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In this column, Joan Boyar et al. from the University of Southern Denmark in Odense present a survey on online algorithms with advice. This is a relatively new area in which we examine the effect of having advice about the optimal solution on the competitive ratio of online algorithms.

As the authors explain, this question is interesting for various reasons. For instance, there is a strong relationship between advice and randomization for online algorithms, and in one case the study of advice for online algorithms has led unexpectedly to the fastest (offline) PTAS that is currently known for a problem.

I would like to invite more contributions to this column, be it surveys, conference reports, or technical articles related to online algorithms and competitive analysis. If you are considering becoming a guest writer, don't hesitate to mail me at `rvs4@le.ac.uk`.

Online Algorithms with Advice: A Survey *

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Abstract

Online algorithms with advice is an area of research where one attempts to measure how much knowledge of the future is necessary to achieve a given competitive ratio. The lower bound results give robust bounds on what is possible using semi-online algorithms. On the other hand, when the advice is of an obtainable form, algorithms using advice can lead to semi-online algorithms. There are strong relationships between advice complexity and randomization, and advice complexity has led to the introduction of the first complexity classes for online problems.

This survey concerning online algorithms with advice explains the models, motivates the study in general, presents some examples of the work that has been carried out, and includes a fairly complete set of references, organized by problem studied.

1 Introduction

Online algorithms solve optimization problems where the input is a finite request sequence, with one request arriving at a time. On receiving a request, the online algorithm must make some irrevocable decision, generally without any knowledge of future requests, attempting to minimize or maximize some objective function. There are various measures for the quality of online algorithms [33, 47], but the most standard is the competitive ratio [74, 99], which is essentially the approximation ratio. The performance of an online algorithm ALG is compared to the performance of an optimal offline algorithm OPT. Let $\text{ALG}(I)$ denote the value of the objective function applied to the output computed by ALG when given the request sequence I as input. Define $\text{OPT}(I)$ similarly. For minimization problems, ALG is *c-competitive* if there exists a constant b, such that for all finite request sequences, I,

$$\text{ALG}(I) \leq c \cdot \text{OPT}(I) + b.$$

Similarly, for maximization problems,

$$\text{OPT}(I) \leq c \cdot \text{ALG}(I) + b.$$

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In both cases, if the inequality holds with $b = 0$, the algorithm is *strictly c-competitive*. The (*strict*) *competitive ratio* of an algorithm is the infimum over all values of c for which the algorithm is (strictly) c-competitive.

Note that competitive analysis is a worst-case measure. Thus, it can be useful to think of the input as being generated by a malicious adversary who knows ALG. When studying an online problem, it is customary to consider both deterministic and randomized online algorithms. A randomized online algorithm ALG is c-competitive if it is c-competitive in expectation, i.e., if there exists a constant b such that $\mathbb{E}[\text{ALG}(I)] \leq c \cdot \text{OPT}(I) + b$ for all inputs I. This corresponds to the adversary being *oblivious* to the random choices made by the algorithm. See [16, 28] for further details.

We use n to denote the length of an input sequence. In some cases the competitiveness, c, is a function of n. When referring to *optimal* algorithms or solutions, we always refer to a solution which could have been produced by an optimal offline algorithm.

Advice complexity. Note that there are three basic assumptions underlying competitive analysis: The input is adversarial, decisions are irrevocable, and an online algorithm knows nothing about the requests before they arrive. Many possible ways of relaxing one or more of these assumptions have been studied. In the advice complexity model, the "no knowledge" assumption is relaxed in a problem-independent and quantitative way (while the two first assumptions remain unaltered). In this model, an *online algorithm with advice* is provided with some bits of advice about the request sequence I. These bits are provided by a trusted *oracle* that knows the entire request sequence and has no computational limitations (the formal definition of the advice complexity model(s) can be found in Section 2). Obviously, an online algorithm with advice may perform better than a traditional online algorithm, but if the amount of advice it receives from the oracle is bounded, it may perform less well than an optimal offline algorithm. The *advice complexity* of an algorithm is the maximum number of bits read by that algorithm on any request sequence of a given length.

Motivation. Given an online problem considered in the advice model, the major question asked is:

> How many bits of advice are necessary and sufficient to obtain a competitive ratio c?

This includes determining the number of bits to become optimal (strictly 1-competitive) or to beat the best deterministic or randomized algorithms. It also includes considerations in the other direction, such as determining what can be obtained using a constant number of bits, for instance.

In what follows, we have attempted to list the most important reasons the advice complexity model is interesting and relevant.

- Lower bounds give *robust bounds on what is possible using semi-online algorithms*. The value of certain upper bounds, on the other hand, may be questioned for the following reason. Since the advice is not restricted except by its size, it can happen that algorithms with advice are not of practical interest; they sometimes rely on advice that we do not expect to possess. However, lower bounds in the advice complexity model are very strong, exactly because we do not impose any restrictions on the type of advice: They apply to *any* possible information

about the request sequence that can be encoded using a sufficiently small number of bits. Thus, they can be very relevant for the study of semi-online algorithms (see Section 3).

- There are *strong connections between online algorithms with advice and randomized online algorithms*. For example, important open problems regarding randomized online algorithms (such as the best possible competitive ratio of a randomized k-Server or List Update algorithm) can be stated equivalently as problems about online algorithms with advice. Some results on online algorithms with advice lead to new lower and/or upper bounds on randomized online algorithms (see Section 4).
- It may be possible to use online algorithms with advice in settings where it is feasible to run multiple algorithms and output the best solution. For example, Boyar et al. [34] gave an algorithm using two bits of advice to choose between three algorithms for List Update, obtaining a competitive ratio better than any deterministic online algorithm. In using a List Update algorithm as a post-processing step of the Burrows-Wheeler Transform, the algorithm performing the compression can compare the results obtained from more than one algorithm, choose the best, and include as part of the compressed data which algorithm was actually used; see Kamali and López-Ortiz [73].
- Suppose that an online algorithm with b bits of advice runs in time $O(T(n))$. Then one may *convert the algorithm into an offline approximation algorithm* with time complexity $O(2^b \cdot T(n))$, by running the algorithm on all possible 2^b advice strings. For Reordering Buffer Management, the currently fastest $(1+\varepsilon)$-approximation algorithm is obtained in exactly this way by using an online algorithm with advice due to Adamaszek et al. [2].
- Online algorithms with advice may be viewed as *non-deterministic* online algorithms since one may think of the online algorithm as non-deterministically guessing the advice which it then uses to compute its output. Thus, the advice complexity of a problem measures the amount of non-determinism an online algorithm needs to achieve a given solution quality. Understanding the power of non-determinism (as compared to determinism and randomization) is one of the main challenges and most well-studied problems in theoretical computer science (P vs. NP, DFA vs. NFA, etc.). It seems natural to try to improve our understanding of how non-determinism may help when solving problems in an online environment.
- The first *complexity classes* for online algorithms have been based on advice complexity (see Section 7.3). The first class, Asymmetric Online Covering (AOC), contains many problems where the algorithm's irrevocable decisions are whether or not to accept or reject each request. All AOC-complete problems, such as Vertex Cover, Independent Set, Dominating Set, Cycle Finding, and Disjoint Path Allocation, have essentially the same advice complexity (linear in $\frac{n}{c}$, where c is the desired competitive ratio). Weighted versions of AOC-complete minimization problems are even harder. These complexity classes are not only interesting with respect to advice; in the online setting without advice, the complete problems are also exceptionally hard.

We now give two examples of simple advice complexity results. Note that the minimum number of bits required to encode the decisions Opt makes is an obvious upper bound for the amount of advice needed to be optimal. Sometimes, though, encoding Opt's decisions requires fewer bits than one first expects. Paging is an example of this.

Example 1: In PAGING, there is a set of N pages. A request sequence arrives online; each request is a page. The algorithm has a cache which starts out empty and can contain up to $k < N$ pages. When a page not in cache is requested, the page must be brought into cache, at a cost of 1. This is referred to as a page fault. If the cache is already full, the algorithm must select another page from its cache to evict to make room for the new page; this is the irrevocable online decision.

The optimal offline PAGING algorithm is Longest Forward Distance (LFD) [15] that always evicts the page, which will not be requested for the longest time. For deterministic online PAGING algorithms without advice, the best attainable competitive ratio is k [99], and for randomized algorithms it is H_k [1, 88], where $H_k \approx \ln k$ is the kth harmonic number.

How many advice bits does an algorithm need to be optimal? Clearly, $\lceil \log k \rceil n$ bits of advice are enough to simulate LFD by specifying the index in cache of the page to evict (if any) at each request. However, using a more clever encoding, one can obtain the following result:

Theorem 1 [Dobrev et al. [44]] There is an optimal PAGING algorithm, ALG, which reads n bits of advice.

Proof Using a fixed optimal solution for the given input, the oracle provides one bit of advice per request. That bit indicates whether or not, in the optimal solution, the page requested is kept in cache until the next time it is requested. ALG only evicts pages which will cause faults on their next request in the optimal solution as well. Thus, ALG is optimal. □

Figure 1 shows that for the question of the number of advice bits necessary and sufficient to achieve a certain competitive ratio, the "phase transitions" are essentially completely understood for PAGING. Mikkelsen [91] proved the following thresholds: For any fixed cache size k, a large but constant total number of advice bits is sufficient to achieve a competitive ratio of $H_k + \varepsilon$ (for any $\varepsilon > 0$), and a linear number of advice bits is necessary to be better than H_k-competitive.

The connection between advice complexity and randomization is key to proving both the upper and lower bounds of H_k. For more details on PAGING, see Section 8.

Another problem with a sharp phase transition is UNIFORM KNAPSACK.

Example 2: In UNIFORM KNAPSACK, a sequence of requests arrives online. Each request is a value in the range $(0, 1]$. When a request arrives, the online algorithm decides irrevocably whether to pack it in the knapsack or reject it. The total size of accepted requests is not allowed to exceed 1, the size of the knapsack. The goal is to maximize the sum of the sizes of the accepted requests. UNIFORM KNAPSACK is the special case of the standard knapsack problem where the size and the value of requests are always equal to each other. The problem is analyzed using strict competitive analysis, since setting the additive constant in the definition of competitiveness equal to 1 would make any algorithm 1-competitive.

A deterministic algorithm without advice for UNIFORM KNAPSACK has unbounded competitive ratio [87]. However, with just one bit of advice it is possible to be 2-competitive. The one bit of advice is used to indicate whether or not there is an item in the input sequence of size at least $\frac{1}{2}$. That information might actually be available in some applications, so it can also be viewed as a *semi-online algorithm*; an online algorithm which knows something about the request sequence in advance.

