Sensorics Exam

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Name:	
Mat.-No.:	
Grade:	

Task	T1	T2	T3	T4	T5	T6	T7	Sum
Score:	20	15	14	13	15	24	19	120
Accomplished:								

Exam duration: 2 hours

Permitted Tools: Pocket Calculator and 4 pages of formula collection

Task 1: Comprehension Questions (20 Points)

a) How large should the internal resistance of a voltmeter and an amperemeter ideally be and why? (3 points)

b) What is the *Student's t-distribution* mainly used for, what is the meaning of its *degrees of freedom* parameter and how does it relate to the Gaussian distribution. (3 points)

c) What is the difference between supervised and unsupervised learning? Give one example of an unsupervised method. (2 points)

d) Describe how a *median filter* works and name one useful application. Is the median filter a linear filter? (3 points)

e) During measurement systematic and random errors can occur. Name one reason for a systematic error and explain if both of these errors can be reduced by averaging over an large number of measured samples. (2 points)

f) Explain briefly what the term *aliasing* means, what can be done to prevent it and how it relates to the *Shannon frequency.* (4 points)

g) Where in the z-plane must the poles of a time-discrete system lie to be stable? There is a formula to approximately transform a system from the continuous to the discrete time domain by replacing s with $\frac{2(1-z^{-1})}{T_0(1+z^{-1})}$. How is this transformation called and what is its special feature? (3 points)

Task 2: Bilinear Transformation (15 Punkte)

The following transfer function is given in the s-domain:

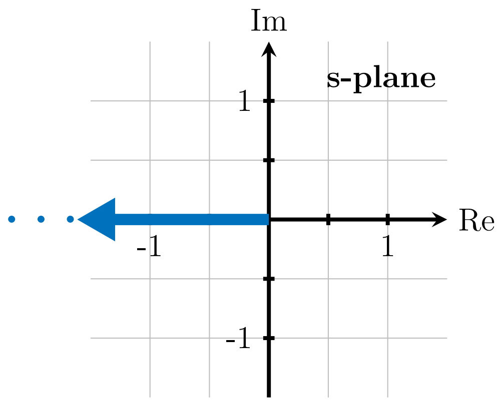
$$G(s) = \frac{s-b}{s-a}.$$

a) Determine the gain K_s, the pole p_s and the zero n_s of the system in the s-domain.

b) Transform the transfer function into the z-domain by using bilinear transformation. Assume $T_0 = 1\,\mathrm{sec}$.

c) Determine the new gain K_z, the pole p_z and the zero n_z of the system in the z-domain after using the transformation.

The following graph shows the position of possible stable poles for a certain system in s-domain. Notice that this ranges from 0 to $-\infty$.



d) Mark the area in the z-domain in which these stable poles lie after applying the bilinear transformation.

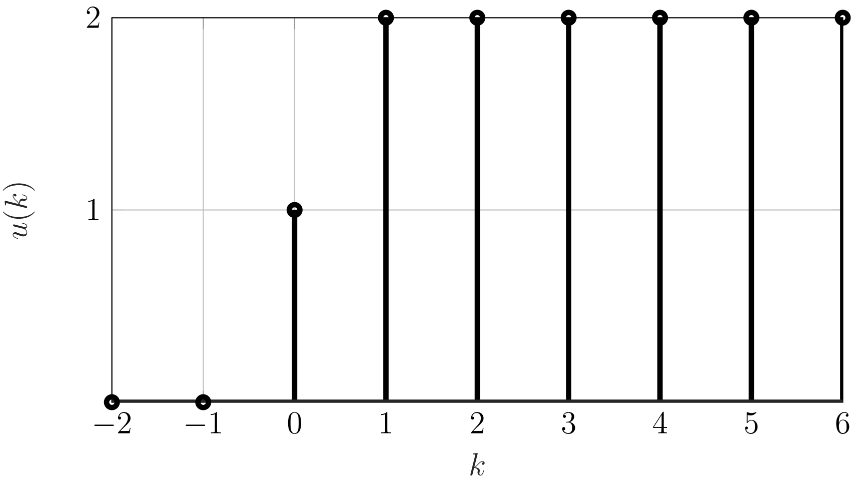
Now let the new transfer function be:

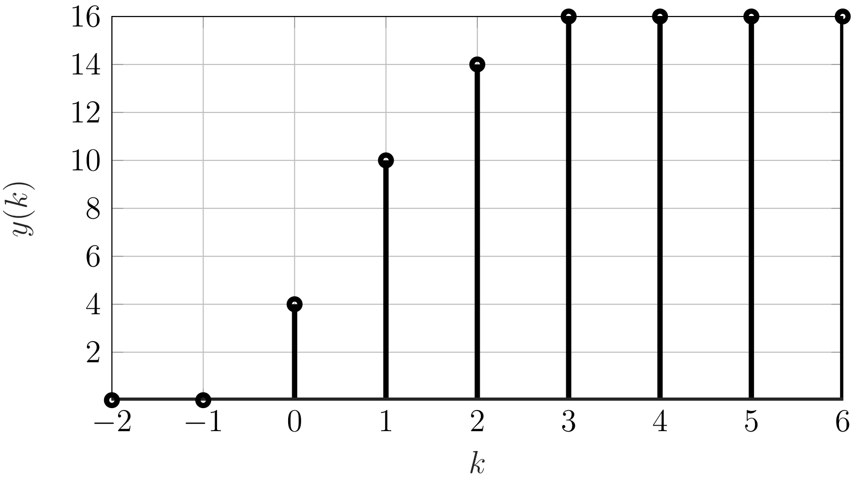
$$G(s) = \frac{1}{s^2 + \omega^2}.$$

e) Transform the transfer function into the z-domain by using the bilinear transformation. For this, assume $T_0 = 1\,\mathrm{sec}$ and $\omega = 2\,\mathrm{sec}^{-1}$. Draw the calculated poles in a diagram of the s-domain and the z-domain.

Task 3: Systemidentification FIR (14 Points)

The following figure shows the response $y(k)$ of an unknown system to the input signal $u(k)$.





The system can be described by a FIR-filter of finite order. Goal of the task is to determine the coefficients of the FIR-filter using the given response.

a) Does the system have a direct feedthrough? Justify your answer.

b) The system can be represented in the form of an FIR filter. Determine the order of the FIR filter needed to result in the system response shown? Justify your answer.

c) Give the difference equation of a FIR filter with the order determined in part b). Determine the coefficients of this filter based on the given system response.

d) Give the transfer function of the determined filter.
If you were unable to determine a filter, use the following difference equation to determine the transfer function:

$$y(k) = 3u(k-1) + 5u(k-2) + 2u(k-3)$$

e) Draw the block diagram of the FIR filter determined.

