The optimal absolute ratio for online bin packing*

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Abstract

We present an online bin packing algorithm with absolute competitive ratio 5/3, which is optimal.

1 Introduction

In the online bin packing problem, a sequence of *items* with sizes in the interval $(0, 1]$ arrive one by one and need to be packed into *bins*, so that each bin contains items of total size at most 1. Each item must be irrevocably assigned to a bin before the next item becomes available. The algorithm has no knowledge about future items. There is an unlimited supply of bins available, and the goal is to minimize the total number of used bins (bins that receive at least one item).

Bin packing is a classical and well-studied problem in combinatorial optimization. The *offline* version, where all the items are given in advance, is well-known to be NP-hard [7]. Extensive research has gone into developing approximation algorithms for this problem. Such algorithms have provably good performance for any possible input and work in polynomial time. In fact, the bin packing problem was one of the first for which approximation algorithms were designed. The (absolute) approximation ratio of an algorithm is the worst case ratio, over all possible inputs, of its cost for a particular input divided by the optimal cost for the same input. Simchi-Levi [15] showed that First Fit Decreasing and Best Fit Decreasing have the best possible absolute approximation ratio of 3/2. For surveys, see [3, 4].

The focus of the research into approximation algorithms is on the question of how much performance degrades if an algorithm is constrained to work in polynomial time. In practical packing problems, however, it happens frequently that the input is not known completely before the algorithm starts working. It is therefore very natural to consider the *online* version of this problem. In online problems, we ask how much performance degrades as a result of not knowing the future. In general, there is no restriction on the amount of computation time used by an online algorithm. However, most online algorithms, including all the ones we consider in this paper, are very efficient.

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For an input L, let $ALG(L)$ be the number of bins used by algorithm ALG to pack this input. Let $\textsc{Opt}(L)$ denote the number of bins in an optimal solution.

The absolute and asymptotic competitive ratios of the algorithm are defined as

$$R_{ABS}(A) := \sup_L \left\{ \frac{A(L)}{\textsc{Opt}(L)} \right\}, \tag{1}$$

and

$$R_{ASY}(A) := \limsup_{n\to\infty} \left\{ \max_{L:\textsc{Opt}(L)=n} \left\{ \frac{A(L)}{n} \right\} \right\}, \tag{2}$$

respectively.

We note that definition (2) focuses on the long-term behavior of online algorithms. For small inputs, the relative performance of an online algorithm might be worse than the asymptotic competitive ratio suggests. Hence, if we want to have a performance guarantee relative to the optimal solution for every possible input, we need to consider the absolute competitive ratio.

In both cases, for any input L, the number of bins used by an online algorithm A is compared to the optimal number of bins needed to pack the same input. Note that calculating the optimal number of bins might take exponential time; moreover, it requires that the entire input is known in advance.

One of the most famous algorithms for bin packing is an online algorithm called First Fit (FF). It packs each item into the first bin where it fits, ordering the bins by when they were opened (i.e., received their first item). First, Ullmann [16] proved that the asymptotic competitive ratio of FF is 1.7. Later, Garey et al. [8] and Johnson et al. [11] extended this work. Among other results, they proved that FF works much better if the elements of the input are sorted in decreasing order. In this case the asymptotic competitive ratio is $\frac{11}{9}$. Of course, this algorithm cannot be used for the online problem. Later, online algorithms improving on FF were given, for example the algorithm of Refined First Fit [18], with an asymptotic competitive ratio of 5/3.

Harmonic-type online algorithms were designed by Lee and Lee [12]. Lee and Lee gave a sequence of Harmonic Fit algorithms, and the asymptotic competitive ratio of their algorithms tends to approximately 1.69103. Further improvement on the asymptotic performance is also presented by Lee and Lee [12], giving an algorithm Revised-Harmonic (RH). The asymptotic competitive ratio of RH is $\frac{373}{228} = 1.6359$. Ramanan et al. [13] presented new Harmonic-based algorithms Modified Harmonic (MH) and Modified Harmonic–2 (MH–2) with asymptotic upper bounds 1.615 and 1.612, respectively. For a long time the best online algorithm Harmonic++ was given by Seiden in 2002 [14]. It is based on the idea of Harmonic Fit and it has an asymptotic competitive ratio of 1.58889. Recently two improved algorithms have been presented: SonOfHarmonic with asymptotic upper bound 1.5816 by Heydrich and van Stee [9, 10] and Advanced Harmonic (AH) by Balogh et al. [1] with asymptotic upper bound 1.57829. Both of these algorithms are Harmonic-based, but go beyond the family of Super Harmonic algorithms introduced in [14].

Van Vliet [17] proved that there is no online algorithm with asymptotic competitive ratio below 1.54014. Balogh et al. [2] improved this to $\frac{248}{161} = 1.54037$.

The absolute competitive ratio of FF was only recently determined to be 1.7 (equal to the asymptotic competitive ratio) by Dósa and Sgall [5]. Afterwards, the absolute competitive ratio of Best Fit (BF), which packs each item into the bin where it leaves the least amount of space unused, was also shown to be exactly 1.7 [6]. Before our work, 1.7 was the best known absolute competitive ratio of any algorithm. There is a simple lower bound, consisting of only 18 items, which shows that no algorithm can be better than 5/3-competitive. This folklore result is included in Section 1.1 for completeness.

The natural question is then whether an algorithm with better competitive ratio than FF or BF exists. In this paper, we answer this question in the affirmative by presenting an online algorithm with absolute competitive ratio 5/3.

1.1 Lower bound of 5/3

First, 6 items of size 1/7 arrive. If they are packed in more than one bin, the competitive ratio is at least 2 and the input stops.

Otherwise the 6 items are packed in a single bin. Then 6 items of size $1/3+\varepsilon$ arrive for some $\varepsilon \in (0, 1/84)$. They do not fit in the first bin. If they are packed into four or more bins, the competitive ratio is at least 5/3 and the input stops.

Otherwise the only possibility is to pack the 6 items of size $1/3+\varepsilon$ into three bins, two per bin. Then, 6 items of size $1/2+\varepsilon$ arrive. These items must be packed into 6 separate new bins, for a total of 10 bins. However, the entire input can be packed into only 6 bins with one item of each size, proving the lower bound.

1.2 Basic definitions

The *level* of a bin is the sum of the sizes of the items in it.

A bin is called a *k-bin* if there are exactly k items packed into it. A *k^+-bin* contains *at least k* items.

Items of size more than 1/2 are called *large*, others are called *small*.

1.3 Idea of the algorithm

Our algorithm, which we call Five-Thirds (FT), behaves like FF whenever possible, while avoiding certain bad situations.

To give some intuition, consider the worst case for FF which is given by instances of the following form. The input starts with $10k$ items of sizes very close to 1/6, for some integer k. FF packs these items into $2k$ bins. Then, $10k$ items of sizes very close to 1/3 arrive. These items are packed in pairs into $5k$ bins by FF. Finally, $10k$ items of size slightly more than 1/2 arrive, that FF packs into individual bins. In the end, $FF = 17k$, while the items can be packed into $10k$ bins. In these instances, it is notable that all bins used by FF are relatively full, apart from the last $10k$ bins.

In our algorithm FT, we try to avoid the bad situation where, at the end of the input, the algorithm has to open many new bins that are only half full. More specifically, FT avoids long sequences of 2-bins. It can be shown easily that 3^+-bins are generally fuller than 2-bins, and this compensates for 1-bins that are only half full at the end. However, 2-bins are problematic, since they may be only about 2/3 full on average. Therefore, whenever FT is about to put an item into any bin that has one item so far, it will from time to time put such an item into a new, empty bin instead, creating a *special bin* which is *specifically reserved for a large item*; no other item will be packed into it.

