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Subsistence level and theory of the welfare state

by

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Abstract

In a broader sense the welfare state ex ante can be seen as a social insurance for life-time risks, and ex post as a redistribution mechanism of incomes. Sinn (1995) has developed a normative theory of the welfare state in this view. On a constitutional plain agents determine the amount of optimal redistributive taxation behind a veil of ignorance relative to their life-time incomes. Our paper extends this theory of the welfare state by allowing for incomes which may fall below a subsistence level. If the income distribution is completely above the subsistence level, agents favour the confiscatorial tax. If some income realizations fall below the subsistence level, there are cases in which the laissez-faire-tax is socially preferred to the confiscatorial tax.

JEL Classification No. H 21, H 23
Key words: social insurance, redistribution, subsistence level, risk-taking

1. Introduction

In recent years the welfare state has come under increasing criticism for eroding incentives for working and risk-taking. In the narrowe sense the state as social insurer provides public health insurance, public pension system, unemployment insurance, etc. Citizens are liable to pay fees and taxes, and the state supplies public goods and services, and transfers. In a broader sense the welfare state can be interpreted as encompassing all public institutions redistributing income and property from the rich to the poor. This view can at best be traced back to Friedman (1953), Buchanan and Tullock (1962, chapter 13), and Arrow (1970, p. 185 ff.).

For an individual at her time of birth or at the beginning of her vocational training social insurance has a risk reducing function. It reduces the risk of failing to have a successful carreer or, in a broader sense, the risk to live in poverty. Technically speaking, the social insurance reduces the variance of the individual's real lifetime income. Reasons for poverty are illness, missed opportunities, or unfavourable endowments of innate abilities. Knowing these reasons people may sign an insurance contract against bad luck. Redistribution and insurance are two sides of the same coin.

The basic assumption of the analysis is that all individuals are in a situation where they do not know their future position in society, implying that their lifetime incomes are uncertain. Agents make their choices about investments[1] in human and physical capital behind a veil of ignorance. The veil is perfect in the sense that every agent faces the same risk regarding her lifetime income. This normative veil-of-ignorance concept is part of modern welfare theory, especially the theory of constitutional choice; it was first mentioned by Harsanyi (1953, 1955) and plays a central role in Rawls (1971). The normative relevance of the veil of ignorance consists in the presumption that a redistribution scheme is socially desirable, if and only if it is unanimously preferred to all alternative redistribution schemes behind the veil. The design of fiscal redistribution schemes is important because it affects individual incentives.

The main focus of this paper is the introduction of a subsistence level of income defined as that amount of money which is needed to satisfy a person's basic needs such as drinks and food. If a person's lifetime income falls below the subsistence level, she perishes. The valuation of incomes behind the veil of ignorance and below the subsistence level, which implies that people starve of hunger, is assumed to be equal. The willingness-to-pay to avoid incomes below the subsistence level is the value of human life The value of life taken as a basis for cost-benefit-analysis of public projects, and determined by the human capital approach or the willingness-to-pay method is less than infinity, see Arthur (1981), whereas a person's valuation of her own life, if the putative project will destroy it, is infinity, because money is no good to her when she is dead, see Broome (1978). However, both cases are admitted and discussed. The subsistence level plays an important role in developing countries, but it must not be neglected in developed countries either in view of slums and many (homeless) very poor people.

The present paper is based on Sinn (1995, 1996) with the difference that in Sinn's (1995, 1996) model, the agents' consumption set is not bounded from below. Implicitly there it is assumed that agents can survive with arbitrarily low (and negative!) incomes. Our analysis is in a more general framework and contains the Sinn assumptions as a special case.

We aim at answering the following questions: Firstly, does increasing public redistribution erode the incentives for individual risk-taking? Secondly, which redistribution scheme is unanimously chosen on the constitutional level? We restrict our investigations to the scenario in which all agents take the government's budget constraint into account. For this scenario

[1] Investments and risk-taking are synonymous. Increasing risk-taking is equivalent to decreasing investments.

Sinn (1995) showed that increasing redistribution induces growing incentives for risk-taking, and that society unanimously chooses that redistribution scheme which leads to the egalitarian (ex post) income distribution. This result implies an optimal income tax rate of unity.

The model is constructed in the scale and location parameter methodology developed by Tobin (1958), Meyer (1987) and Sinn (1983). The only restriction imposed on stochastic variables is that they are required to belong to the same linear distribution class. The advantage of the (μ,σ)-approach over the unrestricted expected utility approach is that individual income decisions under uncertainty are immediately linked to national income and changes in national income inequality.

The paper is organized as follows. Section 2 presents the model. In section 3 the utility function in the presense of the subsistence level is introduced, and the properties of the indifference function are derived. Section 4 describes individual behaviour concerning risk-taking and income taxation, and finally in section 5 some concluding remarks are drawn.

2. The basic framework

The model is identical to Sinn (1995). A detailed description of the framework can be found in Sinn (1995, 1996), thus the presentation in this section is as concise as possible. Individuals are assumed to be identical ex ante. The stochastic pre-tax income of the representative agent is given by[2]

$$X(e) = m - L(e) - e. \tag{1}$$

Risk occurs in form of an income loss $L > 0$ whose magnitude can be influenced by the agent's self-insurance effort, e.g. investments in physical or human capital, e. Let m be an exogenously given upper bound of pre-tax income when income losses and efforts are neglected. The connection between income loss and self-insurance effort is determined by

$$L(e) = \lambda(e) \cdot \Phi, \quad \text{for } \Phi \geq 0 \text{ and } \lambda(e) > 0 \quad \forall\, e \in [0, \bar{e}]. \tag{2}$$

Φ is the random state of nature. The ex post realization of Φ depends e.g. on illness, missed opportunities, injuries or unfavourable endowments of innate abilities. The self-insurance function $\lambda(e)$ introduced by Ehrlich and Becker (1972) in the theory of insurance demand specifies the reduction of income losses by increasing efforts, and is assumed to be twice continously differentiable and to satisfy $\lambda_e(e) < 0, \quad \lambda_{ee}(e) \geq 0, \quad \lambda_e(0) = -\infty$.

Apart from the reduction of income losses, efforts cause costs[3] e which together with $L(e)$ have to be substracted from m as shown in (1).

Now suppose the government supplies a social insurance. It requires an insurance premia $\tau \cdot X(e)$, where $\tau \in [0, 1]$ is the tax rate, and provides a transfer p. Social insurance can be viewed as redistributive taxation or, in a general sense, as a redistributive scheme. Post-tax income then is

$$Y(e; p, \tau) = (1 - \tau) \cdot X(e) + p. \tag{3}$$

The transfer p is deterministic whereas the tax liability is stochastic. To balance the fiscal budget, the government has to choose public transfer and tax rate such that the transfer is equal to average tax liability[4]

$$\tau \cdot E[X(e)] = p. \tag{4}$$

[2] Capital letters are stochastic, small letters are deterministic variables.
[3] The price of efforts is standardized to 1.
[4] E is the symbol of the expectation operator.

Equation (4) ensures that the agent receives a fair transfer in the sense that the state distributes everything that it takes in.

In order to compare different incomes Y the representative individual has to recognize that Y is a random variable. If the compared income distributions belong to the same linear distribution class then it is possible to value different income distributions by their mean μ and their standard deviation σ. This is further illustrated in section 3. If the society comprises a large number of citizens and if the distribution of Φ is the same for everyone, then the distribution of Y can be interpreted as the ex post actual realized income distribution. The law of large numbers ensures that the probability of a realization of Y is identical to the relative cumulation of this realization. Therefore the mean μ and the dispersion measure σ coincide with the ex post mean national income and the income inequality, respectively. Formally, μ and σ are expressed as [5]

$$\mu := \mu(e;\tau,p) := E[Y(e;p,\tau)] = (1-\tau)\cdot(m-\lambda(e)\cdot E[\Phi]-e)+p\,, \tag{5}$$

$$\sigma := \sigma(e;\tau) := R[Y(e;p,\tau)] = \lambda(e)\cdot R[\Phi]\cdot(1-\tau)\,, \tag{6}$$

$$p := p(e;\tau) = \tau\cdot(m-\lambda(e)\cdot E[\Phi]-e)\,. \tag{7}$$

In the literature there are diverging opinions about individual behaviours concerning the fiscal budget constraint. Some economists assume that agents 'see-through' the working of the government and consider the fiscal budget constraint in their behaviour. Other economists reject the see-through assumption. The discussion of agents' behaviour is led back to the size of the population. If the size is sufficiently small, the see-through assumption is usually applied. If the population consists of many citizens, economists assume that citizens do not have a significant effect on the fiscal budget constraint. Boadway, Pestieau and Wildasin (1989), Bernheim (1986) are familiar examples for see-through, Bernheim, Schleifer and Summers (1985) or Konrad and Lommerud (1995) for non-see-through. Apart from the population size it plays a role in our approach that individuals decide behind a veil of ignorance. It seems unplausible that agents are so clever to check their decision situation under uncertainty, but not to check the job of the government. Thus we take the see-through assumption. Equations (5) and (6) contain the policy parameters τ and p. Under consideration of the see-through assumption we eliminate p in (5) through (7) in order to establish

$$\mu = m-\lambda(e)\cdot E[\Phi]-e\,. \tag{8}$$

In order to simplify the analysis, and to have a better understanding of the model implications it is expedient to introduce the pre-tax standard deviation of income

$$\sigma_G := \sigma_G(e) := \sigma(e;\tau=0) = \lambda(e)\cdot R[\Phi] \tag{9}$$

as an endogenous variable. Then (8) and (6) can be rearranged to read[6]

$$\mu = m-\frac{E[\Phi]}{R[\Phi]}\cdot\sigma_G-\lambda^{-1}\left(\frac{\sigma_G}{R[\Phi]}\right) =: M(\sigma_G) \tag{10}$$

$$\sigma = (1-\tau)\cdot\sigma_G\,. \tag{11}$$

The geometrical locus of (μ,σ)-combinations satisfying (10) and (11) for $\tau = 0$ $(\tau > 0)$ will be called self-insurance line (redistribution line). To close the model we introduce the representative agent's utility function in the next section.