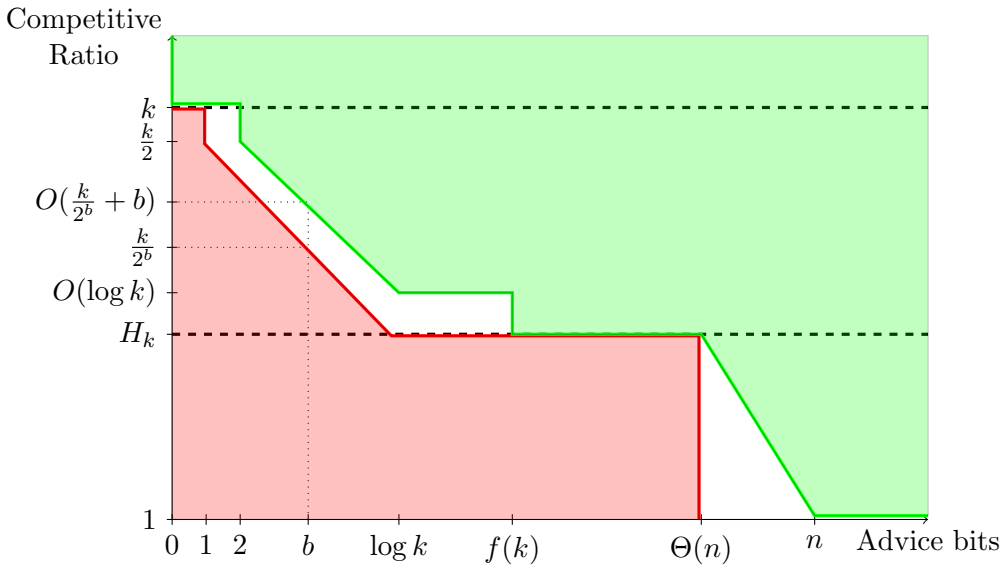

Figure 1: The (asymptotic) trade-off between competitive ratio and advice for Paging. The function $f(k)$ is a rapidly growing function of k (but does not depend on n). Consider a trade-off point (b, c) where b is a number of advice bits and c is a competitive ratio. The red area shows those trade-offs which provably cannot be achieved. The green area shows those trade-offs that we currently have algorithms achieving. It is an open problem if trade-offs in the white area are achievable or not. The horizontal dashed lines are the best possible competitive ratios of deterministic and randomized algorithms without advice.

Theorem 2 [Böckenhauer et al. [26]] There exists a 2-competitive Uniform Knapsack algorithm which reads one bit of advice.

Proof The oracle writes a 0 on the advice tape if no request of size at least $\frac{1}{2}$ will arrive and a 1 otherwise. The algorithm reads this one bit, b, of advice. If $b = 0$, it packs each request if it has enough space left for it. If $b = 1$, it rejects everything until it encounters an item of size at least $\frac{1}{2}$, which it packs (it may pack additional items that fit after this point).

For $b = 0$, if the total size of all requests arriving is less than 1, the algorithm will be optimal; otherwise, its knapsack will be at least half full the first time it rejects a request. If $b = 1$, the knapsack will again be at least half full. Thus, the algorithm is 2-competitive. □

Böckenhauer et al. [26] also prove that to obtain a competitive ratio better than 2, $\Omega(\log n)$ advice bits are required. See Section 9 for more about the knapsack problem.

Organization of survey. First, we introduce the advice models in Section 2. Then, we discuss the relationship between advice and semi-online algorithms in Section 3.

The strong connections between advice complexity and randomization, showing that results in either area can often be carried over to the other, is discussed in Section 4. Some of the techniques which can be used in designing online algorithms with advice are discussed in Section 5 and lower

bound techniques are discussed in Section 6.

A specific frequently used lower bound technique is based on String Guessing and its variants, which can also sometimes be used for proving upper bounds. These problems are discussed in Section 7, along with the first complexity classes for online algorithms, developed based on String Guessing results.

Note that all problems discussed in this survey are online unless explicitly stated otherwise. Metrical Task System problems, including k-Server and Paging, are discussed in Section 8. Bin Packing, Scheduling, and further results on Uniform Knapsack are discussed in Section 9. Graph Coloring is discussed in Section 10 and Graph Exploration in Section 11. Problems studied using advice complexity, along with references, are listed in the appendix.

2 Advice Models

In this section, we define advice models and describe the historical development. We compare the models and also discuss alternative views on what an advice model represents.

All models make use of a trusted oracle that knows the entire request sequence and has unlimited computational power. Bits that we refer to as *advice bits* are supplied to the algorithm by the oracle in some manner. These bits can be assumed to give a correct answer to any question the online algorithm poses. For example, an online algorithm could pose the question of how many future requests there are of a certain type and interpret the bits that are made available as the number of interest. Note that the oracle knows the online algorithm, so the questions are not explicitly asked; the oracle simply writes the answers and the algorithm reads them.

The term *advice complexity* for online algorithms was coined by Dobrev, Královič, and Pardubská [44]. They suggest two models, referred to as the *helper mode* and the *answerer mode*. In the helper mode, the online algorithm receives a number of advice bits, which could be zero, prior to processing each request. The advice complexity is defined to be the total number of bits received from a perfectly designed oracle for the online algorithm to be optimal. The answerer mode is similar, except that advice bits are only given when requested by the online algorithm, in which case at least one bit is given. Note that the length itself of the bit sequence given as response to a request for advice may transfer information in both the helper and the answerer mode.

Allowing the online algorithm to gain knowledge from *not* receiving any bits (or, in general, receiving a varying number of bits) may be reasonable in some applications (see [44] for a discussion), but it also introduces an additional complication which is not always desirable. Following the introduction of online algorithms with advice in [44], two other models were suggested, both avoiding this complication in different ways.

One model was introduced by Böckenhauer, Komm, Královič, Královič, and Mömke [25]. They suggest using an infinite advice tape, written by the oracle; we refer to this model as the *Tape Model*. The online algorithm may consult this advice tape at its discretion, and the advice complexity is simply the number of bits read. The term "tape" is likely suggested by tradition; the important properties are that the algorithm has an a priori unbounded supply of bits that it can receive one at a time on request, and that there is no indication of an "end", i.e., it is the algorithm that stops

asking for bits, not the supply that runs out. In other words, the Tape Model is similar to the answerer mode of [44], except that the algorithm must specify how many bits it wants to receive when asking for advice.

Another model was introduced by Emek, Fraigniaud, Korman, and Rosén [51]. They define a universe, $\mathcal{U}$, of all possible answers, assume that $\lceil \log |\mathcal{U}| \rceil$ advice bits are given to the online algorithm with every request, and define the advice complexity to be $\lceil \log |\mathcal{U}| \rceil$. Phrased in terms of the first models from [44], this corresponds to an advice complexity of $\lceil \log |\mathcal{U}| \rceil n$, where n is the length of the request sequence. Thus, to obtain a good advice complexity, the size of the universe must be minimized, which is equivalent to using as few advice bits per request as possible. We refer to this model as the *Per Request Model*. The optimal solution for Paging discussed in the introduction falls naturally into this model.

In the Per Request Model, any algorithm employing advice uses at least a linear number of bits, making it impossible to explore a lack of information that can be overcome using a sublinear number of advice bits (which is possible in the previously discussed models). Algorithms with sublinear advice are of significant interest for Bin Packing (see Section 9) and several other online problems.

Earlier than [44], a similar notion of advice complexity was introduced by Fraigniaud, Ilcinkas, and Pelc [56] in the setting of graph exploration (see Section 11), rather than for traditional online algorithms. Here, all the advice is given in the beginning, and the algorithm learns the length of the advice.

Unless explicitly stated otherwise, the results in this survey are in the Tape Model.

Further Technical Details. Now, we discuss some technical details that, although they can be useful to know, are not essential to get an overview of the models.

Reading the advice from an infinite tape (as opposed to receiving a fixed number of bits) as in the Tape Model comes at a small price. If an online algorithm wants to read a number of bits encoding an integer X without a (good) known upper bound, the number of bits to be read must also be provided as information. The standard technique for this is to use a so-called *self-delimiting* encoding (also known as a *prefix code*), as in [24]. For example, one may write $\lceil \log(X+1) \rceil$ in unary (using ones), then a zero as a delimiter, followed by X in binary, using $2\lceil \log(X+1) \rceil + 1$ bits in total (this is similar to Elias gamma coding [50]). Slightly more efficient encodings may be obtained by iterating this construction. The next iteration (similar to Elias delta coding) uses $\log X + 2\log\log X + O(1)$ bits to encode an integer X. However, by Kraft's inequality [41], there does not exist a self-delimiting encoding of the integers using, for example, $\log X + \log\log X + O(1)$ bits, and so we cannot obtain significantly better encodings.

In the model of [56], using a self-delimiting encoding may be unnecessary, since all of the advice is given at the beginning and the algorithm learns its length. Furthermore, their oracle is able to send $\sum_{i=0}^{b} 2^i = 2^{b+1} - 1$ different advice strings using at most b bits of advice.

Most lower bounds stated in the Tape Model in the literature are in reality shown in a model similar to that of [56], not using that the algorithm does not know the length of the advice. This means that upper bounds often contain a logarithmic lower-order term which is not present in the lower bounds.

Some bounds transfer between the Per Request Model and the Tape Model. An upper bound from the Per Request Model of b bits for each of n requests gives an upper bound of bn bits in the Tape Model (assuming that b is known to the algorithm, otherwise $bn + O(\log b)$ bits may be required). Similarly, a lower bound stating that b bits are necessary in the Tape Model implies that at least $\lceil \frac{b}{n} \rceil$ bits per request are required in the Per Request Model.

Allowing the algorithm random access to the tape in the Tape Model (as opposed to sequential access) does not make a difference: Since the oracle knows both the algorithm and the input when preparing the advice tape, it can predict which bits the algorithm would access in a random access model and simply place them first sequentially on the advice tape.

Comparison to other computational models. The traditional approach of providing an online algorithm with a specific type of knowledge is discussed in detail in Section 3 on semi-online algorithms.

Hromkovič et al. [68] proposed parameterizing the Tape Model with an upper bound on the running time of the algorithm to obtain an analogue of resource-bounded Kolmogorov complexity. These ideas do not appear to have been investigated much yet.

As mentioned in [44, 51], the advice complexity model for online problems is similar to an earlier advice complexity model for distributed computing [57]. There, the question was how much advice the nodes in a network need in order to complete some task using as little communication as possible (such as broadcasting, leader election, or coloring the nodes of the network).

Note that what is traditionally called a Turing machine with advice (see [6], for example) does not correspond well to an online algorithm with advice. A Turing machine with advice receives advice which may only depend on the *length* of the input, not the input itself.

3 Relationship to Semi-Online Algorithms

A major motivation for considering advice complexity is the relationship it has to semi-online algorithms. In the literature, the term "semi-online" is used for many quite different types of problems. For example, a semi-online algorithm may have a look-ahead, i.e., the ability to see some of the future requests; the algorithm may be allowed to postpone some decisions or modify some of them after arrival of more input; or the algorithm may be allowed to make assumptions about the request sequence, such as non-increasing sizes. These types of semi-online problems have little known relation to advice complexity. Those that do are the type that either assume some advance knowledge about the input or maintain more than one solution and choose the best solution at the end.

3.1 Assuming Advance Knowledge

Having advance knowledge available to a semi-online algorithm corresponds to advice from an oracle in the advice complexity setting. Thus, depending on the type of advice an oracle provides, an online algorithm with advice can be seen as a semi-online algorithm. Uniform Knapsack,

mentioned in the introduction, is a good example of where the advice model can lead to potentially practical semi-online algorithms; it is only necessary to know if there exists an item of size at least $\frac{1}{2}$. Similarly, there is an online algorithm with advice for Bin Packing, where only knowledge of the number of items with sizes in $(\frac{1}{2}, \frac{2}{3}]$ is necessary (see Section 9).

