# Sensorics Exam

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01.04.2021

| | |
|---|---|
| Name: | |
| Mat.-No.: | |
| Grade: | |

| Task | T1 | T2 | T3 | T4 | T5 | T6 | T7 | Sum |
|---|---|---|---|---|---|---|---|---|
| Score: | 19 | 17 | 22 | 17 | 17 | 14 | 14 | 120 |
| Accomplished: | | | | | | | | |

Exam duration: 2 hours

Permitted Tools: Pocket Calculator and 4 pages of formula collection

## Task 1: Comprehension Questions (19 Points)

**Every question has one, two or three correct answers!**
Write down the numbers of the correct answers (1: first answer, 2: second answer ...). For every correct answer you will get one point. For every wrong answer a point will be subtracted, but a question will never be rated with negative points.

a) How can speed be measured?

- ☐ By the measurement of the rotational speed of a wheel and a subsequent multiplication by its radius.
- ☐ By the measurement of the acceleration and a subsequent differentiation.
- ☐ By the measurement of the acceleration and subsequent double integration.
- ☐ Utilizing the Doppler-effect.

b) Which statements are true regarding filters?

- ☐ A first type Chebyshev filter of 4th order has a phase shift of -180° for $\omega \to \infty$.
- ☐ Butterworth filters have a considerable amplitude response ripples in the passband.
- ☐ For identical orders, a Cauer/Elliptic filter has a steeper transition band than a Bessel, Butterworth or Chebyshev filter.
- ☐ Even though the Cauer/Elliptic filter has a smoother amplitude response than the Chebyshev filter, the step response of a Cheychef filter (of same order) oscillates stronger.

c) The discrete Fourier transform is periodic ...

- ☐ ... only in frequency.
- ☐ ... only in time.
- ☐ ... in time and frequency.
- ☐ ... neither in frequency nor time.

d) Which statements are true regarding moving average filters?

- ☐ Moving average filters are FIR filters.
- ☐ Moving average filters are IIR filters.
- ☐ Moving average filters are always stable.
- ☐ The output of moving average filters is usually calculated by recursion.

e) Which statements are true regarding aliasing?

- ☐ Increasing the sampling frequency is a common way to deal with aliasing effects.
- ☐ To avoid aliasing, a signal has to be sampled at least at twice the frequency of the highest frequency contained in the signal.
- ☐ To avoid aliasing, a signal has to be sampled at least at half the frequency of the highest frequency contained in the signal.
- ☐ If a periodic signal is sampled at the same rate as it repeats, it appears to be a constant.

f) A temporal sequence of $N$ measurements that is transformed with the DFT results in a number of ...

- [ ] ... $N/2$ discrete frequencies.
- [ ] ... $2 \cdot N$ discrete frequencies.
- [ ] ... $N$ discrete frequencies.
- [ ] ... $2^N$ discrete frequencies.

g) Torque ...

- [ ] ... and energy are both calculated by multiplication of force and length.
- [ ] ... and energy are the same.
- [ ] ... is calculated by multiplying a force with a length parallel to the force.
- [ ] ... is calculated by multiplying a force with a length perpendicular to the force.

h) Which statements are true regarding clustering?

- [ ] Clustering tries to find data points that are grouped together.
- [ ] Clustering is an **unsupervised** learning method.
- [ ] The number of clusters has not to be known in advance, but is determined through the clustering algorithm.
- [ ] Different clustering algorithms exist, that differ in how the distances of data points from the cluster center are determined.

i) Non-causal filters ...

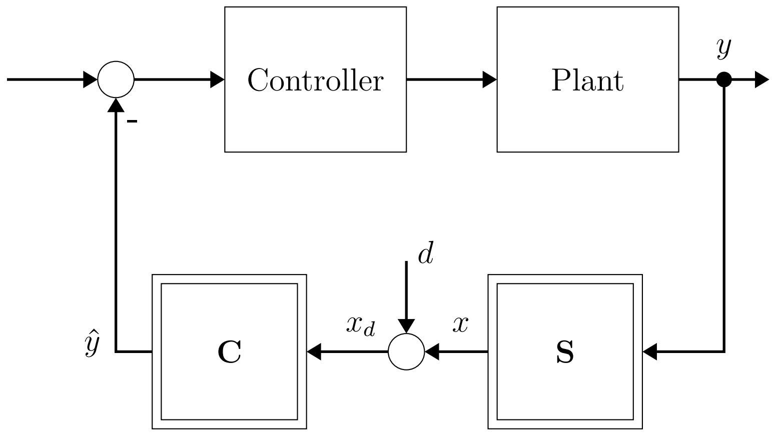
- [ ] ... are only defined in the discrete time domain.
- [ ] ... require only current and previous input values.
- [ ] ... deliver different results each time they are applied to the same data.
- [ ] ... require future input values.

j) PTC resistance thermometers ...

- [ ] ... react slower to temperature change than thermocouples.
- [ ] ... are made from polymer titanium ceramics, hence the acronym PTC.
- [ ] ... have an increasing resistance with increasing temperature.
- [ ] ... have almost a linear characteristic.

## Task 2: Compensator (17 Points)

In order to control a plant, the output of the system has to be measured. For many sensors (e.g. Thermocouples) the measurement of the output results in a voltage value. From this voltage value, the physical value (e.g. a temperature) can be calculated.
In this example we have a sensor in the feedback loop which measures the output $y$ of the plant. The output of the sensor $S$ is $x$. This value can suffer from additional disturbance $d$. $x_d$ is then fed into the compensator $C$. The Compenstor $C$ does not compensate the added noise $d$ it just reconstructs $y$ by $\hat{y}$ from $x_d$ with the inverse characteristic from $S$. So with $d = 0$ the statement $\hat{y} = y$ is valid.




The equation of the sensor is

$$f_S(y) = \sqrt{2e^{0.5y} - 2} = x$$

and it is only operating in the positive range $y > 0$.

a) Find the equation for the Compensator $f_C(x_d) = \hat{y}$.
*Hint: Leave the disturbance $d$ out of the equation.*

b) Find the sensitivity of the Compensator $\dfrac{d\hat{y}}{dx_d}$.

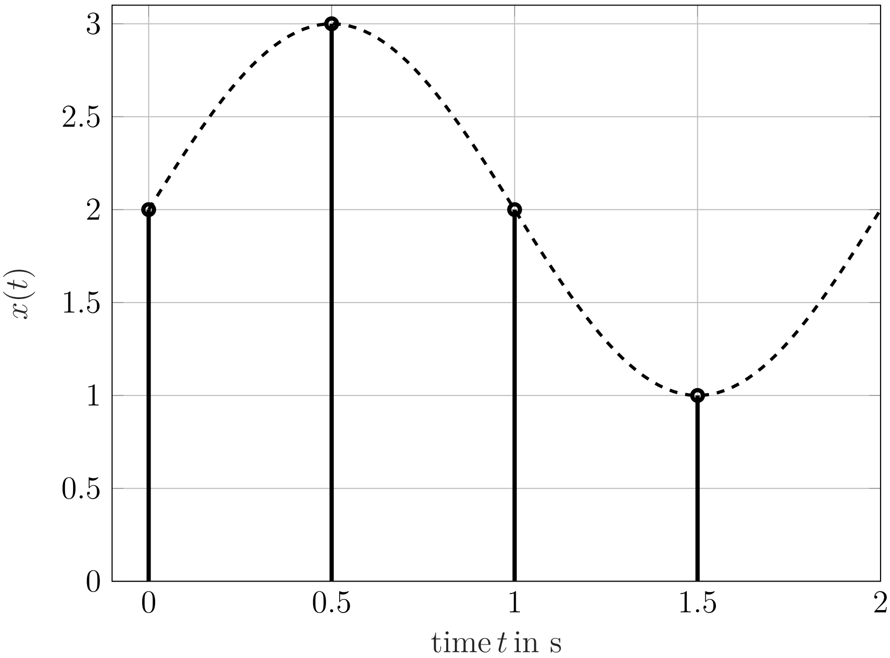
c) At which value of $x_d = x_{set}$ the sensitivity is the highest? Provide your answer with a calculation or explanation.

d) We now have systematic disturbance of $d = 0.1$. What are the values for $\hat{y}$, $x_d$ and $y$ if $x = x_{set}$?

e) What is the relative error $e_{rel}$ and what is the absolute error $e_{abs}$ with the disturbance $d = 0.5$ if $x = x_{set}$ in your measurement of $y$?