Task 4: Filter (13 Points)

A system $G(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$ is given.

a) Sketch the block diagram of the given process.

b) Assume that the feedback part disappears. Set up the difference equation $G_1(z)$ for this system and sketch the block diagram.

c) How is the filter type of $G_1(z)$ from task b) called?

d) With which factor does difference equation $G_1(z)$ from task b) have to be multiplied with in order to get the same gain as $G(z)$? Use the following coefficients:

$$
\begin{aligned}
b_0 &= 0.7 \\
b_1 &= 0.7 \\
b_2 &= 0.6 \\
a_1 &= 0.6 \\
a_2 &= 0.4
\end{aligned}
$$

Task 5: Bode plot for discrete systems (15 Punkte)

Given is the following discrete transfer function $G(z) = \frac{1}{z+0.5}$. The transfer function is determed with a sample time $T_0 = 1\,\text{sec}$.

a) Determine the Shannon frequenz ω_S that belongs to the given sample time.

b) Calculate the magnitude $\mid G(i\omega) \mid_{\text{dB}}$ explicitly for $\omega = 0.1\,\frac{\text{rad}}{\text{sec}}, \omega = 1\,\frac{\text{rad}}{\text{sec}}$ and $\omega = 2\,\frac{\text{rad}}{\text{sec}}$.

c) Calculate the phase $\varphi(\omega)$ explicitly for $\omega = 0.1\,\frac{\text{rad}}{\text{sec}}, \omega = 1\,\frac{\text{rad}}{\text{sec}}$ and $\omega = 2\,\frac{\text{rad}}{\text{sec}}$.

d) Plot a bode plot for the frequency intervall $\omega \in [0.1, 10]\frac{\text{rad}}{\text{sec}}$. Mark the previously calculated values clearly.

e) The sample time has doubled. What is the effect on the Shannon-frequenz? How does the phase and magnitude change for the frequenzes $\omega = 0.1\,\frac{\text{rad}}{\text{sec}}, \omega = 1\,\frac{\text{rad}}{\text{sec}}$ and $\omega = 2\,\frac{\text{rad}}{\text{sec}}$?

Task 6: Probabilities and Sampling (24 Points)

a) Plot the following distribution in a graph, evaluate mean $\bar{x}$, variance s_x^2 and standard deviation s_x and plot samples drawn out of this distribution in a new graph over the number of samples. Also draw the mean as a line in the same graph. Take care that important properties can be seen (at least 10 samples).

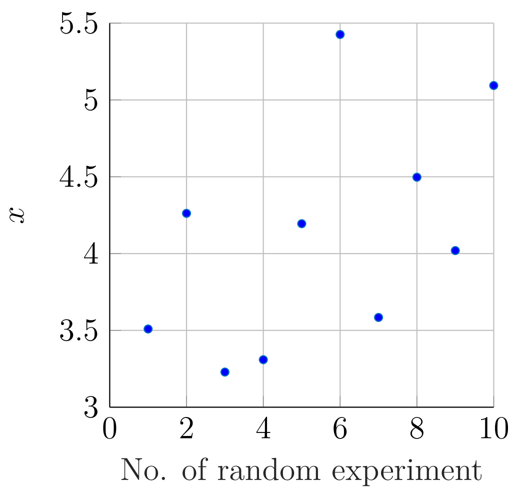
$$p(x) = \frac{1}{1\sqrt{2\pi}} e^{-0.5\left(\frac{x-5}{1}\right)^2}$$

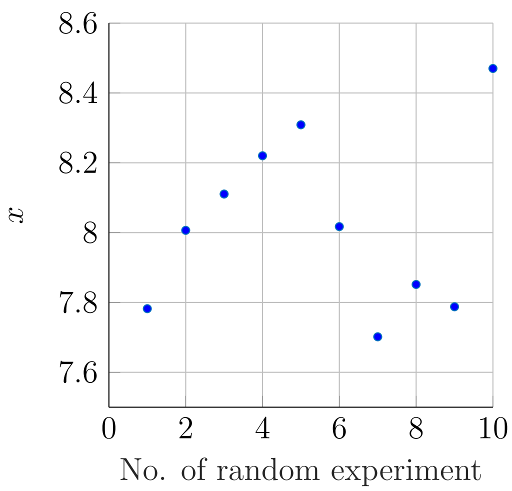
b) Plot the following distribution in a graph, evaluate the mean $\bar{x}$. Plot samples drawn out of this distribution in a new graph over the number of samples taken. Also draw the mean as a line in the same graph. Take care that important properties can be seen (at least 10 samples).

$$p(x) = \frac{0.5}{0.5\sqrt{2\pi}} e^{-0.5\left(\frac{x-3}{0.5}\right)^2} + \frac{0.5}{0.5\sqrt{2\pi}} e^{-0.5\left(\frac{x-6}{0.5}\right)^2}$$

c) Two sets of samples from normal distributions are displayed in the following graphs. Estimate the means $\bar{x}_1, \bar{x}_2$ and standard deviations s_1, s_2 according to the data and plot both probability density functions in one graph.
Hint:

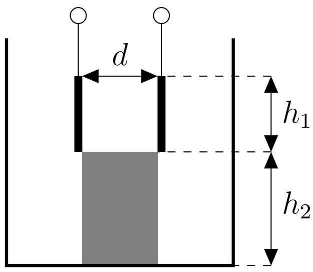
$$s_x^2 = \frac{1}{N-1} \sum_{i=1}^{N} \left[x(i) - \bar{x}\right]^2$$





Task 7: Fill level measurement (19 Points)

A student constructs the measurement for the fill level of a silicone oil basin (height of 0.3 m) with a capacitor. He choses the capacitor plates with a height $h_1 = 0.1\,\mathrm{m}$ and a width $w = 0.2\,\mathrm{m}$. He places a styrofoam swimmer (can fully fill the space between the plates, always floating exactly on top of the oil, no friction etc.) with a height $h_2 = 0.15\,\mathrm{m}$ between the plates. The plates are 0.15m above the ground of the basin and have a distance $d = 0.05\,\mathrm{m}$.



Useful information:
The capacity of a capacitor is calculated by

$$C = \frac{\epsilon_0 \epsilon_r A}{d}$$

with the permittivity of vacuum

$$\epsilon_0 = 8.85 \cdot 10^{-12} \frac{\mathrm{As}}{\mathrm{Vm}},$$

the area A of each of the capacitor plates and the distance of these plates d. ϵ_r is the relative permittivity of a dielectric. We use following constants: $\epsilon_{r,styrofoam} = 1.1$, $\epsilon_{r,styrole\ oil} = 2.7$, $\epsilon_{r,air} = 1$

a) Derive a function or multiple functions for the capacity $C(x)$ depending on the fill level x of the basin.

b) Plot the function $C(x)$ in a diagram for $0\,\mathrm{m} \leq x < 0.4\,\mathrm{m}$.

c) What of this setup should be improved and how?

d) When the min. and max. values for the capacity C are not supposed to change, what needs to be adjusted? Calculate how this needs to be adjusted.

e) Draw the new function $C_{new}(x)$ from subtask d) into the graph from subtask b).