Lower bounds on advice complexity, on the other hand, can give proofs that no good semi-online algorithm (of a certain type) exists. For example, a linear (or even super-logarithmic) lower bound on the advice necessary to obtain a competitive ratio of c shows that knowing the number of requests, which would only require a logarithmic number of bits of advice, cannot be sufficient to obtain a competitive ratio of c. At the same time, it would also rule out many other semi-online algorithmic possibilities.

We present some examples to show the interest in semi-online algorithms assuming advance knowledge and show some of the types of advance knowledge that have been considered. Most of the work of this type focuses on scheduling problems (see Section 9 for definitions of scheduling and the makespan objective), and much of it has been for cases where the number of machines is a small constant. In the examples we give, the number of machines, m, is unbounded.

For Scheduling on identical machines for makespan, Fleischer and Wahl [53] present an upper bound of 1.9201 on the competitive ratio of deterministic algorithms, and Rudin reports a lower bound of 1.88 [96]. However, if a semi-online algorithm knows the total sum of processing times, algorithms can do better. A lower bound of 1.585 is proven in [3], and this lower bound is met by the algorithm in [76] when the number of machines tends to infinity. On the other hand, knowing the value of the optimal makespan, the problem becomes identical to Bin Stretching. This problem was introduced in [8], and the currently best lower bound, $1.\overline{3}$, was proven there. A 1.5-competitive algorithm for Bin Stretching was presented in [27].

For Scheduling preemptively on uniformly related machines for makespan, if the value of the optimal makespan is given in advance, an optimal schedule is possible [49]. If only an approximation to the optimal value is known, even for identical machines, the competitive ratio is increasing with respect to both m and the uncertainty [70]. Note that in terms of advice complexity, more uncertainty would generally imply less advice.

Seiden et al. [98] present a best possible online algorithm for Scheduling preemptively on identical machines for makespan, assuming decreasing job sizes (a competitive ratio of about 1.36603), and remark that the assumption of decreasing job sizes can be replaced with knowledge of the size of the largest job.

As an example where quite a bit of advance knowledge (or advice) is used, fitting well into the Per Request Model, Scheduling parallel batches with known arrival time of the first job among those remaining with the longest processing times for makespan is considered in [108].

For Machine Covering (maximizing the minimum load), it was shown in [107] that the List Scheduling algorithm [59] is m-competitive; it is well known that this is best possible (see [7]). The ratio goes down to $m - 1$ ($m \geq 3$) if either the total sum of processing times or the longest processing time is known [102]. Even if not all machines become available at the same time, the ratio goes down to $m - 2$ ($m > 3$) if both of these are known [69]. If the optimal value is known, the ratio is only $2 - \frac{1}{m}$ [7].

3.2 Parallel Solutions

This model was considered by Albers and Hellwig [4] for SCHEDULING on identical machines for makespan. For example, one of their results is a $(\frac{4}{3}+\varepsilon)$-competitive algorithm using $(\frac{1}{\varepsilon})^{O(\log \frac{1}{\varepsilon})}$ parallel schedules. A corresponding $(\frac{4}{3}+\varepsilon)$-competitive algorithm with advice would receive the index of the best of the $(\frac{1}{\varepsilon})^{O(\log \frac{1}{\varepsilon})}$ parallel schedules from an oracle using $O(\log^2 \frac{1}{\varepsilon})$ bits of advice and perform the same computations as the algorithm with parallel schedules, but only using the schedule indexed by the advice. Similarly, any algorithm with $b(n)$ bits of advice to achieve competitive ratio c can be converted into $2^{b(n)}$ algorithms, each giving a schedule, and choosing the best schedule will give a c-competitive result. Thus, advice complexity can conveniently be used to give lower bounds for parallel solutions approaches.

Maintaining parallel solutions was also considered for the independent set problem in [64] in a slightly different model. Their upper and lower bound results were asymptotically tight for this model. However, using advice complexity techniques, asymmetric string guessing, and the AOC-completeness (see Section 7) of the problem, both the upper and lower bounds were improved in [31], determining the exact constant for the high-order term in the number of parallel solutions.

4 Advice vs. Randomization

Before covering algorithmic techniques for advice more broadly, we discuss the strong connection to randomization as further motivation for studying advice complexity.

Derandomization using advice. It is trivial to see that if an online algorithm uses b random bits, then an at least as good deterministic algorithm using b advice bits also exists: The oracle chooses the random bits giving the best performance. However, it seems reasonable to ask for derandomization results not depending on the number of random bits used by the algorithm. Using derandomization techniques, Böckenhauer et al. [24] obtained the following result: Let $I(n)$ denote the number of inputs of length n to some minimization online problem (later extended to maximization problems [22, 48, 91]). If there exists a randomized c-competitive algorithm without advice, then for every constant $\varepsilon > 0$, there exists a deterministic $(c+\varepsilon)$-competitive algorithm with advice complexity $O(\log n + \log\log I(n))$. For a large number of online problems, the number of possible inputs of length n is at most $2^{n^{O(1)}}$. Thus, for these problems, it is possible to convert any randomized algorithm into an (almost) equally good deterministic algorithm with advice complexity $O(\log n)$.

We remark that this result is essentially tight. It is shown by Mikkelsen [91] that for any increasing function $I(n)$, there exists (pathological) online problems where $\Omega(\log\log I(n))$ bits of advice are indeed needed for such a conversion. Thus, for online problems with large input spaces, it is possible that a lot of advice is required to simulate randomization. However, so far no one has stumbled upon a "natural" online problem (that is, a problem not specifically constructed for this purpose) where more than $O(\log n)$ bits of advice are needed to simulate randomization.

Finally, we note that this derandomization result can, of course, also be used to convert an algorithm

which uses both advice and randomization into a deterministic algorithm with advice (see [91]). Therefore, randomized algorithms with advice are rarely studied explicitly.

Replacing advice bits with random bits. Intuitively, it might appear that having access to even a rather small number of advice bits provided by an omniscient oracle knowing the entire input should often be more powerful than simply having access to (any number of) random bits. Perhaps surprisingly, it turns out that for many important online problems, this is not the case.

Let us first consider the naive idea of simply running an algorithm with advice, ALG, with a tape full of random bits (instead of bits provided by an oracle). Call the resulting randomized algorithm RAND. It is easy to construct a pathological minimization problem where a single bit of advice yields an optimal algorithm while no randomized algorithm can achieve any meaningful competitive ratio (consider a problem where one of the first two requests should be chosen over the other, and either can have arbitrarily larger weight than the other). On the other hand, for a maximization problem with non-negative weights, the naive conversion will turn a c-competitive algorithm reading b bits of advice into a $(c \cdot 2^b)$-competitive randomized algorithm. Indeed, for every input I, we have $\text{RAND}(I) = \text{ALG}(I)$ with probability at least $\frac{1}{2^b}$. Since scores cannot be negative, this implies that $\mathbb{E}[\text{RAND}(I)] \geq \frac{\text{ALG}(I)}{2^b}$.

It is possible to do significantly better than the naive conversion for a large class of important online minimization problems. In particular, it is possible to do better for any problem which can be modeled as a METRICAL TASK SYSTEM (see Section 8). Before the introduction of advice models, this was studied as the problem of "combining online algorithms online". In [20], Blum and Burch showed how to use the celebrated machine learning algorithm Randomized Weighted Majority to obtain the following result: For every $\varepsilon > 0$, it is possible to combine m algorithms for a METRICAL TASK SYSTEM, $\text{ALG}_1, \ldots, \text{ALG}_m$, into a single randomized algorithm, RAND, such that for every input I,

$$\mathbb{E}[\text{RAND}(I)] = (1+\varepsilon) \cdot \min_{1 \leq i \leq m} A_i(I) + O(\Delta \log m).$$

Here, Δ is the normalized diameter of the underlying metric space. Note that if $m \in O(1)$, then $O(\Delta \log m)$ is just an additive constant. Thus, using our terminology, Blum and Burch show that for any METRICAL TASK SYSTEM, a c-competitive algorithm with advice complexity $O(1)$ can be converted into a $(c+\varepsilon)$-competitive randomized algorithm without advice! The result of Blum and Burch was later extended in [91] by showing that such a conversion is also possible if the algorithm uses $o(n)$ bits of advice instead of constant advice. Together with the derandomization result, this gives a striking equivalence between advice and randomization for many online problems, including those mentioned in the following theorem:

Theorem 3 [Mikkelsen [91]] Let P be METRICAL TASK SYSTEM, k-SERVER, LIST UPDATE, PAGING, or DYNAMIC BINARY SEARCH TREE and assume that the underlying metric/node set is finite. Let c be a constant independent of the input length. The following are equivalent

- For every $\varepsilon > 0$, there exists a $(c+\varepsilon)$-competitive P algorithm with advice complexity $o(n)$.
- For every $\varepsilon > 0$, there exists a $(c+\varepsilon)$-competitive randomized P algorithm without advice.

Note that for k-Server, for example, determining the best possible competitive ratio of a randomized algorithm is a long-standing open problem. In particular, the randomized k-Server conjecture states that for every metric space, there exists an $O(\log k)$-competitive randomized algorithm [83]. It was noted in [24] that, due to the derandomization result, a sufficiently large advice complexity lower bound would disprove this conjecture. Theorem 3 shows that the randomized k-Server conjecture is in fact equivalent to the conjecture that there exists an $O(\log k)$-competitive deterministic algorithm with advice complexity $o(n)$ (assuming the underlying metric space is finite). See Section 8 for more information on k-Server.

5 Algorithmic Techniques

We discuss general techniques for designing algorithms with advice.

Derandomization using advice. It is often possible to convert a randomized online algorithm into a deterministic online algorithm reading $O(\log n)$ bits of advice. Section 4 was devoted to the treatment of the relationship between advice and randomization.

Adapting offline algorithms. It is sometimes possible to convert an existing (exact or approximation) offline algorithm into an online algorithm using a relatively small number of advice bits. This has been done for Bin Packing and Scheduling [95] and Multi-Coloring [39] (see Sections 9 and 10.2). It can also be possible to convert streaming algorithms, for example, into online algorithms with advice, as has been done for bipartite matching [48].

The now-or-later technique. The now-or-later technique is based on giving one bit of advice per request. The technique has been used for Paging as described in Example 1 in the introduction: Each time a page is requested, one bit of advice is given, indicating whether the requested page can safely be evicted the next time a page fault occurs, or if the algorithm should keep the page in cache until it has been requested at least once more.

Reordering Buffer Management is similar to paging: A buffer of a certain size is given, and the input is a sequence of items. For each request, if the buffer is full, an item must be removed from the buffer. Each item has a color, and if the evicted item has a color different from the previously evicted item, a cost of 1 is incurred. A slightly more complicated version of the now-or-later technique (using two advice bits per request to also include a "soon, but not now"-option) was applied to Reordering Buffer Management in [2] (see also [93]), resulting in a $\frac{3}{2}$-competitive algorithm, which was extended to a $(1+\varepsilon)$-competitive algorithm using $O(\log \frac{1}{\varepsilon})$ bits per request.

The follow-OPT technique. This technique was introduced in [51] and has been used for Metrical Task System and k-Server [24,51,94]. In these problems, there is a bounded number of possible states. With a lot of advice, it is possible to specify exactly which state the algorithm should be in after each request. With fewer bits, the idea is to specify the exact state as often as possible, ensuring that the state of the algorithm often coincides with the state of Opt. When

serving those requests for which the precise state of OPT is not specified, the algorithm tries to be conservative and not make risky decisions.