**Task 3: Discrete Fourier Transformation (22 Points)**

In the following figure you can see a sinusoidal signal $x(t)$ in continous time and its sampled version $x(k)$ for $k = 0, 1, 2, 3$:




a) What is the frequency $f_{\mathrm{x}}$ of the signal $x(t)$? What is the sampling frequency $f_{\mathrm{s}}$?

b) Calculate the matrix $\underline{W}_N$ with the phasors of the discrete frequencies $W_N^{nk}$ for the equation $\underline{X}(n) = \underline{W}_N\, \underline{x}(k)$.
Hint: $\mathrm{DFT}\{x(k)\} = \underline{X}(n) = \sum_{k=0}^{N-1} x(k)\mathrm{e}^{-i2\pi nk/N} = \sum_{k=0}^{N-1} x(k) W_N^{nk}$
$W_N = \mathrm{e}^{-i2\pi/N}$

c) Calculate $\underline{X}(n)$.

d) Plot the amplitudes of $X(n)$ in a diagram as a function of the discrete frequencies.

e) Draw he phasors of the discrete frequencies $W_N^{nk}$ from part **b)** in the Gaussian plane (complex plane).

f) How does the computational workload rise, if you double the sampling frequency $f_{\mathrm{s}}$? What method is there to reduce the computational workload without changing the sampling frequency? How does it work?

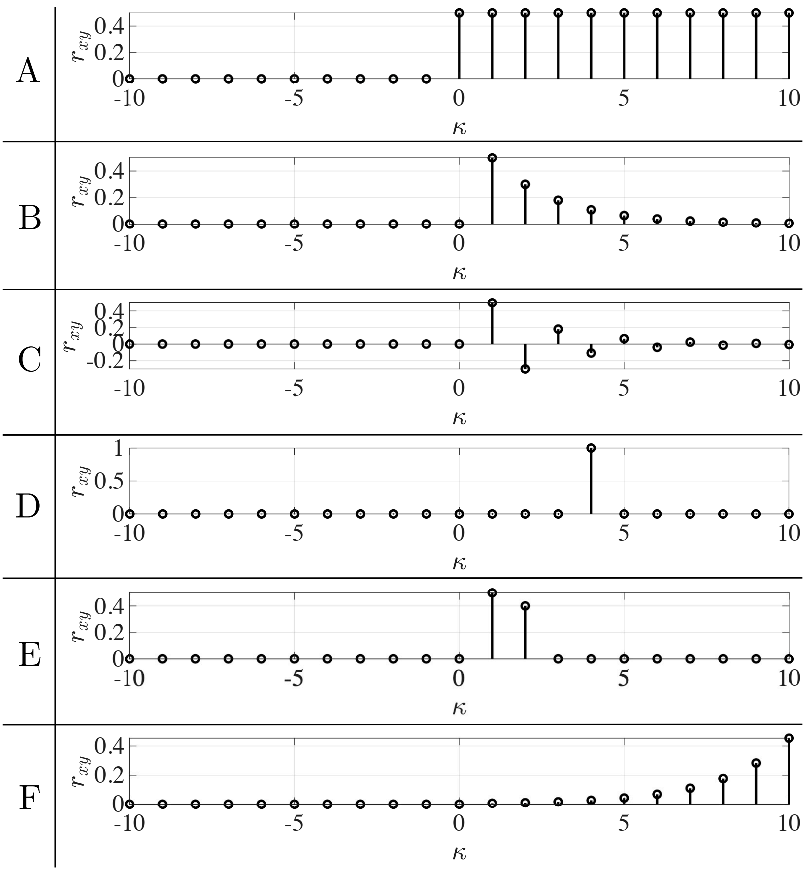
g) Plot the amplitudes of $X(n)$ in a diagram as a function of the discrete frequencies, when the frequency $f_{\mathrm{x}}$ of the signal $x(t)$ is doubled. The sampling frequency and time interval from part **a)** remain.

## <u>Task 4:</u> Correlation Function (17 Points)

a) Three systems are given

$$G_1(z) = \frac{0.5z^{-1}}{1 - 0.6z^{-1}}$$
$$G_2(z) = 0.5z^{-1} + 0.4z^{-2}$$
$$G_3(z) = z^{-4}.$$

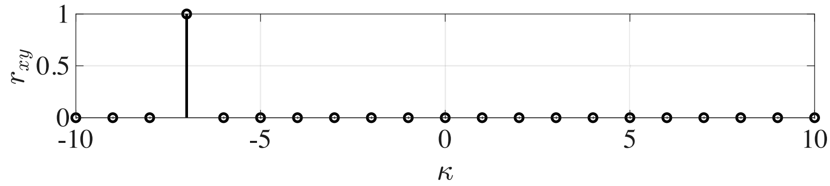
The systems were excited with an Kronecker delta $x(k)$ and the resulting output $y(k)$ was recorded. Subsequently, the cross-correlation function $r_{xy}(\kappa)$ was calculated. Assign the correct cross-correlation functions A,B,...,F to the right system $G_1, G_2, G_3$.




b) Another cross-correlation function H is given (figure shown on the next page), calculated exactly as in the previous task. What is the transfer function of the underlying system?

c) How does the system from task b) differ from the systems in task a)?

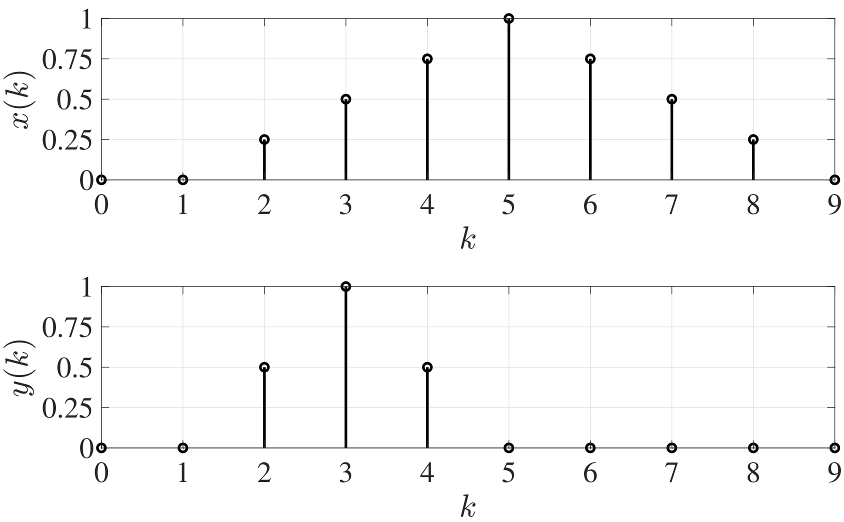
H

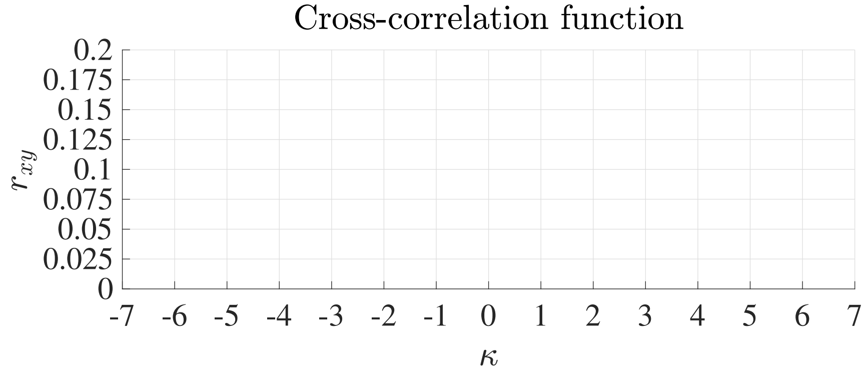



d) Is it possible to represent a real physical system with the transfer function from task b)?

e) The time-discrete signals $x(k)$ and $y(k)$ are given. Calculate the cross-correlation function $r_{xy}$ for $\kappa = -7, -6, ..., 7$ and sketch the cross-correlation function either in the given diagram or on your exam sheet.
Hint:

$$r_{xy}(\kappa) = \begin{cases} \frac{1}{N} \sum\limits_{k=0}^{N-1-\kappa} x(k) \cdot y(k+\kappa), & \text{for } \kappa \geq 0 \\ \frac{1}{N} \sum\limits_{k=-\kappa}^{N-1} x(k) \cdot y(k+\kappa), & \text{else.} \end{cases}$$



