Sensorics Exam

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Name:	
Mat.-No.:	
Grade:	

Task:	T1	T2	T3	T4	T5	T6	Sum
Scores:	17	20	23	21	23	16	120
Accomplished:							

Duration of examination: 2 hours

You are allowed to use a calculator and four pages of notes

Task 1: Measurement of Speed

For the translational speed measurement of a car, angular position sensors are used at each of the four wheels. In order to get the speed value, the angular position measurement should be done every δt seconds. For the speed calculation the radius of the tire has to be known. Figure 1 illustrates several tire radii for one tire. Note that the dynamic tire radius depends on the translational speed of the car due to an increasing centrifugal force acting on the tire.

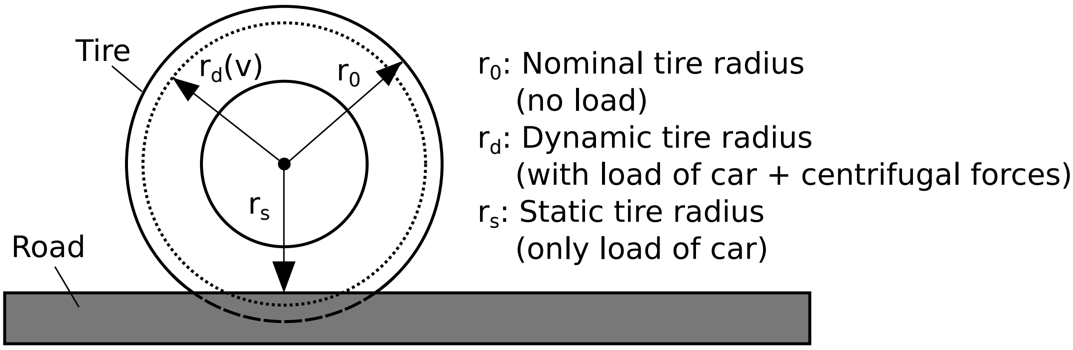


Fig. 1: Illustration of several tire radii

Assumptions for this task: The load as well as the inflation pressure is exactly the same for all tires.

a) Write down the equation to calculate a car's translational speed v depending on the change of the tire angle $\delta\varphi$, the time period between two angle measurements δt and one of the tire radii shown in Fig. 1. Choose which tire radius should be used in order get the most accurate translational speed value.

b) Determine the required time period δt, if the wheel angle measurement should be done at least twice during one whole revolution of a wheel. Therefore assume a maximum speed of $v_{max} = 216$ Kilometers per hour and a constant tire radius of $r = 0.24$ m.

c) If the velocity dependency of the dynamic tire radius $r_d(v)$ is ignored, what type of error would arise (assuming all other quantities are measured without any errors)? You can assume to know the sign of the error you are making by ignoring the velocity dependency. Pick the corresponding error propagation equation from below and calculate Δv (velocity error propagation).

d) Now assume that the tire radius is known at any time without any errors, but the time period δt as well as the change of the tire angle $\delta\varphi$ are corrupted by Gaussian distributed noise. Again, pick the corresponding error propagation equation from below and calculate Δv (velocity error propagation).

e) As already mentioned, you get measurements from all four wheels of the car. Is there a benefit in averaging the signals of all four wheels for the scenarios described in subtasks c) and d)?

Equations that can be used for the error propagation, if a physical quantity y depends on other measurements $y = f(x_1, x_2, \ldots, x_n)$:

- Gaussian error propagation for systematic errors:

$$\Delta y = \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial f}{\partial x_n}\Delta x_n$$

- Gaussian error propagation for maximal errors:

$$\Delta y = \left|\frac{\partial f}{\partial x_1}\Delta x_1\right| + \left|\frac{\partial f}{\partial x_2}\Delta x_2\right| + \cdots + \left|\frac{\partial f}{\partial x_n}\Delta x_n\right|$$

- Approximation for random errors in practice:

$$\Delta y = \sqrt{\left(\frac{\partial f}{\partial x_1}\Delta x_1\right)^2 + \left(\frac{\partial f}{\partial x_2}\Delta x_2\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\Delta x_n\right)^2}$$

Task 2: Inductive Displacement Measurement

Note that all subtasks can be solved independently of each other!

For an inductive displacement measurement the inductance of a coil should be used. The equation that describes the inductance L in dependence of the displacement d is:

$$f(d) = L = \frac{c}{d_1 + d} \,. \tag{1}$$

The values of $c = 4 \cdot 10^{-6}$ Hm and $d_1 = 0.2$ m can be assumed to be constant, i.e. they do not change during the displacement measurement.

a) Linearize equation (1) around the operating point $d_0 = -0.0165$. Use a Taylor series expansion for the linearization. In general the Taylor series for a function $f(x)$ around an operating point x_0 is given by

$$Tf(x; x_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \tag{2}$$

Here, $n!$ denotes the factorial of n and $f^{(n)}(x_0)$ denotes the n-th derivative of $f(x)$ evaluated at x_0.

