask d1)?

Even though the estimated and the true standard deviation are exactly equal, the **resulting interval will be different**. In the case, where the estimated standard deviation is used to calculate the range, the factor c has to be determined from a **t-student distribution**, and therefore c **will be a greater value**. 3

$\sum 28$

Task 5: Time discrete system

a) The step response is the integral of the impulse response:

$$h(t) = \int\limits_0^t g(\tau)d\tau = \int\limits_0^t \frac{K}{T} \cdot e^{-\frac{\tau}{T}} d\tau = \frac{K}{T} \cdot \int\limits_0^t e^{-\frac{\tau}{T}} d\tau$$
$$= \frac{K}{T} \cdot T \cdot \left(e^0 - e^{-\frac{t}{T}}\right) = K \cdot \left(1 - e^{-\frac{t}{T}}\right) \ .$$ $\boxed{2}$

Considering the given values leads to:

$$h(t) = 4 \cdot \left(1 - e^{-\frac{t}{2sec}}\right) \ .$$

b) The first thing to do is to discretize the step response with the sampling time $T_0 = 0.25$ sec:

$$h(k) = h(t = T_0 \cdot k) = K \cdot \left(1 - e^{-\frac{T_0 \cdot k}{T}}\right) = 4 \cdot \left(1 - e^{-\frac{0.25\text{sec}\cdot k}{2\text{sec}}}\right) = 4 \cdot \left(1 - e^{-0.125 \cdot k}\right) \ .$$ $\boxed{1}$

Used in the general equation of the z-transformation results in $H(z)$:

$$H_S(z) = \sum_{k=0}^{\infty} \left(h(k) \cdot z^{-k}\right) = \sum_{k=0}^{\infty} \left(4 \cdot \left(1 - e^{-0.125 \cdot k}\right) \cdot z^{-k}\right)$$
$$= 4 \cdot \left[\sum_{k=0}^{\infty} \left(z^{-k}\right) - \sum_{k=0}^{\infty} \left(e^{-0.125 \cdot k} \cdot z^{-k}\right)\right]$$
$$= 4 \cdot \left[\sum_{k=0}^{\infty} \left(z^{-1}\right)^k - \sum_{k=0}^{\infty} \left(e^{-0.125} \cdot z^{-1}\right)^k\right] \ .$$ $\boxed{3}$

Using the equation of a geometric series $\sum\limits_{k=0}^{\infty} x^k = \frac{1}{1-x}$ leads to:

$$\sum_{k=0}^{\infty} \left(z^{-1}\right)^k = \frac{1}{1 - z^{-1}}$$
$$\sum_{k=0}^{\infty} \left(e^{-0.125} \cdot z^{-1}\right)^k = \frac{1}{1 - e^{-0.125} \cdot z^{-1}}$$
$$\Rightarrow H_S(z) = 4 \cdot \left[\frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-0.125} \cdot z^{-1}}\right] \ .$$ $\boxed{2}$

Next step is converting the transfer function of the step response $H_S(z)$ into the transfer function $G_S(z)$.

$$\begin{aligned}
H_S(z) &= G_S(z) \cdot \underbrace{\frac{1}{1-z^{-1}}}_{\text{step } \sigma(k)} \\
G_S(z) &= H_S(z) \cdot \left(1 - z^{-1}\right) \\
G_S(z) &= 4 \cdot \left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-0.125} \cdot z^{-1}}\right) \cdot \left(1 - z^{-1}\right) \\
G_S(z) &= 4 \cdot \frac{\cancel{1} - e^{-0.125} \cdot z^{-1} \cancel{-1} + z^{-1}}{1-e^{-0.125} \cdot z^{-1}} \\
G_S(z) &= 4 \cdot \frac{\left(1-e^{-0.125}\right) \cdot z^{-1}}{1-e^{-0.125} \cdot z^{-1}} \ .
\end{aligned}$$
[4]

c) Similar to c) the first step is the discretization of the time continuous impulse response with the sampling time $T_0 = 0.25$ sec:

$$g(k) = g(t = T_0 \cdot k) = \frac{K}{T} \cdot e^{-\frac{T_0 \cdot k}{T}} = \frac{4}{2} \cdot e^{-\frac{0{,}25\text{sec} \cdot k}{2\text{sec}}} = 2 \cdot e^{-0.125k} \ .$$
[1]