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[5] R is the symbol of the standard deviation.

[6] λ^{-1} is the inverse image of λ.

3. The utility and indifference function

Meyer (1987) and Sinn (1983) have shown that the (μ, σ)-approach is in accordance with the expected utility approach if the compared random variables belong to the same linear distribution class. These authors take as the basis for deriving the indifference functions increasing and concave utility functions. The new aspect in our paper as mentioned in the introduction is the existence of an exogenously given subsistence level c. Consider post-tax incomes as described in (3). If $y < c$, then the agent is not able to satisfy basic needs and perishes. The representative consumer is indifferent between all incomes below the subsistence level. The distance between the utility of the subsistence level and the utility of incomes below c is the value of life, denoted with d, or in other words the willingness-to-pay to avoid the death. The domain of d is assumed to be $[0, \infty[$ and in order to be as general as possible we study the limit case in which d converges to infinity. Formally, the utility function $u(\mathrm{y})$ and the absolute Arrow-Pratt measure $r(\mathrm{y})$ have the following properties:

$$u(\mathrm{y}) > 0,\ u'(\mathrm{y}) > 0,\ u''(\mathrm{y}) < 0,\ r'(\mathrm{y}) < 0 \qquad \text{for } \mathrm{y} > c, \tag{A1}$$
$$u(c) = 0,$$
$$u(\mathrm{y}) = -d \qquad \text{for } \mathrm{y} < c,$$
$$u'(\mathrm{y}) = u''(\mathrm{y}) = 0 \qquad \text{for } \mathrm{y} < c.$$

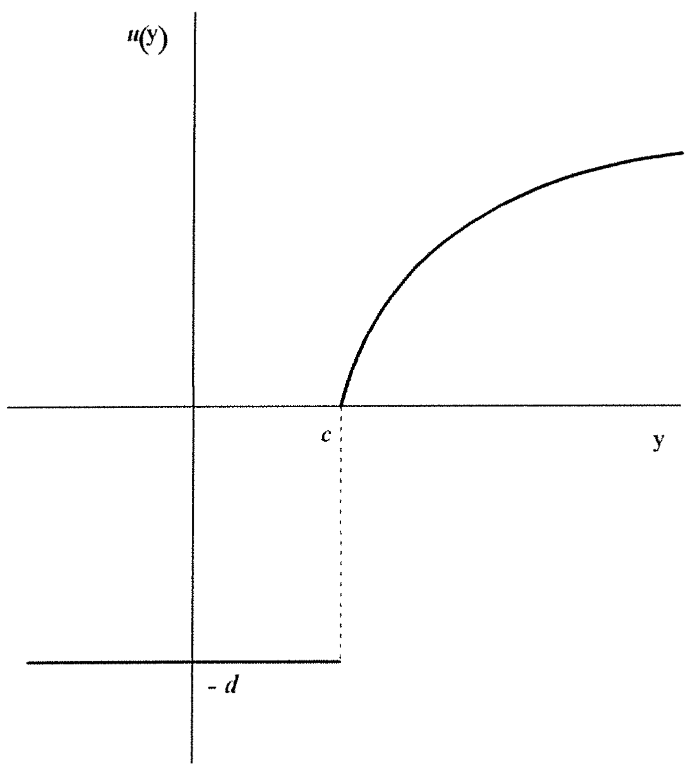

Figure 1: The utility function

Figure 1 illustrates assumption (A1). For incomes $\mathrm{y} > c$ the utility function is positive with positive, decreasing marginal utility which implies that the agent is risk averse. $r'(\mathrm{y}) < 0$ states that she has decreasing absolute risk aversion which is the usual assumption made in studies on behaviour under risk. The utility curve cuts the abscissa at point c and corresponds with the straight line *-d* for $y < c$. Note that the first derivative of u at point $y = c$ does not exist, the utility curve and the marginal utility curve have a jump there.

Since the assumptions about the utility function are different in comparison to those in the works of Sinn (1983, 1989) and Meyer (1987), it is necessary to derive the properties of the indifference function for utility functions satisfying (A1). For that we need more information. The random variables are assumed to belong to a linear distribution class. Let Y be a random variable which is a function y of the state of nature Φ and some vector of choice variables e. $E[Y] = \mu$ and $R[Y] = \sigma$ denote, respectively, the expectation and standard deviation of Y. Distributions form a linear distribution class, if $y(\Phi, e) = \mu(e) + \sigma(e) \cdot z(\Phi)$, i.e. if the random variables Y have a common standardized form $Z := \frac{Y - \mu}{\sigma}$ whose properties are independent of the choice variable e. Then the expected utility of Y can be transformed into the (μ, σ)-space by the relation[7]:

$$E(u(Y)) = \int_{\mu-\sigma\cdot b}^{\mu+\sigma\cdot b} u(y) \cdot dF(y) = \int_{-b}^{b} u(\mu + \sigma \cdot z) \cdot dF(z) =: U(\mu, \sigma). \tag{13}$$

$U(\mu, \sigma)$ is called mean-standard deviation (MS) preference function. To keep the analysis simple, we make in addition to the linear distribution class the following assumption which is somewhat technical in nature.

$$f_Z(z) \text{ is continuous, } f_Z'(z) \leq 0 \quad \text{for} \quad z \geq 0, \text{ and} \tag{A2}$$
$$[f(b) > 0 \quad \text{or there exists a} \quad z \geq 0 \quad \text{such that} \quad f_Z'(z) < 0].$$

Assumption (A2) excludes among other things the in our view empirically less relevant left-skewed distributions, but both symmetric, unimodal (e.g. the Gaussian), right-skewed, unimodal (e.g. the Maxwell or the Rayleigh), and the uniform distribution are examples for distributions which satisfy (A2). Then a first statement is that the MS-preference function increases with growing μ and σ if the compared (μ, σ)-combinations have the same $\frac{c - \mu}{\sigma}$-ratio. This is shown by

Property 1: *For all* $\bar{\mu}$, $\hat{\mu}$, $\bar{\sigma} \neq 0, \hat{\sigma} \neq 0$ *with* $\bar{\mu} > c - \bar{\sigma} \cdot b$ *holds:*

If $\frac{c - \bar{\mu}}{\bar{\sigma}} = \frac{c - \hat{\mu}}{\hat{\sigma}}$ *and* $\hat{\mu} > \bar{\mu}$, *then* $U(\hat{\mu}, \hat{\sigma}) > U(\bar{\mu}, \bar{\sigma})$.

Property 1 can be interpreted with regard to indifference curves. An indifference curve never can intersect a straight line $\mu > c + \sigma \cdot a$ (with $a > -b$) twice.

The MS-preference function serves as the starting point for the following investigations. The task is to derive the properties of the indifference function which is implicitly defined by the equation $U(\mu, \sigma) = \text{constant}$. The geometrical pendant of the indifference function is the indifference curve. $i(\mu, \sigma)$ represents the slope of the indifference curve and is given by

$$i(\mu, \sigma) := \left. \frac{d\mu}{d\sigma} \right|_{U(\mu,\sigma)=\text{const.}} = -\frac{U_\sigma(\mu, \sigma)}{U_\mu(\mu, \sigma)}. \tag{14}$$

Property 2 gives further details about the sign of the indifference curve slope.

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[7] Note that b and $-b$ with $0 < b \leq \infty$ are the upper and lower bound of integration of Z, and $\mu + \sigma \cdot z$ and $\mu - \sigma \cdot z$ are the upper and lower bounds of integration of Y.

Property 2:

(a) Let $c + \sigma \cdot b \leq \mu$. Then $i(\mu, \sigma) > 0$ for $\sigma > 0$ and $i(\mu, 0) = 0$.

(b) Let k be implicitly defined by $\int_{-k}^{b} u'(\mu + \sigma \cdot z) \cdot z \cdot dF(z) = 0$ and let

$c + k \cdot \sigma < \mu < c + \sigma \cdot b$. Then $i(\mu, \sigma) > 0$.

(c) Let $c - \sigma \cdot b < \mu < c$. Then $i(\mu, \sigma) < 0$.

(d) Let $\mu \leq c - \sigma \cdot b$. Then $U(\mu, \sigma) = -d$.