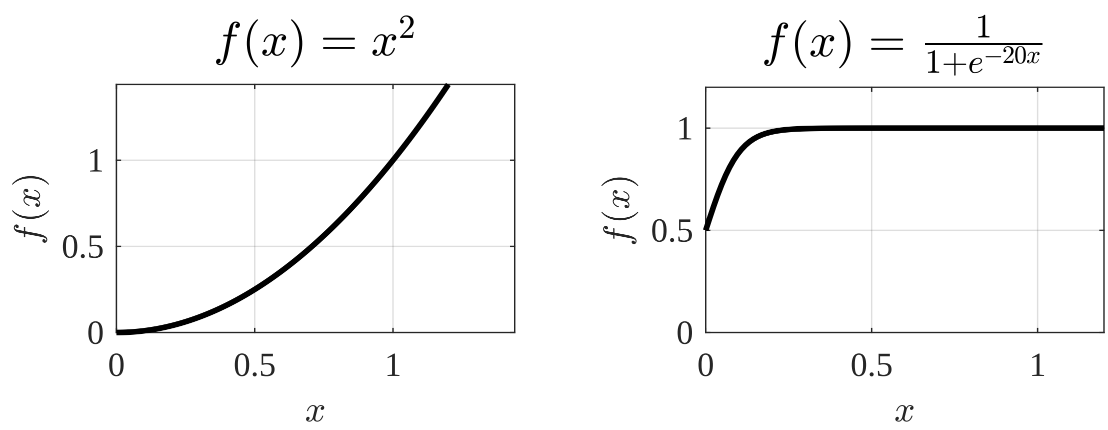
b) Approximate equation 1 by a second order polynomial with the help of a Taylor series expansion around the operating point $d_0 = -0.0165$.

c) Derive the inverse function of equation 1 such that a calculation of the distance in dependence of the inductance is possible ($f^{-1}(L) = d$).

d) Name one other possible way to linearize the characteristic curve given by equation 1.

e) Name one advantage and one disadvantage of the linearization around an operating point with the help of a Taylor series expansion.

f) Sketch the inverse functions $f^{-1}(x)$ qualitatively in the same graph as the function $f(x)$ below.



(a) Quadratic function

(b) Sigmoid function

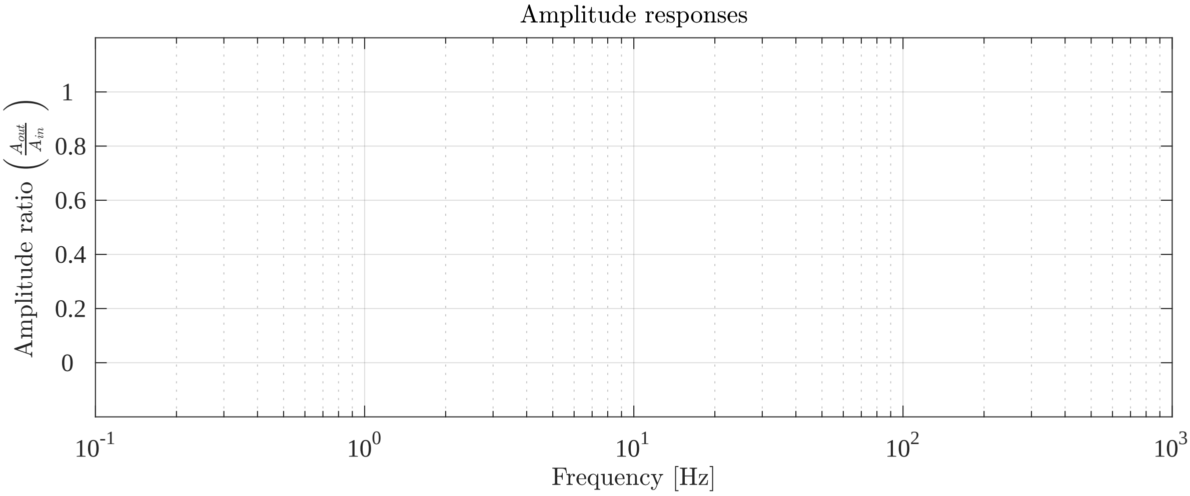
g) What problems might occur if nonlinear characteristics should be inverted? What property is required to allow the inversion of a function in general?

Task 3: Filter

Note that subtask c) can be solved independently from subtasks a) and b).

a) There are two filters G_{BS} and G_{BP}, that should be compared. One filter is a band-stop G_{BS}, the other is a band-pass G_{BP}. The limit (cut-off) frequencies of filter G_{BS} are $f_{BS1} = 10$ Hz and $f_{BS2} = 50$ Hz. The two limit (cut-off) frequencies of filter G_{BP} are $f_{BP1} = 20$ Hz and $f_{BP2} = 60$ Hz.

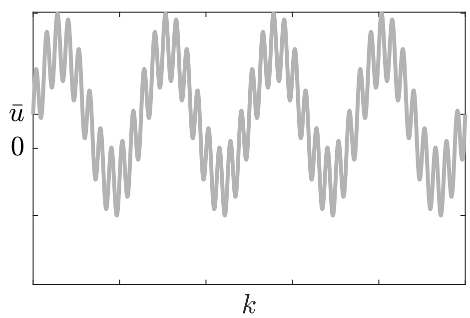
Sketch the ideal amplitude responses of both filters G_{BS} and G_{BP} in the figure below. Label your sketched amplitude responses with the correct transfer function.