Used in the general equation of the z-transformation results in $G(z)$:

$$\begin{aligned}
G_I(z) &= \sum_{k=0}^{\infty} \left(g(k) \cdot z^{-k}\right) = \sum_{k=0}^{\infty} \left(2 \cdot e^{-0.125k} \cdot z^{-k}\right) = 2 \cdot \sum_{k=0}^{\infty} \left(e^{-0.125k} \cdot z^{-k}\right) \\
&= 2 \cdot \sum_{k=0}^{\infty} \left(e^{-0.125} \cdot z^{-1}\right)^k \ .
\end{aligned}$$
[2]

Using the equation of a geometric series $\sum\limits_{k=0}^{\infty} x^k = \frac{1}{1-x}$ leads to:

$$G_I(z) = 2 \cdot \sum_{k=0}^{\infty} \left(e^{-0.125} \cdot z^{-1}\right)^k = \frac{2}{1-e^{-0.125} \cdot z^{-1}} \ .$$
[1]

d) Comparison of the two discrete systems:

1) different gains : $h(k \to \infty) = G(z = 1)$

$$\begin{aligned}
G_I(z=1) &= \frac{2}{1-e^{-0.125}} = 17.02 \\
G_S(z=1) &= 4 \cdot \frac{1-e^{-0.125}}{1-e^{-0.125}} = 4 \ .
\end{aligned}$$
[2]

2) properness :

$$\begin{aligned}
G_I(z) &= \frac{2}{1-e^{-0.125} \cdot z^{-1}} \\
\Leftrightarrow Y(z) - e^{-0.125} \cdot z^{-1} \cdot Y(z) &= 2 \cdot U(z)
\end{aligned}$$

$$\bullet\!\!-\!\!\circ$$

$$y(k) - e^{-0.125} \cdot y(k-1) = 2 \cdot u(k) \ .$$

$\Rightarrow$ The system is proper. $\boxed{1}$

$$
\begin{aligned}
G_S(z) &= 4 \cdot \frac{\left(1 - e^{-0.125}\right) \cdot z^{-1}}{1 - e^{-0.125} \cdot z^{-1}} \\
\Leftrightarrow Y(z) - e^{-0.125} \cdot z^{-1} \cdot Y(z) &= 4 \cdot \left(1 - e^{-0.125}\right) \cdot z^{-1} \cdot U(z) \\
&\bullet\!\!-\!\!\circ \\
y(k) - e^{-0.125} \cdot y(k-1) &= 4 \cdot \left(1 - e^{-0.125}\right) \cdot u(k-1) \, .
\end{aligned}
$$

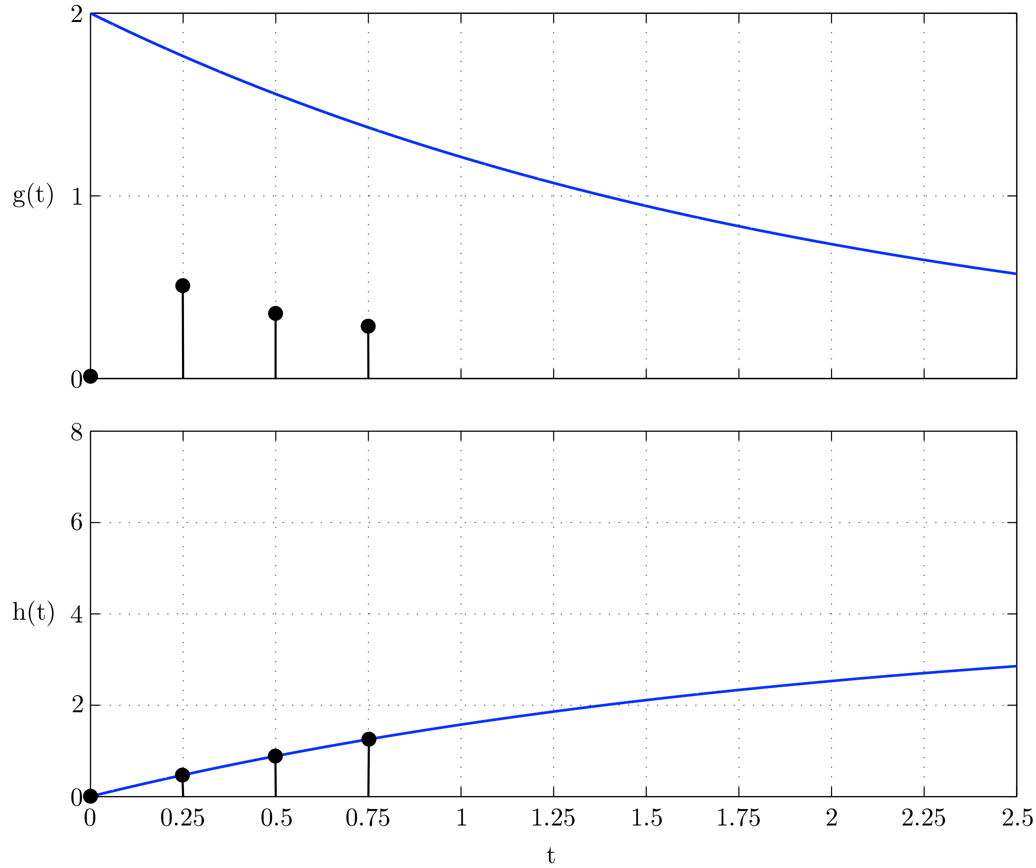
$\Rightarrow$ The system is strictly proper. $\boxed{1}$