Combinatorial designs. In many cases, the amount of advice needed to achieve a given competitive ratio is closely related to the minimum size of certain combinatorial structures. The idea is to "compress" the optimal set S of advice strings into a smaller set S'. The strings in S' have the same length as those in S, and each string in S is "close to" some string S', i.e., each string in S' can be thought of as representing a subset of S. The advice given is an index to a string in the smaller set S'. If the aim is simply to minimize the Hamming distance between each string in S and its representative in S', covering codes can be used. However, in many cases, it must be ensured that all ones (or all zeros) in the string in S be present in its representative in S'. In this case, covering designs can be used. For example, upper bounds on the size of covering designs have been used to obtain algorithms with advice for PAGING (see Section 8) and MINASG (see Section 7.2). Similarly, upper bounds on the size of covering codes have been used to construct algorithms with advice for, for example, STRING GUESSING (see Section 7.1) and MATCHING on paths and trees [75].

Note that since we generally do not restrict the running time of our online algorithms, the upper bounds on the size of the given combinatorial structure need not be constructive. This is important for the applications involving covering designs, for example, where good upper bounds proven via the probabilistic method exist, but where it is not known how to construct such covering designs efficiently (see [31] for details).

The warning signal technique. An obvious technique for designing algorithms with advice is to consider an online algorithm ALG without advice and try to use advice to pinpoint exactly when ALG makes mistakes. The idea is that simply warning the algorithm of mistakes that it is about to make might be much cheaper than telling the algorithm exactly what to do. This has been done for edge coloring of trees (see Section 10.2).

Exponential sparsification. For weighted problems where a good advice algorithm exists for the case where there are only few different weights, exponential sparsification can sometimes be used. The requests are grouped based on their weights into intervals $((1+\varepsilon)^k, (1+\varepsilon)^{k+1}]$ for $k = -\infty \ldots \infty$.

The first idea is to treat requests with weights in the same interval $((1+\varepsilon)^k, (1+\varepsilon)^{k+1}]$ as having weight $(1+\varepsilon)^{k+1}$. For some problems, this gives only a small loss in competitive ratio for the algorithm. This idea was used in [95]. It has also been used for different variants of approximation problems (no advice involved), such as developing a PTAS for minimizing makespan in scheduling; see [103], for example.

The second idea is that requests with weights in an interval $((1+\varepsilon)^k, (1+\varepsilon)^{k+1}]$, for sufficiently small (or large) k (compared to that of the other requests), may be served in some simple way without using any advice with only a small loss in competitive ratio. For example, for WEIGHTED INDEPENDENT SET, a policy could be to always reject vertices with a weight below some threshold. Depending on this threshold, this might only give a small loss in competitive ratio. Note that in the

beginning, some scheme should be used to identify which requests have (relatively) small weights. This could for example involve using $O(\log n)$ bits to give the index of the first request which does not have a small weight.

Combining the two ideas, we now just need an algorithm (for the remaining requests) which solves the problem well when only few different weights are allowed. This approach was used in [32].

6 Lower Bound Techniques

We discuss general techniques for establishing lower bounds against algorithms with advice.

The pigeonhole technique. Construct a set of inputs, $\mathcal{I}$, where $|\mathcal{I}| = m$. Suppose that an algorithm reads at most b bits of advice on any input from $\mathcal{I}$. By the pigeonhole principle, this algorithm must read the same advice for at least $\lceil \frac{m}{2^b} \rceil$ of the inputs in $\mathcal{I}$. Thus, it suffices to show that for any subset, $\mathcal{I}' \subset \mathcal{I}$, of size at least $\lceil \frac{m}{2^b} \rceil$ and any fixed deterministic algorithm (without advice), there is an input from $\mathcal{I}'$ on which the algorithm performs poorly. In many cases, this is achieved by designing $\mathcal{I}$ such that all inputs have some common prefix. On this common prefix, a deterministic algorithm selected for $\mathcal{I}'$, based on the advice, will always produce the same output. So, if different inputs in $\mathcal{I}'$ require different outputs for the common prefix, this yields a lower bound. More generally, one may use a partition tree [13], where nodes in the tree represent sets of inputs with a common prefix. The pigeonhole technique is applied in [19, 24, 26, 35, 80, 95], for example.

The multiple algorithms technique. Any algorithm ALG reading b bits of advice can be converted into 2^b algorithms, $\text{ALG}_1, \ldots, \text{ALG}_{2^b}$, without advice such that for every input I, $\text{ALG}(I) = \min_{1 \leq i \leq 2^b} \text{ALG}_i(I)$ for minimization problems (for maximization problems, min is replaced by max). Thus, we can get a lower bound by showing how an adversary can construct an input such that all of the 2^b algorithms perform poorly on that input. One can, for example, create an input in rounds, where each round ensures that some fraction of the algorithms perform poorly. This technique is applied in [31, 40, 81, 82, 90], for example.

The probabilistic method. Suppose that we are able to construct a probability distribution over a set of inputs $\mathcal{I}$ and show that for any deterministic algorithm without advice, the probability that the algorithm performs "well" is very small. Then this gives an advice complexity lower bound. For example, let ALG be an algorithm reading b bits of advice. Then ALG can be converted into 2^b deterministic algorithms, $\text{ALG}_1, \ldots, \text{ALG}_{2^b}$, without advice (as done in the multiple algorithms technique). Assume that for every deterministic algorithm, DET, without advice, it holds that $\Pr[\text{DET}(I) \leq c \cdot \text{OPT}(I)] < \delta$, where I is drawn from $\mathcal{I}$ according to our input distribution. Then, by the union bound, this implies that $\Pr[\text{ALG}(I) \leq c \cdot \text{OPT}(I)] = \Pr[\min_{1 \leq i \leq 2^b}\{\text{ALG}_i(I)\} \leq c \cdot \text{OPT}(I)] \leq 2^b\delta$. If $2^b\delta < 1$, then this implies that there exists an input $I \in \mathcal{I}$ such that $\text{ALG}(I) > c \cdot \text{OPT}(I)$, and hence ALG is not strictly c-competitive. The probabilistic method is applied in [12,58,91], for example. See also Section 7.1 for a simple but useful lower bound obtained via this technique.

Advice-preserving reduction. Suppose that we already have a lower bound on the advice complexity for a problem P. An easy way to obtain a lower bound on the advice complexity for a related problem P′ is to reduce P to P′ in a suitable way. A number of abstract guessing games have been introduced specifically with the purpose of serving as the starting point of such reductions (see Section 7).

Σ-repeatable online problems. It was shown by Mikkelsen [91] that for online problems which are "repeatable", it is often possible to translate lower bounds for algorithms without advice into lower bounds for algorithms with sublinear advice. Informally, an online problem is Σ-*repeatable* if it is possible to combine r inputs $I_1, \ldots, I_r$ into a single input $I = f(I_1, \ldots, I_r)$ such that serving I essentially amounts to serving each I_i independently and adding the costs incurred. In particular, the way an algorithm serves $I_1, \ldots, I_{i-1}$ should not significantly affect how efficiently the algorithm can serve I_i. Paging is Σ-repeatable since one may simply concatenate the inputs $I_1, \ldots, I_r$. The only dependency between the number of page faults of two different rounds is that our initial cache when serving the requests of I_i corresponds to our final cache when serving the requests of I_{i-1}. However, if we make sure that $\textsc{Opt}(I_i)$ is much larger than the cache size, then this small dependency can essentially be ignored when proving lower bounds. A problem which is *not* Σ-repeatable is Bin Packing. Consider inputs $I_1 = (\frac{1}{2} - \varepsilon, \ldots, \frac{1}{2} - \varepsilon)$ and $I_2 = (\frac{1}{2} + \varepsilon, \ldots, \frac{1}{2} + \varepsilon)$, both of length n. While concatenating I_1 and I_2 does give a valid Bin Packing input I, if we pack the items of I_1 two per bin, then we have to open n new bins for serving the items of I_2. On the other hand, if we pack each item of I_1 in a separate bin, we may pack the items of I_2 without opening any new bins at all. Thus, the choice of how to serve the items of I_1 has a significant influence on the number of bins needed to serve the items of I_2. Of course, one might try to construct $I = f(I_1, I_2)$ in a more clever way than just concatenating I_1 and I_2, but it can be shown that no choice of f will work for Bin Packing.

For a Σ-repeatable problem, we have the following result [91] (omitting some minor technical conditions): Let P be a Σ-repeatable online problem, where, for each n, the number of inputs of length n is finite. Suppose that a randomized algorithm without advice cannot be better than c-competitive, where c does not depend on the input length n. Furthermore, suppose that this lower bound holds even if the adversary has to reveal an upper bound on the length of the input in advance. Then, an algorithm reading $o(n)$ bits of advice must have competitive ratio at least c.

The currently best known lower bounds (for algorithms with sublinear advice) for Paging, k-Server, List Update, Max-SAT, Unit Clustering, Bipartite Matching and several other problems have been achieved by combining the result above with the currently best known lower bounds for randomized algorithms without advice [91].

∨-repeatable online problems. For a Σ-repeatable problem, the total cost has to be essentially the sum of costs incurred in each individual round. It is also possible to consider another collection of repeatable problems, where the total cost is the maximum cost incurred in a single round. We call such problems ∨-*repeatable*. For those problems, we have the following lower bound result [91]: Let P be an ∨-repeatable online problem. Suppose that a deterministic algorithm without advice cannot be better than c-competitive, where c does not depend on n. Furthermore, assume that the lower bound holds even if the algorithm knows $\textsc{Opt}(I)$ in advance and knows an upper bound

on the number of requests. Then no (possibly randomized) algorithm reading $o(n)$ bits of advice can be better than c-competitive. This result is similar to the result for Σ-repeatable problems, but note that for $\vee$-repeatable problems, we only need a lower bound for *deterministic* algorithms without advice in order to apply the technique. On the other hand, for $\vee$-repeatable problems, we have an additional assumption regarding the cost of an optimal solution. This assumption turns out to be crucial (see the Machine Covering results in Section 9). Informally, since knowledge about $\textsc{Opt}(I)$ would not help the algorithm, we may essentially assume that the cost of Opt is the same in each round. Intuitively, even if the algorithm uses $o(n)$ bits of advice, there will still be a single round where it has almost no advice available and, hence, will perform as poorly as an algorithm without advice. The main examples of $\vee$-repeatable problems are graph coloring problems.

Direct product theorems. Direct product theorems were introduced as a way to prove lower bounds in [91]. Intuitively, a direct product theorem says that if b bits of advice are needed for an online algorithm to ensure a cost of at most t when faced with requests drawn from a probability distribution p, then $r \cdot b$ bits of advice are needed to ensure a total cost of at most $r \cdot t$ when r independent rounds of requests are drawn from p.

The result for Σ-repeatable online problems discussed earlier is proven by having the requests of each round be an entire input itself (drawn from some hard input distribution) and then applying a direct product theorem. However, it is also sometimes possible to have each round be only a single request of the input. Obviously, this approach will usually require more effort since one no longer treats the hard input distribution just as a black box (as was the case with the result for Σ-repeatable problems). On the other hand, this approach can lead to significantly stronger lower bounds than what can be achieved by only using the general result for Σ-repeatable problems. For example, a super-linear lower bound for Vertex Coloring has been proven using this approach (see Section 10.1).

