# SIGACT News Online Algorithms Column 27: online matching on the line, part 1

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In the online matching problem on the line, requests (points in $\mathbb{R}$) arrive one by one to be served by a given set of servers. Each server can be used only once. This is a variant of the $k$-server problem restricted to the real line. Although easy to state, this problem is stil wide open. The best known lower bound is 9.001 [2], showing that this problem is really different from the well-known cow path problem. Antoniadis et al. [1] recently presented a sublinearly competitive algorithm.

In this column, I present some results by Elias Koutsoupias and Akash Nanavati on this problem with kind permission of the authors. The column is based on Akash' PhD thesis [4], which contains an extended version of their joint WAOA paper [3] which has never appeared in a journal. I have expanded the proofs and slightly reorganized the presentation.

This column contains a proof of a linear upper bound for the generalized work function algorithm and a logarithmic lower bound for the algorithm. A later column will give a more detailed analysis of this algorithm, leading to a more precise (but still linear) upper bound. I conjecture that this algorithm in fact has a logarithmic competitive ratio (which would match the known lower bound for it), but this remains an open question.

# 1 Introduction

In this column, both requests and servers are specified by points on the real line and are multisets, as there can be multiple requests and/or servers at the same location. The same holds for all other sets discussed below. For example, for $x \in X$, we have $X \subsetneq X \cup \{x\}$. We will use the notation $\{x\}^k$ to denote a set which contains $k$ copies of the point $x$. Time is discrete, with one request occurring per time step.

Let $R^t$ denote the set of requests until time $t$ and let $A^t$ denote the set of servers used by the online algorithm to match these requests. Of course $|A^t| = |R^t|$. Let $M(A, R^t)$ denote an optimal way of matching $R^t$ to $A^t$ that minimizes the total offline cost. Clearly, $M(A^t, R^t)$ can be obtained by matching the requests to servers in order from left to right. For simplicity, we will denote the cost of the optimal matching by $M(A^t, R^t)$ as well.

Given this definition of the matching $M(A^t, R^t)$ we can ask how many of its lines cross a point $x \in \mathbb{R}^t$. We denote this number by $\textsc{cross}_x(A^t, R^t)$ or simply $\textsc{cross}_x$. To define it properly let $\textsc{left}_x(X)$ denote the number of elements of the set $X$ to the left of $x$. Then $\textsc{cross}_x(A^t, R^t) = \textsc{left}_x(A^t) - \textsc{left}_x(R^t)$. We have the following property:

$$M(A^t, R^t) = \int_{-\infty}^{\infty} |\textsc{cross}_x(A^t, R^t)| dx \tag{1}$$

**Definition 1 (Generalized Work Function Algorithm)** *The generalized work function algorithm* $\gamma$-WFA *matches a request* $r$ *at time* $t$ *to an unmatched server* $s$ *that minimizes the expression*

$$\gamma \cdot M(A^t \cup \{s\}, R^t \cup \{r\}) + d(s, r). \tag{2}$$

We begin by proving that $\gamma$-WFA is a so-called local algorithm in Section 2, and give some useful properties of local algorithms. In Section 3, we use this to prove that it is sufficient to consider the performance of $\gamma$-WFA compared to a restricted optimal solution which uses the same servers as $\gamma$-WFA. This allows us to show a linear upper bound for its competitive ratio (Section 4). Finally, we give some lower bounds for $\gamma$-WFA in the next section for various values (intervals) of $\gamma$.

# 2 Locality

Let $r$ be a request at time $t$ and let $s$ be the rightmost unmatched server in the interval $(-\infty, r)$. Similarly, let $s'$ be the leftmost unmatched server in $(r, \infty)$. We call the two servers $s$ and $s'$ the *surrounding servers* of $r$. It is easy to see that any online algorithm can be converted into one that serves each request with one of its two surrounding servers with the same (or better) competitive ratio using a simple exchange argument. However, the algorithm's decisions may be based on requests and servers that are *outside* the interval $(s, s')$. Thus we are led to define a restricted and interesting class of algorithms whose decisions are myopic.

**Definition 2** *Let* $s$ *and* $s'$ *be the surrounding servers for request* $r$*. An online algorithm is called* local *it it serves* $r$ *with one of* $s$ *and* $s'$ *and the choice is based only upon the history of servers and requests in the interval* $[s, s']$*.*

Our first aim is to show that $\gamma$-WFA is local.

**Property 1 (Locality)** $\gamma$-WFA *serves each request with one of its two surrounding servers.*

It is intuitively clear that $\gamma$-WFA should have this property: for any other server $\sigma$, clearly $d(\sigma, r) \geq \min(d(s, r), d(s', r))$, and the cost of the optimal matching should also increase if we use a server which is further away. We proceed to prove this formally. In fact we have the following invariant.

**Theorem 1** *Consider a time $t$. Let $s$ and $s'$ be two servers that have not yet been matched by $\gamma$-WFA at time $t$, whereas every server in the interval $(s, s')$ has been matched. Let $A$ and $R$ denote the sets of servers and requests before time $t$ in $(s, s')$, respectively. Then the sets $A$ and $R$ have the same cardinality and*

$$\gamma M(A, R) \leq \gamma M(A \cup \{s\}, R \cup \{s'\}) + d(s, s'). \tag{3}$$

Given this theorem, Property 1 follows from the statement that $A$ and $R$ have the same cardinality and using induction on the requests. In fact we have that at any time, between any two consecutive unused servers, we have that the number of requests is equal to the number of servers.