It is possible that no large items arrive after all, so FT needs to be conservative about creating these special bins, since they are initially less than half full. In fact, we need to be extremely careful about the conditions for creating new special bins, in order to be able to deal with any possible input.

1.4 Overview of the analysis

In Section 2 we present the algorithm FT, the classification of the bins it creates and some basic properties. The analysis of FT then splits into three main cases. A technical problem in all cases is that there may exist a single non-special 1-bin that has a small item (e.g., if this item arrives near the end of the input and is followed only by large items that do not fit with it). This complicates both the size-based and the weight-based analysis methods that we describe below.

If no special bin is ever created, FT behaves as FF throughout, and (due to our conditions for creating special bins), FF is 5/3-competitive in this situation (Section 3). This case is relatively easy, but note that it includes the instances that prove the tight lower bound. The main technical difficulties occur in the next two cases.

If all special bins have large items (Section 4), it means that these bins are relatively full. For this case, we use a weight-based analysis. Each item is assigned a weight which is a measure for how much space this item needs in any packing. In order to prove an upper bound of 5/3 instead of 1.7, we modify the weight function used by Dósa and Sgall [5]. The idea is that each optimal bin has total weight packed into it of at most 5/3, whereas FT packs an average weight of at least 1 per bin, implying the desired result. In the end, the weight-based analysis leaves one case open, where only one special bin is created; for this case, we use size-based arguments.

Finally, if there exists a special bin that does not contain a large item at the end of the input (Section 5), it means that all large items in 1-bins are relatively large, since FT always puts large items in existing special bins and special items in existing 1-bins with large items if they fit. For this case, we use a size-based analysis, showing that the bins of FT are on average at least 3/5 full. The way of proving this depends on how many 1-bins there are compared to the number of special bins. At the end, some small cases need to be examined in detail to get the desired result.

In an abuse of notation, we will use some variables to denote specific items as well as their sizes.

2 The algorithm

2.1 Classification of bins

At each time, FT maintains a partition of all bins into a set of **special** bins and a set of **regular** bins. FT will maintain the following two properties of special bins.

Property 2.1 *A* special bin *holds exactly one small item (which is called a* special item*) and at most one other item, which must be large.*

Property 2.2 *A regular bin can become a special bin, but any special bin stays special until the end of the instance.*

Thus, special bins that have only one item are specifically reserved for large items that may arrive later, and will not receive any other item. A special bin can be created in three ways, all of which we later use in our algorithm: (i) a regular bin with a single small item can be declared a special bin, (ii) upon creating a new bin with a small item, this bin can be declared a special bin, and (iii) when packing a small item in a bin with a single large item, this bin can be declared a special bin.

The following subtypes of regular bins are important in our algorithm.

Definition 2.3 *A regular 2-bin is* **critical** *if it does not contain a large item and has level less than 3/4.*

A regular 2^+*-bin without a large item is* **interesting** *if its first two items have total size less than 3/4, else it is* **quick**.

A regular bin is **good** *if it contains a large item, it has level at least 5/6, or its first two items have total size at least 3/4.*

Figure 1 illustrates the relations of the bin types defined above and their possible evolution during the run of our algorithm.

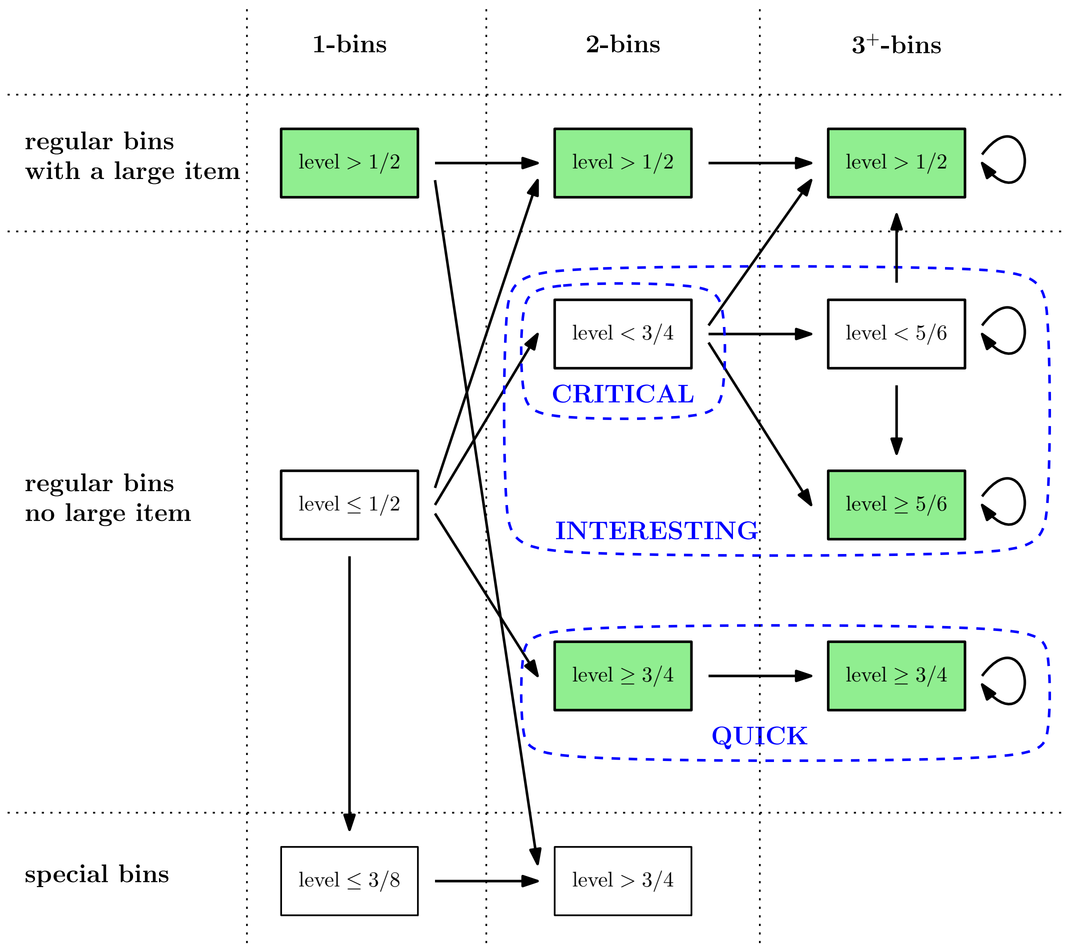

Figure 1: Types of bins. Green (shaded) boxes denote good bins. Arrows depict all possible changes of bin types when an item is packed, or a regular bin is declared special. The bounds on the level of special bins follow from Lemma 2.10.

2.2 Ordering and matching

Our algorithm opens bins one by one. Whenever we speak about first, last, next bin (etc.), or when we consider an ordered list of bins, we will always be referring to the order in which bins were opened by our algorithm.

In our analysis, we will often consider pairs of bins, for which we can prove that the bins in each pair have a total packed size strictly more than 1. We will say that the bins of such a pair are *matched* to each other. Every newly opened bin is initially unmatched.

One type of matching is done by FT itself. This matching additionally guarantees that the matched bins have large total weight (see Section 5 for a definition of the weights). Every new bin is initially unmatched. Whenever FT creates a special bin, it will be matched to an existing unmatched critical bin. To be precise, we always use the *last* such critical bin. In the algorithm, as a necessary condition for creating a special bin we require that an unmatched critical bin exists, so the matching is well-defined. A bin that is matched to a special bin remains matched even if the bin stops being critical at some point, however note that this bin remains interesting.

2.3 Description of the algorithm

A detailed description of the algorithm FT is shown in Figure 2. The variable s counts the number of special bins. In this algorithm and throughout the paper, we abuse notation and let some variables refer both to specific items and to their sizes.