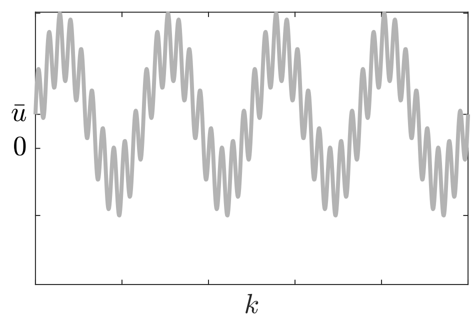
b) Now imagine to filter a signal that follows the equation

$$u(k) = \sin(2\pi \cdot 4\text{Hz}\ kT_0) + 0.5\sin(2\pi \cdot 40\text{Hz}\ kT_0) + \bar{u}$$

with the idealized filters from subtask a) and sketch the filtered output $y(k)$ *qualitatively* into the figures below. Assume that the responses have already reached the steady state. The value $\bar{u}$ labels the mean of the input signal $u(k)$. The sampling frequency is $f_0 = 1000$ Hz.



(a) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_{BS}.



(b) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_{BP}.

The following subtask is completely independent from subtasks a) and b).

c) The following list contains properties of filters. Assign which of these properties are valid for IIR (infinite impulse response) filters, which are valid for FIR (finite impulse response) filters and which are valid for both.

- Typically low order
- Can realize an integrator
- Instability can not occur
- Is a dynamic system
- An equivalent time-continuous system exists
- Can realize a pole at $s = 0$
- Steepness of the amplitude response in the transition band increases with filter order
- A linear phase can be achieved exactly
- Can realize a pole at $z = 0$

Task 4: Confidence Intervals

The temperature of a medium is measured. From a long history of temperature measurements, the standard deviation of measurement disturbances is known to be $\sigma_\nu = 0.3$ for the used thermometer.

Table 1 provides information about the factor c, that corresponds to the chosen confidence level $1-\alpha$, depending on the amount of data samples N and the error probability α.

a) One measurement is carried out and the displayed temperature is $\nu_m = 65.3°$ Celsius. Determine the interval in which the true temperature will be for a requested confidence level of 99.73 %.

b) In order to reduce the size of the temperature interval, $N = 5$ measurements are carried out, while the true temperature can be assumed to be constant.

i	1	2	3	4	5
$\nu_m(i)$	65.2	65.6	64.3	65.3	65.1

Determine the new interval in which the true temperature will be for a requested confidence level of 99.73 %.

c) Calculate the difference between each value $\nu_m(i)$ listed in the above Table and the mean $\bar{\nu}_m(i)$ of all 5 values. Write down the results in the following Table:

i	1	2	3	4	5
$\nu_m(i) - \bar{\nu}_m(i)$					

d) Assume that the $N = 5$ measurements are carried out with a new thermometer, where no information about the measurement disturbances is available (σ_ν unkown). Assume the errors to be Gaussian distributed. Again, determine the interval in which the true temperature will be for a requested confidence level of 99.73 %.

e) An additional measurement is carried out, such that there are 6 measurements in total for the new thermometer. Again, calculate the estimated measurement disturbance s_ν for 6 measurements and recalculate the interval in which the true temperature will be for a requested confidence level of 99.73 %, if $\nu_m(6) = 65.1°$ Celsius.

N	$\alpha = 31.73$ %	$\alpha = 4.55$ %	$\alpha = 0.27$ %
5	1.1	2.65	5.51
6	1.09	2.58	5.2
10	1.05	2.28	3.96
∞	1	2	3

Tab. 1: Factor c for a t-distribution depending on the amount of data samples N and the error probability α.

Task 5: Discrete-Time System

Given is the system below:

$$G(z) = \frac{b_0 + b_2 z^{-2}}{1 + a_2 z^{-2}} \tag{3}$$

The sampling time is T_0.

Note: Use only the following coefficient values of the system in the subtasks b, d, j and k:

$$b_0 = 0 \quad ; \quad b_2 = -3 \quad ; \quad a_2 = -0.64\,. \tag{4}$$

a) Draw the block diagram of the system $G(z)$.

b) Determine the poles of the system $G(z)$ (use only coefficients of (4)).

c) Transform the transfer function in the time domain.

d) Calculate the step response of the system for $k = 0, 1, \ldots, 6$ with $y(k) = 0$ for $k < 0$ (use only coefficients of (4)). Round the values if necessary to the third position after the decimal point.

e) How is the continuous time t linked with the discrete time k?

f) Now the sampling time should be increased to $T_\mathrm{n} = 2T_0$. How is this procedure called?

g) Which problem arises by increasing the sampling time?

h) Determine the transfer function $G_\mathrm{n}(z)$ with $T_\mathrm{n} = 2T_0$.

i) Draw the block diagram of $G_\mathrm{n}(z)$.

j) Determine the poles of the system $G_\mathrm{n}(z)$ (use only coefficients of (4)).

k) Calculate the step response for $k = 0, 1, \ldots, 3$ with $y(k) = 0$ for $k < 0$ (use only coefficients of (4)). Round the values if necessary to the third position after the decimal point. Compare this results with the results of subtask d with one sentence.

Task 6: Properties of Transfer Functions

The following transfer functions in the z-domain are given:

$$G_1 = \frac{z^2 - \frac{1}{4}}{z - \frac{1}{2}} \qquad G_2 = \frac{1}{z^{-1} - \frac{1}{4}} \qquad G_3 = \frac{1 - 0.2z}{z - 0.2}$$

$$G_4 = \frac{1}{1 + \frac{1}{4}z^{-1}} \qquad G_5 = \frac{1 + 2z + z^2}{z} \qquad G_6 = z^2 + 1 + z^{-1}$$

$$G_7 = z^{-5} + z^{-3} \qquad G_8 = \frac{1 + z + z^2}{2 + z}$$

Mark the correct properties in the following table

transfer function	FIR system	stable	causal	linear	all-pass
G_1					
G_2					
G_3					
G_4					
G_5					
G_6					
G_7					
G_8					

For each transfer function two points can be achieved. Each missing or wrong cross reduces the number of points by one. No negative points are given for each transfer function.