Moreover, any time that $\gamma$-WFA matches the leftmost unused server to a new request, that request must be to the left of the second leftmost unused server (and possibly to the left of the leftmost server as well). If not, after the matching the theorem would be violated for the leftmost interval between unmatched servers. Thus, to the left of the leftmost unused server, the number of requests is always the same as the number of servers, and the same holds to the right of the rightmost unused server by symmetry. Recalling that we can match the servers to requests in a left-to-right fashion to get an optimal matching, this leads to the following useful observation.

**Observation 1** *At any time $t$, in the optimal matching $M(A^t, R^t)$, there is no line crossing any unused server.*

Hence the value $M(A^t \cup \{s\}, R^t \cup \{r\})$ in (2) can be decomposed as the sum of a set of optimal matchings of servers to requests, each on a distinct part of the real line. This holds for both possible choices of $s$ (both surrounding servers of $r$) and the only term in the sum which is different between these two choices is the term for the optimal matching in the closed interval between the surrounding servers. Hence the choice of *which* surrounding server to use is indeed based only on the history of requests between these two servers.

Conversely, the theorem follows if we assume that $\gamma$-WFA has Property 1: for a request $r$ to the right of $s'$ (but to the left of the next unused server), $\gamma$-WFA will prefer to use $s'$ rather than $s$ to match $r$ by Property 1. Hence

$$\gamma M(A^t \cup \{s'\}, R^t \cup \{r\}) + d(s', r) \leq \gamma M(A^t \cup \{s\}, R^t \cup \{r\}) + d(s, r)$$

for any such request $r$. By letting $d(s', r)$ tend to 0, we get $M(A^t \cup \{s'\}, R^t \cup \{r\}) \to M(A^t, R^t)$, and $M(A^t \cup \{s\}, R^t \cup \{r\}) \to M(A^t \cup \{s\}, R^t \cup \{s'\})$: the theorem follows by induction on the requests (formally, we should prove Observation 1 by induction on the requests, and use that to split the optimal matching into distinct parts in order to derive (3)).

**Proof** (Theorem 1) We use induction on the number of requests in $R$. At the start of the input (when only the server locations are known), there are no servers or requests between any two consecutive servers $s$ and $s'$, so $A = R = \emptyset$ and $M(A, R) = 0$. For the induction step, let $r$ be the most recent request in $R$ and let $s_r$ be the server matched to $r$ by $\gamma$-WFA. The induction hypothesis is that $s_r$ is in the interval $(s, s')$, so $s_r \in A$. Let $A_1$ and $A_2$ be the set of servers in $(s, s_r)$ and $(s_r, s')$, respectively. Let also $R_1$ and $R_2$ be the associated requests in these intervals that arrived before $r$. By induction, $A_i$ has the same cardinality as $R_i$ for $i = 1, 2$.

**Case 1: $r$ is to the left of $s_r$** By the definition of $\gamma$-WFA we have

$$\gamma M(A_1 \cup \{s_r\}, R_1 \cup \{r\}) + d(s_r, r) \leq \gamma M(A_1 \cup \{s\}, R_1 \cup \{r\}) + d(s, r).$$

By induction, we have

$$\gamma M(A_2, R_2) \leq \gamma M(A_2 \cup \{s_r\}, R_2 \cup \{s'\}) + d(s_r, s').$$

Therefore

$$\begin{aligned}
\gamma M(A, R) &= \gamma M(A_1 \cup \{s_r\}, R_1 \cup \{r\}) + \gamma M(A_2, R_2) \\
&\leq \gamma M(A_1 \cup \{s\}, R_1 \cup \{r\}) + d(s, r) - d(s_r, r) + \gamma M(A_2 \cup \{s_r\}, R_2 \cup \{s'\}) + d(s_r, s') \\
&= \gamma M(A \cup \{s\}, R \cup \{s'\}) + d(s, r) - d(s_r, r) + d(s_r, s') \\
&\leq \gamma M(A \cup \{s\}, R \cup \{s'\}) + d(s, s').
\end{aligned}$$

**Case 2: $r$ is to the right of $s_r$** The proof is completely symmetric to the above, with $A_1$ and $A_2$ switching roles, as well as $R_1$ and $R_2$. □

We next prove a useful lemma about matchings.

**Lemma 1 (Quasi-convexity)** *Let $s$ and $s'$ be two servers and let $A$ and $R$ be two sets of points in $(s, s')$ of equal cardinality. Let also $r_1 \leq r_2$ be two points in $[s, s']$. Then*

*(i)* $M(A \cup \{s\} \cup \{s\}, R \cup \{r_1\} \cup \{r_2\}) - M(A \cup \{s\}, R \cup \{r_1\}) \geq M(A \cup \{s\}, R \cup \{r_2\}) - M(A, R).$

*(ii)* $M(A \cup \{s\} \cup \{s'\}, R \cup \{r_1\} \cup \{r_2\}) - M(A \cup \{s\}, R \cup \{r_1\}) = M(A \cup \{s'\}, R \cup \{r_2\}) - M(A, R).$

**Proof** (i) Because of (1), we will consider the values of $\text{CROSS}_x = \text{CROSS}_x(A, R)$. The inequality to be proved can be rewritten as

$$M(A \cup \{s\} \cup \{s\}, R \cup \{r_1\} \cup \{r_2\}) + M(A, R) \geq M(A \cup \{s\}, R \cup \{r_1\}) + M(A \cup \{s\}, R \cup \{r_2\}). \quad (4)$$

Fix a point $x \in [s, r_1)$. The total number of lines that include $x$ on the right hand side of (4) is

$$|\text{LEFT}_x(A \cup \{s\}) - \text{LEFT}_x(R \cup \{r_1\})| + |\text{LEFT}_x(A \cup \{s\}) - \text{LEFT}_x(R \cup \{r_2\})| = 2|\text{CROSS}_x + 1|$$

and on the left hand side of (4) it is

$$\begin{aligned}
&|\text{LEFT}_x(A \cup \{s\} \cup \{s\}) - \text{LEFT}_x(R \cup \{r_1\} \cup \{r_2\})| + |\text{LEFT}_x(A) - \text{LEFT}_x(R)| \\
=&|\text{CROSS}_x + 2| + |\text{CROSS}_x| \\
\geq&2|\text{CROSS}_x + 1|
\end{aligned}$$

where the inequality follows by concavity of the absolute function.