Initialize s to 0.
For each item a on the input, do the following:

1. If a is large, pack a by FF into the collection of all bins, opening a new bin if needed.
2. Otherwise, let B be the bin that FF would pack a into if the collection of existing bins were restricted to the regular bins. (B is possibly a new bin.) If after packing a into B, there are at most $\max(3, 4s+1)$ *interesting* bins, or B is not critical, or B is the only unmatched critical bin, pack the item into B.
3. Otherwise, if before packing a there exists a regular 1-bin with a large item where a fits, pack a into the first such bin B'; declare B' to be a special bin where a is the special item. Match the special bin to the last unmatched critical bin. Increase s by 1.
4. Otherwise pack a into a new bin A. Let b be the only item previously packed in B. If $a \leq b$, let A be a special bin while B remains regular. Else, change B to a special bin and let A be regular. The single item packed into the special bin is a special item. Match the special bin to the last unmatched critical bin. Increase s by 1.

Figure 2: Algorithm FT

In Step 2, we decide whether we should create a new special bin or not. First we check what FF would do with the current item, while "hiding" the special bins from FF. That is, FF makes its decision based only on the current sequence of regular bins and the item a. The bound $\max(3, 4s+1)$

means that after a slightly irregular initial phase, roughly every fifth time that FT opens a new *interesting* bin (i.e., ignoring bins that quickly reached a level of 3/4 or that contain large items), one bin (the new one, or an existing one) may be turned into a special bin instead. If the bin B used by FF is acceptable (that is, one or more of the conditions listed in Step 2 hold), then we pack the new item into B in this step.

If a new special bin is created, this is done in Step 3 or in Step 4. In Step 3, note that $B' \neq B$, because B' has a large item and B does not (since it would become critical on receiving a). In fact, B' comes after B, because a fits in B' but FF did not suggest it. Regarding Step 4, note that item b is indeed unique, since packing a into B would make B critical, i.e., a 2-bin. Hence, B must have been a 1-bin before.

An item always gets packed in Step 2 (by using FF) if the bin B proposed by FF does not become critical after packing a, e.g., if a opens a new bin or if B was already a 2^+-bin. The earliest possible time that a special bin could be created is the following: FF is about to create the *fourth* interesting bin, this bin becomes a critical 2-bin, and at least one earlier interesting bin is critical. After creating the first special bin, we allow at least $4s + 1 = 5$ regular bins in total before possibly creating another special bin. It is possible that it takes much longer for the first special bin to be created, and there are not always four regular bins between two successive special bins.

As an example, it could happen that first, a large number of non-critical regular bins are created. (These bins are critical when they contain two items but they then receive additional items.) Then, a bin B_1 becomes critical by receiving its second item a_1. At this point, no unmatched critical bin is available, so still no special bin is created. For the next item a_2, a new bin B_2 is opened, and it is turned into the first special bin when item a_3 arrives (which is packed into its own new bin B_3). The bin B_2 is matched with B_1. Item a_4 is then packed into B_3, making it critical, but not special, because again no unmatched critical bin exists. Then, another bin B_4 is opened for item a_5, and it is turned into the second special bin (matched with B_3) when the next item a_6 arrives (which is packed into its own new bin B_5). In the end, the special bins B_2 and B_4 are separated by only one regular bin in this example.

2.4 Properties of the algorithm

Let us recall Figure 1 that summarizes possible development of bin types; it is easy to verify that it covers all the possibilities given by the algorithm. In particular, note that a bin can become critical only when the second item is packed into it, but a critical bin loses this property if the bin receives a third item later. However, it remains interesting throughout the run of FT, unless it receives a large item. An interesting bin with level above 1/2 will therefore always remain interesting. The following observation states the important properties of bin types.

Observation 2.4 *A good bin remains good as long as it remains regular, a regular 2^+-bin remains regular, and an interesting bin that is at least half full remains interesting.*

All critical bins are interesting, and all quick bins are good.

Lemma 2.5 (First Fit property) *If an item is packed into a bin which is regular after receiving it then this item does not fit in any earlier regular bin.*

Proof. Consider an item a that is packed into a bin which is then regular. This does not happen in Step 3. If a is packed in Steps 1 or 2, it is packed by FF into a regular bin and the claim follows. Note that ignoring special bins does not affect this property.

It remains to handle the case when a is packed into a new regular bin in Step 4. FF would pack a into B which contains a single item $b < a$, and FT makes B a special bin instead, packing a into a new bin A. Since FF packs a into B, it follows that a does not fit into any regular bin before B. Now consider any existing (i.e., not A) regular bin C after B and an item c in C. Using Property 2.2, the bins B and C were regular when c was packed into C. Then c does not fit into B, using the lemma inductively on c which comes before a. We have $1 < b + c < a + c$ and thus in turn a does not fit into C, so a indeed does not fit into any existing regular bin. ∎

Corollary 2.6 *At any time, there is at most one regular bin that is at most half full.*

Lemma 2.7 *At any time, if x is a large item in a 1-bin, and y is a special item in a (special) 1-bin, then $x + y > 1$.*

Proof. A large item is always placed in an existing bin if possible in Step 1. Thus if y arrived first, $x + y > 1$.

A small item is always placed in an existing bin if possible in Step 2 or in an existing bin with a large item if possible in Step 3. Thus the lemma holds also if x arrived first. ∎

Lemma 2.8 *When FT creates a special bin in Step 3 or Step 4, each interesting bin precedes bin B defined in Step 2 and will remain interesting throughout the run of FT.*

Proof. Consider a time when FT creates a special bin. Since bin B from Step 2 has a single small item in this case, B is at most half full at this time. So all regular bins after B (if any) contain a large item by the First Fit property (Lemma 2.5). This means all interesting bins are before B. Furthermore, their level is above 1/2, so they will remain interesting. ∎

Lemma 2.9 *Let J be a set of regular bins of cardinality $j = |J| \geq k + 1$, of which at least the last $j - k$ bins are regular k^+-bins. Then the total level of the bins in J is more than*

$$\frac{jk}{k+1} .$$

Proof. Consider any $k + 1$ bins of J with the smallest levels, denoted by $B_1, \ldots, B_{k+1}$ in the order of the packing of FT. It is sufficient to show that their total level is more than k, as any other bin then has level more than $k/(k + 1)$. Note that B_{k+1} is a k^+-bin by the assumption of the lemma. Let the smallest level among $B_1, \ldots, B_k$ be x. Then any item in B_{k+1} is larger than $1 - x$, thus the total level of the $k + 1$ bins is more than $kx + k(1 - x) = k$. ∎

Lemma 2.10 *The total size of a special item and the first two items in the bin it is matched to is more than 1. Each special item has size more than 1/4 but less than 3/8.*

Proof. Consider the situation when a special item is created, on arrival of item a. By Lemma 2.8, the critical bin C that is matched to the new special bin must be before the bin B that FF suggests. Thus a does not fit in C, as otherwise FF would suggest C. This proves the first bound. We have $a > 1/4$, as C has level less than 3/4 by the definition of a critical bin. Furthermore, also the item b in B does not fit into C by the First Fit property (Lemma 2.5). Thus also $b > 1/4$, and the lower bound on the size of the special item follows.

For the upper bound, we know that $a + b < 3/4$ since B would be critical if a were packed into it. If the special bin is created in Step 4, the smaller one of a and b becomes the special item, and we are done. If a is packed into a 1-bin B' in Step 3 and becomes a special item, note that B and B' are 1-bins (before packing a). Thus b does not fit in B' by the First Fit property (Lemma 2.5). It follows that $a < b$, and $a < 3/8$ holds as well. ■

2.5 Additional assumptions and definitions

For the analysis, we assume for contradiction that there exists an instance I for which $FT(I) > \frac{5}{3}\textsc{Opt}(I)$. We fix such an instance I and abbreviate $FT(I)$ by FT and $\textsc{Opt}(I)$ by $\textsc{Opt}$. Let $\textsc{size}$ be the total size of all the items in I.