According to property 2 the (μ, σ)-space can be divided into different areas with regard to the indifference curve slope. We distinguish three cases. First, if (μ, σ) is such that $\mu \leq c - \sigma \cdot b$ then the agents are indifferent between all (μ, σ)-combinations (property 2 (d)), see the grey areas in figure 2. This case is rather unrealistic since the distribution of the random variable income lies completely below the subsistence level, and therefore it will be neglected in the further analysis. Second, if (μ, σ) is such that $\mu \geq c + \sigma \cdot b$ the distribution of incomes is completely above c (property 2 (a)). This is the special case, in which the results of Sinn (1995, 1996) hold. The indifference curves run vertically into the μ-axis $\left(i(\mu, 0) = 0\right)$ and their slope is positive for $\sigma > 0$. In the third case $(c - \sigma \cdot b < \mu < c + \sigma \cdot b)$, which is in our view the economically relevant one, some realizations are above, some are below c. This case has to be split up into three subcases. If $c + \sigma \cdot k < \mu < c + \sigma \cdot b$ only few income realizations fall below c and property 2 (b) points out that the indifference curve slope is positive and has not changed with respect to its sign in comparison to the case in which the distribution is completely above c. Considering $c - \sigma \cdot b < \mu < c$, where the income realizations are largely below the subsistence level, the resulting indifference curve slope is negative (property 2 (c)). The last subcase which is not part of property 2 is $c < \mu < c + \sigma \cdot k$. In this interval the sign of $i(\mu, \sigma)$ depends on the value of d. The one extreme is $d = 0$. For $c < \mu < c + \sigma \cdot k$ and $d = 0$ we get $i(\mu, \sigma) < 0$. The indifference curve slope has the value zero if $\mu = c + \sigma \cdot k$. The other extreme is d converges to infinity. Applying the rule of de l'Hospital we obtain

$$\lim_{d \to \infty} i(\mu, \sigma) = \lim_{d \to \infty} \left(-\frac{\partial_d U_\sigma(\mu, \sigma)}{\partial_d U_\mu(\mu, \sigma)} \right) = -\left(\frac{c - \mu}{\sigma} \right). \tag{15}$$

For $d \to \infty$ the sign of the indifference curve slope is positive if $c < \mu < c + \sigma \cdot k$, and takes the value zero at point $\mu = c$. Both, the results for $d = 0$ and the results for $d \to \infty$ are mapped in figure 2. For $0 < d < \infty$ the sign of the indifference curve slope changes in the area $c \leq \mu \leq c + \sigma \cdot k$ at least once. For the sake of simplicity we assume that the sign changes only once. Then a function $g(d)$ is implicitly defined such that:

$$\text{If } \mu \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} c + \sigma \cdot g(d), \text{ then } i(\mu, \sigma) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0. \tag{A3}$$

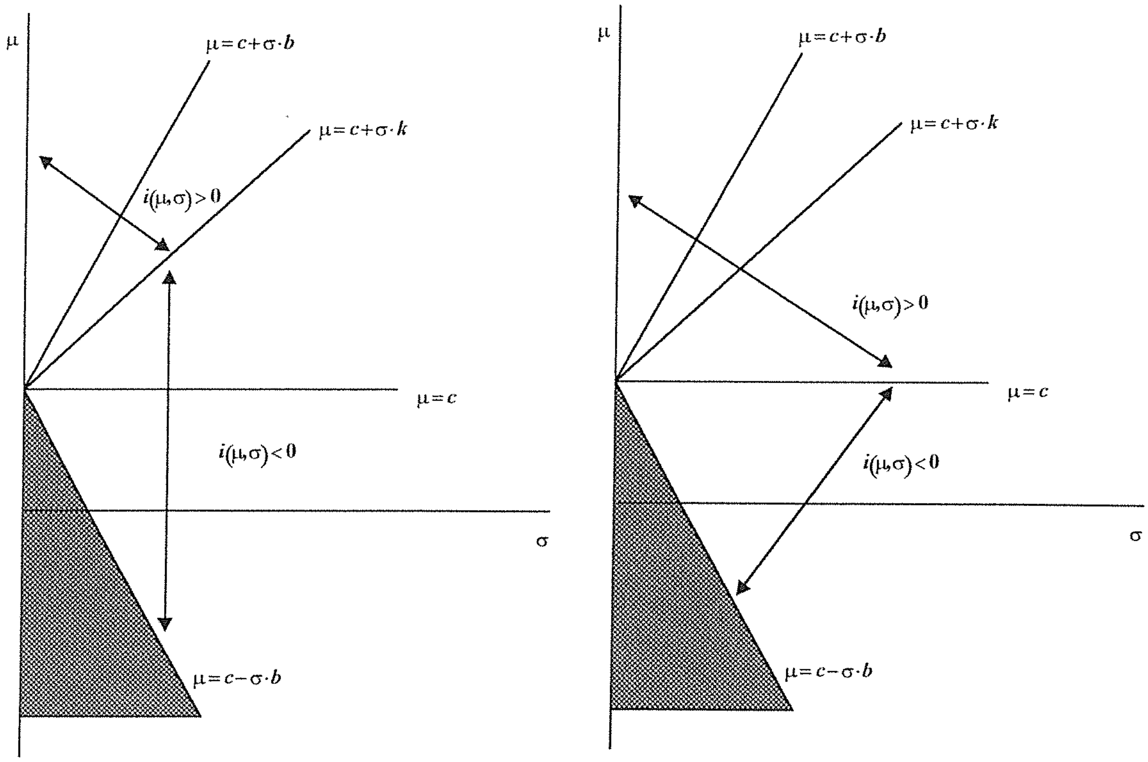

Figure 2: Indifference curve slope areas for $d = 0$ and $d \to \infty$

This simplification allows us to construct two areas with regard to $i(\mu,\sigma)$. In the first one the indifference curve slope is positive, in the second one it is negative. The more income realizations fall below the subsistence level the smaller is the indifference curve slope. The boundary of both areas changes with variation of *d*. The closer *d* is to $d = 0$ the nearer the boundary $\mu = c + \sigma \cdot g(d)$ is to $\mu = c + \sigma \cdot k$, and the closer *d* is to $d = \infty$ the nearer the boundary $\mu = c + \sigma \cdot g(d)$ is to $\mu = c$.

An upper bound of the indifference curve slope is established in the following property.

Property 3: Let $\mu > c - \sigma \cdot b$. Then $i(\mu,\sigma) \leq -\left(\frac{c-\mu}{\sigma}\right)$.

The remainder of this section is addressed to find out the properties of the indifference function's partial derivatives $i_\sigma(\mu,\sigma)$ and $i_\mu(\mu,\sigma)$, and of the indifference function's curvature $\frac{d^2\mu}{d\sigma^2}$. These properties will turn out to have far reaching implications for the economic results in section 4.

Property 4:

(a) Let $c + \sigma \cdot b < \mu$. Then $\frac{d^2\mu}{d\sigma^2} > 0$, $i_\sigma(\mu,\sigma) > 0$, $i_\mu(\mu,\sigma) < 0$.

(b) Let $c - \sigma \cdot b < \mu < c$ and let $d < \infty$. Then $\frac{d^2\mu}{d\sigma^2} < 0$.

(c) Let $c - \sigma \cdot b \le \mu \le c + \sigma \cdot b$ and let $d \to \infty$. Then $\frac{d^2\mu}{d\sigma^2} = 0$,

$$i_\sigma(\mu,\sigma) = \frac{c-\mu}{\sigma^2},\ i_\mu(\mu,\sigma) = \frac{1}{\sigma}.$$

The message of properties 4 (a) and (b) is that indifference curves are convex in the area $c + \sigma \cdot b < \mu$, concave for $c - \sigma \cdot b < \mu < c$. For further analytical relief we assume that the switch from convexity to concavity is only once and a function $h(d)$ is implicitly defined such that:

$$\text{If } \mu \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} c + \sigma \cdot h(d), \text{ then } \frac{d^2\mu}{d\sigma^2} \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 0. \tag{A4}$$

The left side of figure 3 illustrates that indifference curves are convex for $\mu > c + \sigma \cdot h(d)$ and concave for $\mu < c + \sigma \cdot h(d)$. Focusing on $d \to \infty$ the indifference curves are straight lines for $c - \sigma \cdot b \le \mu \le c + \sigma \cdot b$, see property 4 (c) and compare with the right side of figure 2.