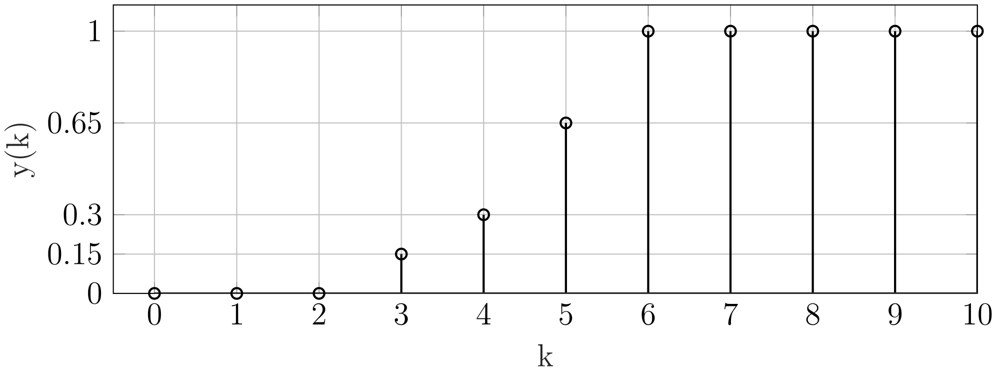
**Task 5:** **FIR and IIR (17 Points)**

The step responses of different filters are given in the following figures. The corresponding step is defined as

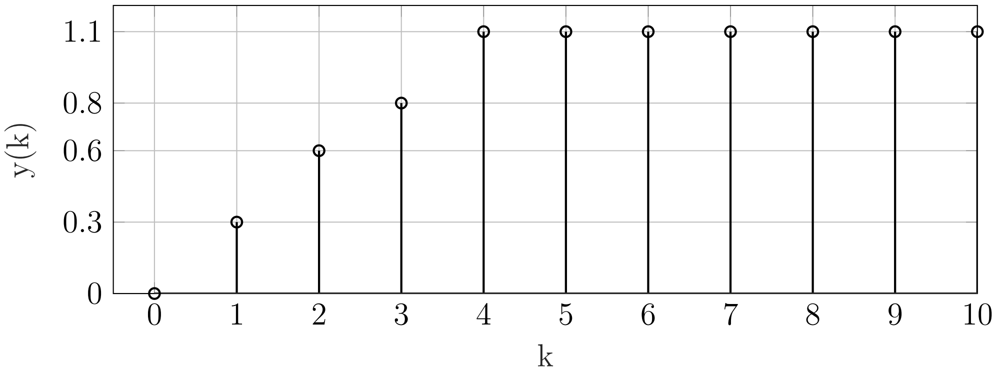
$$u(k) = \begin{cases} 0, & \text{if } k < 2 \\ 1, & \text{if } k \geq 2 \end{cases}.$$

For each task specify a transfer function $G(z)$.

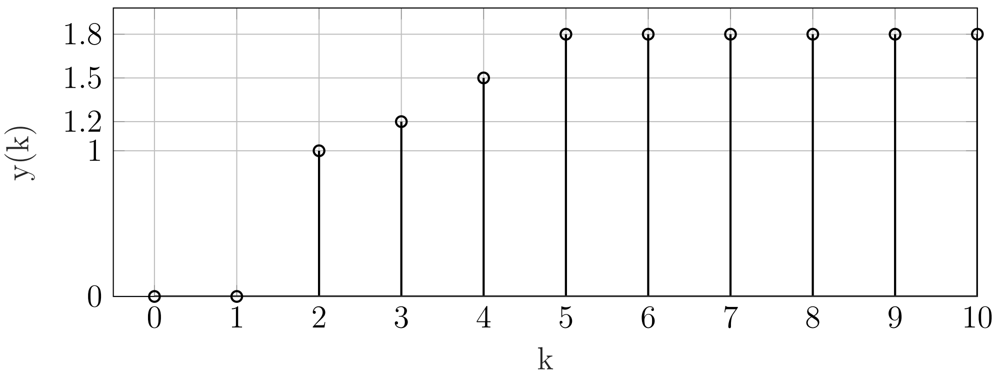
a) Filter 1



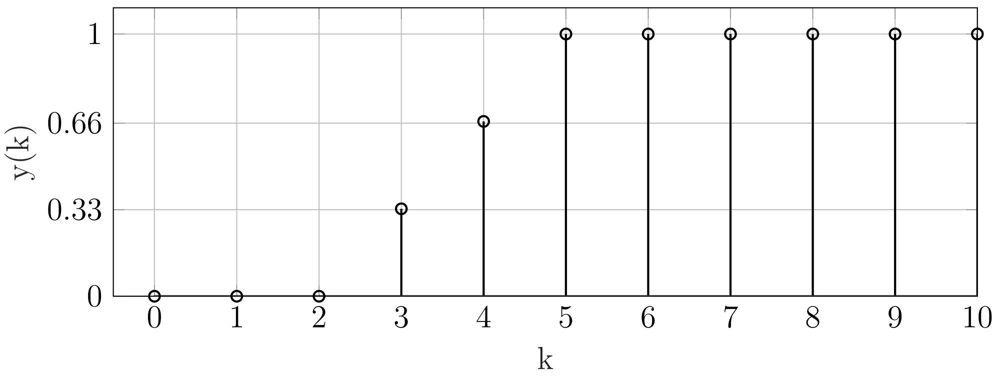

b) Filter 2



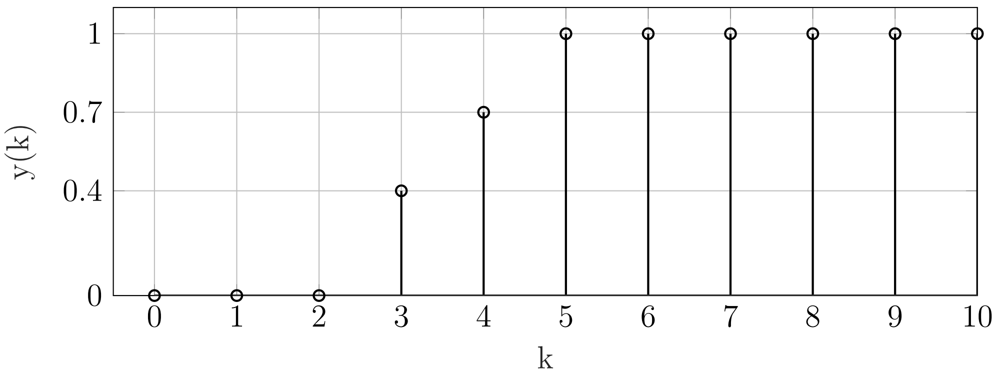

c) Filter 3



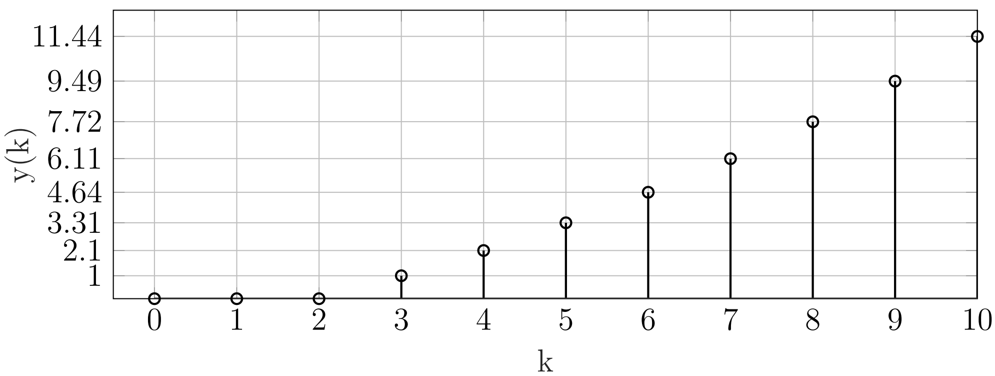
d) Filter 4




e) Filter 5




f) Filter 6




g) Please copy the following table and mark the properties of each filter.

| Filter | FIR | Non-causal | Linear phase | Instable | Gain = 1 |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |
| 6 | | | | | |

## Task 6: Phase of a Filter (14 Points)

a) The following system is given

$$G(z) = \frac{1 + z^{-1}}{2}.$$

Calculate the filter $F(z)$ according to

$$F(z) = G(z) \cdot G(z^{-1}).$$

b) Calculate the phase of the previously determined filter $F(z)$.
*Hint: Set $z = e^{i\omega T_0}$*

c) If $F(z)$ is non-causal, modify $F(z)$ such that it is causal and proper.

d) How is a filter with the previously calculated phase called?

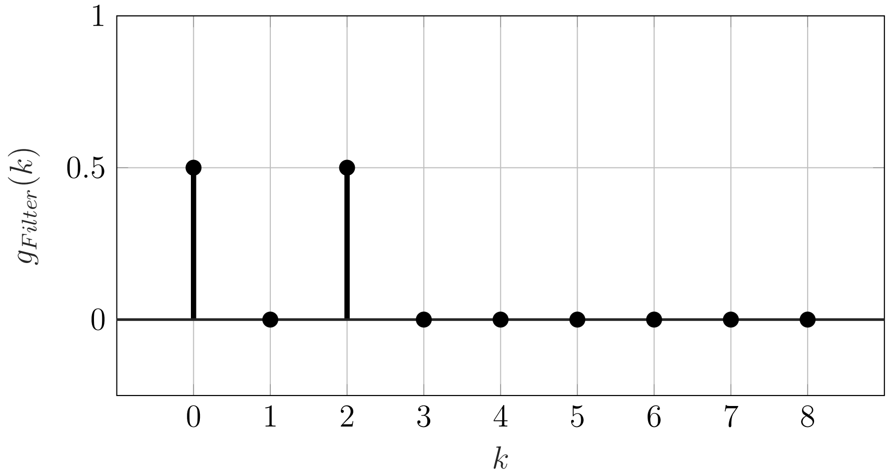
e) What are advantages of such filters?

f) Is it possible to define an IIR filter with linear phase?