Solutions:

Task 1: Measurement of Speed

a) Equation for translational car speed, depending on the change of the wheel angle $\delta\varphi$, the time period δt and the *dynamic* tire radius r_d:

$$v = f(\delta\varphi, \delta t, r_d) = \frac{\delta\varphi}{\delta t} r_d(v)$$

$\boxed{2}$

b) Determination of the time period *deltat* for $v_{max} = 180$ km/h, such that there are at least two measurements during one whole wheel revolution.

At first we calculate the angular tire velocity for the given tire radius and the maximum translational velocity:

$$\begin{aligned} v_{max} &= 216 \text{ km/h} \hat{=} 60 \text{ m/s} \\ \omega &= \frac{60 \text{ m/s}}{0.24 \text{ m}} = 250 \frac{\text{rad}}{\text{s}} \end{aligned}$$

$\boxed{3}$

Because there should be at least two measurements per revolution, we have to calculate how long it takes until the angle of a tire has changed by π rad:

$$\delta t = \frac{\pi \text{ rad}}{250 \frac{\text{rad}}{\text{s}}} \approx 0.0126\text{s}$$

$\boxed{1}$

c) Assumption: The only error that comes into play is due to the ignorance of the changing tire radius.

A systematic error would arise. $\boxed{1}$

The equation for the Gaussian error propagation for systematic errors should be used. Because there are no errors in the time period δt and the change of the wheel angle $\delta\varphi$, only one term is left in that equation:

$$\begin{aligned} \Delta v &= \frac{\partial f}{\partial r_d} \Delta r_d \\ &= \frac{\delta\varphi}{\delta t}(r_d - r_d(v)) \end{aligned}$$

$\boxed{3}$

d) Here the approximation for random errors in practice should be used. Because no error occurs in the tire radius (assumption for this subtask), only two terms are left below the square root.

$$\begin{aligned} \Delta v &= \sqrt{\left(\frac{\partial f}{\partial \delta\varphi}\Delta\delta\varphi\right)^2 + \left(\frac{\partial f}{\partial \delta t}\Delta\delta t\right)^2} \\ &= \sqrt{\left(\frac{r_d}{\delta t}(\delta\varphi - \delta\varphi_{true})\right)^2 + \left(-\frac{\delta\varphi}{\delta t^2}(\delta t - \delta t_{true})\right)^2} \end{aligned}$$

$\boxed{5}$

e) Averaging the signal values from all four wheel only yields benefits in case of the subtask d), since these errors are random errors. No improvements can be expected through the averaging of systematic errors. $\boxed{2}$

$\boxed{\sum 17}$

Task 2: Inductive Displacement Measurement

a) Linearize the equation 1) around the operating point $d_0 = -0.0165$.

To linearize around an operating point, choose $n = 1$:

$$Tf(x; x_0) \approx f_{lin}(d) = f(d_0) + f^{(1)}(d_0) \cdot (d - d_0)$$

For the given equation we get: $\boxed{1}$

$$f^{(1)}(d_0) = -\frac{c}{(d_1 + d_0)^2}$$
$$\Rightarrow f_{lin}(d) = \frac{c}{d_1 + d_0} - \frac{c}{(d_1 + d_0)^2}(d - d_0)$$

$\boxed{3}$

b) 2nd order polynomial approximation via Taylor series expansion around $d_0 = -0.0165$.

The second derivative of the function f evaluated at d_0 is:

$$f^{(2)}(d_0) = 2 \cdot \frac{c}{(d_1 + d_0)^3}$$

As a result, the Taylor series expansion is: $\boxed{1}$

$$f_{\text{2nd Order}}(d) = \frac{c}{d_1 + d_0} - \frac{c}{(d_1 + d_0)^2}(d - d_0) + \frac{c}{(d_1 + d_0)^3}(d - d_0)^2$$

$\boxed{3}$

c) Derive the inverse function $f^{-1}(L) = d$.

$$L = \frac{c}{d_1 + d}$$
$$\Leftrightarrow d_1 + d = \frac{c}{L}$$
$$\Leftrightarrow d = \frac{c}{L} - d_1 = f^{-1}(L)$$

$\boxed{2}$

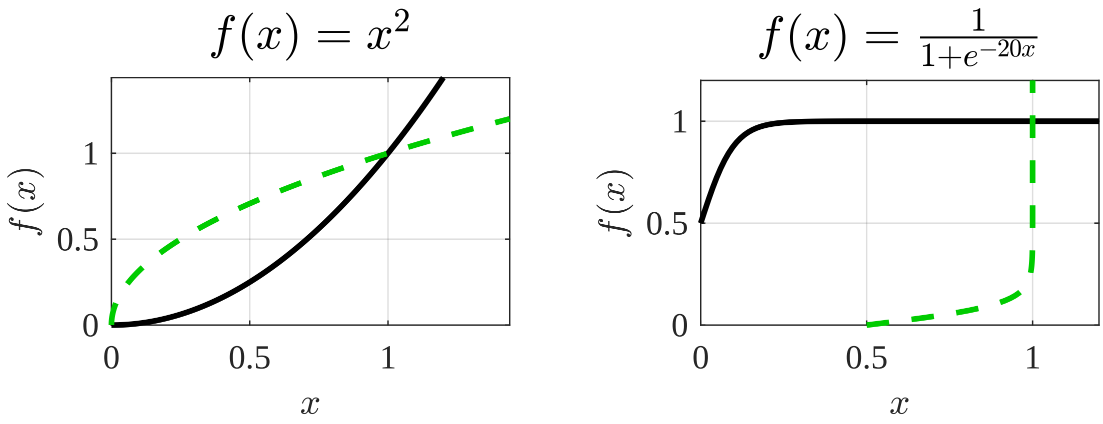
d) Name one other possible way to compensate nonlinearities.