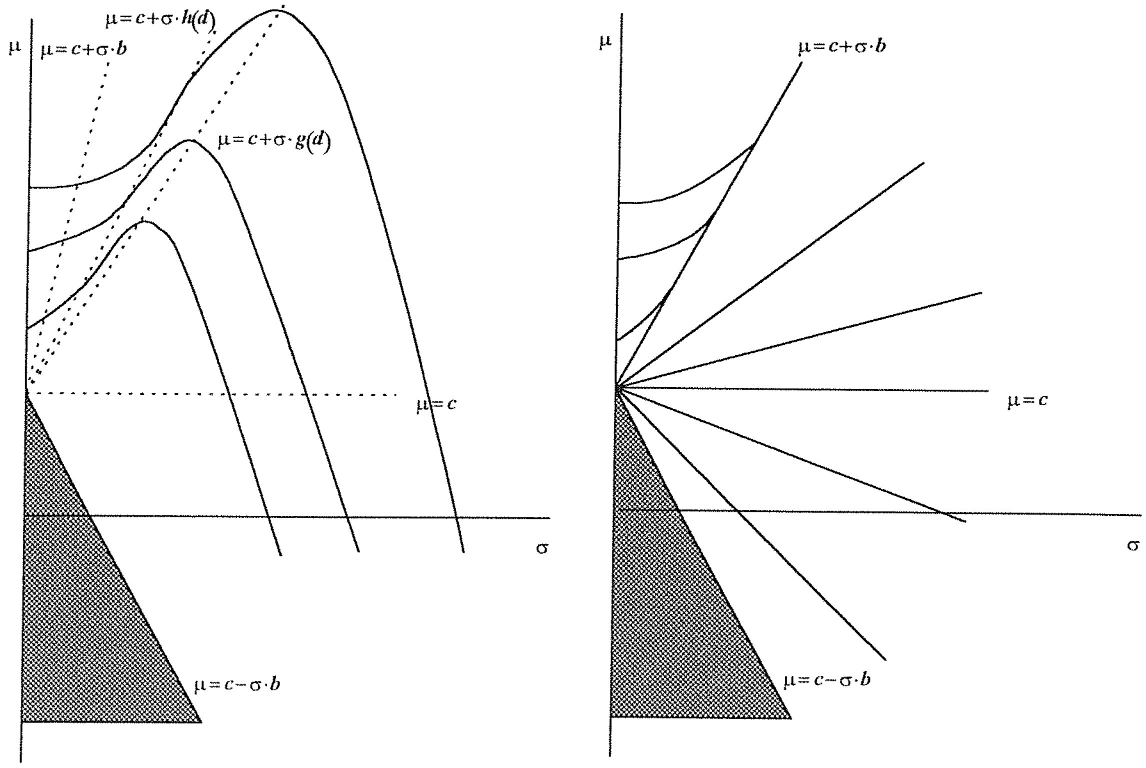

Figure 3: Indifference curves for $d < \infty$ and $d \to \infty$

The determination of the sign of $i_\sigma(\mu,\sigma)$ and $i_\mu(\mu,\sigma)$ is very difficult. Table 1 lists possible constellations for $d < \infty$ and $c - \sigma \cdot b < \mu < c + \sigma \cdot b$:

Case	$\frac{d^2\mu}{d\sigma^2}$	$i(\mu,\sigma)$,	$i_\mu(\mu,\sigma)$	$i_\sigma(\mu,\sigma)$
1	>0	>0	<0	>0
2	>0	>0	>0	<0
3	>0	>0	>0	>0
4	<0	$=0$	<0	<0
5	<0	$=0$	>0	<0
6	<0	<0	<0	<0
7	<0	<0	>0	<0
8	<0	<0	>0	>0

Table 1: Possible $i_\sigma(\mu,\sigma)$ and $i_\mu(\mu,\sigma)$-combinations

The table is to be understood as follows: If $\frac{d^2\mu}{d\sigma^2}>0$ and $i(\mu,\sigma)>0$, then either $i_\mu(\mu,\sigma)<0$ and $i_\sigma(\mu,\sigma)>0$ (case 1) or $i_\mu(\mu,\sigma)>0$ and $i_\sigma(\mu,\sigma)<0$ (case 2) or $i_\mu(\mu,\sigma)>0$ and $i_\sigma(\mu,\sigma)>0$ (case 3). For $i(\mu,\sigma)=0$ in any case we have $i_\sigma(\mu,\sigma)<0$.

Observe that (A3), (A4) and properties 2 and 4 imply $h(d)>g(d)$. Then we obtain the following intervals: (B1) $c+\sigma\cdot b<\mu<c+\sigma\cdot h(d)$ with $i(\mu,\sigma)>0$, $\frac{d^2\mu}{d\sigma^2}>0$,

(B2) $c+\sigma\cdot h(d)<\mu<c+\sigma\cdot g(d)$ with $i(\mu,\sigma)>0$, $\frac{d^2\mu}{d\sigma^2}<0$ and

(B3) $c+\sigma\cdot g(d)<\mu<c-\sigma\cdot b$ with $i(\mu,\sigma)<0$, $\frac{d^2\mu}{d\sigma^2}<0$, and

possible $i_\sigma(\mu,\sigma)$ and $i_\mu(\mu,\sigma)$-constellations are specified in table 1, e.g. for (B3) cases 6-8. It remains an open question what the empirical relevance is of the various cases of table 1. All preliminaries are now completed.

4. Individual and social decisions about risk-taking and income taxes

In this section we first turn to the analysis of the agent's self-insurance effort and second to the analysis of the optimal redistribution scheme. The representative agent chooses the pre-tax standard deviation for exogenously given tax rate by maximizing $U(\mu,\sigma)$ subject to (10) and (11). Differentiation yields a first-order condition which can be conveniently expressed as

$$i\left(M(\sigma_G),(1-\tau)\cdot\sigma_G\right)=\frac{M_{\sigma_G}(\sigma_G)}{(1-\tau)}. \tag{16}$$

The second-order condition requires[8]

$$(1-\tau)^2\cdot\frac{d^2\mu}{d\sigma^2}-M_{\sigma_G\sigma_G}(\sigma_G)>0. \tag{17}$$

[8] The second-order condition is satisfied if the indifference curves are convex, and it is assumed to be satisfied if the indifference curves are concave.

Equation (16) is the standard condition, compare Sinn (1995, 1996), for optimal pre-tax standard deviation and requires the slope of the indifference curve and the slope of the redistribution (self-insurance) line to be equalized. The optimal pre-tax standard deviation inserted in (9) yields the optimal self-insurance effort. The impact of marginal changes in tax rates is spelt out in

Proposition 1:

(a) *Pre-tax standard deviation falls (rises) with the tax rate if and only if* $i(\mu,\sigma)+i_\sigma(\mu,\sigma)\cdot\sigma_G\cdot(1-\tau)<(>)0$.

(b) *Self-insurance effort rises (falls) with the tax rate if and only if* $i(\mu,\sigma)+i_\sigma(\mu,\sigma)\cdot\sigma_G\cdot(1-\tau)>(<)0$.

(c) *Expected income falls (rises) with the tax rate if and only if* $\left(i(\mu,\sigma)+i_\sigma(\mu,\sigma)\cdot\sigma_G\cdot(1-\tau)\right)\cdot M_{\sigma_G}(\sigma_G)<(>)0$.

(d) *Post-tax standard deviation falls (rises) with the tax rate if and only if* $\left[(1-\tau)\cdot\left(i(\mu,\sigma)-\sigma_G\cdot i_\mu(\mu,\sigma)\cdot M_{\sigma_G}(\sigma_G)\right)+\sigma_G\cdot M_{\sigma_G\sigma_G}(\sigma_G)\right]<(>)0$.

(e) *Utility falls (raises) with the tax rate if and only if* $U_\mu(\mu,\sigma)\cdot i(\mu,\sigma)\cdot\sigma_G<(>)0$.

Proposition 1 tells us that the properties of the indifference function have a decisive impact on the effects of a marginal tax rate change. The interpretation of proposition 1 is postponed to subsections 4.1 and 4.2.

4.1 The value of human life is infinite

In this subsection we wish to look at an infinite value of life. We restrict our attention to $\mu>c$. The agent's opportunity set is caracterized by the self-insurance line if $\tau=0$, and by a redistribution line if $\tau>0$. Redistributive taxation induces a shift to the left and a compression of the redistribution line. The properties of indifference curves for $d\to\infty$ are pointed out in equation (15), property 4 (a) and in property 4 (c). Taking these properties into account proposition 1 (b) implies that the agent's self-insurance effort is invariant for tax rate changes if $\mu\le c+\sigma\cdot b$ and increasing with the tax rate if $\mu>c+\sigma\cdot b$. Figure 4 illustrates this result. Point A is the chosen (μ,σ)-combination in the laissez-faire situation, that means in a state without taxation. C and D are the solutions for two tax rates $\tau_2>\tau_1>0$. With increasing tax rate the favoured (μ,σ)-combinations move from A to B along the broken line if $\mu\le c+\sigma\cdot b$. Both self-insurance effort and pre-tax income inequality are constant (propositions 1 (b), (a)). The immediate consequence is that national income stays constant and post-tax inequality declines with respect to tax rate changes (propositions 1 (c), (d)). The reduction of post-tax inequality is utility increasing, thus the tax rate τ_1 is preferred in comparison to $\tau=0$. Point C is on an indifference curve with higher utility than point A. For $\mu>c+\sigma\cdot b$ our economy coincides with Sinn's (1995) economy. Increasing taxation has the effect that agents successively take more risk. As shown in Sinn (1995, p. 506) the effect can be analytically split into two partial effects. First increasing taxation reduces post-tax inequality, which is sacrificed through the second effect for higher national income. The net effect concerning post-tax inequality is ambiguous.