For $x \in [r_1, r_2)$, we get $|\text{CROSS}_x| + |\text{CROSS}_x + 1| = |\text{CROSS}_x + 1| + |\text{CROSS}_x|$, and for $\in [r_2, s)$, we get $|\text{CROSS}_x + 2| + |\text{CROSS}_x| \geq 2|\text{CROSS}_x + 1|$ again.

(ii) We rewrite the equality using sums of positive terms as above and consider the lines crossing $x$. For $x \in (s, r_1] \cup [r_1, s')$, we get $|\text{CROSS}_x + 1| + |\text{CROSS}_x| = |\text{CROSS}_x + 1| + |\text{CROSS}_x|$. For $x \in (r_1, r_2)$, we find $2|\text{CROSS}_x| = 2|\text{CROSS}_x|$: there is no change in this region. □

**Theorem 2 (Generalized Quasi-Convexity)** *Let $s_1, s_2$ be two servers. Let $A, R$ denote sets of servers and requests in $(s_1, s_2)$, where $|A| = |R|$. Let $e$ be a point in $(s_1, s_2)$. Let $R_1$ denote an arbitrary set of points in $[s_1, e]$ and let $R_2$ denote an arbitrary set of points in $[e, s_2]$. Then*

$$M(A \cup \{s_1\}^{|R_1|} \cup \{s_2\}^{|R_2|} \cup s_1, R \cup R_1 \cup R_2 \cup e) - M(A \cup \{s_1\}^{|R_1|} \cup \{s_2\}^{|R_2|}, R \cup R_1 \cup R_2) \\ \geq M(A \cup \{s_1\}, R \cup \{e\}) - M(A, R).$$

**Proof** Order the points in $R_1$ and $R_2$ in some way. Let $R_{1,i}$ denote the first $i$ requests in $R_1$ and let $R_{2,j}$ denote the first $j$ requests in $R_2$. Let $A_{i,j} = A \cup \{s_1\}^i \cup \{s_2\}^j$ and $R_{i,j} = R \cup R_{1,i} \cup R_{2,j}$. By repeatedly applying Lemma 1 (ii), we see that

$$\begin{aligned} & M(A_{i,j} \cup \{s_1\}, R_{i,j} \cup \{e\}) - M(A_{i,j}, R_{i,j}) \\ =& M(A_{i,j-1} \cup \{s_1\}, R_{i,j-1} \cup \{e\}) - M(A_{i,j-1}, R_{i,j-1}) \\ =& \dots \\ =& M(A_{i,0} \cup \{s_1\}, R_{i,0} \cup \{e\}) - M(A_{i,0}, R_{i,0}). \end{aligned}$$

We then repeatedly apply Lemma 1 (i) to find

$$\begin{aligned} & M(A_{i,0} \cup \{s_1\}, R_{i,0} \cup \{e\}) - M(A_{i,0}, R_{i,0}) \\ \geq& M(A_{i-1,0} \cup \{s_1\}, R_{i-1,0} \cup \{e\}) - M(A_{i-1,0}, R_{i-1,0}) \\ \geq& \dots \\ \geq& M(A \cup \{s_1\}, R \cup \{e\}) - M(A, R). \end{aligned}$$

This proves the theorem. □

Theorem 2 allows us to generalize Theorem 1 as follows.

**Theorem 3** *Consider a time $t$. Let $s$ and $s'$ be two servers that have not yet been matched by $\gamma$-wfa at time $t$, whereas every server in the interval $(s, s')$ has been matched. Let $A$ and $R$ denote the sets of servers and requests before time $t$ in $(s, s')$, respectively. Then for any set of points $R' = \{r_1, \dots, r_k\}$ in the interval $(s, s')$ we have*

$$\gamma M(A \cup \{s\}^k, R \cup R') \leq \gamma M(A \cup \{s\}^{k+1}, R \cup R' \cup \{s'\}) + d(s, s').$$

**Proof** By applying Lemma 1(i) repeatedly, we get

$$\begin{aligned} & M(A \cup \{s\}^{k+1}, R \cup R' \cup \{s'\}) - M(A \cup \{s\}^k, R \cup R') \\ \geq& M(A \cup \{s\}^k, R \cup R' \backslash r_k \cup \{s'\}) - M(A \cup \{s\}^{k-1}, R \cup R' \backslash r_k) \\ \geq& \dots \\ \geq& M(A \cup \{s\}, R \cup \{s'\}) - M(A, R) \\ \geq& -\frac{1}{\gamma} d(s, s') \end{aligned}$$

where the last inequality follows from Theorem 2. □

## 3 Optimal vs pseudo-optimal matching

At any time $t$, there is the possibility that the adversary uses a subset $A' \neq A^t$ of the servers $S$ to match $R^t$ and potentially have very small cost. Denote the optimal way of matching $R^t$ using $S$ by $\textsc{opt}_t = M(A', R^t)$, and write $\textsc{pseudo}_t = M(A^t, R^t)$. Our goal in this section is to show that $\textsc{pseudo}_t$ is always relatively close to $\textsc{opt}_t$; how close exactly it is depends on $\gamma$. This means that it is sufficient to compare the cost of $\gamma$-WFA to $\textsc{pseudo}_t$ to bound the competitive ratio.