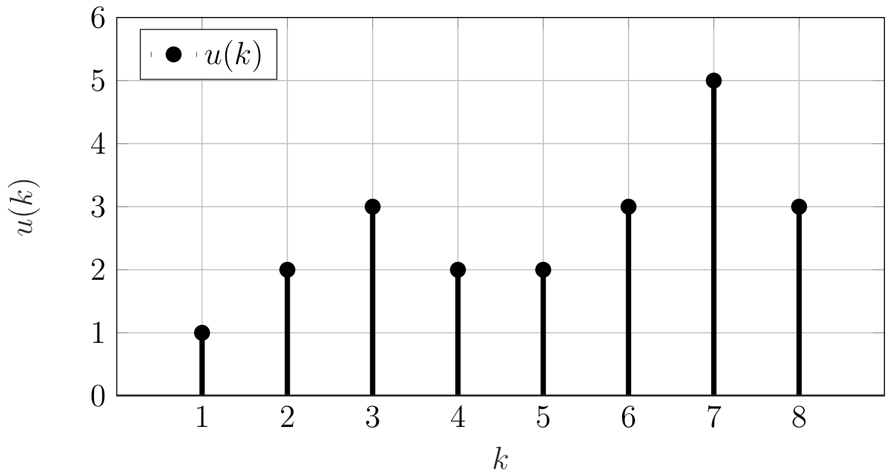
g) Which IIR filter should be used, if a linear phase is required?

## Task 7: Linear filter (14 Points)

The following figure shows the impulse response of a linear filter $g_{Filter}(k)$:




This filter can be used to filter the input signal $u(k)$. For this task assume the following sequence for the input signal $u(k)$.




a) The input signal $u(k)$ is now to be filtered. Calculate the value sequence $y(k)$ of the filter output for $k = 1, 2, \ldots, 8$. Assume $u(k) = 0$ for all $k < 1$.

b) Calculate the transfer function of the filter $G_{Filter}(z)$.

c) Which structure of linear filters is the subject of investigation?

d) Draw the block diagram of the filter.

# Solution:

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f) A temporal sequence of $N$ measurements that is transformed with the DFT results in a number of . . .

- ☐ . . . $N/2$ discrete frequencies.
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- ■ . . . $N$ discrete frequencies.
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h) Which statements are true regarding clustering?

- ■ Clustering tries to find data points that are grouped together.
- ■ Clustering is an **unsupervised** learning method.
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i) Non-causal filters . . .

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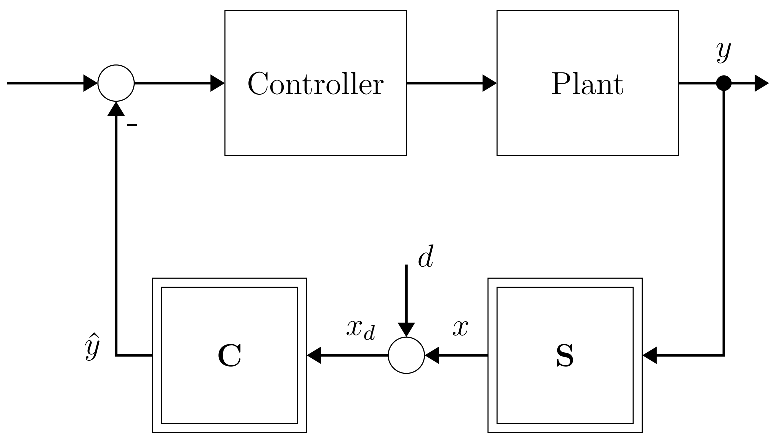
## Task 2: Compensator (17 Points)

In order to control a plant, the output of the system has to be measured. For many sensors (e.g. Thermocouples) the measurement of the output results in a voltage value. From this voltage value, the physical value (e.g. a temperature) can be calculated.
In this example we have a sensor in the feedback loop which measures the output $y$ of the plant. The output of the sensor $S$ is $x$. This value can suffer from additional disturbance $d$. $x_d$ is then fed into the compensator $C$. The Compenstor $C$ does not compensate the added noise $d$ it just reconstructs $y$ by $\hat{y}$ from $x_d$ with the inverse characteristic from $S$. So with $d = 0$ the statement $\hat{y} = y$ is valid.




The equation of the sensor is

$$f_S(y) = \sqrt{2e^{0.5y} - 2} = x$$

and it is only operating in the positive range $y > 0$.

a) Find the equation for the Compensator $f_C(x_d) = \hat{y}$.
*Hint: Leave the disturbance d out of the equation.*
**Answer**: $f_C(x_d) = \hat{y} = 2\ln(0.5x^2 + 1)$ 3

b) Find the sensitivity of the Compensator $\frac{d\hat{y}}{dx_d}$.
**Answer**: $\frac{d\hat{y}}{dx_d} = \frac{4x_d}{x_d^2 + 2}$ 2

c) At which value of $x_d = x_{set}$ the sensitivity is the highest? Provide your answer with a calculation or explanation.
**Answer**:
One way to find the maximum: Take the derivative and set it to 0.

$$\frac{d^2\hat{y}}{dx^2} = \frac{-4x^2 + 8}{(x^2+2)^2} = 0$$
$$x^2 = 2$$
$$x_{d1} = \sqrt{2}; x_{d2} = -\sqrt{2}$$

The maximum must be $x_{d1} = \sqrt{2} = x_{set}$ since the operating range is for positive values of $y$ which also means, that all values of $x$ must be positive. 6

d) We now have systematic disturbance of $d = 0.1$. What are the values for $\hat{y}$, $x_d$ and $y$ if $x = x_{set}$?
**Answer**:

$$\begin{aligned} x_d &= 1.51 \\ \hat{y} &= 1.53 \\ y &= 1.386 \end{aligned}$$

3

e) What is the relative error $e_{rel}$ and what is the absolute error $e_{abs}$ with the disturbance $d = 0.5$ if $x = x_{set}$ in your measurement of $y$?
**Answer**:

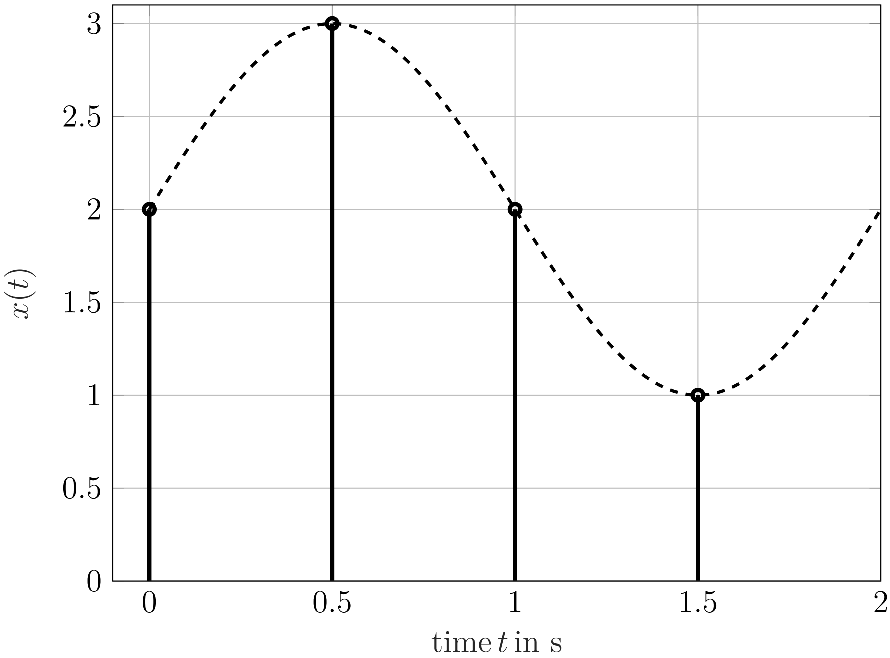
$$\begin{aligned} y &= 1.386 \\ x_d &= \sqrt{2} + 0.5 = 1.91 \\ \hat{y} &= \ln\left(0.5 \cdot (1.91)^2 + 1\right) = 2.08 \\ e_{abs} &= |\hat{y} - y| = 0.70 \\ e_{rel} &= \frac{e_{abs}}{y} = 0.50 \end{aligned}$$