Lösungen **Solution:**

<u>Task 1:</u> Comprehension Questions (20 Points)

a) How large should the internal resistance of a voltmeter and an amperemeter ideally be and why? (3 points)

<u>Answer</u>: For a voltmeter the resistance should be infinitely high, for a amperemeter zero, since the devices should not influence the measured circuit and the voltmeter is used in parallel and the amperemeter in series to the circuit. $\boxed{3}$

b) What is the *Student's t-distribution* mainly used for, what is the meaning of its *degrees of freedom* parameter and how does it relate to the Gaussian distribution. (3 points)

<u>Answer</u>: If you want to calculate confidence intervals for sample mean and you also have to estimate the standard deviation σ_x from the data, the confidence interval is no longer normally distributed (Gaussian) but t-distributed. The shape of the t-distribution depends on the degrees of freedom which correspond to the number of data samples used for calculating σ_x. If the degrees of freedom approach infinity the normal and t-distribution become identical. $\boxed{3}$

c) What is the difference between supervised and unsupervised learning? Give one example of an unsupervised method. (2 points)

<u>Answer</u>: For supervised learning you need input and output data. Unsupervised methods only rely on input data to find information about the data. One example for unsupervised methods would be *clustering*. $\boxed{2}$

d) Describe how a *median filter* works and name one useful application. Is the median filter a linear filter? (3 points)

<u>Answer</u>: A median filter sorts the data samples by magnitude and selects the number in the middle. It is useful for outlier removal. No, it is a nonlinear filter. $\boxed{3}$

e) During measurement systematic and random errors can occur. Name one reason for a systematic error and explain if both of these errors can be reduced by averaging over an large number of measured samples. (2 points)

<u>Answer</u>: Systematic error occur for example if a falsely calibrated instrument is used. Only random errors can be reduced by averaging more samples, systematic errors cause a bias that can not be removed by averaging. $\boxed{2}$

f) Explain briefly what the term *aliasing* means, what can be done to prevent it and how it relates to the *Shannon frequency.* (4 points)

<u>Answer</u>: Aliasing is the effect that occurs if a signal is sampled slower than the Shannon frequency, i.e. slower than twice the highest frequency contained in the signal to be measured. Frequencies higher than the sampling frequency are mirrored into frequencies lower than the sampling frequency resulting in false measurements. Methods to prevent aliasing include the increase of the sampling frequency and filtering the signal with an analog low-pass filter before sampling. $\boxed{4}$

g) Where in the z-plane must the poles of a time-discrete system lie to be stable? There is a formula to approximately transform a system from the continuous to the discrete time domain by replacing s with $\frac{2(1-z^{-1})}{T_0(1+z^{-1})}$. How is this transformation called and what is its special feature? (3 points)

Answer: The poles must lie inside the unit circle $|z| < 1$. The transformation is called *bilinear transformation* or *Tustin formula*. It ensures that a stable system is also stable after the transformation. $\boxed{3}$

Task 2: Bilinear Transformation (15 Punkte)

The following transfer function is given in the s-domain:

$$G(s) = \frac{s-b}{s-a}.$$

a) Determine the gain K_s, the pole p_s and the zero n_s of the system in the s-domain.

Answer:

$K_\mathrm{s} = \lim\limits_{s\to 0} s \cdot \frac{1}{s} G(s) = \frac{b}{a}$

$p_\mathrm{s} = a$

$n_\mathrm{s} = b$ $\boxed{2}$

b) Transform the transfer function into the z-domain by using bilinear transformation. Assume $T_0 = 1\,\mathrm{sec}$.

Answer:

$$s = \frac{2}{T_0}\frac{1-z^{-1}}{1+z^{-1}} = 2\frac{1-z^{-1}}{1+z^{-1}}$$

$$G(z) = \frac{2\frac{1-z^{-1}}{1+z^{-1}} - b}{2\frac{1-z^{-1}}{1+z^{-1}} - a} = \frac{2-2z^{-1}-b-bz^{-1}}{2-2z^{-1}-a-az^{-1}} = \frac{(2-b)-(2+b)z^{-1}}{(2-a)-(2+a)z^{-1}}$$

$$= \frac{(2-b)z-(2+b)}{(2-a)z-(2+a)} = \frac{(2-b)}{(2-a)}\frac{z-\frac{2+b}{2-b}}{z-\frac{2+a}{2-a}}$$

$\boxed{2}$

c) Determine the new gain K_z, the pole p_z and the zero n_z of the system in the z-domain after using the transformation.

Answer:

$K_\mathrm{z} = G(z=1) = \frac{(2-b)1-(2+b)}{(2-a)1-(2+a)} = \frac{2b}{2a} = \frac{b}{a}$

$p_\mathrm{z} = \frac{2+a}{2-a}$

$n_\mathrm{z} = \frac{2+b}{2-b}$ $\boxed{3}$

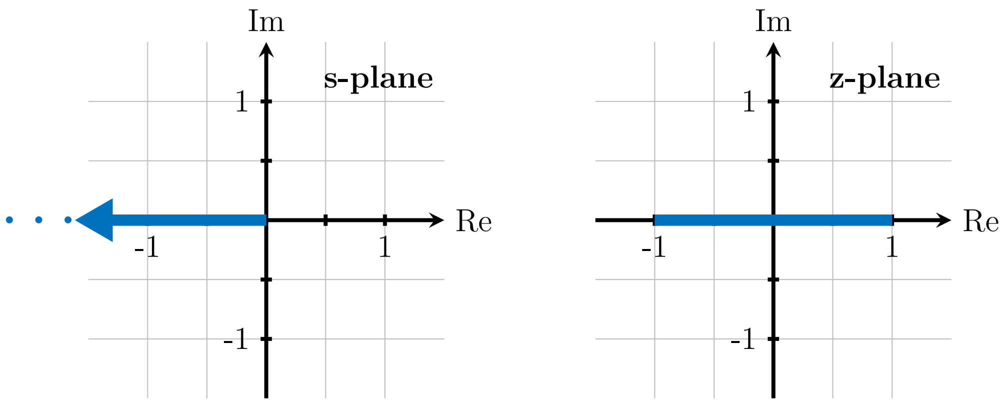
The following graph shows the position of possible stable poles for a certain system in s-domain. Notice that this ranges from 0 to $-\infty$.

d) Mark the area in the z-domain in which these stable poles lie after applying the bilinear transformation.

Answer:

Examples of different poles are given in the table:

p_s	p_z
a	$\frac{2+a}{2-a}$
0	1
-1	$\frac{1}{3}$
-2	0
-10	$-\frac{8}{12}$
$-\infty$	-1

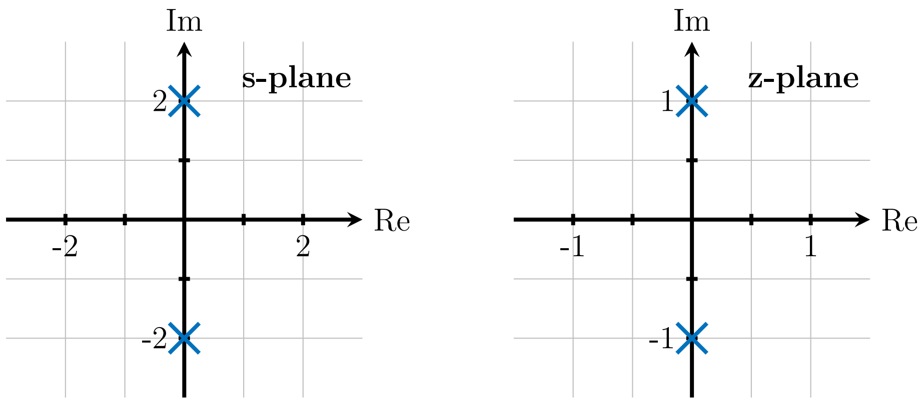


This results in the new poles p_z according to the figure above. $\boxed{2}$

Now let the new transfer function be:

$$G(s) = \frac{1}{s^2 + \omega^2}.$$

e) Transform the transfer function into the z-domain by using the bilinear transformation. For this, assume $T_0 = 1\,\text{sec}$ and $\omega = 2\,\text{sec}^{-1}$. Draw the calculated poles in a diagram of the s-domain and the z-domain.