Global approximation by an affine function or using the differential principle are two possible answers to this question. $\boxed{1}$

e) Name one advantage and one disadvantage of the linearization around an operating point with the help of a Taylor series expansion.

Easy to understand, characteristics can be described by only 1 (or 2 in the affine case) parameter(s), but the linearized function is only valid for small enough deviations from the operating point. $\boxed{2}$

f) Sketch the inverse functions $f^{-1}(x)$ qualitatively in the same graph as the function $f(x)$ below.



(a) Quadratic function (b) Sigmoid function

4

g) What problems might occur if nonlinear characteristics should be inverted? What property is required to allow the inversion of a function in general?

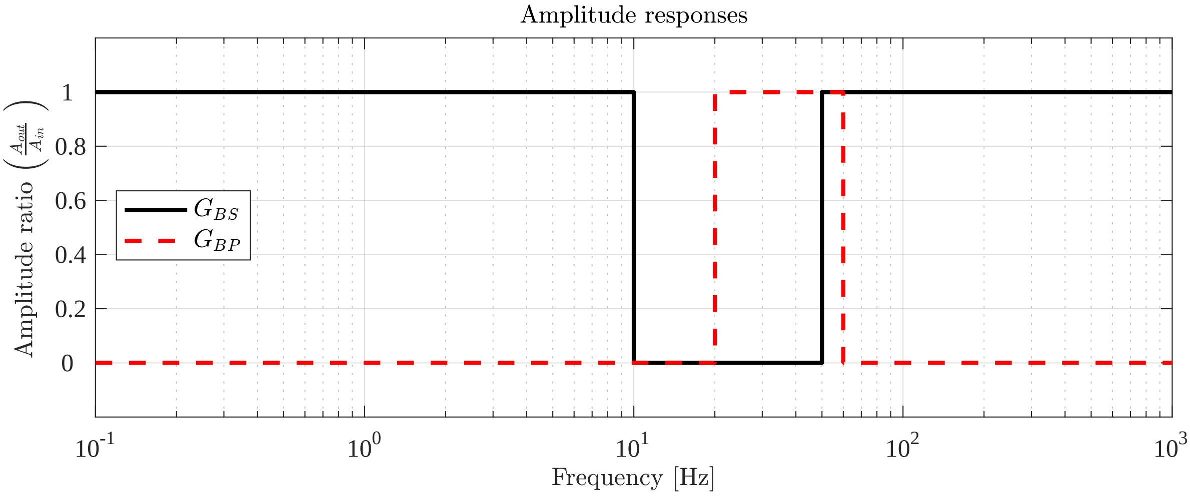
Functions have to be strictly monotonic to be invertible. 1

Very high sensitivities lead to very insensitive behavior in the inverted function and vice versa. 2

$\sum 20$

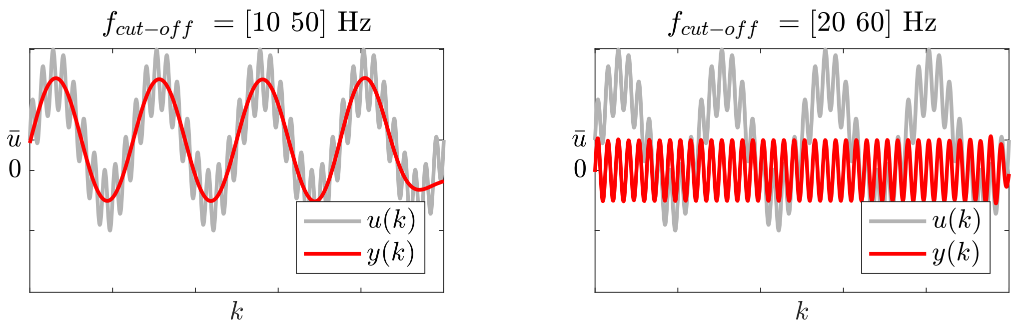
Task 3: Filter

a) Sketch the ideal amplitude responses for both filters G_{LP} and G_{BP} in the figure below.



6

b) Sketch the signals after applying the ideal filters from subtask a).



(a) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_{BS}.

(b) Sketch the output $y(k)$ qualitatively, if this sine wave is sent through filter G_{BP}.

6

c) The following list contains properties of filters. Assign which of these properties are valid for IIR (infinite impulse response) filters and which are valid for FIR (finite impulse response) filters.

- Typically low order → IIR
- Can realize an integrator → IIR
- Instability can not occur → FIR
- Is a dynamic system → IIR and FIR
- An equivalent time-continuous system exists → IIR
- Can realize a pole at $s = 0$ → IIR
- Steepness of the amplitude response in the transition band increases with filter order → IIR and FIR

- A linear phase can be achieved exactly $\rightarrow$ FIR
- Can realize a pole at $z = 0 \rightarrow$ FIR

11

$\sum 23$

Task 4: Confidence Intervals

a) Determine the interval in which the true temperature will be for a requested confidence level of 99.73 %.