We begin by developing some technical terms that help us exploit the locality property of $\gamma$-WFA.

**Definition 3** *Let $s_1, s_2$ be two unmatched servers such that in the interval $(s_1, s_2)$ all servers $A$ are matched to requests $R$. Then the interval $(s_1, s_2)$ is a* balanced *interval.*

By Property 1, we can focus our analysis on balanced intervals. Let $K \subset \mathbb{R}$ be a finite union of finite intervals, and let $||K||$ denote the Lebesgue measure of this set, which is the total sum of the lengths of the intervals in $K$. Let $(s_1, s_2)$ be a balanced interval with servers $A$ and requests $R$. Define $B^j(A, R) = \{x : \textsc{cross}_x(A, R) = j\}$, $B^+ = \cup_{j>0} B^j$ and $B^- = \cup_{j<0} B^j$. We will abuse notation and use these same symbols also to denote the Lebesgue measure of these sets. For any balanced interval $(s_1, s_2)$, we define

$$\textsc{pseudo}(s_1, s_2) = M(A, R) = \int_{x \in (s_1, s_2)} |\textsc{cross}_x(A, R)| dx = \sum_j |j| \cdot B^j.$$

Thus we have for instance $M(A \cup \{s_1\}, R \cup \{s_2\}) = M(A, R) + B^+ + B^0 - B^-$ (visually, the new matching can be obtained by drawing an additional line from $s_1$ to $s_2$; this line gets added to intervals with zero or more existing lines in the same direction, and cancels out one line in all intervals with at least one line in the opposite direction) and $d(s_1, s_2) = B^+ + B^0 + B^-$. Substituting these values in (3) in Theorem 1, we find

$$\begin{aligned} & \gamma M(A, R) \leq \gamma(M(A, R) + B^+ + B^0 - B^-) + B^+ + B^0 + B^- \\ \Rightarrow \quad & 0 \leq (\gamma + 1)(B^+ + B^0) - (\gamma - 1)B^- \\ \Rightarrow \quad & (\gamma - 1)B^- \leq (\gamma + 1)(B^+ + B^0) \\ \Rightarrow \quad & 2\gamma B^- \leq (\gamma + 1)d(s_1, s_2). \end{aligned}$$

Analogously, $M(A \cup \{s_2\}, R \cup \{s_1\}) = M(A, R) - B^+ + B^0 + B^-$ implies $2\gamma B^+ \leq (\gamma + 1)d(s_1, s_2)$. We generalize this result in the next lemma.

**Lemma 2 (Prefix-B-lemma)** *Let $(s_1, s_2)$ be a balanced interval which includes the set of servers $A$ and set of requests $R$. For any $x \in (s_1, s_2)$,*

$$(\gamma - 1)||B^-(A, R) \cap [s_1, x]|| \leq (\gamma + 1)||(B^+(A, R) \cup B^0(A, R)) \cap [s_1, x]||$$
$$(\gamma - 1)||B^+(A, R) \cap [s_1, x]|| \leq (\gamma + 1)||(B^-(A, R) \cup B^0(A, R)) \cap [s_1, x]||.$$

The intuition behind this lemma is that since $s_1$ is an unmatched server, it was rejected in favor of other servers in the interval $(s_1, s_2)$. This indicates that $||B^-(A, R) \cap [s_1, x]||$, which counts the parts of $(s_1, s_2)$ on which there are more requests than servers to the left (alternatively, counts parts where servers are matched to requests to their left), cannot be too large relative to the distance to $s_1$.

**Proof** We use induction on the requests in $(s_1, s_2)$. Before any requests arrive, we have $A = R = \emptyset$ and the lemma holds. For the induction step, suppose that the statement was true before the last request $r \in (s_1, s_2)$ was matched to server $s \in (s_1, s_2)$. Let $A' = A \backslash \{s\}$ and $R' = R \backslash \{r\}$. We consider two cases depending on whether $s$ is to the left of $r$ or not.

If $s < r$, for any $x \in (s_1, s_2)$ we have $\textsc{cross}_x(A, R) \geq \textsc{cross}_x(A', R')$, so $||B^-(A, R) \cap [s_1, x]||$ satisfies the bound by induction.

Suppose $r < s$. For $x \in (s_1, r)$, the lemma holds by induction, as $\textsc{cross}_x(A, R) = \textsc{cross}_x(A', R')$. Suppose $x = s$. Since $\gamma$-$\textsc{wfa}$ serves $r$ with $x = s$ and not $s_1$, we have

$$\gamma M(A' + s, R' + r) + d(r, s) \leq \gamma M(A' + s_1, R' + r) + d(s_1, r) \tag{5}$$

Let $B_1^+ = ||B^+(A', R') \cap [s_1, r]||$, $B_2^+ = ||B^+(A', R') \cap [r, s]||$ and define $B_1^-, B_1^0, B_2^-, B_2^0$ analogously. Finally let $B^+ = B_1^+ + B_2^+$ etc. We have

$$||B^-(A, R) \cap [s_1, s]|| = B^- + B_2^0 \tag{6}$$

since we have $\textsc{left}_x(A) < \textsc{left}_x(R)$ if and only if $x \in B^- \cup B_2^0$. This means that for the complement we have

$$||(B^+(A, R) \cup B^0(A, R)) \cap [s_1, s]|| = B^+ + B^0 - B_2^0 = B^+ + B_1^0. \tag{7}$$