7 String Guessing and Complexity Classes

String Guessing is a rather artificial problem which is used primarily to show linear lower bounds on the advice complexity of certain problems, so most of the problems considered here are hard from an advice complexity point of view, i.e., much advice is needed to obtain a good competitive ratio. Some of the problems are even hard offline.

There are several types of string guessing problems. We start with the simplest version.

7.1 String Guessing

String Guessing was introduced by Böckenhauer et al. [23], and it is essentially the same as Generalized Matching Pennies, defined and studied earlier by Emek et al. [51]. Both of these problems consider strings of length n over an alphabet of size q. The goal is to guess as many of the characters of the input string as possible correctly. There are two versions of the problem: String Guessing with *known history*, where the correct answer to the previous request is revealed with

each new request, and String Guessing with *unknown history*, where the correct answers are revealed only at the end of the input.

Note that an algorithm which answers uniformly at random in each round will guess $\frac{n}{q}$ characters correctly in expectation. Clearly, one can achieve the same guarantee with a deterministic algorithm, reading $\lceil \log q \rceil$ bits of advice (identifying the most frequent character in the input string). The following theorem gives a lower bound on the advice needed to guess more than a fraction of $\frac{1}{q}$ of the input characters correctly.

Theorem 4 [Böckenhauer et al. [23]] Any online algorithm with advice for String Guessing with known history (over an alphabet of size q), guaranteeing guessing γn characters of the input correctly, for some constant $\frac{1}{q} < \gamma < 1$, must read at least

$$\left(1 + (1-\gamma)\log_q\left(\frac{1-\gamma}{q-1}\right) + \gamma \log_q \gamma\right) n \log q \in \Omega(n \log q)$$

advice bits.

The lower bound (Theorem 4) can equivalently be written as $(1 - H_q(1-\gamma))(\log q)n$, where H_q is the q-ary entropy function [23]. Also, it may be useful to know that Theorem 4 is closely related to the Chernoff bound [67]. Indeed, it can be proven using the probabilistic method (see Section 6) as follows: Choose the input string uniformly at random. Let Det be a fixed deterministic algorithm without advice. In each round, the probability that Det guesses the correct character is exactly $\frac{1}{q}$, and this probability is independent of all other rounds. Thus, the number of characters guessed correctly by Det is a sum of independent identically distributed Bernoulli random variables with expected value $\frac{1}{q}$. By the Chernoff bound, the probability that Det guesses γn (or more) characters correctly is at most $2^{-(1-H_q(1-\gamma))(\log q)n}$. It follows that an algorithm with advice needs at least $b \geq (1 - H_q(1-\gamma))(\log q)n$ bits of advice to ensure that it always correctly guesses at least γn characters [91]. This is exactly the lower bound of Theorem 4.

Via reductions, String Guessing with known history has been used to prove many advice complexity lower bounds, including some in [2, 5, 17, 21, 23, 34, 35, 48, 51, 60, 79].

For String Guessing with unknown history, Böckenhauer et al. [23] give (using known bounds on the size of covering codes) an upper bound which matches the lower bound for String Guessing with known history up to an additive $O(\log n)$ term. Note that the lower bound is for the easier of the two problems, and the upper bound is for the harder version. Thus, both bounds are as general as possible.

Other String Guessing Problems. Other string guessing variants were analyzed by Mikkelsen [91]: Anti-String Guessing yields better lower bounds for Paging with advice and for Induced Subgraph [79] and Weighted Binary String Guessing yields a better lower bound for Bin Packing.

7.2 Asymmetric String Guessing

Consider accept/reject minimization problems, i.e., minimization problems where the irrevocable decision for each request is either to accept or reject it. Assume that the problem is such that a

superset of a feasible solution is always feasible. An example one could keep in mind is VERTEX COVER. This is the standard vertex cover problem in the *vertex arrival model*, so the vertices arrive online, and each vertex arrives with a list of all previous vertices to which that vertex is adjacent. The accepted vertices must form a vertex cover, so at least one endpoint of each edge must be chosen. The fact that edges to vertices that have not been seen yet are unknown when a vertex arrives means that the well known 2-approximation algorithm, accepting both endpoints of some edges, cannot be used.

The obvious advice to give is a string of bits, one for each request, with ones indicating acceptance and zeros indicating rejection for an optimal solution. One can also use this idea in a c-competitive algorithm with advice. Suppose that for each request sequence length n and each $t \leq \lceil \frac{n}{c} \rceil$, the algorithm can compute a set of binary strings, $S_{n,t,c}$, such that for every request sequence of length n with a minimum solution of size t, there is a string in $S_{n,t,c}$ which indicates a superset of a minimum solution, where the superset must have size at most ct. Then, the oracle can give the algorithm n, t, and an appropriate index into $S_{n,t,c}$. The algorithm can be c-competitive by using the indexed string and answering "accept" or "reject" based on that string, ignoring the actual request sequence. If $t > \lceil \frac{n}{c} \rceil$, it is safe to answer "accept" for every request. Note that the value n must be given in a self-delimiting encoding, and the total length of the advice is $\lceil \log |S_{n,t,c}| \rceil + O(\log n)$. One can think of the above algorithm as trying to guess a string corresponding to a minimum solution, but being allowed to make a few errors in the direction of guessing ones for some zeros in that optimal solution.

Realizing that many problems exhibit the same characteristics as VERTEX COVER, Boyar et al. [31] study an abstraction of this problem in the form of Minimum Asymmetric String Guessing, MIN-ASG. As with other string guessing problems, MINASG does not appear interesting in its own right, but the above example shows its relation to other problems. In MINASG, the request sequence is a sequence of bits that the algorithm must try to guess (for example indicating a minimum solution to an instance of VERTEX COVER). The cost is the number of ones guessed, unless the algorithm at some point guesses zero, when the correct bit was a one. In the latter case, the cost is infinite (this corresponds to a, possibly, infeasible answer in the VERTEX COVER case). The goal is, of course, to minimize cost.

As with STRING GUESSING, there are two variants of MINASG, known history and unknown history. There is also a maximization version of Asymmetric String Guessing, MAXASG. For that version, INDEPENDENT SET could be the problem to keep in mind. The objective is to guess as many zeros correctly as possible, and guessing a zero where the correct answer is a one gives a profit of $-\infty$. Again, there is a version with known history and one with unknown history.

Using results on covering designs, tight bounds are proven in [31] on all four version of Asymmetric String Guessing, showing that the number of advice bits necessary and sufficient to achieve competitive ratio c is

$$B(n,c) = \log\left(1 + \frac{(c-1)^{c-1}}{c^c}\right) n \pm \Theta(\log n), \tag{1}$$

where

$$\frac{1}{e \ln 2}\frac{n}{c} \leq \log\left(1 + \frac{(c-1)^{c-1}}{c^c}\right) n \leq \frac{n}{c}\,.$$

Returning to the motivating VERTEX COVER example, the closed formula (1) bounds the term

$\lceil \log |S_{n,t,c}| \rceil + O(\log n)$ from that example. VERTEX COVER is not exactly the same problem as either MINASG with known or unknown history, since it may be possible to deduce some but not all information about past mistakes during the processing of vertices. However, MINASG with known history can be used to provide lower bounds, whereas MINASG with unknown history can be used for upper bounds.

7.3 Complexity Classes

Problems such as MINASG and VERTEX COVER led to the definition of the first complexity class for online algorithms, AOC, Asymmetric Online Covering [31], which contains many accept/reject problems, both minimization and maximization problems. The minimization problems have the property that any superset of a feasible solution is a feasible solution, and the maximization problems have the property that any subset of a feasible solution is a feasible solution. In both cases, the cost/profit of a feasible solution is the size of the accepted set, and the cost (profit) of an infeasible solution is ∞ ($-\infty$). Maximization versions of MINASG have the same advice complexity as MINASG. This is used to show an upper bound on the advice complexity of all problems in AOC.

The hardest problems in AOC, those which require

$$\log\left(1 + \frac{(c-1)^{c-1}}{c^c}\right) n \pm \Theta(\log n)$$

bits of advice to be c-competitive, are called AOC-complete [31]. Using reductions from the asymmetric string guessing problems, VERTEX COVER, INDEPENDENT SET, DOMINATING SET, DISJOINT PATH ALLOCATION, SET COVER, and CYCLE FINDING are shown to be AOC-complete. All but the last of these correspond to offline problems which are NP-hard. Note that although these problems are proven to be complete via reductions, there are no unproven assumptions, such as P $\neq$ NP. Tight bounds on the advice complexity of these problems are known. The AOC-complete problems are all hard online problems: Without advice, these problems have competitive ratios which are $\Omega(\frac{n}{\log n})$, and in fact, all the known AOC-complete problems [31] have $\Omega(n)$ competitive ratios (actually, n or $n-1$ for all but one problem). Examples of problems which are in AOC, but not AOC-complete, are UNIFORM KNAPSACK and MATCHING.

Corresponding to each AOC problem is a weighted version of the problem, which is still an accept/reject problem, but the cost/profit of each request may vary due to a weight associated with the request. For example, for WEIGHTED INDEPENDENT SET, the vertex arrival model is used, but each vertex arrives with a weight, in addition to a list of all previous vertices adjacent to it. The goal is to accept a maximum weight independent set. In contrast to the unweighted case, when weights are added to AOC-complete problems, the maximization and minimization problems have different advice complexities. Boyar et al. [32] showed that the weighted versions of the complete maximization problems have advice complexity at most an additive term $O(\log^2 n)$ worse than the unweighted versions, but the weighted versions of the known complete minimization problems all have unbounded competitive ratios with fewer than $n - O(\log n)$ bits of advice. This latter result is proven using length-preserving advice reductions; all known AOC-complete minimization problems were proven complete for AOC using this type of reduction. Thus, the class containing the weighted versions of these complete minimization problems is harder with respect to advice complexity than the class containing the weighted versions of the complete maximization problems.

The maximization (and not the minimization) problems in AOC are examples of problems where the greedy algorithm is best possible according to online bounded analysis, which is defined by Boyar et al. in [30].

In [79], Komm et al. consider the following problem: A graph property, Π, is a set of graphs. It is said to be

- hereditary if for every graph G in Π, all induced subgraphs of G are also in Π.
- cohereditary if for every graph G in Π, all graphs containing G as an induced subgraph are also in Π.
- non-trivial if there are an infinite number of graphs in Π and an infinite number of graphs not in Π.

Examples of non-trivial hereditary graph properties include independent sets, forests, and planar graphs. Examples of non-trivial cohereditary graph properties include graphs containing a cycle and non-planar graphs. Let a non-trivial hereditary graph property Π be given. A graph is presented in the vertex arrival model and the goal is for the algorithm to accept as many vertices as possible, such that the induced subgraph defined by the accepted vertices is in Π. They show that at least

$$\log\left(1+\frac{(c-1)^{c-1}}{c^c}\right)n-\Theta(\log^2 n)$$

bits of advice are required to be c-competitive for these problems (independent of the choice of Π). These problems are, in some sense, shown to be almost AOC-complete. For a cohereditary graph property Π, the problem considered is the same, except that the goal is to accept as few vertices as possible, such that the induced subgraph defined by the accepted vertices at the end is in Π (it is guaranteed that the graph presented is in Π). They show that for this problem, the advice complexity depends crucially on the choice of graph property. For some properties, it is AOC-complete; for others, it is possible for an algorithm to be optimal using only $O(\log n)$ advice bits.