For a set of items A and a set of bins $\mathcal{A}$, let $\textsc{size}(A)$ and $\textsc{size}(\mathcal{A})$ denote the total size of all items in A or $\mathcal{A}$. For a bin B, let $\textsc{level}(B)$ be the total size of the items packed into B.

Because FT and $\textsc{Opt}$ are integers, our assumption is equivalent to assuming

$$FT \geq \frac{5}{3}\textsc{Opt} + \frac{1}{3}\,. \tag{3}$$

Let us denote the number of regular 1-bins in the final packing of FT by δ. Furthermore, let δ_0 and δ_1 be the number of these bins with a small and large item, respectively. By Lemma 2.5, we have $\delta_0 \leq 1$. If such a regular 1-bin with a small item exists, we denote this bin by D_0 and the item in it by d_0. Lemma 2.5 also implies that d_0 does not fit in a bin with any other item from the remaining δ_1 regular 1-bins and thus

$$\textsc{Opt} \geq \delta. \tag{4}$$

We also have the following useful property, which follows immediately from the First Fit property.

Lemma 2.11 *Let $\mathcal{A}$ be a nonempty set of regular bins, $D_0 \notin \mathcal{A}$. Then $\textsc{size}(\mathcal{A} \cup D_0) > (|\mathcal{A}| + 1)/2$.*

In the next three sections, we analyze the three different cases that may occur.

3 No special bin is created

Theorem 3.1 *If no special bin is created, then $FT \leq \frac{5}{3}\textsc{Opt}$.*

Proof. Suppose that no special bin is created. It follows that at any step of algorithm FT we accepted the proposal of FF, thus finally we get just the FF packing. For FF, we already know that FF$\leq 1.7 \cdot \textsc{Opt}$ holds.

If $\textsc{Opt} \leq 9$, the stronger inequality FF$\leq \frac{5}{3}\textsc{Opt}$ is true according to Table 1.

$\textsc{Opt}$	1	2	3	4	5	6	7	8	9
$1.7 \cdot \textsc{Opt}$	1.7	3.4	5.1	6.8	8.5	10.2	11.9	13.6	15.3
$\lfloor 1.7 \cdot \textsc{Opt} \rfloor$	1	3	5	6	8	10	11	13	15

Table 1: Calculation of the competitive ratio of FF for small instances.

Now suppose that $\textsc{Opt} \geq 10$. First observe that the number of 2^+-bins is at least 7. Indeed, the number of 1-bins is at most $\textsc{Opt}$ by (4). From (3) and $\textsc{Opt} \geq 10$, we get that the number of 2^+-bins is at least $\frac{2}{3} \cdot \textsc{Opt} + \frac{1}{3} \geq 7$.

Our plan is to account carefully for the total level of the bins in the final packing of FT and reach a contradiction by showing that it is more than OPT. We partition the bins into three sets: $\mathcal{D}$ containing all 1-bins, with at most OPT bins and average level at least $1/2$, $\mathcal{E}$ containing all critical bins, with exactly three bins and average level at least $2/3$, and $\mathcal{G}$ with bins of average level at least $3/4$. The exact definitions are based on various observations and possible special cases. We first define the sets provisionally and later slightly modify them for easier accounting. The formal proof follows.

We initialize the sets as follows:

- $\mathcal{D}$ contains all 1-bins;
- $\mathcal{G}$ contains all 2^+-bins with level at least $3/4$;
- $\mathcal{E}$ contains all interesting bins with level below $3/4$.

Let B_0 be the first 2^+-bin with a large item and level below $3/4$, if such a bin exists. Any following bin contains only items larger than $1/4$, as they do not fit into B_0, and it follows that it belongs to $\mathcal{D}$, $\mathcal{E}$, or $\mathcal{G}$, depending on the number and size of the items in it.

Hence B_0 is the only 2^+-bin with a large item and level below $3/4$, and it is the only bin which is not in $\mathcal{D}, \mathcal{G}$, or $\mathcal{E}$ (if it exists). If B_0 is the first 2^+-bin, we add it to $\mathcal{D}$, otherwise we add it to $\mathcal{G}$. Now all bins are included in exactly one of the sets $\mathcal{D}$, $\mathcal{E}$, $\mathcal{G}$.

Lemma 3.2 *The set $\mathcal{E}$ contains at most three bins.*

Proof. The first bin in $\mathcal{E}$ has level below $3/4$, so all other bins in $\mathcal{E}$ contain only items larger than $1/4$ by the First Fit property. Since they are not 1-bins and their level is below $3/4$, they have two items. They are critical from the moment their second item arrives and remain unmatched, as no special bin is created. For a contradiction, consider the moment when the fourth bin from $\mathcal{E}$ receives the second item. This bin becomes (at least) the fourth interesting bin, it is critical and unmatched, and there are other unmatched critical bins, contradicting Step 2 of the algorithm. ■

Now we make our two final modifications of the sets. First, if $|\mathcal{D}| = 1$ and this single bin has level at most $1/2$ (i.e., it is the 1-bin D_0 with a small item), we remove the last bin from $\mathcal{G}$ and add it to $\mathcal{D}$. Second, we assure that $|\mathcal{E}| = 3$ by moving up to three last bins from $\mathcal{G}$ to $\mathcal{E}$ (if necessary). Note that $\mathcal{G}$ has sufficiently many bins for both of these modifications, since the number of 2^+-bins is at least 7, as we observed above. Furthermore, the first bin is not removed from $\mathcal{G}$.

Lemma 3.3 *$\text{SIZE}(\mathcal{E}) > 2$.*

Proof. After the second modification, $\mathcal{E}$ contains exactly three 2^+-bins. The claim follows by Lemma 2.9. ■

Lemma 3.4 *$\text{SIZE}(\mathcal{G}) \geq \frac{3}{4}|\mathcal{G}|$.*

Proof. All bins in $\mathcal{G}$ have level at least $3/4$ except possibly B_0. In particular, the lemma holds if $\mathcal{G}$ does not contain B_0.

If $B_0 \in \mathcal{G}$, then let B be the first 2^+-bin; by the condition on putting B_0 into $\mathcal{G}$, bin B is distinct from B_0 and precedes it. Bin B is either in $\mathcal{E}$ or in $\mathcal{G}$. By its definition, B_0 contains a large item and at least one item $x < 1/4$ that does not fit into B. Thus $B \in \mathcal{G}$ (and it is not removed from $\mathcal{G}$

during the modifications, as it is the first bin). Furthermore $\text{LEVEL}(B) + x > 1$ and together with the large item in B_0 this gives $\text{LEVEL}(B) + \text{LEVEL}(B_0) > 3/2$. The lemma now follows by adding the levels of the remaining bins in $\mathcal{G}$. ■

Lemma 3.5 $\text{SIZE}(\mathcal{D}) \geq \frac{1}{2}|\mathcal{D}|$ *and* $|\mathcal{D}| \leq \text{OPT}$.

Proof. Each bin in $\mathcal{D}$ contains a large item, except D_0 if it exists. Due to our first modification, D_0 is never the only bin in $\mathcal{D}$, so we can apply Lemma 2.11, and the size bound follows.