$\boxed{2}$

Answer:

$$
\begin{aligned}
s &= \frac{2}{T_0}\frac{1-z^{-1}}{1+z^{-1}} = 2\frac{1-z^{-1}}{1+z^{-1}} \\
G(z) &= \frac{1}{\left(2\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 2^2} = \frac{1}{4\frac{1-2z^{-1}+z^{-2}}{1+2z^{-1}+z^{-2}}+4} \\
&= \frac{1+2z^{-1}+z^{-2}}{4(1-2z^{-1}+z^{-2})+4(1+2z^{-1}+z^{-2})} \\
&= \frac{1+2z^{-1}+z^{-2}}{8+8z^{-2}} = \frac{z^2+2z^1+1}{8z^2+8}
\end{aligned}
$$

$$p_{\mathrm{z},1/2} = \pm i$$

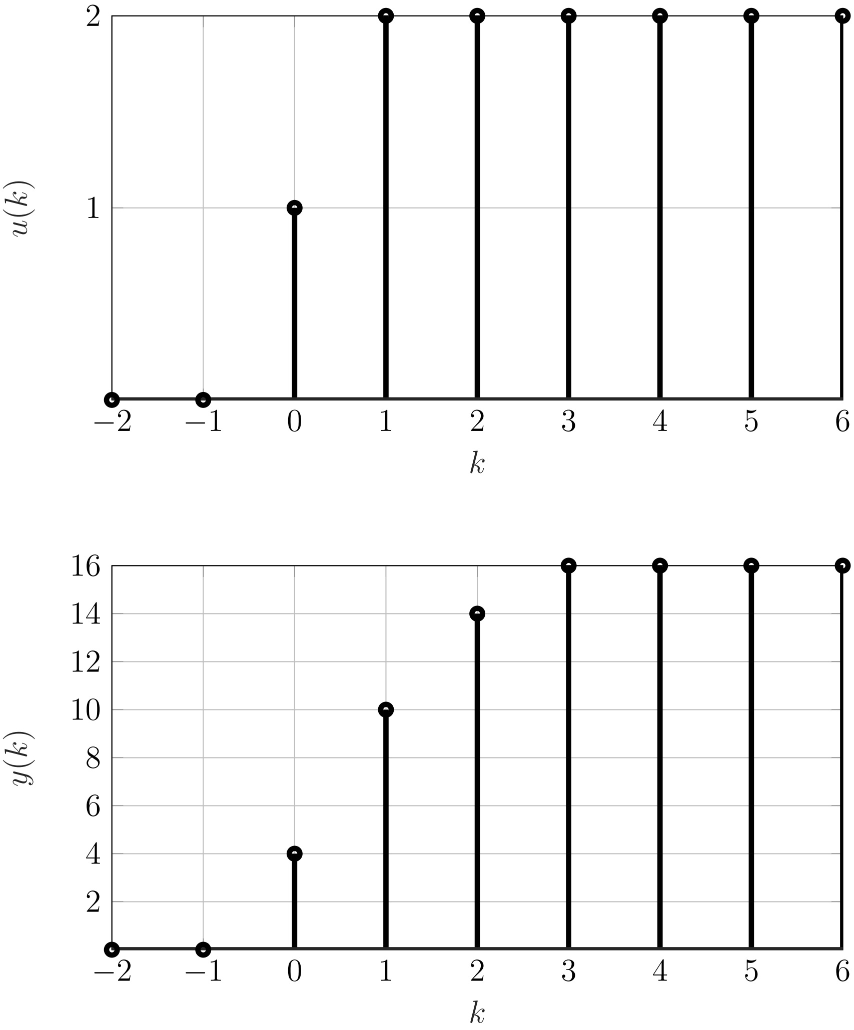
$$
\begin{aligned}
G(s) &= \frac{1}{s^2+\omega^2} = \frac{1}{(s+i\omega)(s-i\omega)} \\
p_{\mathrm{s},1/2} &= \pm i\omega = \pm i2
\end{aligned}
$$

$\boxed{4}$

$\boxed{\sum 15}$

Task 3: **Systemidentification FIR (14 Points)**

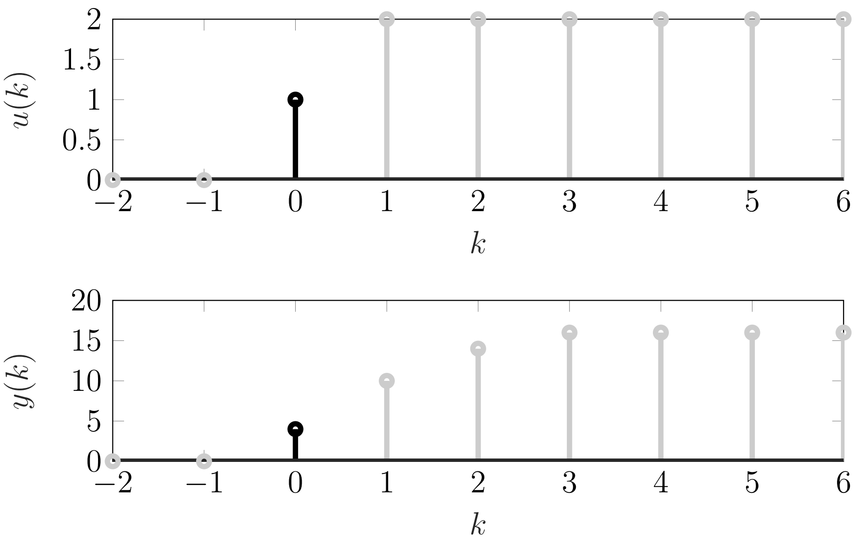
The following figure shows the response $y(k)$ of an unknown system to the input signal $u(k)$.



The system can be described by a FIR-filter of finite order. Goal of the task is to determine the coefficients of the FIR-filter using the given response.

a) Does the system have a direct feedthrough? Justify your answer.