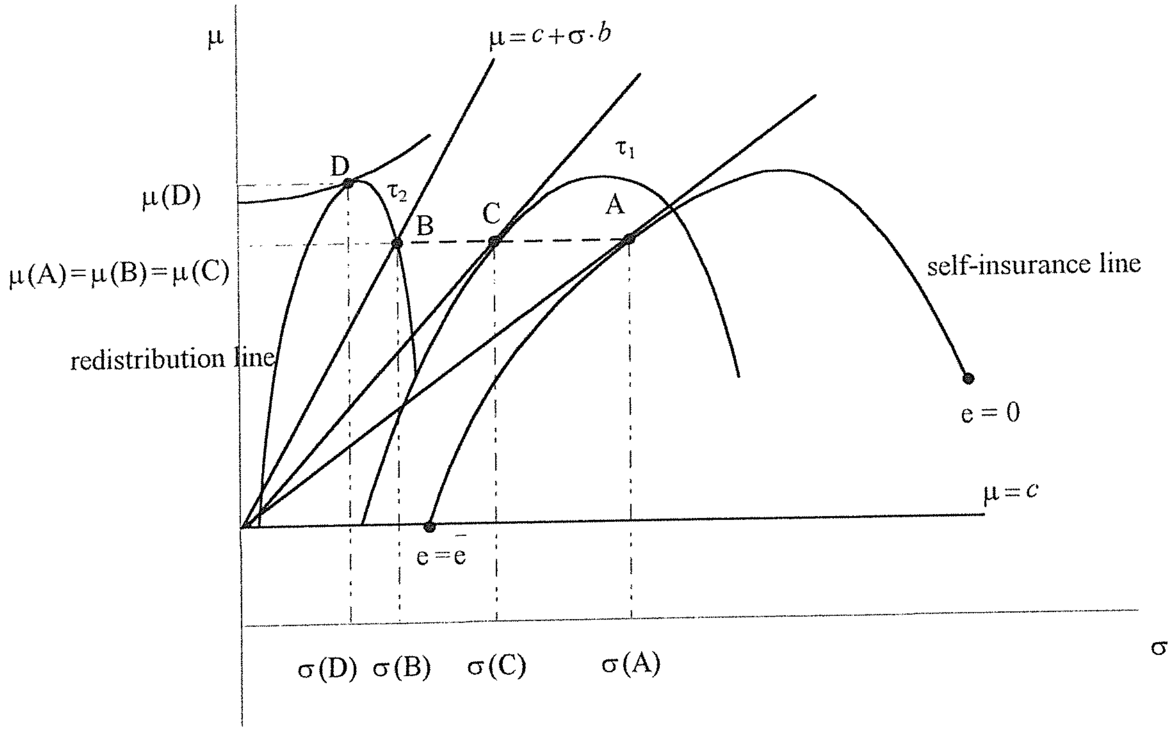

Figure 4: (μ,σ)-movements for $d \rightarrow \infty$

Since indifference curves are straight lines and redistribution lines are concave, there is no possibility for the agent to sacrifice post-tax inequality for higher national incomes if $\mu \leq c + \sigma \cdot b$. Consider point C. If μ is chosen higher and σ is chosen smaller such that the new allocation is still on the redistribution line, then the $\frac{\mu}{\sigma}$-ratio of this new allocation gets smaller and that means that the probability for stochastic income to remain under the subsistence level gets smaller, too, which implies that utility falls in comparison to the allocation C.

On constitutional plain the society votes unanimously for $\tau = 1$ (proposition 1 (e)). The reduction of post-tax inequality for $\mu \leq c + \sigma \cdot b$ and the growth of national income for $\mu > c + \sigma \cdot b$ are the reasons for a continuous utility improvement, which implies that the confiscatorial tax is preferred to all other tax rates.

4.2 The value of human life is finite

The results for $d < \infty$ are manifold. Possible reactions of agents are summarized in table 2. Consider cases IX, X and XI. The relevant area is $c < \mu < c + \sigma \cdot g(d)$. From (A3) it is known that $i(\mu,\sigma)$ is negative (column 3), the FOC (16) ensures $M_{\sigma_G}(\sigma_G) < 0$ (column 4) and property 4 (b) implies $\frac{d^2\mu}{d\sigma^2} < 0$. Both in cases X and XI we assume $i_\sigma(\mu,\sigma) > 0$ (column 6). These cases differ with respect to the sign of $i(\mu,\sigma) + i_\sigma(\mu,\sigma) \cdot \sigma_G \cdot (1-\tau)$ (column 7). The remaining assumption $i_\sigma(\mu,\sigma) < 0$ is handled in case IX. Then the results of columns 9-12 follow by inserting the statements of columns 3-6 in proposition 1. Note that the sign of $\frac{d\sigma}{d\tau}$ is ambiguous for the cases I, II, III, V, VII and XI.

Case	area	i	M_{σ_G}	$\frac{d^2\mu}{d\sigma^2}$	i_σ	$i + i_\sigma \sigma_G (1-\tau)$	Case from table 1	$\frac{d\sigma_G}{d\tau}$	$\frac{de}{d\tau}$	$\frac{d\mu}{d\tau}$	$\frac{d\sigma}{d\tau}$
I	$\mu > c + \sigma \cdot b$	> 0	> 0	> 0	> 0	> 0		> 0	< 0	> 0	
II	$c + \sigma \cdot h(d) \leq \mu \leq c + \sigma \cdot b$	> 0	> 0	≥ 0	> 0	> 0	1, 3	> 0	< 0	> 0	
III		> 0	> 0	≥ 0	< 0	> 0	2	> 0	< 0	> 0	
IV		> 0	> 0	≥ 0	< 0	< 0	2	< 0	> 0	< 0	< 0
V	$c + \sigma \cdot g(d) < \mu < c + \sigma \cdot h(d)$	> 0	> 0	< 0	< 0	> 0	6, 7	> 0	< 0	> 0	
VI		> 0	> 0	< 0	< 0	< 0	6, 7	< 0	> 0	< 0	< 0
VII		> 0	> 0	< 0	> 0	> 0	8	> 0	< 0	> 0	
VIII	$\mu = c + \sigma \cdot g(d)$	$= 0$	$= 0$	< 0	< 0	< 0	4, 5	< 0	> 0	$= 0$	< 0
IX	$c < \mu < c + \sigma \cdot g(d)$	< 0	< 0	< 0	< 0	< 0	6, 7	< 0	> 0	> 0	< 0
X		< 0	< 0	< 0	> 0	< 0	8	< 0	> 0	> 0	< 0
XI		< 0	< 0	< 0	> 0	> 0	8	> 0	< 0	< 0	

Table 2: Possible comparative static results of a marginal tax rate change

We demonstrate social behaviour concerning national income and income inequality by two examples.

Example 1:
The following example is illustrated in figure 5. The starting point is the laissez-faire solution point A. Before analyzing the impacts of taxation observe that an increase of σ is utility increasing if $\mu < c + \sigma \cdot g(d)$.

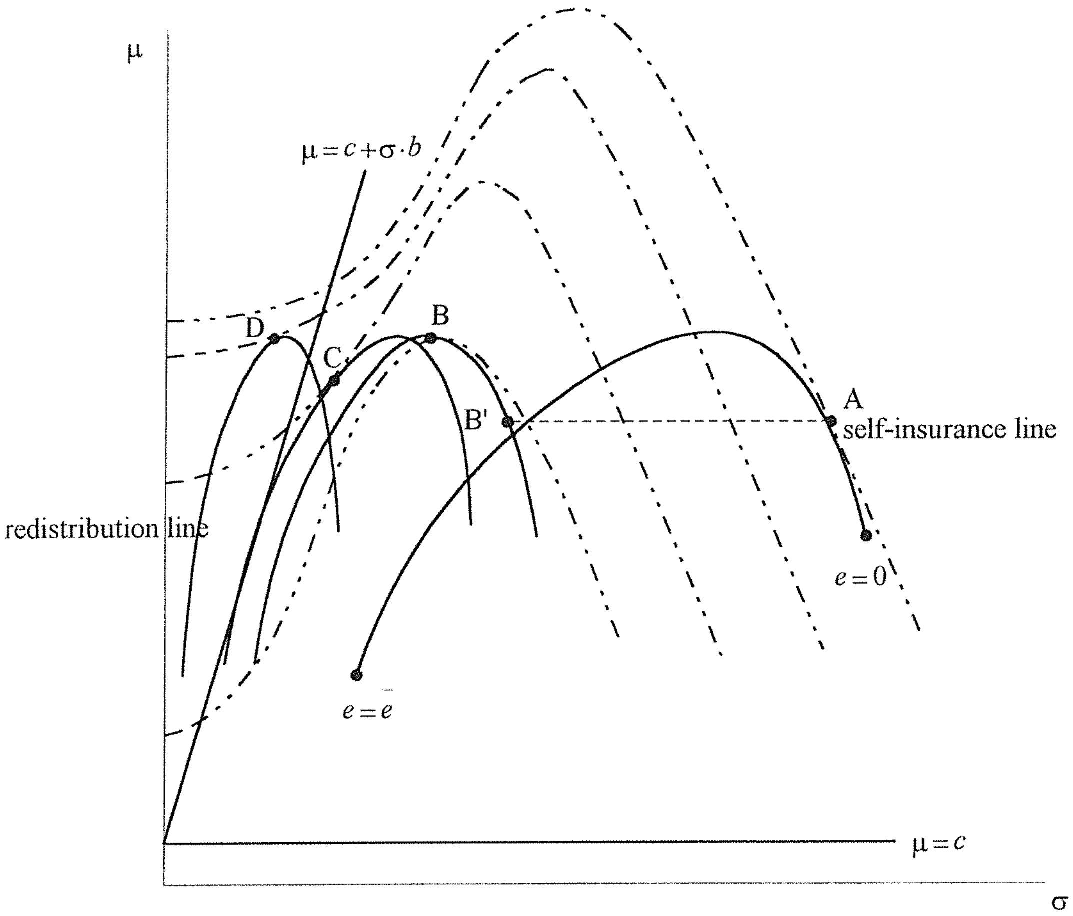

Figure 5: (μ,σ)-movements for $d < \infty$, example 1

A distribution with higher post-tax standard deviation is preferred opposite to a distribution with lower σ, because the probability to achieve high incomes $y >> c$ falls with decreasing standard deviation. As in the previous subsection the effect of taxation is split into two partial effects. Since redistributive taxation shifts and compresses the redistribution line, for constant self-insurance effort the post-tax standard deviation and utility decline (first partial effect). The agent compensates the low utility level through an increase of her self-insurance effort with the consequence to raise national income (second partial effect). However, the net impact is utility decreasing. The first partial effect is illustrated by the movement from A to B', the second partial effect by the movement from B' to B in figure 5. The results coincide with cases IX and X of table 2. In B arrived the probability of incomes to remain below the subsistence level is smaller than in A. That is the reason why now a reduction of post-tax standard deviation caused by redistributive taxation improves utility. The representative agent strengthens the impact of taxation by increasing self-insurance effort with the consequence that national income goes down. Reducing μ and σ the probability of the stochastic income to be complete above the subsistence level is increased. The task to shift the distribution above the c has for riskaverse individuals supreme priority in this boundary area (movement from B to C, cases IV and VI in table 2). For income distributions above c the agent takes more risks with the effect to raise national income (movement from C to D, case I in table 2).