3

$\sum 17$

## Task 3: Discrete Fourier Transformation (22 Points)

In the following figure you can see a sinusoidal signal $x(t)$ in continous time and its sampled version $x(k)$ for $k = 0, 1, 2, 3$:




a) What is the frequency $f_\text{x}$ of the signal $x(t)$? What is the sampling frequency $f_\text{s}$?
**Answer**: $f_x = 0.5\,\text{Hz}$ and $f_s = 2\,\text{Hz}$. 2

b) Calculate the matrix $\underline{W}_N$ with the phasors of the discrete frequencies $W_N^{nk}$ for the equation $\underline{X}(n) = \underline{W}_N\, \underline{x}(k)$.
Hint: $\text{DFT}\{x(k)\} = \underline{X}(n) = \sum_{k=0}^{N-1} x(k)\mathrm{e}^{-i2\pi nk/N} = \sum_{k=0}^{N-1} x(k) W_N^{nk}$
$W_N = \mathrm{e}^{-i2\pi/N}$
**Answer**:

$$\underline{W}_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \tag{1}$$

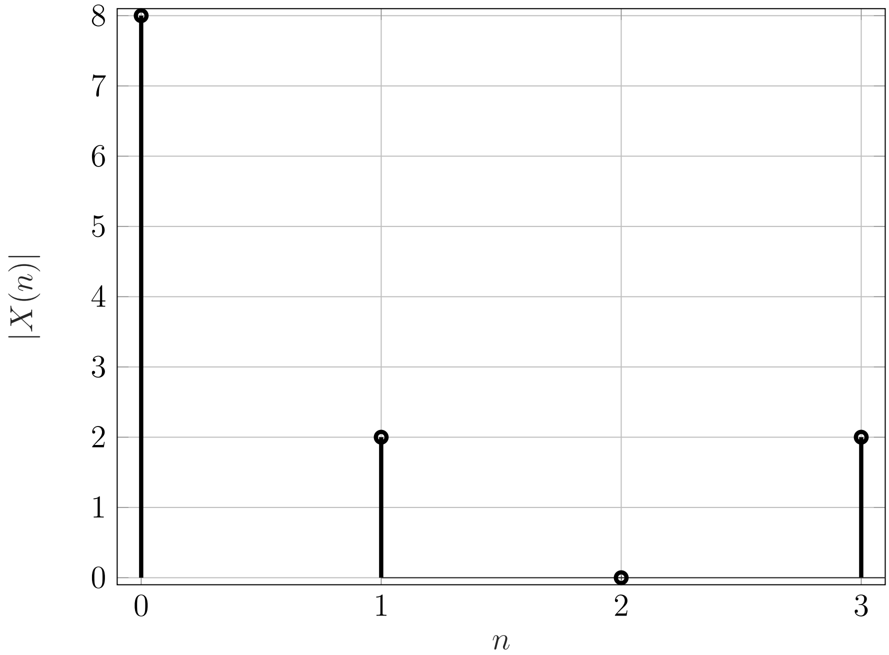
4

c) Calculate $\underline{X}(n)$.
**Answer**:

$$\underline{X}(n) = \underline{W}_N\, \underline{x}(k) = \begin{bmatrix} 8 \\ -2i \\ 0 \\ 2i \end{bmatrix} \tag{2}$$

2

d) Plot the amplitudes of $X(n)$ in a diagram as a function of the discrete frequencies.
**Answer**:

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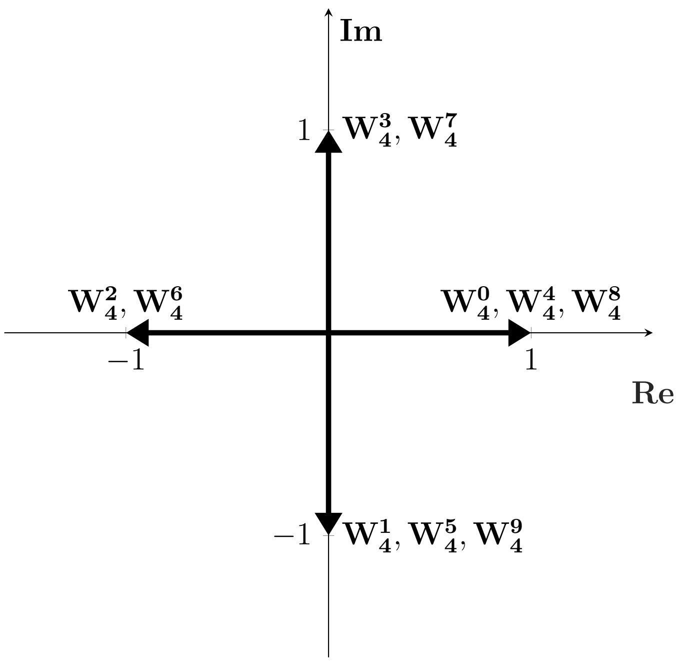
e) Draw he phasors of the discrete frequencies $W_N^{nk}$ from part **b)** in the Gaussian plane (complex plane).
**Answer**:

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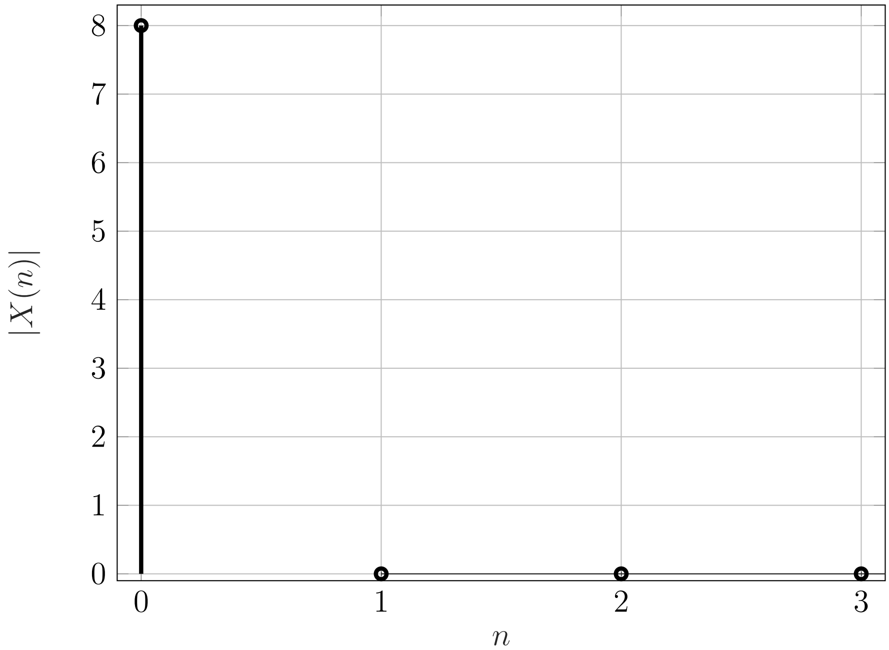
f) How does the computational workload rise, if you double the sampling frequency $f_s$? What method is there to reduce the computational workload without changing the sampling frequency? How does it work?
**Answer**: The entries in $\underline{W}_N$ rise by the factor 4, if the amount of samples is doubled. The computational workload rises almost quadratic with the number of samples.
Method: FFT
In the FFT, the data is enhanced with zero entries, until the number of datapoints is equal to a power of 2. This is called zero padding. Then the data is split into 2

sets of equal size for a DFT of 2 smaller sets. These sets are again divided each into 2 subsets etc... until $N/2^s = 1$ is reached (s = the number of datasplits). Finally the subsets are used for DFTs which are much faster calculated than a DFT of the original dataset. 3

g) Plot the amplitudes of $X(n)$ in a diagram as a function of the discrete frequencies, when the frequency $f_\mathrm{x}$ of the signal $x(t)$ is doubled. The sampling frequency and time interval from part **a)** remain.
**Answer**:
By doubling the frequency of the signal, all measurements measure the mean of the sinusoidal function. Therefore, the signal is indistinguishable from a constant. The sampling theorem is violated.