Now (5) can be rewritten as

$$\begin{aligned}
\gamma(M(A', R') + B_2^- + B_2^0 - B_2^+) + d(r, s) &\leq \gamma(M(A', R') + B_1^+ + B_1^0 - B_1^-) + d(s_1, r) \\
\Rightarrow \gamma(B_2^- + B_2^0 - B_2^+) &\leq \gamma(B_1^+ + B_1^0 - B_1^-) + B_1^+ + B_1^0 + B_1^- - d(r, s) \\
&= (\gamma + 1)(B_1^+ + B_1^0) - (\gamma - 1)B_1^- - d(r, s) \\
\Rightarrow (\gamma - 1)(B^- + B_2^0) + B_2^- + B_2^0 &\leq (\gamma + 1)(B^+ + B_1^0) - B_2^+ - d(r, s) \\
\Rightarrow (\gamma - 1)||B^-(A, R) \cap [s_1, s]|| &\leq (\gamma + 1)((||B^+(A, R) \cup B^0(A, R)) \cap [s_1, s]||) - 2d(r, s) \\
\Rightarrow 2\gamma||B^-(A, R) \cap [s_1, s]|| &\leq (\gamma + 1)d(s_1, s) - 2d(r, s)
\end{aligned} \tag{8}$$

where we have used (6) and (7) in the penultimate line. This means that the lemma also holds for any $x \in [s, s_2)$, from the induction hypothesis for the balanced interval $(s, s_2)$ (which has fewer requests).

Now suppose $x \in [r, s)$. Before request $r$ arrived, $(s_1, s)$ was a balanced interval. Hence by the second part of the induction hypothesis, we have

$$\begin{aligned}
2\gamma||B^+(A', R') \cap [x, s]|| &\leq (\gamma + 1)d(x, s) \\
\Rightarrow ||B^+(A', R') \cap [x, s]|| &\leq \frac{\gamma + 1}{2\gamma}d(x, s) \\
\Rightarrow ||B^-(A, R) \cap [x, s]|| &\geq \frac{\gamma - 1}{2\gamma}d(x, s) \\
\Rightarrow -2\gamma||B^-(A, R) \cap [x, s]|| &\leq -(\gamma - 1)d(x, s)
\end{aligned} \tag{9}$$

where we have used $||B^-(A, R) \cap [x, s]|| = s - x - ||B^+(A', R') \cap [x, s]||$. Summing (8) and (9) we get

$$2\gamma||B^-(A, R) \cap [s_1, x]|| \leq (\gamma + 1)d(s_1, x) - 2d(r, x)$$

and the lemma follows. □

**Theorem 4** *At any time $t$,*

$$\text{OPT}_t \le \text{PSEUDO}_t \le \frac{\gamma+1}{\gamma-1}\text{OPT}_t. \tag{10}$$

The first inequality in (10) holds trivially. Thus, consider the second inequality. By Observation 1, it is sufficient to consider a set of requests in a single balanced interval $(s_1, s_2)$, with servers $A$ and requests $R$. Unlike PSEUDO, OPT may use servers outside this interval to serve some or all of the requests inside it. Let $A'$ be the set of servers used by OPT. This means that for OPT, the servers in $A\backslash A'$ are still unmatched. (If $A = A'$, there is nothing to show.) Hence requests to the locations of these servers can be matched for free. For the analysis, we add a request for each such location, and match them to servers in $A'\backslash A$. We have

$$\text{OPT}_t = M(A', R) = M(A' \cup (A\backslash A'), R \cup (A\backslash A')),$$

whereas $\text{PSEUDO}_t$ increases by the sum of the distances between the servers in $A'\backslash A$ and the server locations in $A\backslash A'$ that they are matched to.

It can be seen that for each $x \notin (s_1, s_2)$, the number of lines that connect points outside $(s_1, s_2)$ to points inside $(s_1, s_2)$ and cross $x$ in both $\text{OPT}_t$ (modified or unmodified) and the modified $\text{PSEUDO}_t$ matching is the same. Therefore the cost of the matchings $\text{OPT}_t$ and $\text{PSEUDO}_t$ outside $(s_1, s_2)$ (restricted to requests in $(s_1, s_2)$) are the same. Since we are interested in upper bounding the ratio $\text{PSEUDO}_t/\text{OPT}_t$, the worst case is that all servers in $A'\backslash A$ are located exactly at the boundaries $s_1$ and $s_2$. Thus, the theorem will follow from the following lemma.

**Lemma 3** *Let $(s_1, s_2)$ be a balanced interval that contains the set of servers $A$ which is matched to requests $R$. For any set of points $\{v_i\}_{i=1}^{k+m}$ in the interval $(s_1, s_2)$, we have*

$$M\left(A \cup \{s_1\}^k \cup \{s_2\}^m, R \cup \{v_i\}_{i=1}^{k+m}\right) \ge \frac{\gamma-1}{\gamma+1} M(A, R).$$

**Proof** Rename the points such that $v_1 \le \cdots \le v_{k+m}$ and denote $v_0 = s_1$ and $v_{k+m+1} = s_2$. The difference between the two matchings under consideration consists of $k+1-i$ lines from left to right in each interval $(v_{i-1}, v_i)$ for $i = 1, \ldots, k$, and $i-k$ lines from right to left in each interval $(v_i, v_{i+1})$ for $i = k+1, \ldots, k+m$. Particularly, there is no difference in the interval $(v_k, v_{k+1})$ (since for $x \in (v_k, v_{k+1})$ we have $\text{LEFT}_x(A) - \text{LEFT}_x(R) = \text{LEFT}_x(A \cup \{s_1\}^k \cup \{s_2\}^m) - \text{LEFT}_x(R \cup \{v_i\}_{i=1}^{k+m})$). Denote by $M(A, R)|_{(a,b)}$ the total cost of the optimal matching of $R$ to $A$ inside the interval $(a, b)$, then

$$M\left(A \cup \{s_1\}^k \cup \{s_2\}^m, R \cup \{v_i\}_{i=1}^{k+m}\right)|_{(v_k, v_{k+1})} = M(A, R)|_{(v_k, v_{k+1})}.$$