Example 2:
Another possible implication of taxation is illustrated in figure 6. Again point A is the laissez-faire allocation. In this example taxation increases at first risk taking if $\mu < c + \sigma \cdot g(d)$. Less self-insurance effort decreases national income and post-tax inequality, see point B in figure 6 and case XI in table 2. The net effect is a loss of utility, since the post-tax inequality effect dominates the countervailing national income effect. The interesting feature of figure 6 is the jump from B to C. In B the indifference curve is negative, whereas in C it is positive. Suppose taxation is such that society chooses B. If then the government increases marginally the tax rate, agents think carefully about the following possibilities: Should we raise σ in order to increase the probability of high incomes or is it favourable to reduce σ and to raise μ in order to shift the distribution as far as possible above the subsistence level. From B to C they decided to reduce the risk that incomes fall below the subsistence level which requires vast efforts in self-insurance. This behaviour is compatible with cases IX and X. In C arrived we pick out case VII and afterwards case III such that increasing tax rates successively imply a movement from C to D. Exceeds the (μ, σ)-opportunity set the straight line $\mu = c + \sigma \cdot b$ the remarks of subsection 4.1 have to be recalled.

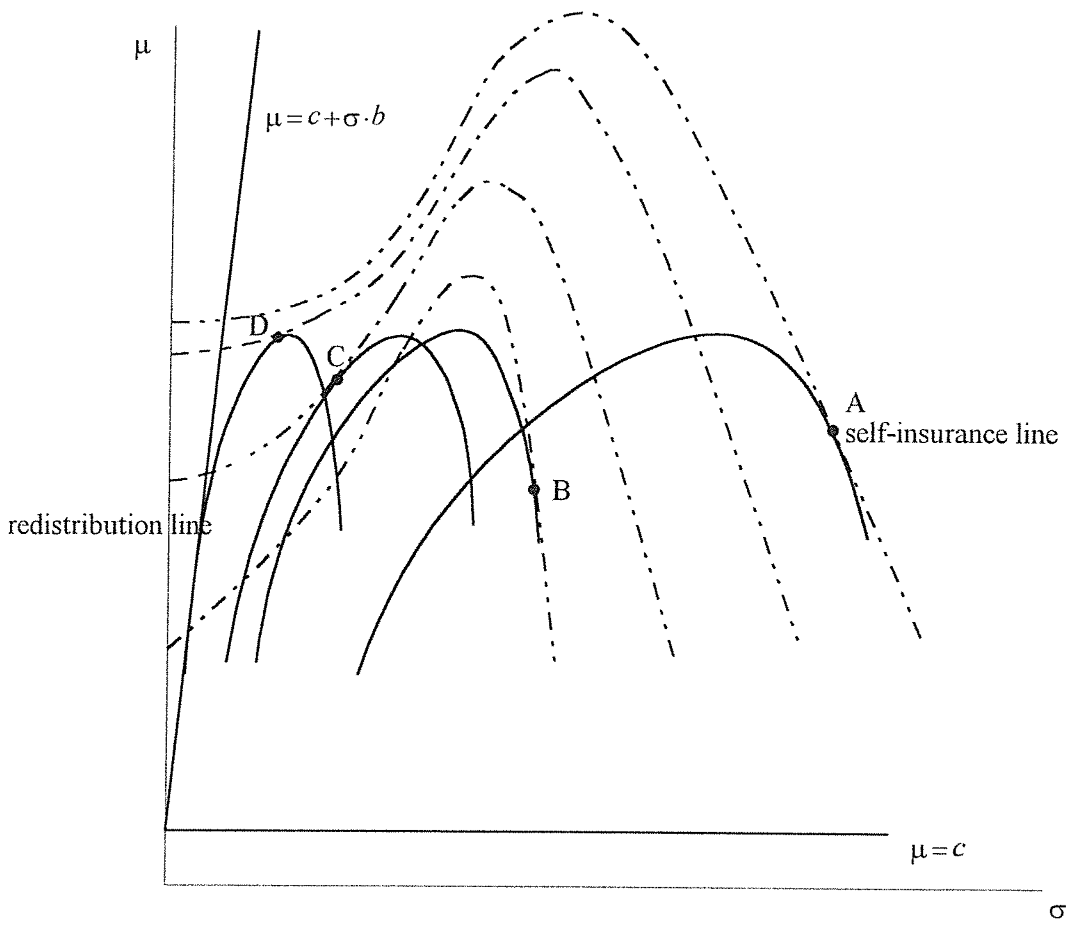

Figure 6: (μ, σ)-developements for $d < \infty$, example 2

The last step in section 4 is the determination of the optimal tax rate. For $d \to \infty$ proposition 1 (e) shows that the society chooses unanimously the confiscatorial tax. In contrast to that it turns out for $d < \infty$:

Proposition 2: *In general, the confiscatorial tax is not socially optimal.*

Taking the subsistence level into account has the economic implication that the justification of taxation begins to sway. Clearly, there are cases in which the laissez-faire tax is socially optimal. Without further assumptions redistributive taxation can improve or make worse welfare, so proposition 2 provides a theoretical base for criticism of opponents of the social state.

5. Concluding remarks

This paper focused on constitutional decisions about self-insurance effort and income taxation under consideration of a subsistence level. For the special case in which the income distribution is completely above the subsistence level our economy is identical to Sinn's (1995) economy. If income realizations fall below the subsistence level, we distinguish two cases. In the first one the worth of human life converges to infinity with the implication to decrease successively self-insurance effort. Then analogous to Sinn (1995) the optimal tax is the confiscatorial tax. For a value of human life less than infinitywhich is the second case we obtain a multitude of subcases in dependence of different μ/σ-ratios. In some subcases agents increase self-insurance effort in order to shift uncertain incomes as far as possible above the subsistence level. The impact on national income and income inequality is in both direction possible. However, the main message of our paper is that the confiscatorial tax is not welfare optimal, in general. Now, remarks are in order to our assumptions. The assumptions (A2)-(A4) do not affect the statement of proposition 2. They ensure that the number of the above mentioned subcases is as small as possible in order to keep the analysis within manageable limits. The assumption (A1) is standard and may be modified from DARA into IARA utility without any consequences on proposition 2. An adequate numerical example can be constructed with the quadratic utility function.

Our theoretical analysis raises the important and challanging question what the empirical relevance is of the various cases studied above. Unfortunately, we are not aware of empirical investigations along the lines of our theoretical framework that would provide clear-cut evidence about the characteristics of optimal welfare state. This remains for future research.

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Appendix

Let $h > 0$, we define $g(a^+) := \lim_{h \to 0} g(a+h)$ as the right-side limit of the function g at point a.

In the proofs of properties 1-4 we use the first and second-order partial derivatives of the MS-preference function $U(\mu,\sigma) = -d \cdot \int_{-b}^{\left(\frac{c-\mu}{\sigma}\right)} dF(z) + \int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u(\mu + \sigma \cdot z) \cdot dF(z)$ and of the indifference function[9] which are given by:

[9] Calculating partial derivatives we applied the following formula. Let $I(y)$ be defined by

$$U_{\mu}(\mu,\sigma)=\frac{d}{\sigma}\cdot f\left(\frac{c-\mu}{\sigma}\right)+\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u'(\mu+\sigma\cdot z)\cdot dF(z), \qquad \text{[A1]}$$

$$U_{\sigma}(\mu,\sigma)=\frac{d\cdot(c-\mu)}{\sigma^{2}}\cdot f\left(\frac{c-\mu}{\sigma}\right)+\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u'(\mu+\sigma\cdot z)\cdot z\cdot dF(z), \qquad \text{[A2]}$$

$$U_{\mu\mu}=-\frac{d}{\sigma^{2}}\cdot f'\left(\frac{c-\mu}{\sigma}\right)+\frac{1}{\sigma}\cdot u'(c^{+})\cdot f\left(\frac{c-\mu}{\sigma}\right)+\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u''(\mu+\sigma\cdot z)\cdot dF(z), \qquad \text{[A3]}$$

$$U_{\mu\sigma}=U_{\sigma\mu}=-\frac{d}{\sigma^{2}}\cdot f\left(\frac{c-\mu}{\sigma}\right)-\frac{d\cdot(c-\mu)}{\sigma^{3}}\cdot f'\left(\frac{c-\mu}{\sigma}\right)+\frac{(c-\mu)}{\sigma^{2}}\cdot u'(c^{+})\cdot f\left(\frac{c-\mu}{\sigma}\right) \qquad \text{[A4]}$$
$$+\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u''(\mu+\sigma\cdot z)\cdot z\cdot dF(z),$$

$$U_{\sigma\sigma}=-\frac{2\cdot d\cdot(c-\mu)}{\sigma^{3}}\cdot f\left(\frac{c-\mu}{\sigma}\right)-\frac{d\cdot(c-\mu)^{2}}{\sigma^{4}}\cdot f'\left(\frac{c-\mu}{\sigma}\right) \qquad \text{[A5]}$$
$$+\frac{(c-\mu)^{2}}{\sigma^{3}}\cdot u'(c^{+})\cdot f\left(\frac{c-\mu}{\sigma}\right)+\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u''(\mu+\sigma\cdot z)\cdot z^{2}\cdot dF(z),$$

$$i_{\sigma}(\mu,\sigma)=-\frac{U_{\sigma\sigma}(\mu,\sigma)\cdot U_{\mu}(\mu,\sigma)-U_{\sigma}(\mu,\sigma)\cdot U_{\mu\sigma}(\mu,\sigma)}{\left(U_{\mu}(\mu,\sigma)\right)^{2}}, \qquad \text{[A6]}$$

$$i_{\mu}(\mu,\sigma)=-\frac{U_{\sigma\mu}(\mu,\sigma)\cdot U_{\mu}(\mu,\sigma)-U_{\sigma}(\mu,\sigma)\cdot U_{\mu\mu}(\mu,\sigma)}{\left(U_{\mu}(\mu,\sigma)\right)^{2}}, \qquad \text{[A7]}$$

$$\frac{d^{2}\mu}{d\sigma^{2}}=i_{\sigma}(\mu,\sigma)+i_{\mu}(\mu,\sigma)\cdot i(\mu,\sigma). \qquad \text{[A8]}$$

Inserting [A3]-[A5] into [A6]-[A8], and integrating by parts leads to:[10]

$$i_{\sigma}(\mu,\sigma)=\frac{1}{U_{\mu}}\cdot\left[\frac{d}{\sigma^{2}}\cdot\left(i+\frac{2\cdot(c-\mu)}{\sigma}\right)\cdot f\left(\frac{c-\mu}{\sigma}\right)+\frac{d\cdot(c-\mu)}{\sigma^{3}}\cdot\left(i+\frac{c-\mu}{\sigma}\right)\cdot f'\left(\frac{c-\mu}{\sigma}\right)\right. \qquad \text{[A9]}$$
$$\left.-\frac{u'(\mu+\sigma\cdot b)}{\sigma}\cdot\left(b^{2}+i\cdot b\right)\cdot f(b)+\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b}\frac{u'(\mu+\sigma\cdot z)}{\sigma}\cdot\left((2\cdot z+i)\cdot f(z)+\left(z^{2}+i\cdot z\right)\cdot f'(z)\right)\cdot d(z)\right],$$

$$i_{\mu}(\mu,\sigma)=\frac{1}{U_{\mu}}\cdot\left[\frac{d}{\sigma^{2}}\cdot f\left(\frac{c-\mu}{\sigma}\right)+\frac{d}{\sigma^{2}}\cdot\left(i+\frac{c-\mu}{\sigma}\right)\cdot f'\left(\frac{c-\mu}{\sigma}\right)\right. \qquad \text{[A10]}$$

$I(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)\cdot dx$. Then $I'(y)$ can be obtained by $I'(y)=\int_{\alpha(y)}^{\beta(y)} f_{y}(x,y)\cdot dx+\beta'(y)\cdot f(\beta(y),y)$
$-\alpha'(y)\cdot f(\alpha(y),y)$.

[10] For the sake of convenience the variables μ, σ as arguments of functions are occasionally suppressed.

$$-\frac{u'(\mu+\sigma\cdot b)}{\sigma}\cdot(b+i)\cdot f(b)+\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b}\frac{u'(\mu+\sigma\cdot z)}{\sigma}\cdot\left(f(z)+(z+i)\cdot f'(z)\right)\cdot d(z)\Bigg],$$

$$\frac{d^2\mu}{d\sigma^2}=\frac{1}{U_\mu}\cdot\left[\frac{d}{\sigma^2}\cdot\left(i+\frac{c-\mu}{\sigma}\right)^2\cdot f'\left(\frac{c-\mu}{\sigma}\right)\quad-\frac{u'(\mu+\sigma\cdot b)}{\sigma}\cdot(b+i)^2\cdot f(b)\right. \qquad [A11]$$

$$\left.+\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b}\frac{u'(\mu+\sigma\cdot z)}{\sigma}\cdot(z+i)^2\cdot f'(z)\cdot d(z)\right].$$

Proof of Property 1:

$\hat{\mu},\hat{\sigma}$ and $\overline{\mu},\overline{\sigma}$ inserted in $U(\mu,\sigma)=-d\cdot\int_{-b}^{\left(\frac{c-\mu}{\sigma}\right)}dF(z)+\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b}u(\mu+\sigma\cdot z)\cdot dF(z)$ and summed up yield $U(\hat{\mu},\hat{\sigma})-U(\overline{\mu},\overline{\sigma})=\int_{\left(\frac{c-\hat{\mu}}{\hat{\sigma}}\right)}^{b}\left[u(\hat{\mu}+\hat{\sigma}\cdot z)-u(\overline{\mu}+\overline{\sigma}\cdot z)\right]\cdot dF(z)$.

We first consider $\frac{c-\overline{\mu}}{\overline{\sigma}}>0$. $\frac{c-\overline{\mu}}{\overline{\sigma}}=\frac{c-\hat{\mu}}{\hat{\sigma}}$ and $\hat{\mu}>\overline{\mu}$ implies $\hat{\sigma}>\overline{\sigma}$ and it holds $\hat{\mu}+\hat{\sigma}\cdot z>\overline{\mu}+\overline{\sigma}\cdot z$ for $z>\frac{c-\overline{\mu}}{\overline{\sigma}}$. From $u'(y)>0$ for $y>c$ we obtain $u(\hat{\mu}+\hat{\sigma}\cdot z)>u(\overline{\mu}+\overline{\sigma}\cdot z)$ for $z>\frac{c-\overline{\mu}}{\overline{\sigma}}$, and integration yields

$$\int_{\left(\frac{c-\hat{\mu}}{\hat{\sigma}}\right)}^{b}\left[u(\hat{\mu}+\hat{\sigma}\cdot z)-u(\overline{\mu}+\overline{\sigma}\cdot z)\right]\cdot dF(z)>0.$$

Now let $\frac{c-\overline{\mu}}{\overline{\sigma}}\le 0$. $U(\hat{\mu},\hat{\sigma})-U(\overline{\mu},\overline{\sigma})$ can be rearranged to

$$U(\hat{\mu},\hat{\sigma})-U(\overline{\mu},\overline{\sigma})=\int_{\left(\frac{c-\hat{\mu}}{\hat{\sigma}}\right)}^{0}\left[u(\hat{\mu}+\hat{\sigma}\cdot z)-u(\overline{\mu}+\overline{\sigma}\cdot z)\right]\cdot dF(z)+\int_{0}^{b}\left[u(\hat{\mu}+\hat{\sigma}\cdot z)-u(\overline{\mu}+\overline{\sigma}\cdot z)\right]\cdot dF(z).$$

$\int_{0}^{b}\left[u(\hat{\mu}+\hat{\sigma}\cdot z)-u(\overline{\mu}+\overline{\sigma}\cdot z)\right]\cdot dF(z)>0$ follows by the same line of argument used for $\frac{c-\overline{\mu}}{\overline{\sigma}}>0$. Observe that $\hat{\mu}+\hat{\sigma}\cdot z>\overline{\mu}+\overline{\sigma}\cdot z$ for $0<z<\frac{c-\overline{\mu}}{\overline{\sigma}}$ and $\hat{\mu}+\hat{\sigma}\cdot z=\overline{\mu}+\overline{\sigma}\cdot z$ for $z=\frac{c-\overline{\mu}}{\overline{\sigma}}$. Then integrating we have

$\int_{\left(\frac{c-\hat{\mu}}{\hat{\sigma}}\right)}^{0}\left[u(\hat{\mu}+\hat{\sigma}\cdot z)-u(\overline{\mu}+\overline{\sigma}\cdot z)\right]\cdot dF(z)>0$ and together with

$\int_0^b [u(\hat{\mu}+\hat{\sigma}\cdot z)-u(\overline{\mu}+\overline{\sigma}\cdot z)]\cdot dF(z)>0$ we have proven $U(\hat{\mu},\hat{\sigma})-U(\overline{\mu},\overline{\sigma})>0$. □

Proof of Property 2:

Note that $U_\mu(\mu,\sigma)>0$ for all $\mu>c-\sigma\cdot b$. Thus the sign of $i(\mu,\sigma)$ is opposite to the sign of $U_\sigma(\mu,\sigma)$ for the proofs of (a)-(c).