8 *K*-Server, Paging, and Friends

A Metrical Task System [29] is defined by a tuple $(\mathcal{S},\mathcal{T},d)$ where $\mathcal{S}$ is a set of N *states*, $\mathcal{T}$ is a set of *tasks*, and $d\colon S\times S\to[0,\infty)$ is a metric distance function. A *task* is a mapping $t\colon\mathcal{S}\to[0,\infty]$ satisfying that there exists at least one state $s\in\mathcal{S}$ such that $t(s)\neq\infty$. An input consists of an initial state $s_0\in\mathcal{S}$ and n tasks $t_1,\ldots,t_n$. Immediately after a task t_i arrives, the online algorithm must choose a state s_i for serving t_i: The online algorithm moves from its current state s_{i-1} to the state s_i at a cost of $d(s_{i-1},s_i)$ and serves the task t_i at a cost of $t_i(s_i)$. The goal is to minimize the total cost incurred. Each of the classic online problems of Paging, k-Server, and List Update can be modeled as a Metrical Task System (see [28], for example).

For the classic online scenario, a matching upper and lower bound of $2N-1$ is known for deterministic Metrical Task System algorithms [29]. For the randomized case, there exists a randomized $O(\log^2 N\log\log N)$-competitive algorithm [52], whereas the best known lower bound on the competitive ratio is the $\Omega(\log N)$ lower bound arising from Paging.

The advice complexity of Metrical Task System is well understood. We know that sublinear advice is equivalent to randomization (Theorem 3). Furthermore, it was shown in [51] that b bits of advice per request are both necessary and sufficient to be $\Theta(\frac{\log N}{b})$-competitive. The upper bound is achieved using the follow-OPT technique. The matching lower bound is proven via a reduction from Generalized Matching Pennies (see Section 7.1).

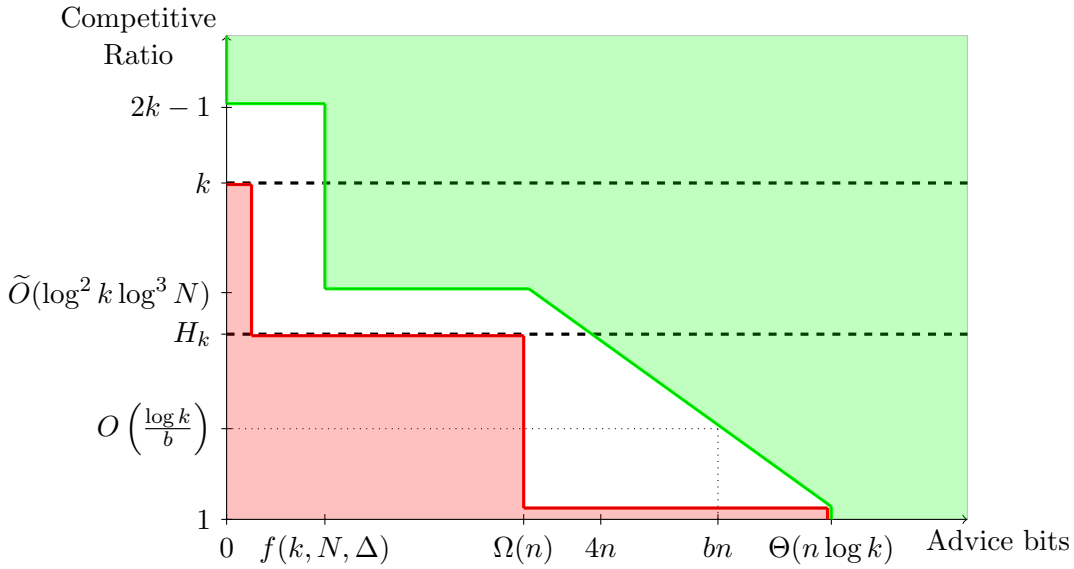

Figure 2: The asymptotic trade-off between competitive ratio and advice for k-Server. N is the number of points in the metric space and Δ the (normalized) diameter. The function $f(k, N, \Delta)$ is a rapidly growing function of k, N and Δ (but does not depend on n). For the randomized algorithm with a competitive ratio depending on N, we assume that N is relatively small; polynomial in k, for example.

The advice complexity of k-Server (see Figure 2) is not as well understood as for Metrical Task System. Again, we know that randomization is equivalent to sublinear advice. Depending on the size N of the metric space, the currently best known randomized algorithm without advice for k-Server is either the $O(\log^2 k \log^3 N \log\log N)$-competitive algorithm due to Bansal et al. [10] or simply the deterministic $(2k-1)$-competitive Work Function Algorithm by Koutsoupias and Papadimitriou [84]. By [91], randomized k-Server algorithms (on finite metric spaces) can be simulated using a number of advice bits depending only on k and the metric space. The current best upper bound for algorithms using $b \geq 3$ bits of advice per request is $O(\frac{\log k}{b})$, using the follow-OPT technique [24, 94]. However, no matching lower bound is known. The lower bound used for Metrical Task System does not seem to be applicable to k-Server. For k-Server, we only know that $\Omega(n \log k)$ bits of advice are needed to be optimal [24] and that $\Omega(n)$ bits of advice are needed to be better than H_k-competitive (the last lower bound follows since Paging is a special case of k-Server; see the next paragraph for details). In particular, it is an intriguing open problem whether or not it is possible to be $(1+\varepsilon)$-competitive using $O(n)$ bits of advice for arbitrarily small ε. It was shown in [24] that this is in fact the case if the underlying metric space is the Euclidean plane: Along with every request, one may use $O(1)$ bits of advice to indicate as precisely as possible in which "direction" the server used by Opt for serving this request is currently located. Also, for

various sparse metric spaces (such as paths, trees, and planar graphs), algorithms which are better than the algorithm for the general case are known [60, 94].

The asymptotic advice complexity of PAGING is essentially completely understood (see Figure 1). Recall that the best possible competitive ratio for a randomized PAGING algorithm without advice is $H_k \in \Theta(\log k)$. Using b bits of advice, it is possible to be $(\frac{2k+2}{2^b} + 3b)$-competitive, while any algorithm using only b bits of advice must have a competitive ratio of at least $\frac{k}{2^b}$ [25]. In particular, $O(\log k)$ bits of advice suffice to be $O(\log k)$-competitive while $o(\log k)$ bits of advice is not enough to be, for example, $k^{0.99}$-competitive. Furthermore, it is possible to be $(H_k + \varepsilon)$-competitive using a number of advice bits depending only on k and ε (and not the input length n) [91]. In order to achieve a competitive ratio better than H_k, we need $\Omega(n)$ bits of advice (since reading $o(n)$ bits of advice is equivalent to randomization, according to Theorem 3). On the other hand, n bits of advice suffice to be optimal using the algorithm described in the introduction.

The exact trade-off between advice and the competitive ratio for PAGING is still open. For constant competitive ratios, the current best upper bound is (perhaps a bit surprisingly) achieved by using the upper bound for the AOC-complete problem MINASG (see Section 7.2 and in particular Equation (1)): Let $x = x_1 \dots x_n$ be a binary string such that x_i is 0 if and only if the page requested in round i will be requested once more before it is removed from the cache of OPT. As already mentioned in the introduction, a PAGING algorithm which is given x as advice can be optimal. It was observed in [25] that if an algorithm is given an n-bit string x' such that $x_i = 1 \Rightarrow x'_i = 1$ and such that $|x'| \leq c\,|x|$ (where $|x|$ is the Hamming weight of x), then a PAGING algorithm which knows x' can be c-competitive. This means that (for all cache sizes) there exists a c-competitive PAGING algorithm reading $B(n, c) + O(\log n)$ bits of advice on inputs of length n. In particular, $(\log \frac{5}{4})n + O(\log n) > 0.3219n + O(\log n)$ bits of advice suffice to be 2-competitive. The best known lower bound on the exact advice complexity of PAGING was proven in [91] by a reduction from ANTI-STRING GUESSING. This lower bound is quite far from the AOC-based upper bound. For example, it only shows that at least $0.00877n - O(\log n)$ bits are needed to be 2-competitive.

LIST UPDATE has been studied with advice in [34]. The main result is a $\frac{5}{3}$-competitive algorithm using just two bits of advice (in total). The advice tells which of the three classic algorithms TIMESTAMP, MOVETOFRONT-EVEN, and MOVETOFRONT-ODD is the best algorithm for the current input. An interesting application of the idea of choosing the better of two classic algorithms for LIST UPDATE to a data compression problem is described in [73].

9 Bin Packing, Machine Scheduling, and Knapsack

In this section, we consider three related problems.

Bin Packing. In BIN PACKING, requests are sizes in the range $(0, 1]$. Bins of size one are available, and items must be placed in a bin such that the total volume of items placed in that bin does not exceed one. The objective is to minimize the number of bins used.

The ultimate advice for any online problem is to be informed of exactly how OPT behaves on the request sequence. For BIN PACKING, OPT uses $\text{OPT}(I)$ bins on a request sequence I, so with

$n \lceil \log \textsc{Opt}(I) \rceil$ bits of advice, it is possible to mimic the behavior of Opt. This was observed by Boyar et al. [35], where it was also established that this is essentially tight, in that a lower bound of $(n - 2\,\textsc{Opt}(I)) \log \textsc{Opt}(I)$ was given. They employed the pigeonhole technique, giving a long prefix which has to be packed exactly right, depending on the unknown suffix, in order to pack all the items in the optimal number of bins.

To beat the best known lower bound for Bin Packing of 1.54037 [9] (the best known upper bound is 1.5815 [66]), a ratio of $\frac{3}{2}$ was obtained using $\log n + o(\log n)$ bits of advice [35]. The observation underlying this result is that large items fill bins sufficiently and small items are easy to pack effectively, so we need to know about medium-sized items (concretely in the range $(\frac{1}{2}, \frac{2}{3}]$), and $\lceil \log(n+1) \rceil$ bits are sufficient to specify the number of such items in the input. Different categorization schemes by Angelopoulos et al. [5] led to a competitive ratio of $1.47012 + \varepsilon$, for any fixed ε, using a constant number of advice bits, dependent on ε. They also show that 16 bits of advice are sufficient to beat the best algorithm without advice, obtaining a competitive ratio of 1.530.

Using a linear number of bits, $2n + o(n)$, to get limited information regarding Opt's packing, a ratio of $\frac{4}{3} + \varepsilon$, for any ε, was obtained in [35]. Asymptotically, for quite large input, Renault et al. [95] proved that one can get arbitrarily close to optimal, establishing $(1 + \varepsilon)$-competitiveness using $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$ bits of advice per request.

A further improvement of the $\frac{4}{3}$ result is claimed in [109], but we have not been able to verify the result. An example problematic sequence for their algorithm is $\frac{n}{2}$ items of size $\frac{1}{3} - \varepsilon$ followed by $\frac{n}{2}$ items of size $\frac{2}{3} + \varepsilon$.

For negative results, Boyar et al. [35] showed that $\frac{9}{8} - \delta$ is a lower bound for any δ and sublinear advice. Refining those methods, Angelopoulos et al. [5] raised this lower bound to $\frac{7}{6} = 1.1\overline{6}$ and Mikkelsen [91] to $4 - 2\sqrt{2} > 1.1715$.