We now proceed to prove

$$M\left(A \cup \{s_1\}^k, R \cup \{v_i\}_{i=1}^{k}\right)|_{(s, v_k)} \ge \frac{\gamma-1}{\gamma+1} M(A, R)|_{(s, v_k)}.$$

The symmetric statement involving $s_2$ then follows immediately, and altogether this proves the lemma (note that $M\left(A \cup \{s_1\}^k, R \cup \{v_i\}_{i=1}^{k}\right)|_{(s,v_k)} = M\left(A \cup \{s_1\}^k \cup \{s_2\}^m, R \cup \{v_i\}_{i=1}^{k+m}\right)|_{(s,v_k)}$.) Define $L_i^j = ||\{x : \text{CROSS}_x(A, R) = j\} \cap (v_{i-1}, v_i]||$ to be the measure of points between $v_{i-1}$ and $v_i$ which the optimal matching crosses $j$ times (in the direction indicated by the sign of $j$).

Let $t \in \{1, \dots, k\}$. Applying Lemma 2 to interval $(s_1, v_t]$ we obtain

$$\frac{\gamma - 1}{\gamma + 1} \sum_{i=1}^{t} \sum_{j<0} L_i^j \le \sum_{i=1}^{t} \sum_{j \ge 0} L_i^j.$$

Summing for $t = 1, \dots, k$, we get

$$\begin{aligned} & \frac{\gamma - 1}{\gamma + 1} \sum_{t=1}^{k} \sum_{i=1}^{t} \sum_{j<0} L_i^j \le \sum_{t=1}^{k} \sum_{i=1}^{t} \sum_{j \ge 0} L_i^j \\ \Rightarrow \quad & \frac{\gamma - 1}{\gamma + 1} \sum_{i=1}^{k} \sum_{j<0} (k+1-i) L_i^j \le \sum_{i=1}^{k} \sum_{j \ge 0} (k+1-i) L_i^j \end{aligned} \tag{11}$$

Now we can bound

$$\begin{aligned} & M(A, R) - M\left(A \cup \{s_1\}^k, R \cup \{v_i\}_{i=1}^k\right) \\ = & \sum_{i=1}^{k} \sum_{j} |j| L_i^j - \sum_{i=1}^{k} \sum_{j} |j + k + 1 - i| L_i^j \\ = & \sum_{i=1}^{k} \sum_{j<0} |j| L_i^j - \sum_{i=1}^{k} \sum_{j<0} |j + k + 1 - i| L_i^j - \sum_{i=1}^{k} \sum_{j \ge 0} (k+1-i) L_i^j \\ \le & \sum_{i=1}^{k} \sum_{j \le -(k+1-i)} (k+1-i) L_i^j + \sum_{i=1}^{k} \sum_{j=1}^{k+1-i} (2j - k - 1 + i) L_i^j \\ & - \frac{\gamma - 1}{\gamma + 1} \sum_{i=1}^{k} \sum_{j<0} (k+1-i) L_i^j \qquad \text{using (11)} \\ = & \sum_{i=1}^{k} \sum_{j \le -(k+1-i)} \frac{2}{\gamma + 1} (k+1-i) L_i^j + \sum_{i=1}^{k} \sum_{j=1}^{k+1-i} \left(2j - \frac{2\gamma}{\gamma + 1}(k+1-i)\right) L_i^j \\ \le & \sum_{i=1}^{k} \sum_{j \le -(k+1-i)} \frac{2}{\gamma + 1} |j| L_i^j + \sum_{i=1}^{k} \sum_{j=1}^{k+1-i} \frac{2}{\gamma + 1} j L_i^j \\ = & \frac{2}{\gamma + 1} \sum_{i=1}^{k} \sum_{j<0} |j| L_i^j \\ \le & \frac{2}{\gamma + 1} M(A, R). \end{aligned}$$

Here the penultimate inequality follows since

$$j \le k + 1 - i \Rightarrow \left(2 - \frac{2}{\gamma + 1}\right) j = \frac{2\gamma}{\gamma + 1} j \le \frac{2\gamma}{\gamma + 1} (k + 1 - i).$$

Thus the lemma follows for $m = 0$. When $m > 0$ the proof is similar but now the right-hand side includes also the terms with $j > 0$; this is still bounded by $\frac{2}{\gamma+1} M(A, R)$ and the lemma holds. $\square$

## 4 A linear upper bound

Let $r_i$ be the $i$-th request and $s_i$ be the server it is matched to. Generally, let $R_i$ be the set of the first $i$ requests and denote by $S_i$ the set of servers that $\gamma$-WFA uses to match them. Using the $\text{CROSS}_x$ values, we have

$$d(r_n, s_n) \leq M(S_{n-1}, R_{n-1}) + M(S_n, R_n).$$

This can be seen as follows. Let $B^0 = B^0(S_{n-1}, R_{n-1})$. If $B_0 = \emptyset$, we are done. Else, we have two cases. If $r_n < s_n$, we have $B^-(S_n, R_n) \geq B_0$. If $s_n > r_n$, we have $B^+(S_n, R_n) \geq B_0$. These inequalities follow because in intervals where there were previously no lines (i.e., $\text{LEFT}_x(S_{n-1}) = \text{LEFT}_x(R_{n-1})$) we now do have a line, because $\text{LEFT}_x(S_n) \neq \text{LEFT}_x(R_{n-1})$. In both cases, the desired result follows.