If $\mathcal{D}$ does not contain B_0, then (4) implies $|\mathcal{D}| = \delta \leq \text{OPT}$ and the second part of the lemma follows. If $B_0 \in \mathcal{D}$ then we show that $\delta < \text{OPT}$; this implies the second part of the lemma as $|\mathcal{D}| = \delta + 1$ in this case. The condition on putting B_0 into $\mathcal{D}$ implies that B_0 is the first 2^+-bin. Each item in the remaining 2^+-bins has size more than $1/4$, since B_0 has level less than $3/4$. By (4), there are at most OPT 1-bins, so by assumption (3) the number of 2^+-bins is more than $\frac{2}{3}\text{OPT} > \frac{1}{2}\text{OPT} + 1$ (using $\text{OPT} \geq 10$). Hence the 2^+-bins after B_0 contain more than OPT items larger than $1/4$. We also have at least δ large items (including the one in B_0). As the large items are in distinct bins of the optimum and each can combine with at most one item larger than $1/4$, this implies that $\text{OPT} > \delta$ and the lemma follows. ■

Since each bin is in exactly one set and $|\mathcal{E}| = 3$, we have $|\mathcal{G}| = FT - 3 - |\mathcal{D}|$. Using this, the previous three lemmata and also (3) for the final inequality, we obtain

$$\begin{aligned}\text{OPT} &\geq \text{SIZE} = \text{SIZE}(\mathcal{E}) + \text{SIZE}(\mathcal{D}) + \text{SIZE}(\mathcal{G}) > 2 + \frac{1}{2}|\mathcal{D}| + \frac{3}{4}(FT - 3 - |\mathcal{D}|) \\ &\geq 2 - \frac{9}{4} - \frac{1}{4}|\mathcal{D}| + \frac{3}{4}\left(\frac{5}{3}\text{OPT} + \frac{1}{3}\right) \geq -\frac{1}{4} - \frac{1}{4}\text{OPT} + \frac{5}{4}\text{OPT} + \frac{1}{4} = \text{OPT}\,,\end{aligned}$$

a contradiction. ■

4 Each special bin contains a large item

In this and the following section, we have $\text{OPT} \geq 3$, since no special bin is created until there are at least three interesting bins and any three interesting bins have total level larger than 2 by Lemma 2.9.

In this section all 1-bins are regular. So, recalling the notation, the number of 1-bins is $\delta = \delta_1 + \delta_0$, where δ_1 is the number of 1-bins with a large item. If $\delta_0 > 0$, then $\delta_0 = 1$ and there exists a single bin with a small item, D_0 and d_0 denote that bin and its item.

We introduce a weight function that is a natural modification of the one that was used in the tight analysis of FF [5]. This modification ensures that every optimal bin (i.e., a bin in an optimal solution) has weight at most $5/3$ (see Lemma 4.2). We no longer have that FF has a weight of 1 per bin (because it is not $5/3$-competitive, after all), but we can show that the *amortized* weight of a bin of FT is at least 1.

Definition 4.1 *For any item a we define its* regular weight *as $r(a) = \frac{6}{5}a$. We also define the* bonus *of the item that is denoted by $b(a)$ as follows:*

$$b(a) = \begin{cases} 0 & \text{if } 0 < a \leq 1/6 \\ \frac{2}{5}(a - \frac{1}{6}) & \text{if } 1/6 < a \leq 1/3 \\ 1/15 & \text{if } 1/3 < a \leq 1/2 \\ 2/5 & \text{if } a > 1/2. \end{cases}$$

The weight *of the item a is defined as $w(a) = r(a) + b(a)$.*

Note that the bonus function is monotonically non-decreasing in a, and large items always have weight larger than 1.

For a set of items A and a set of bins $\mathcal{A}$, let $w(A)$ and $w(\mathcal{A})$ denote the total weight of all items in A or $\mathcal{A}$; similarly for r and b.

Note that if we have a set A of k items with sizes in $(1/6, 1/3]$, then the definition implies that its bonus is exactly $b(A) = \frac{2}{5}\left(\text{SIZE}(A) - \frac{k}{6}\right)$. If A contains k items, each of size $\in (1/6, 1/2]$, then we get the upper bounds $b(A) \leq \frac{2}{5}\left(\text{SIZE}(A) - \frac{k}{6}\right)$ and $b(A) \leq \frac{k}{15}$.

First we analyze the weight of the optimal bins, which is the easy part of the proof. This proof is what we based the definition of our weight function on.

Lemma 4.2 *For every optimal bin A its weight $w(A)$ can be bounded as follows:*

(i) *$w(A) \leq 5/3$.*

(ii) *If A contains no large item, then $w(A) \leq 7/5$.*

Proof. In all cases $r(A) \leq 6/5$, thus it remains to bound $b(A)$.

If A contains no large item, we distinguish two cases. If A contains at least four items with non-zero bonus, then their total bonus is at most

$$b(A) \leq \frac{2}{5}\left(\text{LEVEL}(A) - \frac{4}{6}\right) \leq \frac{2}{5} \cdot \frac{1}{3} = \frac{2}{15}\,.$$

If A contains at most three items with non-zero bonus, then $b(A) \leq 3/15 = 1/5$. In both cases, (ii) and thus also (i) holds.

If A contains a large item, note that the bonus of the large item is $2/5$. In addition, A contains at most two items larger than $1/6$ of total size $y < 1/2$. If there are two such items then we have

$$b(A) \leq \frac{2}{5} + \frac{2}{5}\left(y - \frac{2}{6}\right) < \frac{2}{5} + \frac{2}{5} \cdot \frac{1}{6} = \frac{7}{15}\,.$$

If there is at most one such item then $b(A) \leq 2/5 + 1/15 = 7/15$ again. In both cases $w(A) \leq 6/5 + 7/15 = 5/3$ and (i) holds. ■

Throughout this section, we will assume (3) and derive a contradiction. Together with (3), by adding up the weight of all the optimal bins, Lemma 4.2 implies that

$$w(I) \leq \frac{5}{3} \cdot \text{OPT} \leq FT - \frac{1}{3}\,. \tag{5}$$

In the following lemma, we exclude some extreme cases by a simple calculation of total volume.

Lemma 4.3 *The following three properties hold.*

(i) *If d_0 exists, then $d_0 > 1/3$.*

(ii) *There exists at least one 1-bin with a large item.*

(iii) *No 2^+-bin has level $1/2$ or smaller.*

Proof. For all three statements, we will use that the level of any special bin is more than $3/4$ by Lemma 2.10 and because each special bin has a large item.

(i) Suppose $d_0 = 1/3 - x$ with some $0 \le x \le 1/12$. Then the level of any other regular or special bin is bigger than $2/3 + x$. By $FT \ge 2$ and (3), we get for the total size that

$$\begin{aligned}
\textsc{Opt} \ge \textsc{size} &> \left(\frac{2}{3} + x\right)(FT - 1) + \left(\frac{1}{3} - x\right) \\
&= \frac{2}{3}FT - \frac{1}{3} + (FT - 2)x \\
&\ge \frac{2}{3}\left(\frac{5}{3}\textsc{Opt} + \frac{1}{3}\right) - \frac{1}{3} = \textsc{Opt} + \frac{1}{9}(\textsc{Opt} - 1) \ge \textsc{Opt},
\end{aligned}$$

a contradiction. If d_0 is even smaller, i.e. $d_0 \le 1/4$, then the level of any other bin is bigger than $3/4$, and hence we get $\textsc{Opt} \ge \textsc{size} > \frac{3}{4}(FT - 1) \ge \frac{3}{4}\left(\frac{5}{3}\textsc{Opt} - \frac{2}{3}\right) = \textsc{Opt} + \frac{1}{4}(\textsc{Opt} - 2) \ge \textsc{Opt}$, a contradiction.