For known disturbance variances, the true temperature lays in the interval:

$$\bar{x} - c\frac{\sigma}{\sqrt{N}} < x < \bar{x} + c\frac{\sigma}{\sqrt{N}} \ ,$$

where $\bar{x}$ denotes the estimated mean of the taken measurements. Here only one measurement is carried out, such that $N = 1$ and $\bar{\nu}_m = \nu_m = 65.3°$ Celsius. For the requested confidence level of $1 - \alpha = 99.73$ %, $c = 3$.

$$65.3 - 3\frac{0.3}{1} < x < 65.3 + 3\frac{0.3}{1}$$
$$64.4 < x < 66.2$$

4

b) In order to reduce the size of the temperature interval, $N = 5$ measurements are carried out, while the true temperature can be assumed to be constant.

i	1	2	3	4	5
$\nu_m(i)$	65.2	65.6	64.3	65.3	65.1

Determine the new interval in which the true temperature will be for a requested confidence level of 99.73 %.

Here the approximated mean of all measurements equals $\bar{\nu}_m = 65.1$. Therefore the 1
interval is

$$65.1 - 3\frac{0.3}{\sqrt{5}} < x < 65.1 - 3\frac{0.3}{\sqrt{5}}$$
$$64.7 < x < 65.5$$

1

c) Write down the results in the following Table:

i	1	2	3	4	5
$\nu_m(i) - \bar{\nu}_m(i)$	0.1	0.5	-0.8	0.2	0

5

d) Again, determine the interval in which the true temperature will be for a requested confidence level of 99.73 % for the new thermometer.

The standard deviation s_ν has to be estimated. Based on the 5 measurements, the estimated variance is $s_\nu = 0.48$. 1

Because the standard deviation of the thermometer is unknown, the factor c has to be determined with the help of Tab. 1: $c = 5.51$. 1

The mean estimation has not changed, because we assumed to have the same measurements as in subtask b). $\rightarrow \bar{\nu}_m = 65.1$ 1

$$65.1 - 5.51\frac{0.48}{\sqrt{5}} < x < 65.1 - 5.51\frac{0.48}{\sqrt{5}}$$
$$63.9 < x < 66.3$$

$\boxed{1}$

e) Again, calculate the interval in wich the true temperature will be for a requested confidence level of 99.73 %, if $\nu_m(6) = 65.1°$ Celsius is added.

Because measurement $\nu_m(6)$ equals the mean value of the already existing 5 measurements, this value stays constant and has not to be calculated again $\rightarrow \bar{\nu}_m = 65.1$. $\boxed{1}$

The Table from subtask c) can be utilized to estimate the new standard deviation more quickly: $\sigma_\nu = 0.43$. $\boxed{2}$

Because the number of samples has increased, an other c factor has to be used: $c = 5.2$ $\boxed{2}$

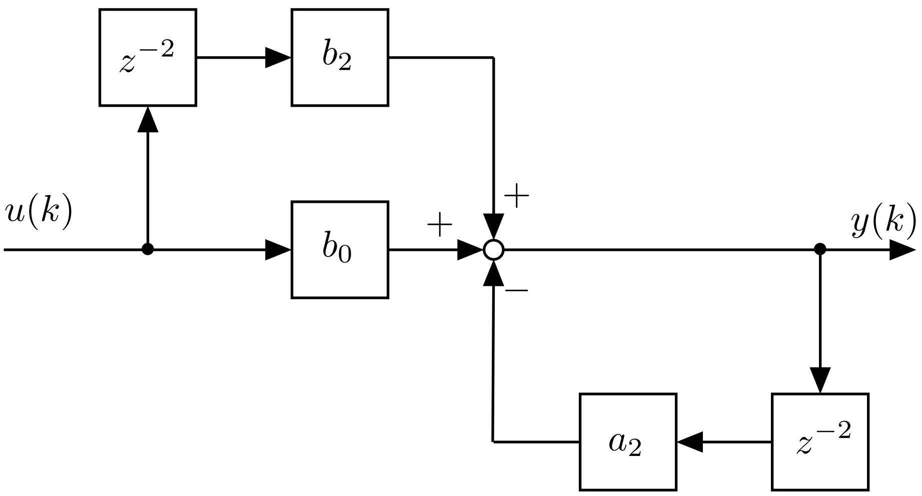
$$65.1 - 5.2\frac{0.43}{\sqrt{6}} < x < 65.1 - 5.2\frac{0.43}{\sqrt{6}}$$
$$64.2 < x < 66.0$$

$\boxed{1}$

$\boxed{\sum 21}$

Task 5: Discrete-Time System (23 points)

a) Block diagram of the system: $\boxed{3}$



b) The poles of the transfer function are determined by:

$$1 + a_2 z^{-2} = 0 \tag{1}$$
$$z^2 + a_2 = 0 \tag{2}$$
$$\Rightarrow \quad z_{1,2} = \pm\sqrt{-a_2} \tag{3}$$
$$\Rightarrow \quad z_{1,2} = \pm 0.8 \tag{4}$$

$\boxed{1}$

c) Transformation in time domain:

$$U(z)\left(b_0 + b_2 z^{-2}\right) = Y(z)\left(1 + a_2 z^{-2}\right) \quad \bullet\!\!-\!\!\circ \quad b_0 u(k) + b_2 u(k-2) = y(k) + a_2 y(k-2) \tag{5}$$
$$\Rightarrow \quad y(k) = b_0 u(k) + b_2 u(k-2) - a_2 y(k-2) \tag{6}$$