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## Task 4: Correlation Function (17 Points)

a) Three systems are given

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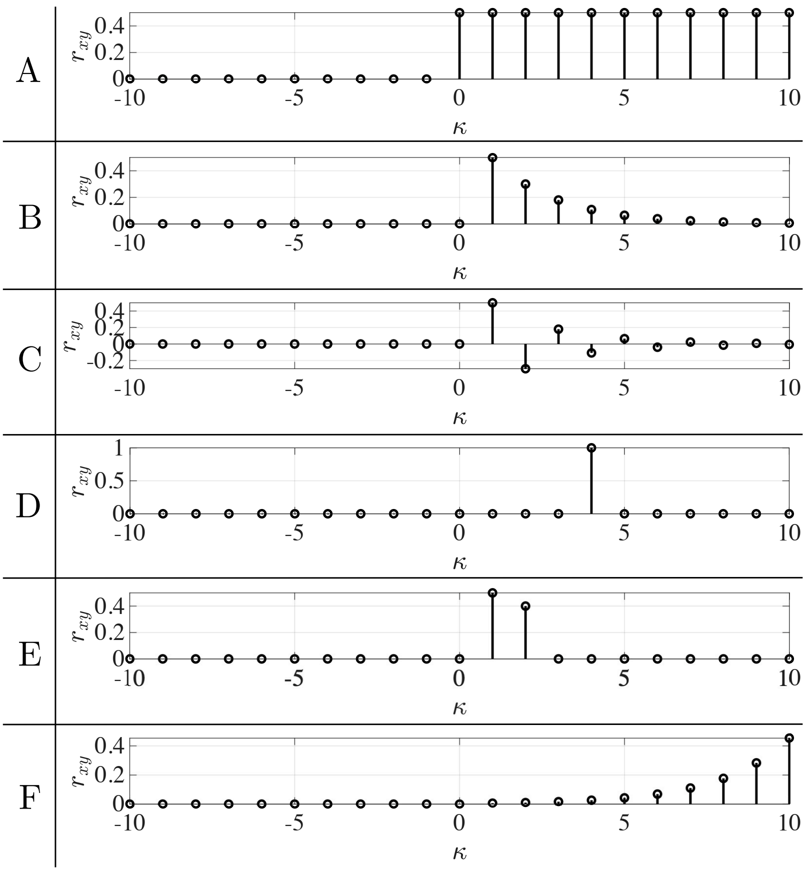
$$G_2(z) = 0.5z^{-1} + 0.4z^{-2}$$
$$G_3(z) = z^{-4}.$$

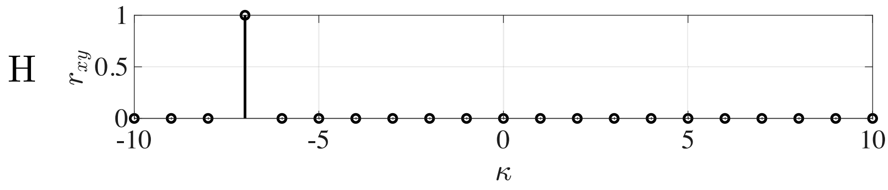
The systems were excited with an Kronecker delta $x(k)$ and the resulting output $y(k)$ was recorded. Subsequently, the cross-correlation function $r_{xy}(\kappa)$ was calculated. Assign the correct cross-correlation functions A,B,...,F to the right system $G_1, G_2, G_3$. **Answer**:




| System | Cross-correlation-function |
|---|---|
| 1 | B |
| 2 | E |
| 3 | D |

6

b) Another cross-correlation function H is given (figure shown on the next page), calculated exactly as in the previous task. What is the transfer function of the underlying system? **Answer**:




$G_H = z^7$ 1

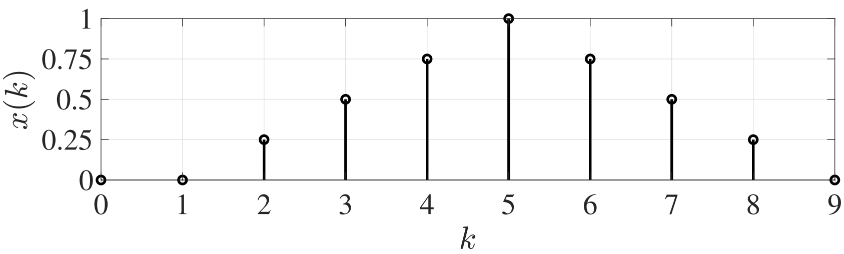
c) How does the system from task b) differ from the systems in task a)?
**Answer**:
The system is acausal. 1

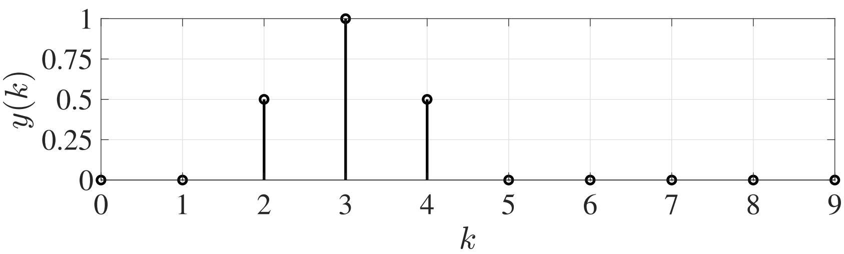
d) Is it possible to represent a real physical system with the transfer function from task b)? **Answer**:
No. 1

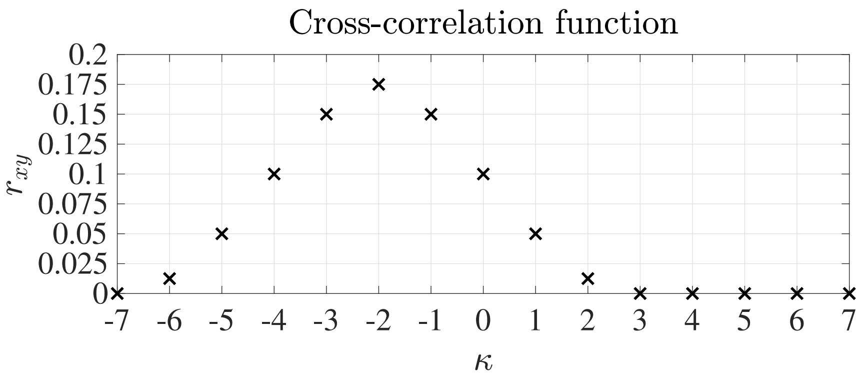
e) The time-discrete signals $x(k)$ and $y(k)$ are given. Calculate the cross-correlation function $r_{xy}$ for $\kappa = -7, -6, ..., 7$ and sketch the cross-correlation function either in the given diagram or on your exam sheet.
Hint:

$$r_{xy}(\kappa) = \begin{cases} \frac{1}{N} \sum\limits_{k=0}^{N-1-\kappa} x(k) \cdot y(k+\kappa), & \text{for } \kappa \geq 0 \\ \frac{1}{N} \sum\limits_{k=-\kappa}^{N-1} x(k) \cdot y(k+\kappa), & \text{else.} \end{cases}$$

**Answer**:






$\boxed{3}$

$$
\begin{aligned}
r_{xy}(7) &= \frac{1}{10}\left(0.5 \cdot 0 + 0.25 \cdot 0 + 0 \cdot 0\right) = 0 \\
r_{xy}(6) &= \frac{1}{10}\left(0.75 \cdot 0 + 0.5 \cdot 0 + 0.25 \cdot 0 + 0 \cdot 0\right) = 0 \\
r_{xy}(5) &= \frac{1}{10}\left(1 \cdot 0 + 0.75 \cdot 0 + 0.5 \cdot 0 + 0.25 \cdot 0 + 0 \cdot 0\right) = 0 \\
r_{xy}(4) &= \frac{1}{10}\left(0.75 \cdot 0 + 1 \cdot 0 + 0.75 \cdot 0 + 0.5 \cdot 0 + 0.25 \cdot 0 + 0 \cdot 0\right) = 0 \\
r_{xy}(3) &= 0 \\
r_{xy}(2) &= \frac{1}{10}\left(0.25 \cdot 0.5\right) = 0.0125 \\
r_{xy}(1) &= \frac{1}{10}\left(0.5 \cdot 0.5 + 1 \cdot 0.25\right) = 0.05 \\
r_{xy}(0) &= \frac{1}{10}\left(0.5 \cdot 0.75 + 1 \cdot 0.5 + 0.5 \cdot 0.25\right) = 0.1 \\
r_{xy}(-1) &= \frac{1}{10}\left(0.5 \cdot 1 + 1 \cdot 0.75 + 0.5 \cdot 0.5\right) = 0.15 \\
r_{xy}(-2) &= \frac{1}{10}\left(0.5 \cdot 0.75 + 1 \cdot 1 + 0.5 \cdot 0.75\right) = 0.175 \\
r_{xy}(-3) &= \frac{1}{10}\left(0.5 \cdot 1 + 1 \cdot 0.75 + 0.5 \cdot 0.5\right) = 0.15 \\
r_{xy}(-4) &= \frac{1}{10}\left(0.5 \cdot 0.75 + 1 \cdot 0.5 + 0.5 \cdot 0.25\right) = 0.1 \\
r_{xy}(-5) &= \frac{1}{10}\left(0.5 \cdot 0.5 + 1 \cdot 0.25\right) = 0.05 \\
r_{xy}(-6) &= \frac{1}{10}\left(0.25 \cdot 0.5\right) = 0.0125 \\
r_{xy}(-7) &= 0
\end{aligned}
$$