(ii) Suppose there is no 1-bin with a large item. If D_0 exists, we consider it together with the first special bin; their total level is above 1 using (i) and the first line of this proof. There are at least three regular 2^+-bins since there exists a special bin. We can apply Lemma 2.9, the first line of this proof and (3), and we get for the total size that

$$\textsc{Opt} \ge \textsc{size} > \frac{2}{3}(FT - 2) + 1 = \frac{2}{3}FT - \frac{1}{3} \ge \frac{2}{3}\left(\frac{5}{3}\textsc{Opt} + \frac{1}{3}\right) - \frac{1}{3} = \frac{10}{9}\textsc{Opt} - \frac{1}{9} \ge \textsc{Opt},$$

a contradiction.

(iii) Suppose to the contrary that there exists a 2^+-bin, say B_0, such that the level of B_0 is at most $1/2$. Then there is an item a in B_0 with size at most $1/4$, and B_0 is regular. By Lemma 2.5, the level of any earlier regular bin is bigger than $3/4$. Moreover, any later regular bin is a 1-bin since any item in these bins must be large.

There is at least one 1-bin by (ii). The total level of any 1-bin and B_0 is bigger than 1. We get that there are $\delta + 1 \ge 2$ bins with total level bigger than $(\delta + 1)/2$, and the level of any other bin is bigger than $3/4$. Then the next estimation is valid for the total size:

$$\begin{aligned}
\textsc{Opt} \ge \textsc{size} &> \frac{3}{4}(FT - \delta - 1) + \frac{1}{2}(\delta + 1) \\
&= \frac{3}{4}FT - \frac{1}{4}\delta - \frac{1}{4} \ge \frac{5}{4}\textsc{Opt} + \frac{1}{4} - \frac{1}{4}\textsc{Opt} - \frac{1}{4} = \textsc{Opt},
\end{aligned}$$

which is a contradiction. Here we have used $\delta \le \textsc{Opt}$.

∎

For the analysis of the bins of FT, we partition them in several sets. We use s to denote the final value of s, after all items are packed. Recall that we assume there are no special 1-bins in this section. Let $\mathcal{D}$ be the set of 1-bins. Let $\mathcal{B}$ be the set of all 2^+-bins. The set $\mathcal{B}$ is further partitioned into three parts:

- Let $\mathcal{S}$ be the set of special bins and their matches; note that $|\mathcal{S}| = 2s$.
- Let $\mathcal{G}$ be the set of good bins in $\mathcal{B} \setminus \mathcal{S}$.

- Let $\mathcal{C} = \mathcal{B} \setminus (\mathcal{S} \cup \mathcal{G})$. These bins are either critical bins or interesting 3^+-bins with level below 5/6. We number the bins in $\mathcal{C}$ according to their order in the packing of FT: $\mathcal{C} = \{C_1, \ldots, C_{|\mathcal{C}|}\}$. Let τ be the number of critical bins among $C_2, \ldots, C_{|\mathcal{C}|}$ ($\tau = 0$ if $|\mathcal{C}| \leq 1$).

Note that τ counts only bins that are still critical at the end of the execution of FT, excluding C_1. We start by estimating the weight of $\mathcal{D}$, $\mathcal{G}$, and $\mathcal{S}$, which are the easy sets.

Lemma 4.4 *We have $w(\mathcal{D}) - \delta > -\delta_0/3$.*

Proof. If there is no item d_0, each bin in $\mathcal{D}$ has weight more than 1, since each large item has weight more than 1 by the definition, and we are done. Else, by Lemma 4.3, $d_0 > 1/3$ and $\delta_1 \geq 1$. Now the total size of the items in 1-bins is greater than $\delta/2$, thus the total regular weight of these bins is greater than $3\delta/5$. The bonus of any of the δ_1 large items in 1-bins is 2/5, and the bonus of d_0 is 1/15. Thus the total weight is $w(\mathcal{D}) > 3\delta/5 + \frac{2}{5}(\delta - 1) + 1/15 = \delta - 1/3$. ■

Lemma 4.5 *For every bin $G \in \mathcal{G}$, we have $w(G) \geq 1$, thus $w(\mathcal{G}) - |\mathcal{G}| \geq 0$.*

Proof. If G contains a large item, or the level of the bin is at least 5/6, the weight is at least 1. Else, G has two items of combined size at least 3/4 but no large item, so the largest item has size in $[3/8, 1/2]$ and bonus 1/15 and the second largest item has size at least 1/4 and thus bonus at least 1/30. Thus $w(G) = r(G) + b(G) \geq \frac{6}{5} \cdot \frac{3}{4} + \frac{1}{15} + \frac{1}{30} = 1$. ■

Lemma 4.6 *Let S be a special bin with a special item a and let M be its match. Let b and c denote the first two items in bin M. Then the total weight of these three items is more than 4/3. Consequently $w(S) + w(M) > 7/3$ and $w(\mathcal{S}) - |\mathcal{S}| > s/3$.*

Proof. By Lemma 2.10, $a + b + c > 1$, so the regular weight of these three items is more than 6/5.

We claim that the bonus of the three items is at least 2/15. If there is a large item among them, or there are two items of size at least 1/3 among them, the claim holds. Otherwise each item is of size at most 1/2, and there is exactly one of size at least 1/3. Denote the largest size by x and the other two by y and z. Since $x + y + z > 1$, $1/3 < x \leq 1/2$, thus $y + z > 1/2$, and both y and z are in $(1/6, 1/3)$. For the total bonus, we get $b(x) + b(y) + b(z) = \frac{1}{15} + \frac{2}{5}(y - \frac{1}{6}) + \frac{2}{5}(z - \frac{1}{6}) = \frac{2}{5}(y + z) - \frac{1}{15} > \frac{2}{5} \cdot \frac{1}{2} - \frac{1}{15} = \frac{2}{15}$.

Thus the total weight of the three items is more than $6/5 + 2/15 = 4/3$.

Regarding the second claim we only need to recall that in any special bin there exists also a large item, and this item has weight more than 1. Hence, for every pair (S, M), the total weight of the two bins is more than $4/3 + 1 = 7/3$. Finally, recall that $|\mathcal{S}| = 2s$. ■

In the previous parts, we have shown that the weight per bin is typically at least 1. One exception is D_0, in which we have less weight and this constitutes the hard case later. Another exception are the special bins in which together with their matches we have 1/3 extra weight per matched pair. Later it turns out that on each critical bin in $\mathcal{C}$ we have about 1/15 too little weight. So, the next claim which relates τ to s is essential in our proof and in fact gives some justification for creating special bins at regular intervals. Recall that τ is the number of critical bins in $\mathcal{C} \setminus \{C_1\}$.

Claim 4.7 *$\tau \leq 3s$.*

Proof. There are at most $4s+1$ interesting bins and s of them are matched, and therefore contained in $\mathcal{S}$ instead of $\mathcal{C}$. As C_1 is not counted in τ, we have $\tau \leq |\mathcal{C}| - 1 \leq (4s + 1) - s - 1 = 3s$. ■

Now we estimate the weight of the bins in $\mathcal{C}$. We need the following amortization lemma.