An overview of these results is given in Figure 3.

Finally, we mention some special cases. For a limited number m of different items, by using $m \lceil \log(n+1) \rceil + o(\log n)$ bits of advice to inform the algorithm in advance of how many items to expect of the different types, one can be essentially optimal, achieving a packing of $(1+\varepsilon)\,\textsc{Opt}(I)+1$ bins. This is essentially tight, since $(m-1) \log n - 2m \log m$ bits of advice are required to be optimal [35]. If all items are known to be larger than $\frac{1}{3}$, one bit of advice is sufficient to be 1.3904-competitive [5].

Machine scheduling. Consider Scheduling on m machines, where requests are real numbers, referred to as job sizes, and the irrevocable decision is to assign a request to a particular machine.

In Section 3, we discuss a parallel solutions algorithm from [4], where the objective is to minimize the makespan, i.e., the maximum sum of job sizes assigned to any one machine. The parallel solutions algorithm can be viewed as a $(\frac{4}{3} + \varepsilon)$-competitive algorithm using $O(\log^2 \frac{1}{\varepsilon})$ bits of advice in the Tape Model. The same paper gives a $(1 + \varepsilon)$-competitive algorithm which can be viewed as using $O(\frac{1}{\varepsilon} \log \frac{m}{\varepsilon} \log \frac{1}{\varepsilon})$ advice bits.

Boyar et al. [32] give $(1 + \varepsilon)$-competitive algorithms for weighted scheduling problems with various objective functions: For minimizing a norm (the makespan, for example) on related machines, an

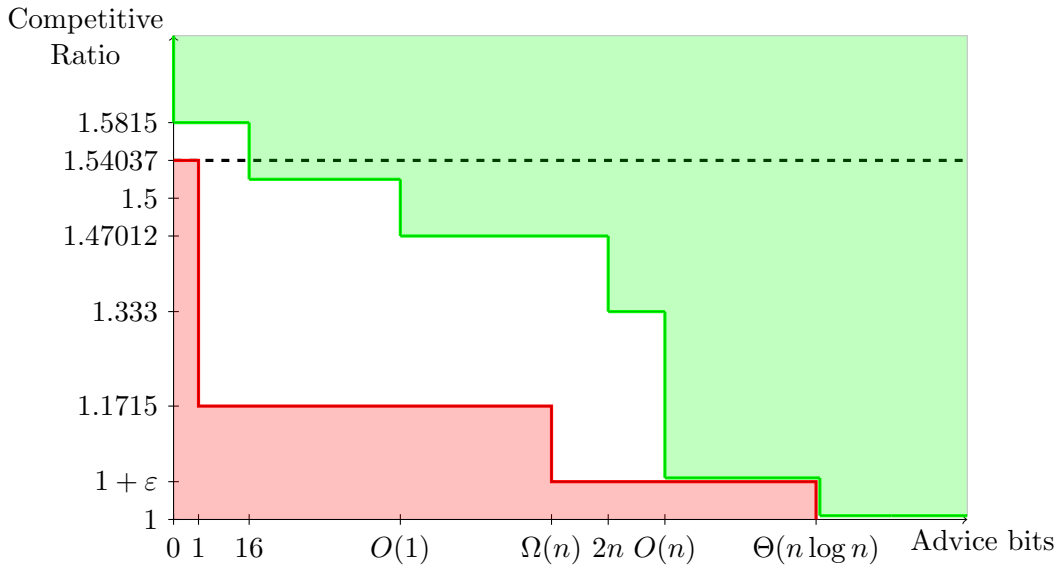

Figure 3: The best known bounds on the advice complexity of Bin Packing. The horizontal dashed line is the currently best lower bound on (possibly randomized) Bin Packing algorithms without advice.

algorithm reading $O(\frac{1}{\varepsilon}\log^2 n)$ advice bits is given. For minimizing a norm on a constant number of unrelated machines, an algorithm reading $O(\frac{1}{\varepsilon^m}\log^{m+1} n)$ bits of advice is given. The same advice complexity is obtained for maximizing a semi-norm (the minimum load as in Machine Covering, for example) on a constant number of unrelated machines. For a non-constant number of unrelated machines, the expressions for the advice complexity are more complicated; see the paper for details.

For the Per Request Model, Renault et al. [95] obtain a competitive ratio of $1+\varepsilon$ for minimizing makespan using $O(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon})$ bits of advice per request. Similar results are obtained for Machine Covering and minimization of the L_p norm, $p \geq 2$. Complementing these results, using the pigeonhole technique, they establish a $(1-\frac{2m}{n})\log m$ lower bound on advice per request in order to obtain optimality, i.e., almost as much advice is required as is used by the trivial optimal algorithm with advice that receives $\lceil \log m \rceil$ bits of advice per request, indicating which machine to place a job on.

Knapsack. In the introduction, Uniform Knapsack was used as an example (Example 2). An algorithm from [26] using one advice bit was described: the advice bit indicates whether the input sequence contains an item of size at least $\frac{1}{2}$. If it does, the first item accepted by the algorithm is the first item of size at least $\frac{1}{2}$. Otherwise, the algorithm accepts items greedily. In this section, we describe other results from this paper.

The competitive ratio of the above algorithm cannot be improved using a few additional advice bits; no algorithm reading fewer than $\lfloor \log(n-1) \rfloor$ bits of advice has a competitive ratio better than 2. On the other hand, for any constant $\varepsilon > 0$, there is a $(1+\varepsilon)$-competitive algorithm reading $O(\frac{1}{\varepsilon}\log n)$ bits of advice. For optimality, $n-1$ bits of advice are necessary and sufficient.

If one considers the obvious randomized algorithm based on the above 2-competitive algorithm, its competitive ratio is 4: Simply flip a coin instead of reading an advice bit. There is a related 2-competitive randomized algorithm using only one bit of randomness: One of the deterministic algorithms to choose between is again just accepting everything possible; the other is rejecting until the first item, which would have been rejected had everything before it been accepted, and then accepting from there when possible. This is best possible, since no randomized algorithm can have competitive ratio better than 2.

For the weighted version, where each item has both a size and a value, any (possibly randomized) algorithm reading fewer than $\log n$ bits of advice has unbounded competitive ratio. On the other hand, if all values and weights can be represented within polynomial space, then $O(\frac{\sqrt{1+\varepsilon}}{\sqrt{1+\varepsilon}-1}\log n)$ advice bits suffice to be $(1+\varepsilon)$-competitive.

10 Graph Coloring

Being of both practical and theoretical interest, graph coloring problems have been extensively studied from an online perspective. In fact, some of the earliest results on online graph coloring predate the formal introduction of competitive analysis. We refer to [78] for a good (although slightly dated) survey on Vertex Coloring. In this section, we survey some of the results obtained on various graph coloring problems in the advice complexity model. With a single notable exception, it generally turns out that a lot of advice is needed in order to obtain good online graph coloring algorithms.

10.1 Vertex Coloring

The most classic graph coloring problem is Vertex Coloring, where the vertices of a graph must be colored such that no two neighbors receive the same color. The aim is to use as few colors as possible. In the most studied online model, the *vertex-arrival* model, vertices arrive one by one, each with information about its edges to vertices that have already arrived. Usually, the colors are enumerated starting from one.

Without advice, Vertex Coloring is an extremely difficult problem; Halldórsson and Szegedy [65] showed that any (possibly randomized) online algorithm has a competitive ratio of $\Omega(\frac{n}{\log^2 n})$. The hardness of the problem carries over to the advice setting; applying a direct product theorem (see Section 6) to the lower bound of [65], Mikkelsen showed in [91] that any $O(n^{1-\varepsilon})$-competitive Vertex Coloring algorithm must read $\Omega(n\log n)$ bits of advice. This is an unusually strong advice complexity lower bound. Vertex Coloring is so far the only known example of a natural online problem where linear advice is not enough to obtain a truly sublinear competitive ratio. Also, note that $O(n\log n)$ bits of advice trivially suffice to achieve optimality. In fact, $n\log n - n\log\log n + O(n)$ advice bits are necessary and sufficient for an optimal coloring, even necessary if the vertices arrive in a breadth-first order [55]. Thus, Vertex Coloring has a sharp phase transition in its advice complexity.

On trees, First-Fit (the greedy algorithm using the lowest available color) uses at most $\lfloor\log n\rfloor + 1$ colors [62], thus obtaining a competitive ratio of $\frac{1}{2}\log n$. This is a best possible result, since, even

on trees, any deterministic online algorithm can be forced to use $\lfloor \log n \rfloor + 1$ colors [63], while OPT, of course, only needs two. Since VERTEX COLORING is $\vee$-repeatable, and since the lower bound of $\frac{1}{2} \log n \in \omega(1)$ does not depend on the algorithm not knowing n or $\text{OPT}(I)$, it follows that no VERTEX COLORING algorithm with $o(n)$ bits of advice can achieve a constant competitive ratio, even on trees [91] (see Section 6).

For bipartite graphs, any deterministic online algorithm without advice can be forced to use $2 \log n - 10$ colors [61]. Thus, coloring bipartite graphs is harder than coloring trees. On the other hand, the online algorithm (without advice) Bipartite First-Fit (BFF) uses at most $2 \log n$ colors [77] for $n \geq 2$. For each vertex v, BFF simply uses the smallest color not used in the opposite partition of the connected component containing v.

Building on BFF, a family of algorithms, A_k, with advice for coloring bipartite graphs was given by Bianchi et al. [18], obtaining a trade-off between competitive ratio and advice. For $k \geq 2$, the algorithm A_k uses advice to ensure that the color $k-1$ is only used in one partition of the final graph, and that the color k is only used in the other partition. For each vertex v, if BFF would use a color no larger than $k-2$, A_k uses this color. Otherwise, if at least one of the colors $k-1$ and k is already used in the connected component containing v, the algorithm can deduce which color to use. If this is not the case, the algorithm reads one bit of advice to decide which of the colors $k-1$ and k to use. Since BFF uses color $k-1$ only if the requested vertex is contained in a connected component of at least $2^{\frac{k-1}{2}}$ vertices, and since the algorithm does not use advice for more than one vertex within a connected component, the number of advice bits used is at most $\frac{n-1}{2^{\frac{k-1}{2}}} = \frac{n-1}{\sqrt{2^{k-1}}}$. This shows that $O(\sqrt{n})$ advice bits suffice to use fewer than $\log n$ colors, beating the lower bound for deterministic online algorithms without advice. For 2 and 3 colors, the upper bound is complemented with essentially tight lower bounds of $n-3$ and $\frac{n}{2}-4$ bits, respectively.

Note that the approach taken by the algorithms A_k resembles the warning signal technique described in Section 5. However, a sublinear number of advice bits is obtained, because the algorithms can detect themselves when they need advice.

In [101], Steffen specialized the algorithm A_k from [18] to trees, using First-Fit instead of Bipartite First-Fit. Since, on trees, First-Fit uses the color $k-1$, only if the requested vertex belongs to a component of at least 2^{k-2} vertices [63], a k-coloring is obtained using at most $\frac{n-1}{2^{k-2}}$ bits of advice. Thus, for each additional color, the number of advice bits is halved.

For 3-coloring of trees, a linear lower bound of approximately $0.0328n$ advice bits is given in [101]. The dissertation also contains linear lower (and upper) bounds for coloring combs and caterpillars with 2 or 3 colors. Note that caterpillars can be 4-colored without advice, since no vertex has more than three neighbors.