$\boxed{5}$

$\boxed{\sum 17}$

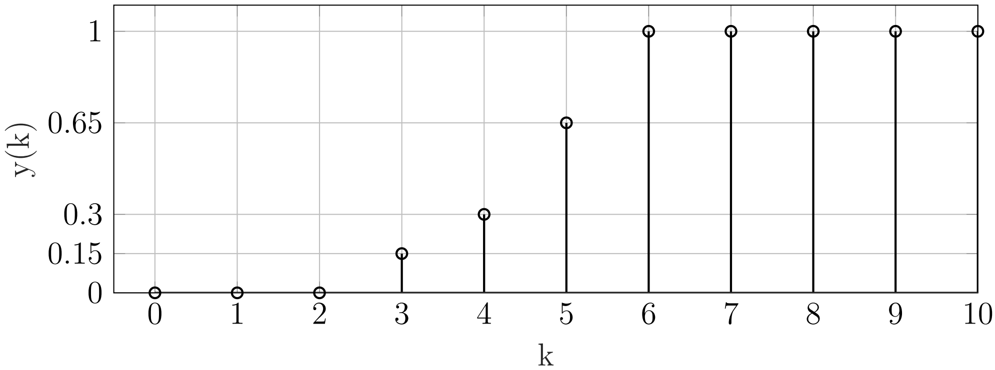
## Task 5: FIR and IIR (17 Points)

The step responses of different filters are given in the following figures. The corresponding step is defined as

$$u(k) = \begin{cases} 0, & \text{if } k < 2 \\ 1, & \text{if } k \geq 2 \end{cases}.$$

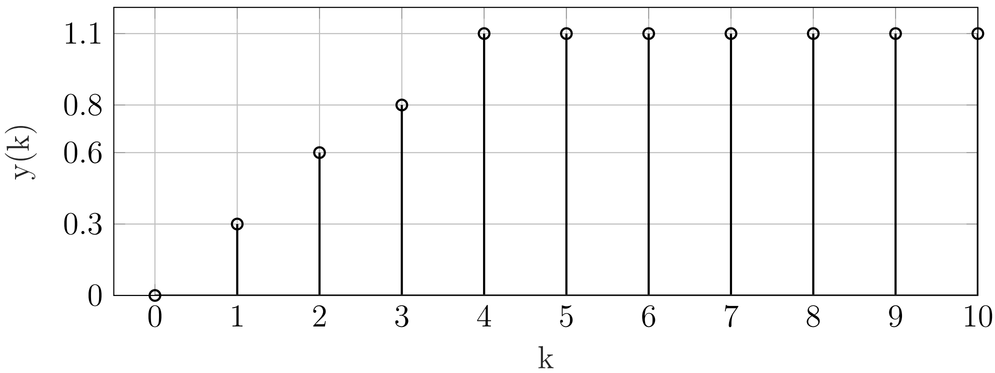
For each task specify a transfer function $G(z)$.

a) Filter 1



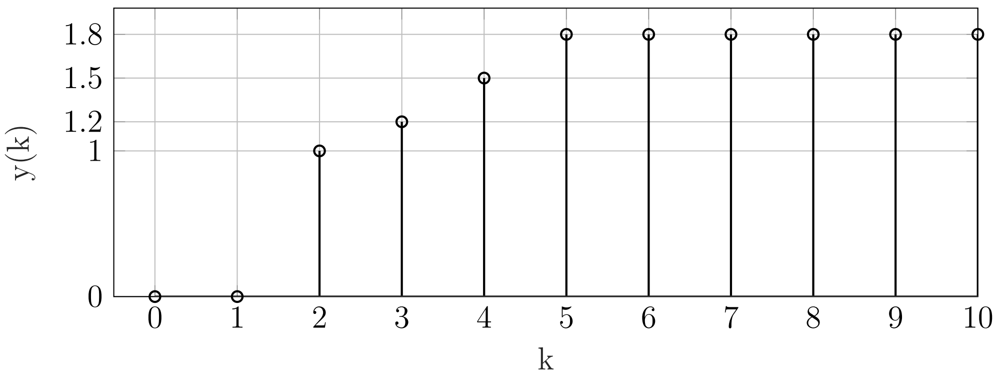

**Answer**: FIR causal: $G(z) = 0.15z^{-1} + 0.15z^{-2} + 0.35z^{-3} + 0.35z^{-4}$ 2

b) Filter 2



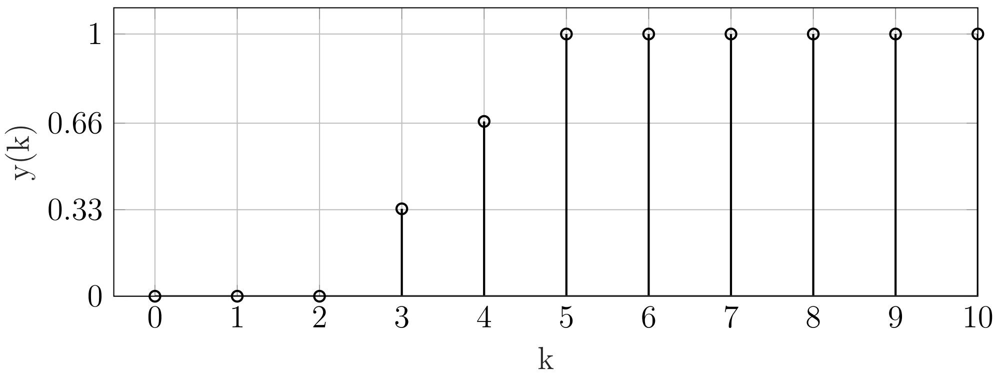

**Answer**: FIR non-causal: $G(z) = 0.3z^{1} + 0.3 + 0.2z^{-1} + 0.3z^{-2}$ 2

c) Filter 3



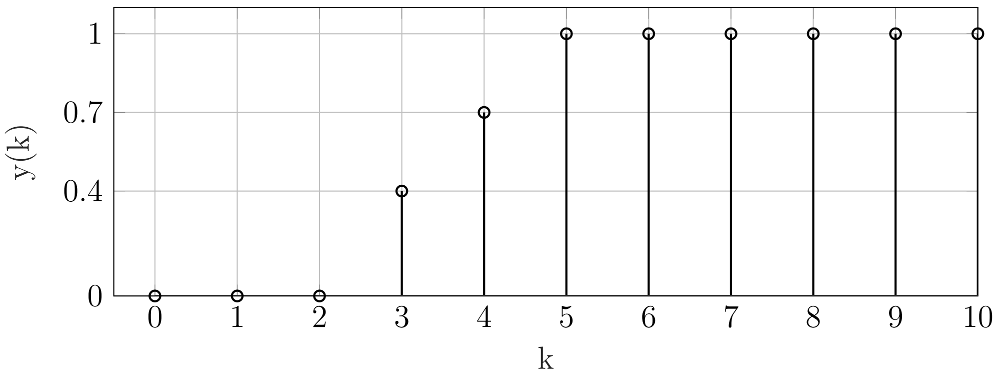

**Answer**: Proper system: $G(z) = 1 + 0.2z^{-1} + 0.3z^{-2} + 0.3z^{-3}$ 2

d) Filter 4



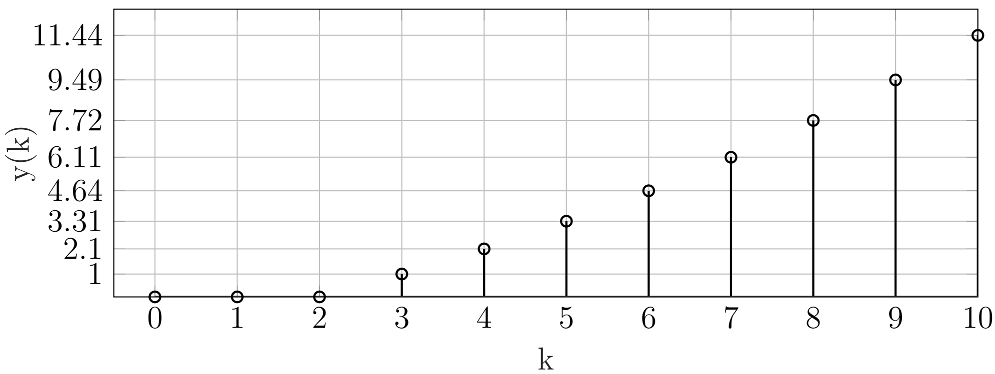

**Answer**: Linear phase: $G(z) = \frac{1}{3}z^{-1} + \frac{1}{3}z^{-2} + \frac{1}{3}z^{-3}$ 2

e) Filter 5




**Answer**: Gain = 1: $G(z) = 0.4z^{-1} + 0.3z^{-2} + 0.3z^{-3}$ 2

f) Filter 6




**Answer**: Instable: $G(z) = \frac{z^{-1}}{1-1.1z^{-1}}$ 2

g) Please copy the following table and mark the properties of each filter.
**Answer**:

| Filter | FIR | Non-causal | Linear phase | Instable | Gain = 1 |
|---|---|---|---|---|---|
| 1 | X | | | | X |
| 2 | X | X | | | |
| 3 | X | | | | |
| 4 | X | | X | | X |
| 5 | X | | | | X |
| 6 | | | | X | |