Lemma 4.8 *Let C_i and C_j be two bins in $\mathcal{C}$, $i < j$.*

(i) *If the level of C_i is at least $2/3$ and C_j is a 3^+-bin, then $r(C_i) + b(C_j) \geq 1$.*

(ii) *If the level of C_i is at least $2/3$, then $r(C_i) + b(C_j) \geq 14/15$.*

(iii) *If the level of C_i is $2/3 + \varepsilon/2$, for some $\varepsilon > 0$, and C_j is a 2-bin with level at least $2/3$, then $r(C_i) + b(C_j) \geq 14/15 + 2\varepsilon/5$.*

(iv) *If C_i has level $2/3 - \varepsilon$, for some $\varepsilon > 0$, then C_j is critical and $w(C_j) \geq 14/15 + 12\varepsilon/5$.*

Proof. (i) Since C_i is not good, its level is less than $5/6$. Let its level be $5/6 - x$ for some $0 < x \leq 1/6$. Then each item in the 3^+-bin C_j is larger than $1/6 + x$, so

$$r(C_i) + b(C_j) \geq \frac{6}{5}\left(\frac{5}{6} - x\right) + 3 \cdot \frac{2}{5}x = 1.$$

(ii) We follow the proof of (i), and we now get a weaker bound as follows:

$$r(C_i) + b(C_j) \geq \frac{6}{5}\left(\frac{5}{6} - x\right) + 2 \cdot \frac{2}{5}x = 1 - \frac{2}{5}x \geq \frac{14}{15}\,.$$

(iii) Items in C_j must have size more than $1/3 - \varepsilon/2$. Additionally, at least one of them must have size more than $1/3$. Then

$$r(C_i) + b(C_j) \geq \frac{6}{5}\left(\frac{2}{3} + \frac{\varepsilon}{2}\right) + \frac{2}{5}\left(\frac{1}{6} - \frac{\varepsilon}{2}\right) + \frac{1}{15} = \frac{14}{15} + \frac{2\varepsilon}{5} \geq \frac{14}{15} + \frac{2\varepsilon}{5}\,.$$

(iv) C_j contains exactly two items each of size larger than $1/3 + \varepsilon$, so it is a 2-bin and therefore must be critical. Its weight is at least $w(C_j) = r(C_j) + b(C_j) > 12/15 + 12\varepsilon/5 + 2/15$. ■

Lemma 4.9 *If $\tau = 0$, then*

$$w(\mathcal{C}) - |\mathcal{C}| \geq -\frac{2}{5}\,. \tag{6}$$

If the level of $C_{|\mathcal{C}|}$ is at least $2/3$ then

$$w(\mathcal{C}) - |\mathcal{C}| \geq \frac{-3 - \tau}{15}\,. \tag{7}$$

If $\tau > 0$ and the level of $C_{|\mathcal{C}|}$ is $2/3 - \varepsilon$ for $\varepsilon > 0$ then

$$w(\mathcal{C}) - |\mathcal{C}| \geq \frac{-3 - \tau}{15} + \left(\frac{2}{5}\tau - \frac{8}{5}\right)\varepsilon. \tag{8}$$

Proof. First note, that in case $|\mathcal{C}| = 0$, set $\mathcal{C}$ is empty, so $\tau = 0$ and (6) and (7) hold trivially.

If $|\mathcal{C}| = 1$, again $\tau = 0$, and both of (6) and (7) follow from Lemma 4.3(iii) by considering the weight of the only bin in $\mathcal{C}$. Thus let us suppose that $|\mathcal{C}| \geq 2$.

To bound $w(\mathcal{C})$, we apply Lemma 4.8 $|\mathcal{C}|-1$ times for $j = 2, \ldots, |\mathcal{C}|$, typically with $i = j-1$, and to the sum of the inequalities we add the regular weight of $C_{|\mathcal{C}|}$. We use Lemma 4.8(i) $|\mathcal{C}| - \tau - 1$ times, for all pairs of consecutive bins where the second bin is a 3^+-bin.

In case $\tau = 0$ we add these $|\mathcal{C}| - 1$ inequalities; then we use $\text{LEVEL}(C_{|\mathcal{C}|}) > 1/2$ and $r(C_{|\mathcal{C}|}) \geq \frac{6}{5} \cdot \frac{1}{2} = \frac{3}{5}$ to obtain (6) and $r(C_{|\mathcal{C}|}) \geq \frac{6}{5} \cdot \frac{2}{3} = \frac{4}{5}$ to obtain (7).

It remains to deal with the case when $\tau > 0$, meaning at least one bin in $C_2, \ldots, C_{|\mathcal{C}|}$ is critical. The situation is a bit more complicated if there is a bin of size smaller than $2/3$. According to this, we will distinguish three cases.

Case 1: *C_i has level at least $2/3$ for $i = 1, \ldots, |\mathcal{C}|$.* We need to prove (7). We apply Lemma 4.8(ii) τ times, and for the final bin use that $r(C_{|\mathcal{C}|}) \geq \frac{6}{5} \cdot \frac{2}{3} = \frac{4}{5}$. Thus we get

$$w(\mathcal{C}) \geq (|\mathcal{C}| - \tau - 1) + \frac{14}{15}\tau + \frac{4}{5} = |\mathcal{C}| - \frac{1}{5} - \frac{1}{15}\tau,$$

and (7) holds.

Case 2: *C_k has level $2/3 - \varepsilon$ for some $1 \leq k < |\mathcal{C}|$ and $\varepsilon > 0$.* Note that k is unique, as any following bin must contain two items larger than $1/3 + \varepsilon$. (We can also conclude from this that $\varepsilon < 1/6$.) Again, we need to prove (7).

We use Lemma 4.8(iv) to bound the total weight of C_j, for any $j > k$ (no amortization here), i.e. we use this inequality $|\mathcal{C}| - k$ times, where $|\mathcal{C}| > k$ by case assumption. We get the regular weight of C_k which is $4/5 - 6\varepsilon/5$.

The number of applications of Lemma 4.8(ii) is now $\tau - (|\mathcal{C}| - k)$, since all bins in $\mathcal{C}$ following C_k are critical. As previously noted, we use Lemma 4.8(i) $|\mathcal{C}| - \tau - 1$ times. We get

$$\begin{aligned} w(\mathcal{C}) &\geq |\mathcal{C}| - \tau - 1 + \frac{14}{15}(\tau - (|\mathcal{C}| - k)) + (|\mathcal{C}| - k)\left(\frac{14}{15} + \frac{12}{5}\varepsilon\right) + \left(\frac{4}{5} - \frac{6}{5}\varepsilon\right) \\ &= |\mathcal{C}| - \tau - 1 + \frac{4}{5} + \frac{14}{15}\tau + (|\mathcal{C}| - k)\frac{12}{5}\varepsilon - \frac{6}{5}\varepsilon \\ &\geq |\mathcal{C}| - \frac{1}{5} - \frac{\tau}{15}. \end{aligned}$$

using $|\mathcal{C}| - k \geq 1$ in the last inequality, and (7) holds.

Case 3: *$C_{|\mathcal{C}|}$ has level $2/3 - \varepsilon$ for some $\varepsilon > 0$.* Suppose $C_{|\mathcal{C}|}$ is not critical. Then an earlier bin is critical since $\tau > 0$, but then any bin following that critical bin contains only items larger than $1/4$, meaning that $C_{|\mathcal{C}|}$ would have level more than $3/4$ if it contained three items. So $C_{|\mathcal{C}|}$ contains two items and is critical. (Since this is the last critical bin, it could even be that $\varepsilon > 1/6$.) We need to show (8). Since $C_{|\mathcal{C}|}$ contains at least two items, it contains an item of size at most $1/3 - \varepsilon/2$ and thus C_i has level at least $2/3 + \varepsilon/2$ for all $i = 1, \ldots, |\mathcal{C}| - 1$.

We use Lemma 4.8(iii) $\tau - 1$ times, namely for every pair of consecutive bins where the second bin is critical apart from the last such pair, which involves $C_{|\mathcal{C}|}$. We apply Lemma 4.8(i) $|\mathcal{C}| - \tau - 1$ times as usual, noting that $j < |\mathcal{C}|$ in every pair for which this lemma is applied. Finally, we apply Lemma 4.8(ii) once for $j = |\mathcal{C}|$. We note that $r(C_{|\mathcal{C}|}) = 4/5 - 6\varepsilon/5$, and obtain

$$\begin{aligned} w(\mathcal{C}) &\geq |\mathcal{C}| - \tau - 1 + (\tau - 1)\left(\frac{14}{15} + \frac{2}{5}\varepsilon\right) + \frac{14}{15} + \frac{4}{5} - \frac{6}{5}\varepsilon \\ &= |\mathcal{C}| - \frac{\tau}{15} - \frac{1}{5} + \left(\frac{2}{5}\tau - \frac{8}{5}\right)\varepsilon. \end{aligned}$$

Thus (8) holds. ■

Note that the bound (8) is stronger than the bound (7) only in the case that $\tau > 4$, it is the same if $\tau = 4$, otherwise it is weaker. We are now ready to derive the desired contradiction to (5) in almost all cases.