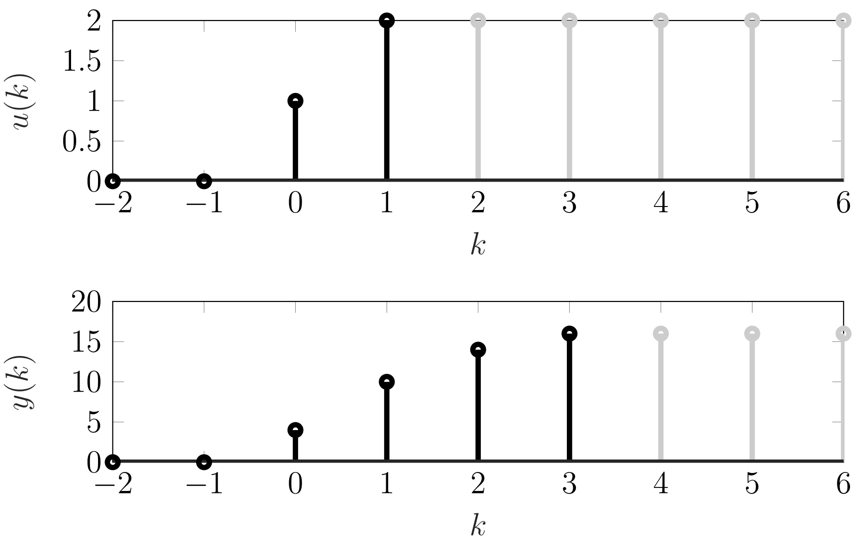
Answer: Whether a system has a direct feedthrough can be seen by the fact that the system reacts immediately to a change in the input variable in the same time step.



The figure shows that the system reacts immediately to the step in the input signal. Therefore the system has a direct feedthrough. 2

b) The system can be represented in the form of an FIR filter. Determine the order of the FIR filter needed to result in the system response shown? Justify your answer.

Answer: The order of the FIR filter describing the system can be determined by the time steps required for the system output to reach a constant value after the system input is no longer changed.



From the figure it can be seen that the system does not change its output $y(k = 3) = 16$ from time step $k = 3$. The input variable $u(k)$ is constant from time step $k = 1$ onward. Thus, the FIR filter must have an order of 2. 2

c) Give the difference equation of a FIR filter with the order determined in part b). Determine the coefficients of this filter based on the given system response.

Answer: The FIR filter, which describes the system, has order 2 and also has a direct feedthrough. This results in the following difference equation of the FIR filter:

$$y(k) = b_0u(k) + b_1u(k-1) + b_2u(k-2)$$

To determine the coefficients of the FIR filter, the values of the system response can be substituted into the difference equation. Since the values of the input variable $u(k<0)=0$, the coefficients can be determined stepwise.
For $k=0$:

$$y(0)=4 \qquad u(0)=1 \qquad u(-1)=0 \qquad u(-2)=0$$

$$\begin{aligned} y(0) &= b_0u(0)+b_1u(-1)+b_2u(-2) \\ &= b_0\cdot 1+b_1\cdot 0+b_2\cdot 0 \\ &= b_0 \\ \rightarrow b_0 &= 4 \end{aligned}$$

For $k=1$

$$y(1)=10 \qquad u(1)=2 \qquad u(0)=1 \qquad u(-1)=0$$

$$\begin{aligned} y(1) &= b_0u(1)+b_1u(0)+b_2u(-1) \\ &= 4\cdot 2+b_1\cdot 1+b_2\cdot 0 \\ &= 8+b_1 \\ \rightarrow b_1 &= 10-8=2 \end{aligned}$$

For $k=2$

$$y(2)=14 \qquad u(2)=2 \qquad u(1)=2 \qquad u(0)=1$$

$$\begin{aligned} y(2) &= b_0u(2)+b_1u(1)+b_2u(0) \\ &= 4\cdot 2+2\cdot 2+b_2\cdot 1 \\ &= 8+4+b_2 \\ \rightarrow b_2 &= 14-12=2 \end{aligned}$$

This results in the following difference equation:

$$y(k)=4u(k)+2u(k-1)+2u(k-2)$$

5

d) Give the transfer function of the determined filter.
If you were unable to determine a filter, use the following difference equation to determine the transfer function:

$$y(k)=3u(k-1)+5u(k-2)+2u(k-3)$$

Answer: To determine the transfer function, the difference equation can be transformed into the z-domain.

$$\begin{aligned} &y(k)=4u(k)+2u(k-1)+2u(k-2) \\ &\circ\!\!-\!\!\bullet \quad Y(z)=4U(z)+2U(z)z^{-1}+2U(z)z^{-2} \end{aligned}$$

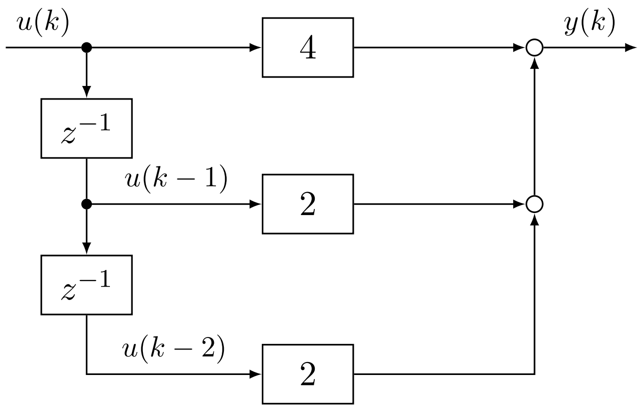
By rearranging the transformed equation, the transfer function of the FIR filter is obtained.

$$\rightarrow G(z) = \frac{Y(z)}{U(z)} = 4 + 2z^{-1} + 2z^{-2}$$

2

e) Draw the block diagram of the FIR filter determined.

Answer: A possible block diagram is shown in the following figure:



3

$\sum 14$

Task 4: **Filter (13 Points)**

A system $G(z) = \dfrac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$ is given.

a) Sketch the block diagram of the given process.
Answer:

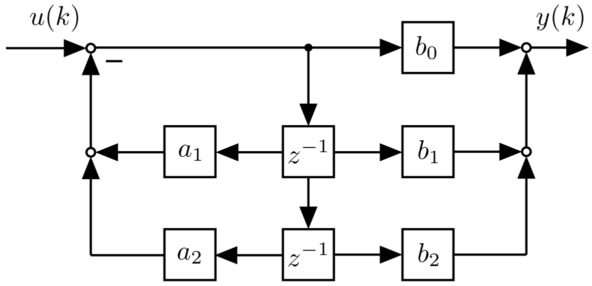


Figure 1: Block diagram of the system.

3

b) Assume that the feedback part disappears. Set up the difference equation $G_1(z)$ for this system and sketch the block diagram. **Answer**:

$$G_1(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

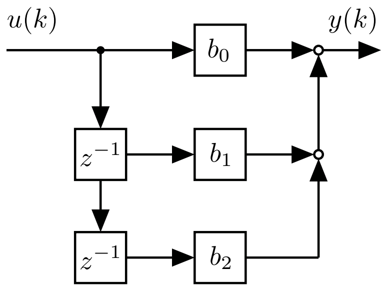


Figure 2: Block diagram of the system without feedback.