3) stability:

$$
\begin{aligned}
G_I(z) &= \frac{2}{1 - e^{-0.125} \cdot z^{-1}} = \frac{2 \cdot z}{z - e^{-0.125}} \\
G_S(z) &= 4 \cdot \frac{\left(1 - e^{-0.125}\right) \cdot z^{-1}}{1 - e^{-0.125} \cdot z^{-1}} = \frac{4 \cdot \left(1 - e^{-0.125}\right)}{z - e^{-0.125}} \, .
\end{aligned}
$$
$\boxed{2}$

$\Rightarrow$ both systems have the same pole at $z = e^{-0.125} \approx 0.8825 < 1$ and therefore both are stable.

e) The impulse and step response for the system $G_S(z)$ is:



$\boxed{2}$

$\boxed{2}$

The step response of the system $G_S(z)$ has the same values on the discrete points as the continuous step response of the system. To get the value of the impulse response it is necessary to calculate the first points k=0,1,2,3 of the difference equation of the system $G_S(z)$

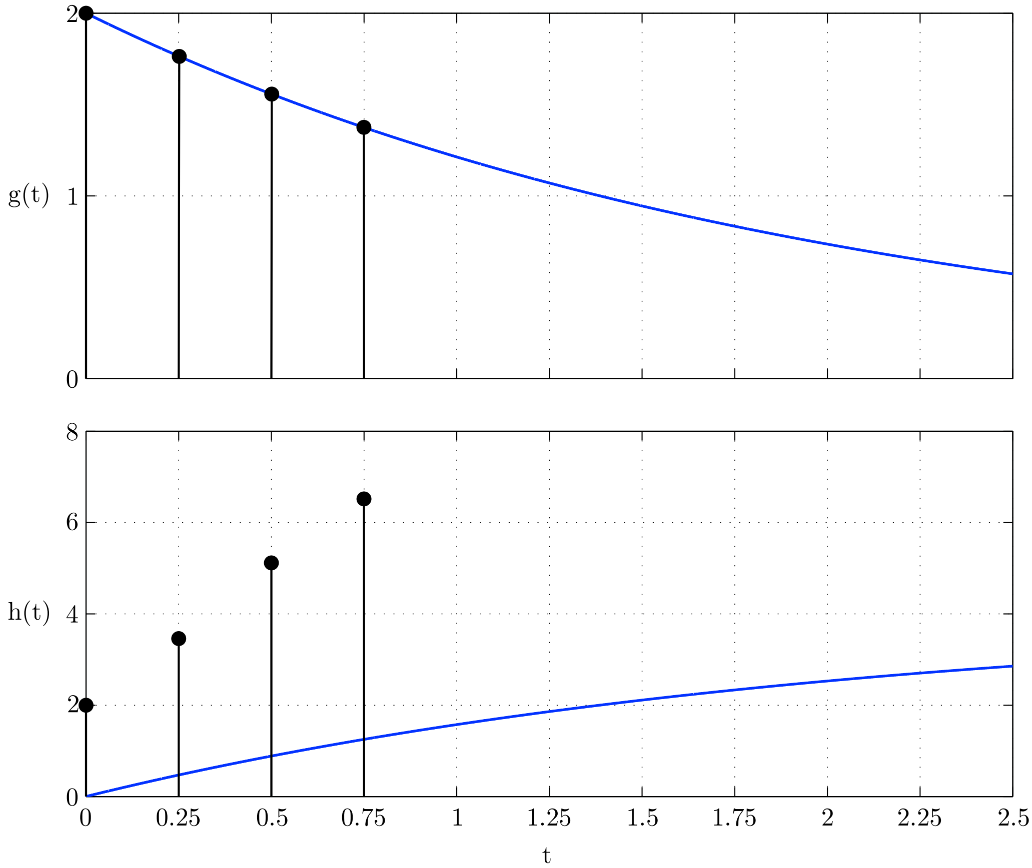
$$G_S(z) = 4 \cdot \frac{(1 - e^{-0.125}) \cdot z^{-1}}{1 - e^{-0.125} \cdot z^{-1}}$$