Together with Theorem 1, this implies that the cost to service request at time $i$ is bounded from above by

$$\text{PSEUDO}_{i-1} + \text{PSEUDO}_i \leq \frac{\gamma+1}{\gamma-1}(\text{OPT}_{i-1} + \text{OPT}_i).$$

Since $\text{OPT}_i \leq \text{OPT}_j$ for all $i \leq j$, the claim follows. (Note that the cost of PSEUDO is not necessarily non-decreasing over time.)

## 5 Lower bounds for $\gamma$-wfa

In this section, we will repeatedly use the so-called cruel adversary, which gives each request (after the first one) at the position of the last server used by $\gamma$-WFA. Let $n$ be the number of requests.

**Theorem 5** *Let $\alpha = 2/(\gamma+1)$. Then $\gamma$-WFA has a competitive ratio of at least*

$$2\frac{\alpha^{n-1}-1}{\alpha-1} + 1$$

*on the line for $0 \leq \gamma < 1$.*

**Proof** We have $\alpha > 1$. Let the positions of the servers be $-1-\varepsilon$ and $(\alpha^i - 1)/(\alpha - 1)$ for $i = 1, \ldots, n-1$. The requests will be at the locations $(\alpha^i - 1)/(\alpha - 1)$ for $i = 0, \ldots, n-1$. We show by induction that each request is served from the right. The first request is at position 0, and the surrounding servers are at positions $-1-\varepsilon$ and 1. Thus, $\gamma$-WFA serves it from the right since the distance is smaller. For the $i$th request, by induction the surrounding servers are at positions $-1-\varepsilon$ and $(\alpha^i - 1)/(\alpha - 1)$, and the total cost so far is exactly the position of this request, which is $(\alpha^{i-1} - 1)/(\alpha - 1)$. We have that $\gamma$-WFA serves it from the right if

$$\begin{aligned}
\gamma \cdot (1+\varepsilon) + \left(\frac{\alpha^{i-1}-1}{\alpha-1} + 1 + \varepsilon\right) &> \gamma\left(\frac{\alpha^i-1}{\alpha-1}\right) + \left(\frac{\alpha^i-1}{\alpha-1} - \frac{\alpha^{i-1}-1}{\alpha-1}\right) \\
\Leftrightarrow \gamma(\alpha-1)(1+\varepsilon) + \alpha^{i-1} - 1 + (\alpha-1)(1+\varepsilon) &> \gamma(\alpha^i - 1) + (\alpha^i - \alpha^{i-1}) \\
\Leftrightarrow 2\alpha^{i-1} - \alpha^i + \alpha - 2 &> \gamma(\alpha^i - 1 - (\alpha-1)(1+\varepsilon)) - \varepsilon(\alpha-1) \\
&= \gamma(\alpha^i - \alpha) - \varepsilon(\gamma+1)(\alpha-1) \\
\Leftrightarrow (\alpha^{i-1}-1)(2-\alpha) &> \gamma\alpha(\alpha^{i-1}-1) - \varepsilon(\gamma+1)(\alpha-1) \\
\Leftrightarrow 2 - \alpha &> \alpha\gamma - \varepsilon\gamma(\alpha-1)/(\alpha^i-1) \\
\Leftrightarrow 2 &> \alpha(\gamma+1) - \varepsilon(\gamma+1)(\alpha-1)/(\alpha^i-1).
\end{aligned}$$

Since $\alpha = 2/(\gamma+1) > 1$, this holds for any $\varepsilon > 0$.

The final request at position $(\alpha^{n-1}-1)/(\alpha-1)$ is served from the left, using the last remaining server, giving a total online cost of $2(\alpha^{n-1}-1)/(\alpha-1)+1+\varepsilon$. The optimal solution is to serve the request at 0 from the left and all others with cost 0, for an optimal cost of $1+\varepsilon$. □

**Theorem 6** *For $\gamma = 1$, $\gamma$-WFA has a competitive ratio of at least $2n-1$ on the line.*

**Proof** Let the positions of the servers be $-1-\varepsilon$ and $i-1$ for $i = 2, \ldots, n$. As in the previous proof, the first request at position 0 is served from the right. We continue using the cruel adversary and prove by induction that each request is served from the right. By the induction hypothesis, the total cost to serve the first $i-1$ requests is $i-1$. The $i$th request is at position $i-1$ and the surrounding servers are at positions $-1-\varepsilon$ and $i$. $\gamma$-WFA uses the server at position $i$ if $\gamma(1+\varepsilon)+i+\varepsilon > \gamma \cdot i+1$, which is true for $\gamma = 1$ and any $\varepsilon > 0$.

The final request at position $n-1$ is served from the left, for a total cost of $2n-1$. The optimal cost to serve this sequence is $1+\varepsilon$ as before. □

**Theorem 7** *For $\gamma = \infty$, $\gamma$-WFA has a competitive ratio of at least $n$ on the line.*

**Proof** Let $n$ be even and $\varepsilon \in (0,1)$. Let the positions of the servers be $i$ for $i = 1, \ldots, n/2$ and $-i-\varepsilon$ for $i = 1, \ldots, n/2$. Again the first request at position 0 is served from the right. We claim that the cruel adversary gives the requests in the order $0, 1, -1-\varepsilon, 2, -2-\varepsilon, \ldots$.