4

c) How is the filter type of $G_1(z)$ from task b) called?
Answer:
FIR-filter 1

d) With which factor does difference equation $G_1(z)$ from task b) have to be multiplied with in order to get the same gain as $G(z)$? Use the following coefficients:

$$b_0 = 0.7$$
$$b_1 = 0.7$$
$$b_2 = 0.6$$
$$a_1 = 0.6$$
$$a_2 = 0.4$$

Answer:

$$h(t \to \infty) = \lim_{z \to 1} (z-1) \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \frac{z}{z-1} = \frac{2}{2} = 1$$

$$h_1(t \to \infty) = \lim_{z \to 1} (z-1) b_0 + b_1 z^{-1} + b_2 z^{-2} \frac{z}{z-1} = 2$$

$G_1(z)$ has to be multiplied by 0.5 to get the same gain as $G(z)$. $\boxed{5}$

$\boxed{\sum 13}$

Task 5: Bode plot for discrete systems (15 Punkte)

Given is the following discrete transfer function $G(z) = \frac{1}{z+0.5}$. The transfer function is determed with a sample time $T_0 = 1\,\text{sec}$.

a) Determine the Shannon frequenz ω_S that belongs to the given sample time.
Answer:

$$\omega_S = \frac{\pi}{T_0}$$
$$\omega_S = \pi\,\frac{\text{rad}}{\text{sec}}$$

1

b) Calculate the magnitude $\mid G(i\omega) \mid_{\text{dB}}$ explicitly for $\omega = 0.1\,\frac{\text{rad}}{\text{sec}}, \omega = 1\,\frac{\text{rad}}{\text{sec}}$ and $\omega = 2\,\frac{\text{rad}}{\text{sec}}$.

Answer:

$$G(z) = \frac{1}{z+0.5}$$
$$G(i\omega) = \frac{1}{e^{i\omega T_0}+0.5}$$
$$\mid G(i\omega) \mid = \frac{1}{\sqrt{\sin^2(\omega T_0) + (\cos(\omega T_0)+0.5)^2}}$$
$$\mid G(i\omega) \mid_{\text{dB}} = 20\log\frac{1}{\sqrt{\sin^2(\omega T_0) + (\cos(\omega T_0)+0.5)^2}}$$
$$\omega = 0.1\frac{\text{rad}}{\text{sec}} : \mid G(i\omega) \mid_{\text{dB}} = -3.5122\,\text{dB}$$
$$\omega = 1\frac{\text{rad}}{\text{sec}} : \mid G(i\omega) \mid_{\text{dB}} = -2.5293\,\text{dB}$$
$$\omega = 2\frac{\text{rad}}{\text{sec}} : \mid G(i\omega) \mid_{\text{dB}} = 0.7891\,\text{dB}$$

4

c) Calculate the phase $\varphi(\omega)$ explicitly for $\omega = 0.1\,\frac{\text{rad}}{\text{sec}}, \omega = 1\,\frac{\text{rad}}{\text{sec}}$ and $\omega = 2\,\frac{\text{rad}}{\text{sec}}$.
<u>**Answer**</u>:

$$
\begin{aligned}
G(z) &= \frac{1}{z+0.5} \\
G(i\omega) &= \frac{1}{e^{i\omega T_0}+0.5} \\
G(i\omega) &= \frac{1(e^{-i\omega T_0}+0.5)}{(e^{i\omega T_0}+0.5)(e^{-i\omega T_0}+0.5)} \\
G(i\omega) &= \frac{e^{-i\omega T_0}+0.5}{1.25+0.5e^{i\omega T_0}+0.5e^{-i\omega T_0}} \\
G(i\omega) &= \frac{e^{-i\omega T_0}+0.5}{1.25+\cos(\omega T_0)} \\
G(i\omega) &= \frac{0.5+\cos(\omega T_0)-i\sin(\omega T_0)}{1.25+\cos(\omega T_0)} \\
\text{Re}\{G(i\omega)\} &= \frac{0.5+\cos(\omega T_0)}{1.25+\cos(\omega T_0)} \\
\text{Im}\{G(i\omega)\} &= \frac{-\sin(\omega T_0)}{1.25+\cos(\omega T_0)} \\
\varphi(\omega) &= \arctan\frac{-\sin(\omega T_0)}{0.5+\cos(\omega T_0)}
\end{aligned}
$$

$$
\begin{aligned}
\omega = 0.1\,\frac{\text{rad}}{\text{sec}} &: \varphi(\omega) = -3.8204\,^\circ \\
\omega = 1\,\frac{\text{rad}}{\text{sec}} &: \varphi(\omega) = -38.9684\,^\circ \\
\omega = 2\,\frac{\text{rad}}{\text{sec}} &: \varphi(\omega) = -84.7312\,^\circ
\end{aligned}
$$

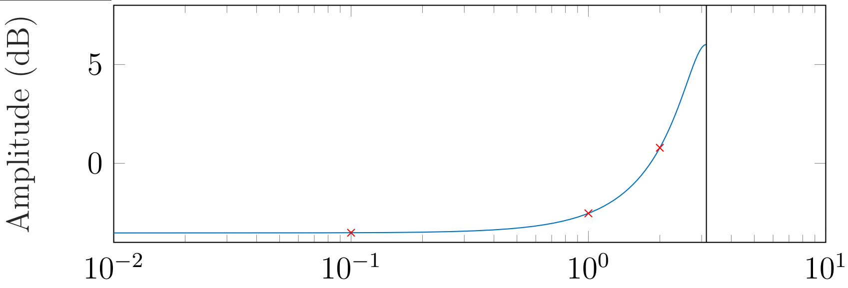
Or with this relation:

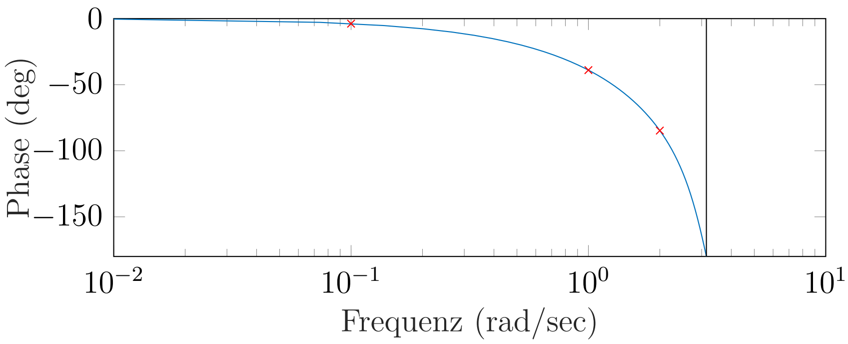
$$
\begin{aligned}
\varphi\left(\frac{A}{B}\right) &= \varphi(A) - \varphi(B) \\
\varphi\left(\frac{1}{B}\right) &= -\varphi(B) \\
\varphi(\omega) &= -\arctan\frac{\sin(\omega T_0)}{0.5+\cos(\omega T_0)}
\end{aligned}
$$

$\boxed{4}$

d) Plot a bode plot for the frequency intervall $\omega \in [0.1, 10]\frac{\text{rad}}{\text{sec}}$. Mark the previously calculated values clearly.