(a) $c+\sigma\cdot b\le\mu$ is equivalent to $\frac{c-\mu}{\sigma}\le -b$. Thus we get:

$U_\sigma(\mu,\sigma)=\int_{-b}^{b} u'(\mu+\sigma\cdot z)\cdot z\cdot dF(z)$. Meyer (1987, Property 3) shows that $u'(y)>0$ and $u''(y)<0$ imply $i(\mu,\sigma)>0$ for $\sigma>0$, and for $i(\mu,0)$ we obtain:

$$i(\mu,0)=-\frac{u'(\mu)\cdot\int_{-b}^{b} z\cdot dF(z)}{u'(\mu)\cdot\int_{-b}^{b} dF(z)}=0.$$

(b) From Meyer's Property 2 it is known that $\int_{-b}^{b} u'(\mu+\sigma\cdot z)\cdot z\cdot dF(z)<0$ if

$u''(\mu+\sigma\cdot z)<0$ for all $(\mu+\sigma\cdot z)\in[-b,b]$.

$u'(\mu+\sigma\cdot z)>0$ for $(\mu+\sigma\cdot z)\in[0,b]$ ensures $\int_0^b u'(\mu+\sigma\cdot z)\cdot z\cdot dF(z)>0$.

Continuity of the marginal utility function for $y>c$, and continuity of the density function imply that there exists a k with $0<k<b$ implicitly defined by $\int_{-k}^{b} u'(\mu+\sigma\cdot z)\cdot z\cdot dF(z)=0$. For $-b<\frac{c-\mu}{\sigma}<-k$ the terms $\frac{d\cdot(c-\mu)}{\sigma^2}\cdot f\left(\frac{c-\mu}{\sigma}\right)$ and $\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u'(\mu+\sigma\cdot z)\cdot z\cdot dF(z)$ are both negative and ensure $i(\mu,\sigma)>0$.

(c) $0<\frac{c-\mu}{\sigma}<b$ implies $\frac{d\cdot(c-\mu)}{\sigma^2}\cdot f\left(\frac{c-\mu}{\sigma}\right)>0$ and $\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u'(\mu+\sigma\cdot z)\cdot z\cdot dF(z)>0$.

The immediate consequence is $i(\mu,\sigma)<0$.

(d) For $b\le\frac{c-\mu}{\sigma}$ the integral $\int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u(\mu+\sigma\cdot z)\cdot dF(z)$ is zero and we obtain

$$U(\mu,\sigma)=-d\cdot\int_{-b}^{\left(\frac{c-\mu}{\sigma}\right)} dF(z)=-d\cdot\int_{-b}^{b} dF(z)=-d. \ \square$$

Proof of Property 3:

The proof is taken by contradiction. Recall

$$i(\mu,\sigma) = -\frac{\frac{d\cdot(c-\mu)}{\sigma^2}\cdot f\left(\frac{c-\mu}{\sigma}\right) + \int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u'(\mu+\sigma\cdot z)\cdot z\cdot dF(z)}{\frac{d}{\sigma}\cdot f\left(\frac{c-\mu}{\sigma}\right) + \int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} u'(\mu+\sigma\cdot z)\cdot dF(z)}.$$

Rearranging this term and applying the mean value theorem of integral calculus, there exists a $z^* \in \left[\frac{c-\mu}{\sigma}, b\right]$ such that:

$$\frac{d}{\sigma}\cdot f\left(\frac{c-\mu}{\sigma}\right)\cdot\left(i(\mu,\sigma)+\frac{c-\mu}{\sigma}\right) + \left(i(\mu,\sigma)+z^*\right)\cdot \int_{\left(\frac{c-\mu}{\sigma}\right)^*}^{b} u'(\mu+\sigma\cdot z)\cdot dF(z) = 0. \qquad \text{[A12]}$$

Suppose $i(\mu,\sigma)+\frac{c-\mu}{\sigma} > 0$. This implies $\frac{d}{\sigma}\cdot f\left(\frac{c-\mu}{\sigma}\right)\cdot\left(i(\mu,\sigma)+\frac{c-\mu}{\sigma}\right) \geq 0$.

[A12] and $\frac{d}{\sigma}\cdot f\left(\frac{c-\mu}{\sigma}\right)\cdot\left(i(\mu,\sigma)+\frac{c-\mu}{\sigma}\right) \geq 0$ require $i(\mu,\sigma)+z^* \leq 0$. Because of $i(\mu,\sigma)+z^* \leq 0$ for $z^* \in \left[\frac{c-\mu}{\sigma}, b\right]$ the infinimum $i(\mu,\sigma)+\frac{c-\mu}{\sigma}$ is required to be non-positive, too, which contradicts the supposition. □

Proof of Property 4:

The proof of (a) can be found in Meyer (1987, Property 4 and 5).

(b) $\frac{d^2\mu}{d\sigma^2}$ is specified in [A11]. (A2) implies $f'(z) \leq 0$ for all $\frac{c-\mu}{\sigma} \leq z \leq b$, and $f'(z) < 0$ for at least one $\frac{c-\mu}{\sigma} \leq z \leq b$ (or $f(b) > 0$). Thus we get:

$$\frac{d}{\sigma^2}\cdot\left(i+\frac{c-\mu}{\sigma}\right)^2\cdot f'\left(\frac{c-\mu}{\sigma}\right) \leq 0,\ \int_{\left(\frac{c-\mu}{\sigma}\right)}^{b} \frac{u'(\mu+\sigma\cdot z)}{\sigma}\cdot(z+i)^2\cdot f'(z)\cdot d(z) < (\leq) 0 \text{ and}$$

$$-\frac{u'(\mu+\sigma\cdot b)}{\sigma}\cdot(b+i)^2\cdot f(b) \leq (<) 0 \text{ which imply } \frac{d^2\mu}{d\sigma^2} < 0.$$

(c) Consider (15). Differentiation leads to: $i_\sigma(\mu,\sigma) = \frac{c-\mu}{\sigma^2}$, $i_\mu(\mu,\sigma) = \frac{1}{\sigma}$, and $\frac{d^2\mu}{d\sigma^2} = 0$. □

Proof of Proposition 1:

(a) Implicit differentiation of $A := i\left(M(\sigma_G), (1-\tau)\cdot\sigma_G\right)\cdot(1-\tau) - M_{\sigma_G}(\sigma_G)$ with respect to τ yields:

$$\frac{d\sigma_G}{d\tau} = -\frac{A_\tau}{A_{\sigma_G}} = \frac{i + i_\sigma\cdot\sigma_G\cdot(1-\tau)}{(1-\tau)^2\cdot\frac{d^2\mu}{d\sigma^2} - M_{\sigma_G\sigma_G}(\sigma_G)}.$$

The second-order condition ensures that the denominator is positive.

(b) $\frac{de}{d\tau} = \lambda^{-1}{}_{\sigma_G}\left(\sigma_G / R[\Phi]\right) \cdot \frac{d\sigma_G}{d\tau} \cdot \frac{1}{R[\Phi]}$.

$R[\Phi]$ is positive and the first derivative of the inverse of the self-insurance function $\left(\lambda_e^{-1}(e) < 0\right)$ is negative.

(c) $\frac{d\mu}{d\tau} = M_{\sigma_G}(\sigma_G) \cdot \frac{d\sigma_G}{d\tau}$.

(d) $\frac{d\sigma}{d\tau} = -\sigma_G + (1-\tau) \cdot \frac{d\sigma_G}{d\tau}$

$$= \frac{-\sigma_G \cdot \left[(1-\tau) \cdot \left\{i_\mu \cdot M_{\sigma_G}(\sigma_G) + i_\sigma \cdot (1-\tau)\right\} - M_{\sigma_G\sigma_G}(\sigma_G)\right] + (1-\tau) \cdot \left(i + \sigma_G \cdot i_\sigma \cdot (1-\tau)\right)}{(1-\tau)^2 \cdot \frac{d^2\mu}{d\sigma^2} - M_{\sigma_G\sigma_G}(\sigma_G)}.$$

The nominator can be summarized to

$(1-\tau) \cdot \left(i - \sigma_G \cdot i_\mu \cdot M_{\sigma_G}(\sigma_G)\right) + \sigma_G \cdot M_{\sigma_G\sigma_G}(\sigma_G)$.

(e) $\frac{dU}{d\tau} = U_\mu \cdot M_{\sigma_G}(\sigma_G) \cdot \frac{d\sigma_G}{d\tau} + U_\sigma \cdot \frac{d\sigma}{d\tau} = U_\mu \cdot i \cdot \sigma_G$. □

Proof of Proposition 2:

To establish proposition 2 consider the numerical example $\lambda(e) = 1 - \frac{1}{13} \cdot \sqrt{e}$, $E[\Phi] = 100$, $R[\Phi] = 100$. The utility function is specified as $u(y) = \sqrt{y + 100} - \sqrt{100}$ for $y > 0$ and $u(y) = 0$ for $y \le 0$ ($c = d = 0$), and random variables Y are assumed to be uniformly distributed such that the standardized random variable Z has density $f_Z(z) = \frac{1}{2 \cdot \sqrt{3}}$ for $-\sqrt{3} < z \le \sqrt{3}$. With the help of the programme Mathematica the values of μ, σ and $U(\mu, \sigma)$ are calculated for $\tau = 0$ and $\tau = 1$.

$\tau = 0$:	$\mu = 62.8051$,	$\sigma = 81.2595$,	$U(62.8051, 81.2585) = 2.9259$;
$\tau = 1$:	$\mu = 64.7929$,	$\sigma = 0$,	$u(64.7929) = 2.8372$.

Since $U(62.8051, 81.2585) = 2.9259 > u(64.7929) = 2.8372$ the statement of proposition 2 follows immediately. □