$\boxed{1}$

d) Step response for $k = 0, 1, \ldots, 6$:

$$k=0 \qquad y(0) = b_0 \underbrace{\sigma(0)}_{=1} + b_2 \underbrace{\sigma(-2)}_{=0} - a_2 \underbrace{y(-2)}_{=0} \tag{7}$$
$$y(0) = b_0 \tag{8}$$
$$y(0) = 0 \tag{9}$$

$$k=1 \qquad y(1) = b_0 \underbrace{\sigma(1)}_{=1} + b_2 \underbrace{\sigma(-1)}_{=0} - a_2 \underbrace{y(-1)}_{=0} \tag{10}$$
$$y(1) = b_0 \tag{11}$$
$$y(1) = 0 \tag{12}$$

$$k=2 \qquad y(2) = b_0 \underbrace{\sigma(2)}_{=1} + b_2 \underbrace{\sigma(0)}_{=1} - a_2 \underbrace{y(0)}_{=b_0} \tag{13}$$
$$y(2) = b_0 + b_2 - a_2 b_0 \tag{14}$$
$$y(2) = 0 + (-3) - (-0.64)\cdot 0 = -3 \tag{15}$$

$$k=3 \qquad y(3) = b_0 \underbrace{\sigma(3)}_{=1} + b_2 \underbrace{\sigma(1)}_{=1} - a_2 \underbrace{y(1)}_{=b_0} \tag{16}$$
$$y(3) = b_0 + b_2 - a_2 b_0 \tag{17}$$
$$y(3) = 0 + (-3) - (-0.64)\cdot 0 = -3 \tag{18}$$

$$k=4 \qquad y(4) = b_0 \underbrace{\sigma(4)}_{=1} + b_2 \underbrace{\sigma(2)}_{=1} - a_2 \underbrace{y(2)}_{=b_0+b_2-a_2b_0} \tag{19}$$
$$y(4) = b_0 + b_2 - a_2\left(b_0 + b_2 - a_2 b_0\right) \tag{20}$$
$$y(4) = 0 + (-3) - (-0.64)\left(0 + (-3) - (-0.64)\cdot 0\right) = -4.92 \tag{21}$$

$$k=5 \qquad y(5) = b_0 \underbrace{\sigma(5)}_{=1} + b_2 \underbrace{\sigma(3)}_{=1} - a_2 \underbrace{y(3)}_{=b_0+b_2-a_2b_0} \tag{22}$$
$$y(5) = b_0 + b_2 - a_2\left(b_0 + b_2 - a_2 b_0\right) \tag{23}$$
$$y(5) = 0 + (-3) - (-0.64)\left(0 + (-3) - (-0.64)\cdot 0\right) = -4.92 \tag{24}$$

$$k=6 \qquad y(6) = b_0 \underbrace{\sigma(6)}_{=1} + b_2 \underbrace{\sigma(4)}_{=1} - a_2 \underbrace{y(4)}_{=b_0+b_2-a_2(b_0+b_2-a_2b_0)} \tag{25}$$
$$y(6) = b_0 + b_2 - a_2\left(b_0 + b_2 - a_2\left(b_0 + b_2 - a_2 b_0\right)\right) \tag{26}$$
$$y(6) = 0 + (-3) - (-0.64)\ldots$$
$$\ldots\left(0 + (-3) - (-0.64)\left(0 + (-3) - (-0.64)\cdot 0\right)\right) \tag{27}$$
$$y(6) = -6.1488 \tag{28}$$
$$y(6) \approx -6.149 \tag{29}$$

The step response stays constant for two time steps ($y(0) = y(1)$, $y(2) = y(3)$, …). $\boxed{5}$

e) Linkage t and k:

$$t = kT_0, \quad k \in \mathbb{Z}\,. \tag{30}$$

$\boxed{1}$

f) Increasing the sampling time:

The process of increasing the sampling time is called down sampling. $\boxed{1}$

g) By down sampling aliasing may occur. $\boxed{1}$

h) Transfer function with increased sampling time $G_\mathrm{n}(z)$:

By increasing the sampling time to $T_\mathrm{n} = 2T_0$, the following equations arise.

$$\begin{aligned}
z &= e^{sT_0} && (31)\\
z_\mathrm{n} &= e^{sT_{0,\mathrm{n}}} && (32)\\
z_\mathrm{n} &= e^{s2T_0} && (33)\\
\Rightarrow \quad z_\mathrm{n} &= z^2 && (34)\\
\Rightarrow \quad z_\mathrm{n}^{\frac{1}{2}} &= z\,. && (35)
\end{aligned}$$