Answer:





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e) The sample time has doubled. What is the effect on the Shannon-frequenz? How does the phase and magnitude change for the frequenzes $\omega = 0.1\,\frac{\text{rad}}{\text{sec}}, \omega = 1\,\frac{\text{rad}}{\text{sec}}$ and $\omega = 2\,\frac{\text{rad}}{\text{sec}}$?
<u>Answer</u>:
Shannon-frequenz

$$\omega_S = \frac{\pi}{T_0}$$
$$\omega_S = 1.571\,\frac{\text{rad}}{\text{sec}}$$

$\omega = 2\,\frac{\text{rad}}{\text{sec}} > \omega_S$ Evaluation not reasonable (periodicity)

$$\mid G(i\omega) \mid_{\text{dB}} = 20 \log \frac{1}{\sin^2(\omega T_0) + (\cos(\omega T_0) + 0.5)^2}$$
$$\omega = 0.1 \frac{\text{rad}}{\text{sec}} : \mid G(i\omega) \mid_{\text{dB}} = -3.4832\,\text{dB}$$
$$\omega = 1 \frac{\text{rad}}{\text{sec}} : \mid G(i\omega) \mid_{\text{dB}} = 0.7891\,\text{dB}$$

$$\varphi(\omega) = \arctan \frac{-\sin(\omega T_0)}{0.5 + \cos(\omega T_0)}$$
$$\omega = 0.1\,\frac{\text{rad}}{\text{sec}} : \varphi(\omega) = -7.6451\,^\circ$$
$$\omega = 1\,\frac{\text{rad}}{\text{sec}} : \varphi(\omega) = -84.7312\,^\circ$$

$\boxed{3}$

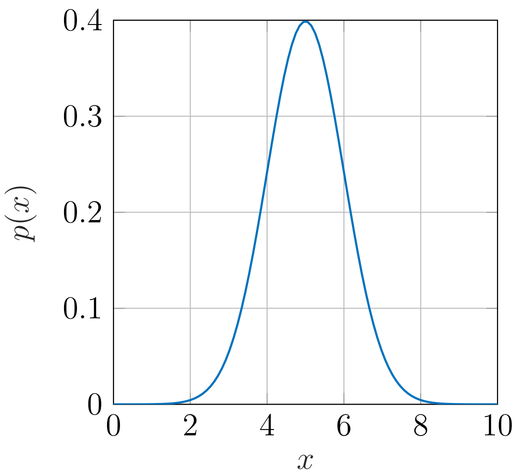
$\boxed{\sum 15}$

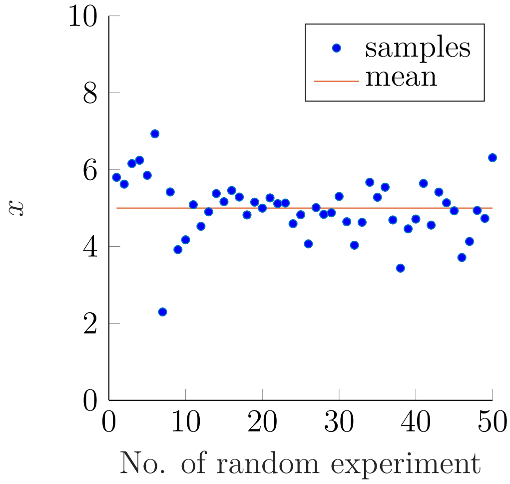
Task 6: Probabilities and Sampling (24 Points)

a) Plot the following distribution in a graph, evaluate mean $\bar{x}$, variance s_x^2 and standard deviation s_x and plot samples drawn out of this distribution in a new graph over the number of samples. Also draw the mean as a line in the same graph. Take care that important properties can be seen (at least 10 samples).

$$p(x) = \frac{1}{1\sqrt{2\pi}} e^{-0.5\left(\frac{x-5}{1}\right)^2}$$

Answer:
mean $\bar{x} = 5$, variance $s_x^2 = 1$ and standard deviation $s_x = 1$. These values can be taken from the given equation. 1





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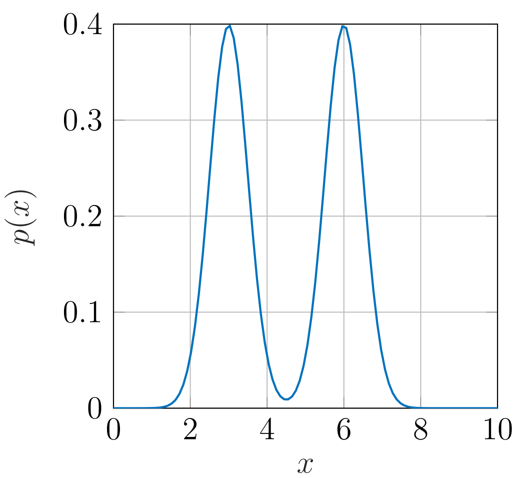
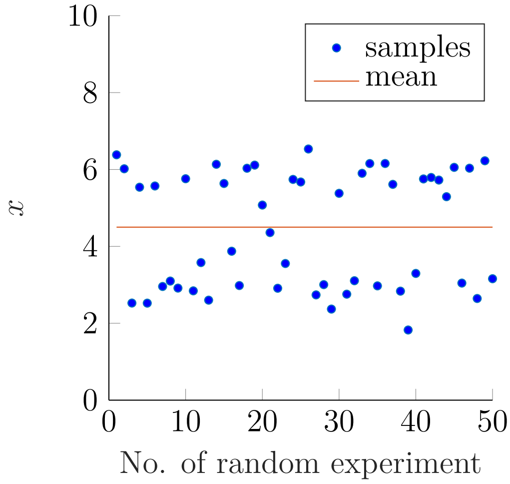
b) Plot the following distribution in a graph, evaluate the mean $\bar{x}$. Plot samples drawn out of this distribution in a new graph over the number of samples taken. Also draw the mean as a line in the same graph. Take care that important properties can be seen (at least 10 samples).

$$p(x) = \frac{0.5}{0.5\sqrt{2\pi}} e^{-0.5\left(\frac{x-3}{0.5}\right)^2} + \frac{0.5}{0.5\sqrt{2\pi}} e^{-0.5\left(\frac{x-6}{0.5}\right)^2}$$

Answer:
$\bar{x} = \frac{6+3}{2} = 4.5$ Can be computed from the means of both Gaussians. 2