$$y(k) = 4 \cdot \left(1 - e^{-0.125}\right) \cdot u(k-1) + e^{-0.125} \cdot y(k-1) \; .$$

with $u(k) = \delta_K(k)$ and $u(k-1) = y(k-1) = 0$:

$$\begin{aligned}
g(0) &= 4 \cdot \left(1 - e^{-0.125}\right) \cdot 0 + e^{-0.125} \cdot 0 = 0 \\
g(1) &= 4 \cdot \left(1 - e^{-0.125}\right) \cdot 1 + e^{-0.125} \cdot 0 = 4 \cdot \left(1 - e^{-0.125}\right) = 0.4700 \\
g(2) &= 4 \cdot \left(1 - e^{-0.125}\right) \cdot 0 + e^{-0.125} \cdot 4 \cdot \left(1 - e^{-0.125}\right) = e^{-0.125} \cdot 4 \cdot \left(1 - e^{-0.125}\right) = 0.4148 \\
g(3) &= 0 + e^{-0.125} \cdot e^{-0.125} \cdot 4 \cdot \left(1 - e^{-0.125}\right) = e^{-0.125 \cdot 2} \cdot 4 \cdot \left(1 - e^{-0.125}\right) = 0.3660 \; .
\end{aligned}$$

f) The impulse and step response for the system $G_I(z)$ is:



$\boxed{2}$

$\boxed{2}$

The impulse response of the system $G_I(z)$ has the same values on the discrete points as the continuous impulse response of the system. To get the value of the step response it is necessary to calculate the first points k=0,1,2,3 of the difference equation of the system $G_I(z)$

$$G_I(z) = \frac{2}{1 - e^{-0.125} \cdot z^{-1}}$$

$$y(k) = 2 \cdot u(k) + e^{-0.125} \cdot y(k-1) \; .$$

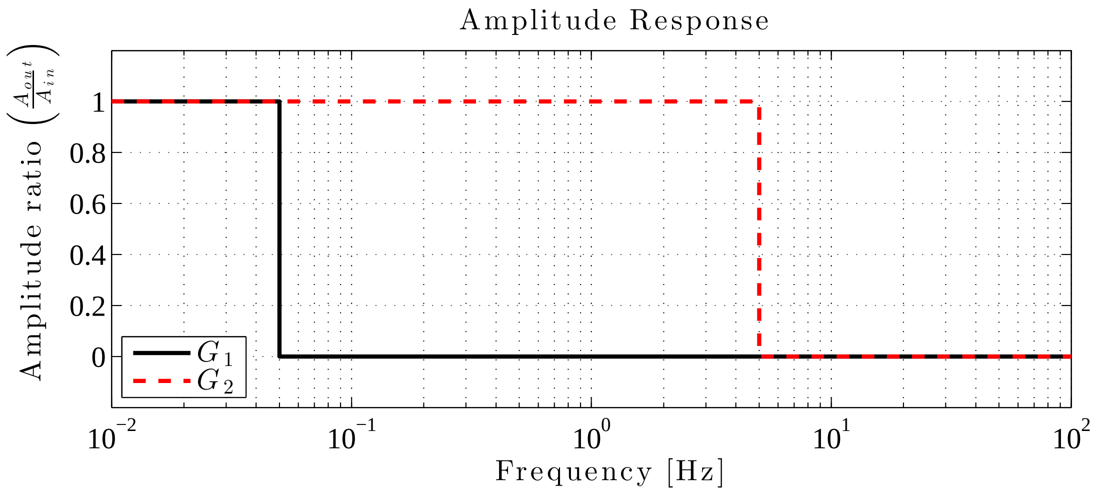
with $u(k) = \sigma(k)$ and $y(-1) = 0$:

$$\begin{aligned}
h(0) &= 2 \cdot 1 + e^{-0.125} \cdot 0 = 2 \\
h(1) &= 2 \cdot 1 + e^{-0.125} \cdot 2 = 2 \cdot \left(1 + e^{-0.125}\right) = 3.765 \\
h(2) &= 2 + e^{-0.125} \cdot 4 \cdot \left(1 + e^{-0.125}\right) = 2 \cdot \left(1 + e^{-0.125} + e^{-0.125 \cdot 2}\right) = 5.3226 \\
h(3) &= 2 + e^{-0.125} \cdot 4 \cdot \left(1 + e^{-0.125} + e^{-0.125 \cdot 2}\right) \\
&= 4 \cdot \left(1 + e^{-0.125} + e^{-0.125 \cdot 2} + e^{-0.125 \cdot 3}\right) = 6.6972 \; .
\end{aligned}$$

$\boxed{\sum 30}$

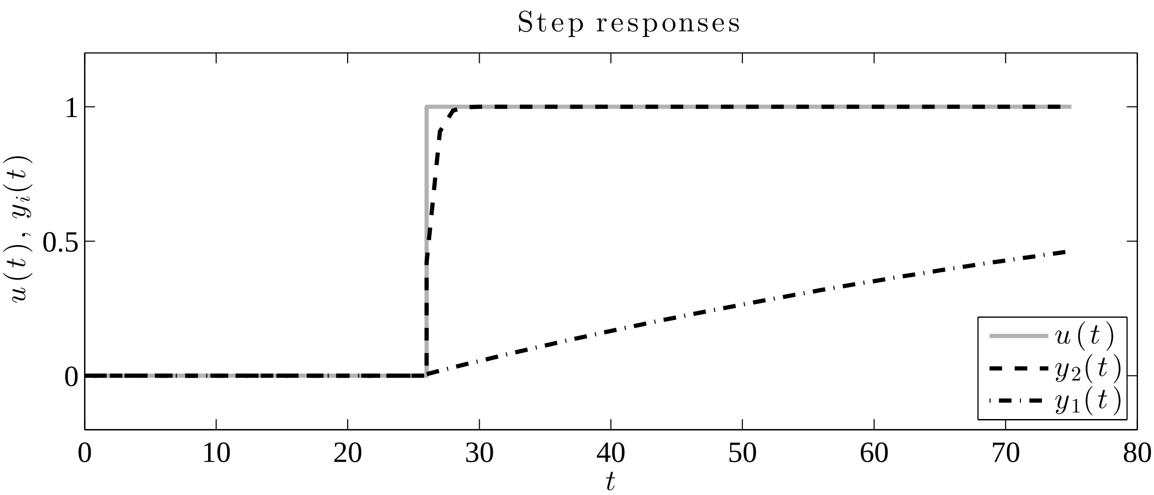
Task 6: Filter

a) Sketch the idealized amplitude responses of both filters G_1 and G_2 in the figure below.



4

b) Label the two step responses correctly.