For a positive request at location $i$, by the induction hypothesis, the optimal cost using the server at $-i-\varepsilon$ is $-i-\varepsilon$, and the optimal cost using the server at $i+1$ is $i+1$. Hence $\gamma$-WFA will indeed use the server at position $-i-\varepsilon$.

For a negative request at location $-i-\varepsilon$, by the induction hypothesis, the optimal cost using the server at $i+1$ is $i+1$ and the optimal cost using the server at $-i-1-\varepsilon$ is $i+1+\varepsilon$. Hence $\gamma$-WFA will indeed use the server at position $i+1$.

The last request is at position $n/2$ and will be served by the last unused server at position $-n/2-\varepsilon$. The total online cost is $1+(2+\varepsilon)+3+(4+\varepsilon)+\cdots+(n+\varepsilon) = n(n+1+\varepsilon)/2$. The optimal matching moves all servers at negative locations to the right by 1 and the last one to 0 for a total cost of $n/2+\varepsilon$, and the lemma follows. □

**Theorem 8** *For any value of $\gamma$, $\gamma$-WFA has a competitive ratio of $\Omega(\log n)$.*

**Proof** For simplicity, assume that $n$ is of the form $n = 2^t$. Place 1 server at locations 0 and $n$, and two servers at each location $i = 2, 4, \ldots, n-2$. There will be $t$ phases. In the first phase, requests arrive at locations $2i \pm (1-\varepsilon)$ for $i = 1, 3, \ldots, n/2-1$, for some small positive value of $\varepsilon$. $\gamma$-WFA will serve these requests using the pairs of servers at the locations $i = 2, 6, \ldots, n-2$ (from each pair, one server will move left and one server will move right). We now have unused servers left at locations $i = 4, 8, \ldots, n-4$, as well as the single servers at locations 0 and $n$. Hence, the structure of the server locations is exactly the same as at the beginning, but they are now twice as far apart.

In phase $p = 2, \ldots, t-1$, requests at locations $i \cdot 2^p \pm (2^{p-1}-\varepsilon)$ will arrive for $i = 1, 3, \ldots, 2^{t-p}-1$, and due to symmetry it is easy to see that they will be served using the servers at locations $i2^p$ for $i = 1, 3, \ldots, 2^{t-p}-1$. In phase $t$, there are two requests at location $2^{t-1}$, and the only servers remaining are at locations 0 and $2^t$.

Ignoring $\varepsilon$, it is easy to see that at the end there is one request at each location $1, 2, \ldots, n-1$, plus one additional request at location $n/2$. Hence one server at locations $0, 2, \ldots, n/2-2$ can be

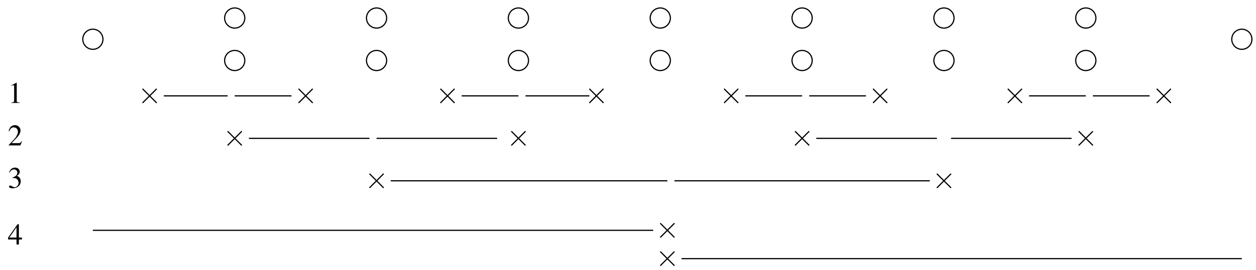

Figure 1: The lower bound for $n = 16$. We have $t = 4$, so there are four phases; the fourth phase is slightly different than the others. Circles denote servers and crosses denote requests.

used to serve a request at a distance of 1 to its right, and one server at locations $2, \ldots, n/2$ can be used to serve a request at distance 0. We can use a symmetric version of this solution on the interval $[n/2, n]$ for a total cost of $n/2 = 2^{t-1}$.

In contrast, $\gamma$-WFA incurs a cost of $n/2$ in each phase $1, \ldots, t-1$ and $n$ in phase $t$. Since there are $t = \log n$ phases, the theorem follows. □

## References


[1] Antonios Antoniadis, Neal Barcelo, Michael Nugent, Kirk Pruhs, and Michele Scquizzato. A o(n) -competitive deterministic algorithm for online matching on a line. In Evripidis Bampis and Ola Svensson, editors, *Approximation and Online Algorithms - 12th International Workshop, WAOA 2014, Wrocław, Poland, September 11-12, 2014, Revised Selected Papers*, volume 8952 of *Lecture Notes in Computer Science*, pages 11–22. Springer, 2014.

[2] Bernhard Fuchs, Winfried Hochstättler, and Walter Kern. Online matching on a line. *Theor. Comput. Sci.*, 332(1-3):251–264, 2005.

[3] Elias Koutsoupias and Akash Nanavati. The online matching problem on a line. In Klaus Jansen and Roberto Solis-Oba, editors, *Approximation and Online Algorithms, First International Workshop, WAOA 2003, Budapest, Hungary, September 16-18, 2003, Revised Papers*, volume 2909 of *Lecture Notes in Computer Science*, pages 179–191. Springer, 2003.

[4] Akash Nanavati. *Coordination Mechanisms and Online Matching.* PhD thesis, University of California, Los Angeles, 2004.