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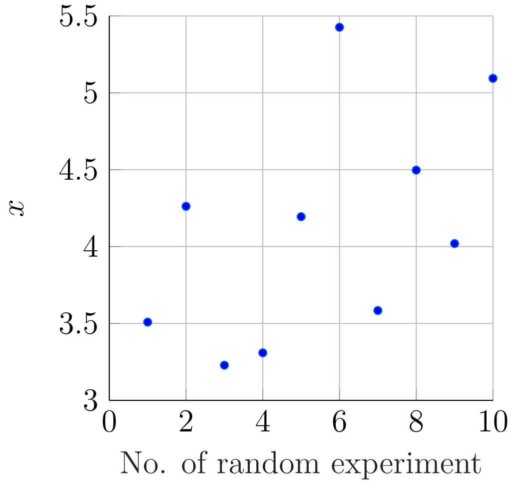
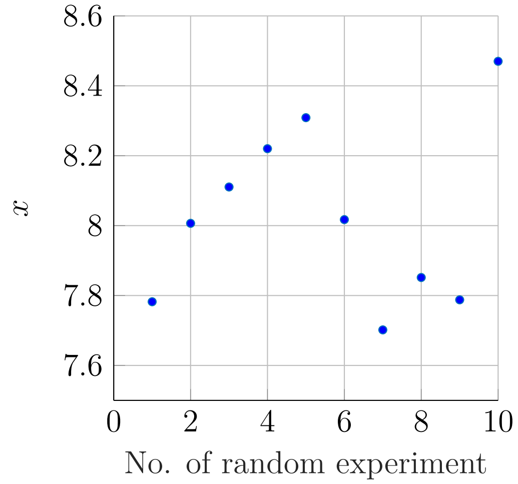
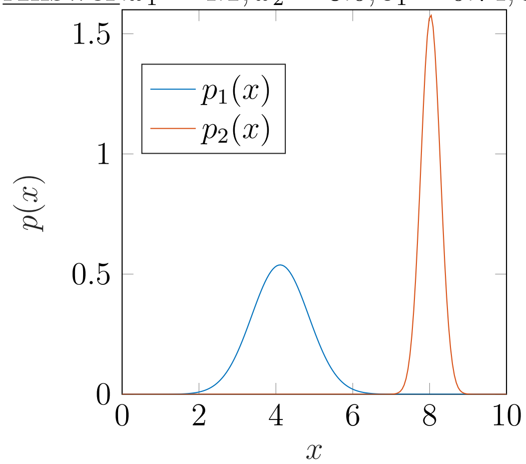
c) Two sets of samples from normal distributions are displayed in the following graphs. Estimate the means $\bar{x}_1, \bar{x}_2$ and standard deviations s_1, s_2 according to the data and plot both probability density functions in one graph. 3
<u>Hint</u>:

$$s_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left[x\left(i\right) - \bar{x}\right]^2$$





<u>Answer</u>: $\bar{x}_1 = 4.1, \bar{x}_2 = 8.0, s_1 = 0.74, s_2 = 0.25$ 6



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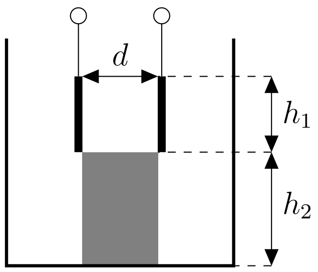
Task 7: Fill level measurement (19 Points)

A student constructs the measurement for the fill level of a silicone oil basin (height of 0.3 m) with a capacitor. He choses the capacitor plates with a height $h_1 = 0.1\,\text{m}$ and a width $w = 0.2\,\text{m}$. He places a styrofoam swimmer (can fully fill the space between the plates, always floating exactly on top of the oil, no friction etc.) with a height $h_2 = 0.15\,\text{m}$ between the plates. The plates are 0.15m above the ground of the basin and have a distance $d = 0.05\,\text{m}$.



Useful information:
The capacity of a capacitor is calculated by

$$C = \frac{\epsilon_0 \epsilon_r A}{d}$$

with the permittivity of vacuum

$$\epsilon_0 = 8.85 \cdot 10^{-12} \frac{\text{As}}{\text{Vm}},$$

the area A of each of the capacitor plates and the distance of these plates d. ϵ_r is the relative permittivity of a dielectric. We use following constants: $\epsilon_{r,styrofoam} = 1.1$, $\epsilon_{r,styrole\ oil} = 2.7$, $\epsilon_{r,air} = 1$

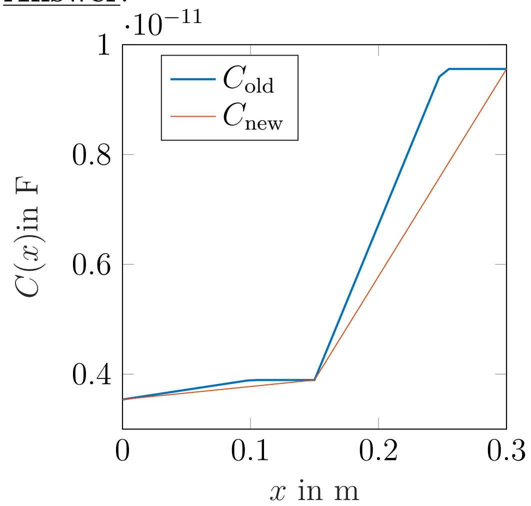
a) Derive a function or multiple functions for the capacity $C(x)$ depending on the fill level x of the basin.
   Answer: $A = 0.2\text{ m} \cdot 0.1\,\text{m}$, $d = 0.05\,\text{m}$, $w = 0.2\,\text{m}$
   1) $x = 0$ m: Only air
   $$C(x) = \frac{\epsilon_0 \epsilon_{r,air} A}{d}$$
   2) $0\text{ m} < x \leq 0.1$ m: air and foam
   $$C(x) = \frac{\epsilon_0 \epsilon_{r,air}\left(A - xw\right)}{d} + \frac{\epsilon_0 \epsilon_{r,styrofoam} xw}{d}$$
   3) $0.1\text{ m} < x \leq 0.15$ m: only foam
   $$C(x) = \frac{\epsilon_0 \epsilon_{r,styrofoam} A}{d}$$
   4) $0.15\text{ m} < x \leq 0.25$ m: foam and oil
   $$C(x) = \frac{\epsilon_0 \epsilon_{r,styrofoam}\left(A - (x - h_2)\, w\right)}{d} + \frac{\epsilon_0 \epsilon_{r,oil}\left(x - h_2\right) w}{d}$$

5) $0.25 \text{ m} < x$: only oil

$$C(x) = \frac{\epsilon_0 \epsilon_{r,styrole\ oil} A}{d}$$

8

b) Plot the function $C(x)$ in a diagram for $0\,\text{m} \leq x < 0.4\,\text{m}$.
Answer:



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c) What of this setup should be improved and how?
Answer: The swimmer is longer than the plates. Therefore, for $0.1\,\text{m} < x \leq 0.15\,\text{m}$ no change of the fill level can be measured. The capacitor plates could be swapped for 0.05 m longer plates. Other Ideas that make the dead zone of measurement disappear, without reducing the max. oil level are also valid. 2

d) When the min. and max. values for the capacity C are not supposed to change, what needs to be adjusted? Calculate how this needs to be adjusted.
Answer: $l_{new} = 0.15\,\text{m}$, $A_{new} = 0.15\,\text{m} \cdot 0.2\,\text{m}$ Since the Area is increasing, the distance needs to be increased as well. Other concepts which don't decrease the total height or exceed the limits for the capacity are also valid.

$$C_{old} = \frac{\epsilon_0 \epsilon_{r,air} A}{d} = \frac{\epsilon_0 \epsilon_{r,air} A_{new}}{d_{new}} = C_{new}$$

$$\frac{A}{d} = \frac{A_{new}}{d_{new}}$$

$$d_{new} = \frac{A_{new}}{A} d = \frac{0.2 \cdot 0.15\text{m}^2}{0.2 \cdot 0.1\text{m}^2} 0.05\,\text{m} = 0.075\,\text{m}$$

Since the permittivities cancel themselves in the equations, this calculation is valid for min. and max values of C. 3

e) Draw the new function $C_{new}(x)$ from subtask d) into the graph from subtask b). 2