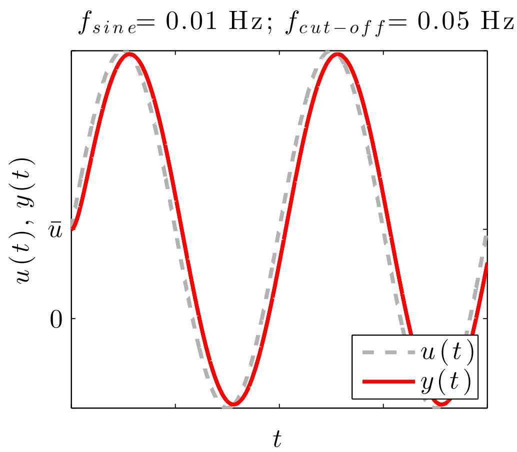


2

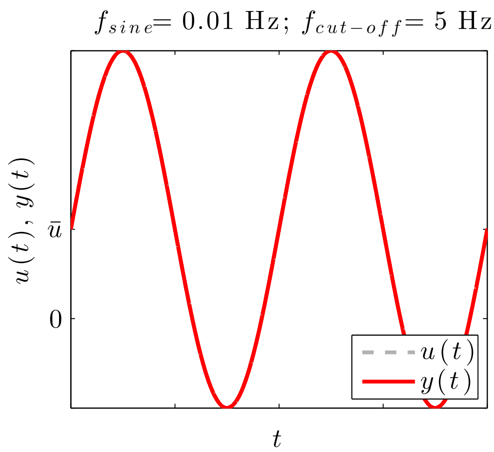
c) Sketch the filtered outputs qualitatively.
The shown solution below is the exact result, if a first order Butterworth filter is used. Notice that you get the full score, if the output is either a completely horizontal line or lays exactly over the input signal depending on the specific case.

6

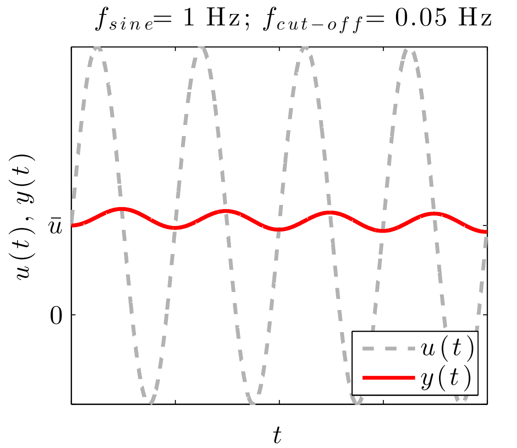
$\sum$ 12



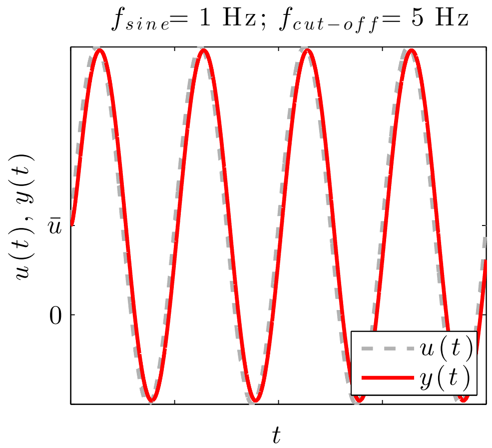
(a) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_1.



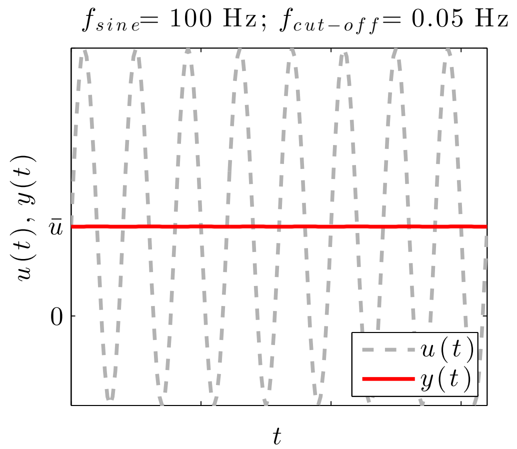
(b) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_2.



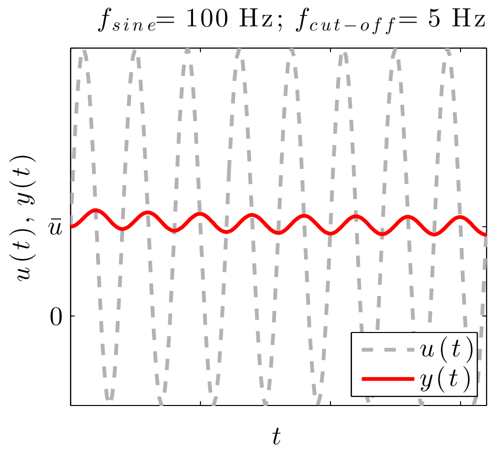
(c) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_1.



(d) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_2.



(e) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_1.



(f) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_2.