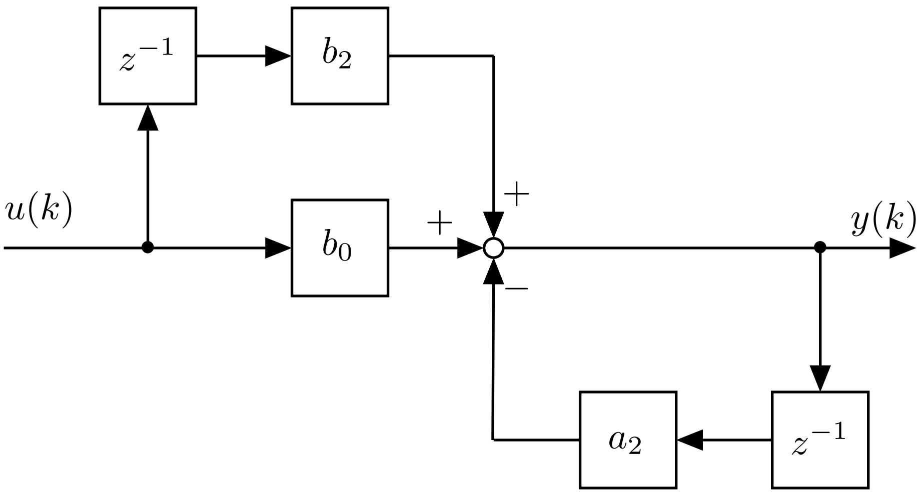
For the down sampled transfer function follows:

$$G_\mathrm{n}(z_\mathrm{n}) = \frac{b_0 + b_2 z_\mathrm{n}^{\frac{1}{2}\cdot(-2)}}{1 + a_2 z_\mathrm{n}^{\frac{1}{2}\cdot(-2)}} \tag{36}$$

$$G_\mathrm{n}(z_\mathrm{n}) = \frac{b_0 + b_2 z_\mathrm{n}^{-1}}{1 + a_2 z_\mathrm{n}^{-1}}\,. \tag{37}$$

$\boxed{3}$

i) Block diagram of $G_\mathrm{n}(z)$.



$\boxed{3}$

j) Poles of $G_\mathrm{n}(z)$:

$$\begin{aligned}
1 + a_2 z^{-1} &= 0 && (38)\\
z^1 + a_2 &= 0 && (39)\\
\Rightarrow \quad z_1 &= -a_2\,. && (40)
\end{aligned}$$

$\boxed{1}$

k) Step response of $G_\mathrm{n}(z)$ for $k = 0, 1, \ldots, 3$:

$$U(z)\left(b_0 + b_2 z^{-1}\right) = Y(z)\left(1 + a_2 z^{-1}\right) \quad \bullet\!\!-\!\!\circ \quad b_0 u(k) + b_2 u(k-1) = y(k) + a_2 y(k-1) \tag{41}$$

$$\Rightarrow \quad y(k) = b_0 u(k) + b_2 u(k-1) - a_2 y(k-1) \tag{42}$$

$$k=0 \qquad y(0) = b_0 \underbrace{\sigma(0)}_{=1} + b_2 \underbrace{\sigma(-1)}_{=0} - a_2 \underbrace{y(-1)}_{=0} \tag{43}$$

$$y(0) = b_0 \tag{44}$$

$$y(0) = 0 \tag{45}$$

$$k=1 \qquad y(1) = b_0 \underbrace{\sigma(1)}_{=1} + b_2 \underbrace{\sigma(0)}_{=1} - a_2 \underbrace{y(0)}_{=b_0} \tag{46}$$

$$y(1) = b_0 + b_2 - a_2 b_0 \tag{47}$$

$$y(1) = 0 + (-3) - (-0.25) \cdot 0 = -3 \tag{48}$$

$$k=2 \qquad y(2) = b_0 \underbrace{\sigma(2)}_{=1} + b_2 \underbrace{\sigma(1)}_{=1} - a_2 \underbrace{y(1)}_{=b_2 - a_2 b_0} \tag{49}$$

$$y(2) = b_0 + b_2 - a_2 \left(b_0 + b_2 - a_2 b_0\right) \tag{50}$$

$$y(2) = 0 + (-3) - (-0.64)\left(0 + (-3) - (-0.64) \cdot 0\right) = -4.92 \tag{51}$$

$$k=3 \qquad y(3) = b_0 \underbrace{\sigma(3)}_{=1} + b_2 \underbrace{\sigma(2)}_{=1} - a_2 \underbrace{y(2)}_{=b_0 + b_2 - a_2(b_0 + b_2 - a_2 b_0)} \tag{52}$$

$$y(3) = b_0 + b_2 - a_2 \left(b_0 + b_2 - a_2 \left(b_0 + b_2 - a_2 b_0\right)\right) \tag{53}$$

$$y(3) = 0 + (-3) - (-0.64)\left(0 + (-3) - (-0.64)\left(0 + (-3) - (-0.64) \cdot 0\right)\right) \tag{54}$$

$$y(3) = -6.1488 \tag{55}$$

$$y(3) \approx -6.149 \tag{56}$$

In comparison to the original system $G(z)$ the step response does not stay constant for two time steps. Each $y(k)$ is different two the previous and following. $\boxed{3}$

Task 6: Properties of Transfer Functions

The following transfer functions in the z-domain are given:

$$G_1 = \frac{z^2 - \frac{1}{4}}{z - \frac{1}{2}} \qquad G_2 = \frac{1}{z^{-1} - \frac{1}{4}} \qquad G_3 = \frac{1 - 0.2z}{z - 0.2}$$

$$G_4 = \frac{1}{1 + \frac{1}{4}z^{-1}} \qquad G_5 = \frac{1 + 2z + z^2}{z} \qquad G_6 = z^2 + 1 + z^{-1}$$

$$G_7 = z^{-5} + z^{-3} \qquad G_8 = \frac{1 + z + z^2}{2 + z}$$

Mark the correct properties in the following table

transfer function	FIR system	stable	causal	linear	all-pass
G_1	x	x		x	
G_2			x	x	
G_3		x		x	x
G_4		x	x	x	
G_5	x	x		x	
G_6	x	x		x	
G_7	x	x	x	x	
G_8				x	