For 3-colorable graphs, the trivial upper bound of $(\log 3)n$ is essentially tight, even if the graphs are chordal [97].

10.2 Edge Coloring and Variants of Vertex Coloring

Many variants of VERTEX COLORING have been studied. Here we mention two of them.

For $L(i,j)$-Coloring, each pair of neighboring vertices must receive colors that are at least i apart, and each pair of vertices at distance two must receive colors that are at least j apart. The aim is to minimize the span of the coloring, i.e., the difference, λ, between the largest and smallest color used (thus, potentially, $\lambda + 1$ colors are used).

For Multi-Coloring, a graph is given from the beginning and the requests are vertices, with possible repetitions. For each request, an (additional) color must be assigned to the requested vertex. The colors assigned to a vertex and its neighbors must all be distinct.

Though not as famous as Vertex Coloring, many papers have been devoted to Edge Coloring. Analogous to Vertex Coloring, the edges of a graph must be colored such that no two adjacent edges receive the same color, and the aim is to use as few colors as possible. In the online version, one typically uses the *edge arrival model*, where the edges arrive one by one, each with information about adjacent edges among those that have already arrived. Adhering to standard notation in graph theory, where n denotes the number of vertices and m the number of edges, we let m denote the sequence length for this particular problem.

Bianchi et al. [19] studied $L(2,1)$-Coloring of paths. In the offline setting, the color range $0, 1, \ldots, \lambda = 4$ is sufficient. For the best possible online algorithm without advice, the color span is $\lambda = 6$ in the worst case, resulting in a competitive ratio of $\frac{3}{2}$. To obtain a better competitive ratio, a linear number of advice bits are necessary (a lower bound of approximately $3.9402 \cdot 10^{-10}n$ bits for obtaining $\lambda = 5$ is given in [19]). This was the first example of a natural online problem with the property that beating the best deterministic online algorithm without advice requires a linear number of advice bits. Note that linear advice trivially suffices to be optimal (in fact, approximately $0.6955n$ bits of advice are sufficient [19]). Since $\lambda = 4$ is obtainable in the offline setting, the linear lower bound for $\lambda = 5$, together with the derandomization technique of [24] mentioned in Section 4, implies a lower bound of $\frac{5}{4}$ on the competitive ratio of any randomized online algorithm for the problem.

For edge coloring of a graph with m vertices and maximum degree Δ, $m \lceil \log(\Delta + 1) \rceil$ bits of advice trivially suffice for an optimal solution (by Vizing's Theorem [104]). Mikkelsen [90] showed that, in the Per Request Model, this bound is asymptotically tight. On the other hand, the paper also shows that, for graphs of bounded degeneracy (including planar graphs), $O(m)$ advice bits are sufficient to be optimal. For trees, the warning signal technique applied to First-Fit yields an optimal algorithm for trees using exactly one bit of advice per edge: If the advice bit is a 0, then First-Fit colors the current edge as usual. If the advice bit is a 1, then First-Fit will skip the lowest numbered color available and instead use the second lowest numbered color available.

For edge coloring without advice, the competitive ratio is 2, on trees as well as in general [11]. In [90], Mikkelsen showed that, even for trees, linear advice is necessary to beat the best deterministic online algorithm without advice. Comparing the proof and the proof of the corresponding result for $L(2,1)$-Coloring, it turns out that they are in fact quite similar. Based on this observation, Mikkelsen showed in [91] that these problems are indeed hard for essentially the same reason; they are both $\vee$-repeatable (see Section 6).

Recall that a problem being $\vee$-repeatable is not enough for a lower bound to carry over from deterministic online algorithms to online algorithms with sublinear advice; it is required that the lower bound does not depend on the online algorithm not knowing $\textsc{Opt}(I)$ (or n). It turns out that

this requirement is vital for the lower bound technique to work. In fact, Christ et al. showed in [39] that sublinear advice suffices to be optimal for Multi-Coloring on a path whereas it is known that an algorithm without advice cannot be better than $\frac{4}{3}$-competitive [38]. This may seem at odds with the previously mentioned result, but the reason is that the $\frac{4}{3}$ lower bound relies heavily on the algorithm not knowing $\textsc{Opt}(I)$. In fact, it is shown in [39] that if $\textsc{Opt}(I)$ is known (note that $\textsc{Opt}(I)$ can be encoded using $O(\log n)$ bits of advice), then it is easy for an online algorithm to be optimal. The case where the exact value of $\textsc{Opt}(I)$ is not known (or not communicated to the algorithm) is also considered, resulting in a trade-off, where the competitive ratio ranges from 1 to $\frac{9}{8}$ and the number of advice bits ranges from $\log n + O(\log \log n)$ to $O(\log \log n)$.

On hexagonal graphs, no Multi-Coloring algorithm without advice can be better than $\frac{3}{2}$-competitive [37]. In [39], it is shown that $\Omega(n)$ bits are necessary for obtaining a ratio better than $\frac{5}{4}$, $n + 2|V|$ bits are sufficient to obtain a ratio of $\frac{4}{3}$, and $\log n + O(\log \log n)$ bits suffice to obtain a ratio of $\frac{3}{2}$.

11 Graph Exploration

Graph Exploration is a family of problems where an agent (sometimes called a robot) with a fixed starting point explores an unknown graph. The goal is usually to visit each vertex of the graph, minimizing the total cost of following edges. Sometimes assumptions are made on the structure of the graph.

These problems are unusual online problems in the following sense: For most other online problems, it is possible to fix an input sequence, $I = x_1, \ldots, x_n$, such that x_i is revealed in round i no matter how the algorithm behaves (of course, if it is deterministic, we know what it will do). In Graph Exploration, even when an input is fixed, the new information the algorithm gains in each step still depends on what it has done in previous steps. Thus, an input sequence cannot be defined independently of an algorithm.

In [71], Kalyanasundaram and Pruhs present an algorithm for Graph Exploration which is 16-competitive on planar graphs. Megow, Mehlhorn, and Schweitzer [89] show that this algorithm does not have a constant competitive ratio on general graphs, but is $16(1 + 2g)$-competitive for graphs with genus at most g. Furthermore, [89] give an algorithm with constant competitive ratio for general graphs with a bounded number of distinct weights. The main open question is whether there exists an algorithm which has a constant competitive ratio for arbitrary graphs with arbitrary weights.

In [56], Tree Exploration with advice is considered by Fraigniaud et al. A robot explores an unknown undirected tree and its goal is to visit every vertex at least once. Each move incurs a cost of one. When the robot is at a given vertex, it can see the labels of the neighboring vertices, but the advice is only allowed to depend on the structure of the tree and not the labels (which are assigned adversarially after the advice is given). Without advice, the best possible competitive ratio for deterministic online algorithms is 2. This is achieved by depth-first-search, DFS. It is shown that roughly $\log \log D$ bits of advice are necessary and sufficient to achieve a better competitive ratio (D is the diameter of the graph). For the upper bound, one bit is used to choose between two algorithms; one is DFS and the other is a more sophisticated algorithm using an approximation

of D. The model used is the Tape Model, except that the length of the advice is known to the algorithm (see Section 2). The lower bound is shown on paths.

The more general case, Graph Exploration, is studied by Dobrev et al. in [43]. Here, the unknown undirected graph is arbitrary and edges have non-negative weights. When the robot is at a vertex, it can see the weight of each adjacent edge and the label of its other endpoint. The goal is to visit each vertex and return to the starting point. Each time an edge is traversed, it costs the weight of that edge. Here the advice is allowed to depend on the labels. It is shown that $\Theta(n \log n)$ bits are necessary and sufficient to be optimal. A $(6+\varepsilon)$-competitive algorithm with $O(n)$ advice is also given. The algorithm works by traversing edges of a minimum spanning tree and some additional light edges.

A related problem, Treasure Hunt, is studied by Komm et al. in [80]. The model is the same as in [43] with the following difference: The robot is given the label of a target vertex and the goal is to visit that vertex. It is observed that a simple greedy algorithm has competitive ratio $\Theta(n)$ and this is best possible for online algorithms without advice (even on unweighted graphs and if randomization is allowed). It is shown that there is an optimal algorithm reading n bits of advice. For each vertex, one bit of advice indicates if that vertex is on a fixed shortest path. For the unweighted case, it is shown that $\Theta(\frac{n}{c})$ bits are necessary and sufficient to achieve a competitive ratio of c (where c has to be of a certain form, but may depend on n).

12 Open Problems

We end the survey with a few open problems:

- Can advice complexity be used to build a complexity theory for online computation?

  The study of online algorithms with advice has led to the first complexity classes in online algorithms and to new possibilities for proving results on randomized online algorithms and semi-online algorithms. Further study may lead to additional meaningful complexity classes and new fundamental insights into the properties of online problems.

- Is it possible for a k-Server algorithm to be $(1+\varepsilon)$-competitive with $O(1)$ bits of advice per request?

  Currently, it is known that the answer to this problem is "yes" if the underlying metric space is the Euclidean plane. It is also known that $\Omega(\log k)$ bits per request are required to be 1-competitive.

- How small a competitive ratio can be achieved for Bin Packing using constant advice?

- Are there further connections between advice and randomization in online computation which have not yet been discovered?

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A Problems Studied in Advice Complexity Models

We list problems explicitly studied in advice complexity models.

- Inherently online problems
  - K-server [24, 51, 91, 94]
  - K-server on sparse graphs [60]
  - K-server on a path [100]
  - List update [34, 91]; application in [73]
  - Paging [25, 91]
  - Metrical task systems [51]
  - Sleep state management [21, 91]
  - Online search [40]
- Scheduling and packing problems
  - Scheduling on identical machines with constant advice [4, 45]
  - Scheduling with sublinear advice [32]
  - Job shop scheduling [81, 105, 106]
  - Job shop with randomized adversary [105]
  - Linear advice approximation schemes for bin packing and scheduling [95]
  - Bin packing with sublinear advice [5, 35, 91]
  - Dual bin packing [93]
  - Bin packing [109] (see Section 9, though)
  - Square packing [72]
  - Reordering buffer management [2, 91]
  - Buffer management [46]
  - Knapsack [26]
  - Set cover [82]
- Coloring problems
  - 2-vertex coloring [18, 91]

  - 3-vertex coloring [97]
  - Graph coloring, general graphs [55, 91]
  - Graph coloring on paths [54]
  - Multi-coloring paths and grids [39]
  - Edge coloring [90]
  - $L(2,1)$-coloring on paths [19]
- Other graph problems
  - Tree exploration with advice [56]
  - Graph exploration [14, 43]
  - Treasure hunt [80]
  - Bipartite matching [48, 91, 92]
  - Independent set [31, 64]
  - Independent set with known supergraph [42]
  - Vertex cover on restricted graph classes [101]
  - Steiner trees [12]
  - Disjoint path allocation [13, 58]
  - Minimum spanning tree [17]
  - Matching on restricted graph classes [75]
- Asymmetric online covering
  - AOC [31] (complexity class comprising, among other problems, independent set, vertex cover, dominating set, disjoint path allocation)
  - Induced subgraph [79]
  - Weighted AOC [32]
- Miscellaneous
  - String guessing/generalized matching pennies [23, 51, 86]
  - Repeated matrix games [91]
  - Graph coloring with randomized adversary [36]
  - Brief survey [85]