$\boxed{5}$

$\boxed{\sum 17}$

## Task 6: Phase of a Filter (14 Points)

a) The following system is given

$$G(z) = \frac{1+z^{-1}}{2}.$$

Calculate the filter $F(z)$ according to

$$F(z) = G(z) \cdot G(z^{-1}).$$

**Answer**:

$$\begin{aligned} F(z) &= G(z) \cdot G(z^{-1}) \\ &= (\frac{1}{2} + \frac{z^{-1}}{2}) \cdot (\frac{1}{2} + \frac{z^{1}}{2}) \\ &= \frac{1}{4} + \frac{z^1}{4} + \frac{z^{-1}}{4} + \frac{z^1 z^{-1}}{4} \\ &= \frac{z}{4} + \frac{1}{2} + \frac{z^{-1}}{4} \\ F(z) &= \frac{z+2+z^{-1}}{4} \end{aligned}$$

3

b) Calculate the phase of the previously determined filter $F(z)$.
*Hint: Set* $z = e^{i\omega T_0}$
**Answer**:

$$\begin{aligned} F(z = e^{i\omega T_0}) &= \frac{e^{i\omega T_0} + 2 + e^{-i\omega T_0}}{4} \\ &= \frac{(\cos(\omega T_0) + i\sin(\omega T_0)) + 2 + (\cos(-\omega T_0) + i\sin(-\omega T_0))}{4} \\ &= \frac{\cos(\omega T_0) + 2 + \cos(-\omega T_0)}{4} \end{aligned}$$

$F(z = e^{i\omega T_0})$ purely real, therfore phase = 0. 3

c) If $F(z)$ is non-causal, modify $F(z)$ such that it is causal and proper.
**Answer**:
$\tilde{F}(z) = z^{-1}F(z) = \frac{1+2z^{-1}+z^{-2}}{4}$ 2

d) How is a filter with the previously calculated phase called?
**Answer**:
Filter with linear phase 2

e) What are advantages of such filters?
**Answer**:
No phase shift and a constant group propagation delay 2

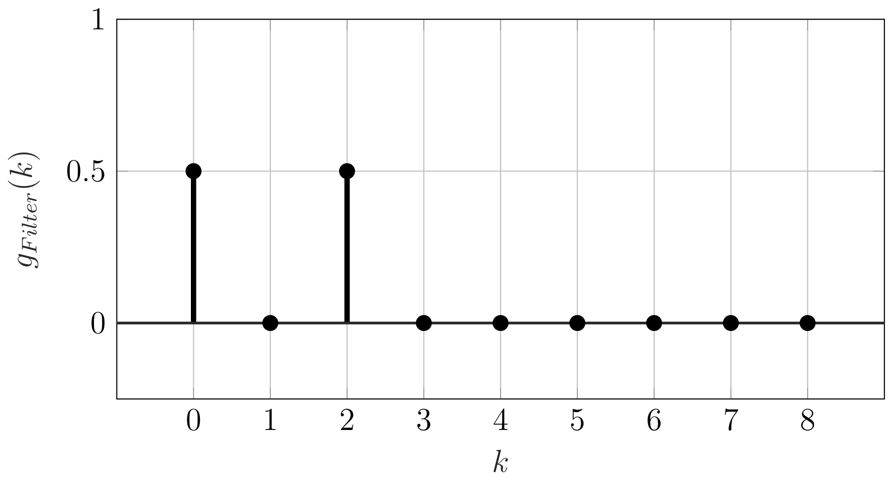
f) Is it possible to define an IIR filter with linear phase?
**Answer**:
No, only approximately 1

g) Which IIR filter should be used, if a linear phase is required?
**Answer**:
Bessel filter 1

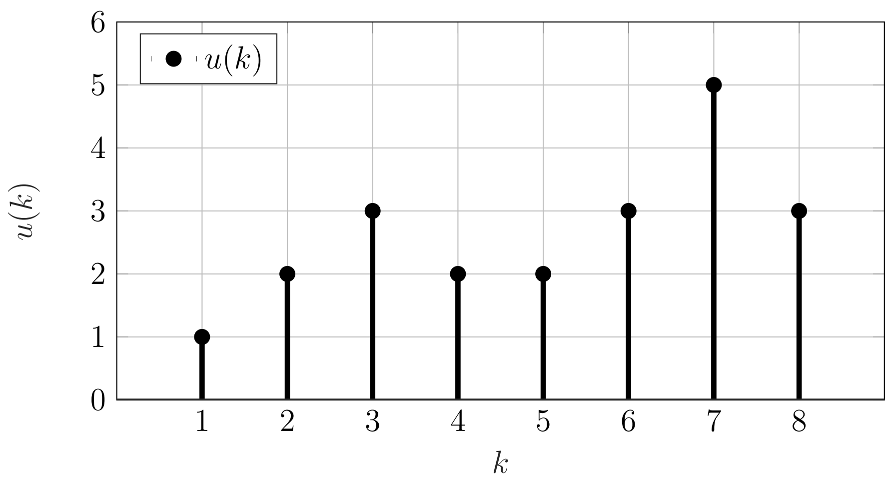
$\sum 14$

**Task 7: Linear filter (14 Points)**

The following figure shows the impulse response of a linear filter $g_{Filter}(k)$:




This filter can be used to filter the input signal $u(k)$. For this task assume the following sequence for the input signal $u(k)$.




a) The input signal $u(k)$ is now to be filtered. Calculate the value sequence $y(k)$ of the filter output for $k = 1, 2, \ldots, 8$. Assume $u(k) = 0$ for all $k < 1$.
**Answer**: The filter output can be easily calculated using the difference equation:

$$y(k) = \frac{1}{2}u(k) + \frac{1}{2}u(k-2) = \frac{1}{2}\left(u(k) + u(k-2)\right)$$

| $k$ | $0.5 \cdot u(k)$ | $0.5 \cdot u(k-2)$ | $y(k)$ |
|---|---|---|---|
| 1 | 0.5 | 0 | 0.5 |
| 2 | 1 | 0 | 1 |
| 3 | 1.5 | 0.5 | 2 |
| 4 | 1 | 1 | 2 |
| 5 | 1 | 1.5 | 2.5 |
| 6 | 1.5 | 1 | 2.5 |
| 7 | 2.5 | 1 | 3.5 |
| 8 | 1.5 | 1.5 | 3 |

8

b) Calculate the transfer function of the filter $G_{Filter}(z)$.
**Answer**:
The transfer function of the filter can be calculated from the difference equation given in subtask a).

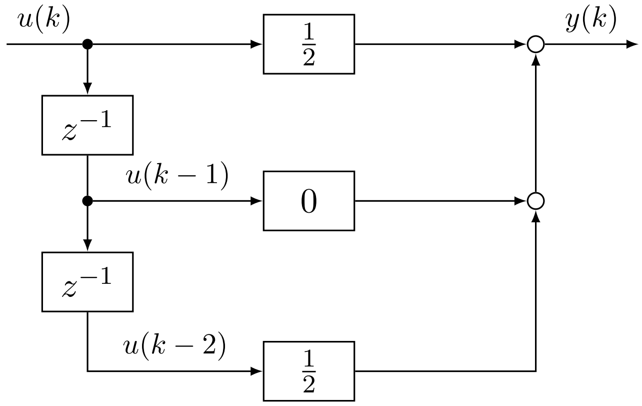
$$y(k) = \frac{1}{2}u(k) + \frac{1}{2}u(k-2) \quad \circ\!\!-\!\!\bullet \quad Y(z) = \left(\frac{1}{2} + \frac{1}{2}z^{-2}\right) U(z)$$

$$G_{Filter}(z) = \frac{Y(z)}{U(z)} = \frac{1}{2} + \frac{1}{2}z^{-2}$$

2

c) Which structure of linear filters is the subject of investigation?
**Answer**:
The filter structure is of FIR-type. 1

d) Draw the block diagram of the filter.
**Answer**:
A possible block diagram is shown in the following figure.




3

$\sum$ 14