Lemma 4.10 (i) *If $s \geq 2$ or $\delta_0 = 0$ then $w(I) - FT > -1/3$ (and consequently $FT \leq \frac{5}{3} \cdot \text{OPT}$).*

(ii) *If* $s = \delta_0 = 1$ *then* $w(I) - FT > -7/15$.

Proof. We split the proof into three cases.

Case 1: $\tau = 0$. By Lemma 4.9 combined with the bounds for $\mathcal{D}$ (Lemma 4.4), $\mathcal{S}$ (Lemma 4.5) and $\mathcal{G}$ (Lemma 4.6), we obtain (i) as follows:

$$w(I) - FT > s/3 - 2/5 - \delta_0/3 \geq -2/5\,.$$

Furthermore, for $\delta_0 = 0$ or $s \geq 2$ we get

$$w(I) - FT > 1/3 - 2/5 > -1/3.$$

Case 2: $C_{|\mathcal{C}|}$ has size at least $2/3$ or $\tau \geq 4$. We have $w(\mathcal{C}) - |\mathcal{C}| \geq \frac{-3-\tau}{15}$ by Lemma 4.9. Together with the other bounds we get

$$w(I) - FT > \frac{s}{3} + \frac{-3-\tau}{15} - \frac{\delta_0}{3} = \frac{5s-3-\tau}{15} - \frac{\delta_0}{3} \geq \frac{2s-3}{15} - \frac{\delta_0}{3} \geq -\frac{6}{15}\,,$$

using that $\tau \leq 3s$ by Claim 4.7. Furthermore, using $\delta_0 = 0$ or $s \geq 2$ in the last inequality, we get an improved bound

$$w(I) - FT > -4/15 > -1/3.$$

Case 3: $C_{|\mathcal{C}|}$ has size $2/3 - \varepsilon$ and $0 < \tau \leq 3$. We obtain, using $\varepsilon < 1/6$ (Lemma 4.3),

$$\begin{aligned} w(I) - FT &> \frac{s}{3} + \frac{-3-\tau}{15} + \left(\frac{2}{5}\tau - \frac{8}{5}\right)\varepsilon - \frac{\delta_0}{3} \\ &> \frac{s-\delta_0}{3} + \frac{-3-\tau}{15} + \frac{1}{15}\tau - \frac{4}{15} = \frac{s-\delta_0}{3} - \frac{7}{15} \geq -\frac{7}{15}\,. \end{aligned}$$

Again, using $\delta_0 = 0$ or $s \geq 2$ in the last inequality, we get an improved bound $w(I) - FT > 1/3 - 7/15 > -1/3$. ■

Lemma 4.10 shows that FT is 5/3-competitive if each special bin contains a large item, except for the single remaining case $s = \delta_0 = 1$, where we have to work a bit harder.

4.1 The case $s = \delta_0 = 1$

In this remaining case, we typically achieve a contradiction by computing the total size of the instance and also by carefully counting the number of items of size at least 1/4. To be able to count these items, we first revisit Lemma 4.10 to further restrict the remaining case and prove that all optimal bins contain a large item. Later in the proof we also use some of the specific choices of FT, namely the fact that a special bin is matched to the last critical bin and also the exact timing of the creation of the first two special bins.

For $s = 1$, using Lemma 4.2(i) and Lemma 4.10, we have

$$FT < w(I) + \frac{7}{15} \leq \frac{5}{3}\text{OPT} + \frac{7}{15}.$$

If $\textsc{Opt} \not\equiv 1 \pmod 3$ then this and the integrality of $\textsc{Opt}$ and FT is enough to conclude that $FT \leq \frac{5}{3}\textsc{Opt}$. Thus in the remaining case $\textsc{Opt} = 3k+1$ for some $k \geq 0$; fix such a k. We have

$$FT < \frac{5}{3}(3k+1) + \frac{7}{15} = 5k + 2 + \frac{2}{15}, \tag{9}$$

implying that $FT \leq 5k+2$. In fact, we have $FT = 5k+2$ since $(5k+1)/(3k+1) < 5/3$ for $k \geq 0$.

Claim 4.11 *Each optimal bin contains a large item, so there are exactly* $3k+1$ *large items in* I*. There are at most* $3k+1$ *items with size in* $[1/4, 1/2]$ *in* I*.*

Proof. If there is an optimal bin without a large item, then by comparing Lemma 4.2(ii) to Lemma 4.2(i), $w(I)$ decreases by 4/15 compared to (9), implying that $FT < 5k+2$, i.e., $FT \leq 5k+1$. It follows immediately that each optimal bin can contain at most one item of size in $[1/4, 1/2]$. ■

There is a large item in the special bin, but no large item in D_0. Using Claim 4.11 and Lemma 4.3(ii), the final packing of FT therefore has the following bins:

- one special bin with a large item, and
- $3k+1$ bins that are either regular 2^+-bins with a large item or 1-bins (including D_0), and
- $2k$ bins that are either interesting or quick (i.e., regular 2^+-bins without large items).

Since $s = 1$, we have at least three interesting bins in the final packing by Lemma 2.8. Thus $k \geq 2$. Also note that each quick bin contains at least two items with size in $[1/4, 1/2]$, as its first two items are not large and have total size at least 3/4.

Claim 4.12 *The first interesting bin in the final packing has level more than* 3/4*.*

Proof. Suppose for a contradiction that the first interesting bin has level at most 3/4. Using the First Fit property, the remaining interesting bins contain each two items with size in $[1/4, 1/2]$, as do all the quick bins. Thus the number of items with size in $[1/4, 1/2]$ in interesting and quick bins is at least $2(2k-1)$, there is one such item in D_0 (Lemma 4.3(i)) and one in the special bin by Lemma 2.10. Altogether we have $4k$ such items, contradicting Claim 4.11, as $k \geq 2$ implies $4k > 3k+1$. ■

Claim 4.13 *The final packing of FT contains at least four interesting bins with level at most* 3/4*. At least three of these bins are critical and contain two items with size in* $[1/4, 1/2]$*.*

Proof. Suppose to the contrary that there are at most three interesting bins with level at most 3/4. We apply Lemma 2.9 to the three interesting or quick bins with lowest level to conclude that their total size is at least 2; note that the total number of the considered bins is $2k \geq 4$, so we indeed can apply the lemma. There are $3k+1$ bins with large items and D_0 with average level more than 1/2 by Lemma 2.11. All other bins, i.e., the special bin and $2k-3$ interesting and quick bins, have level at least 3/4. Thus we get for the total size that $\textsc{Opt} \geq \textsc{size} > 2 + \frac{1}{2}(3k+1) + \frac{3}{4}(2k-2) = 3k+1 = \textsc{Opt}$, a contradiction. We conclude that there are at least four interesting bins with level at most 3/4.

Each of these four interesting bins apart from possibly the first one is critical, as each bin after the first contains only items larger than 1/4. It can therefore receive only two items, none of them large, as otherwise its level would be more than 3/4. ■

Claim 4.14 *The final packing of FT contains exactly five interesting bins.*

Proof. By Claim 4.12 and Claim 4.13, there are at least five interesting bins.

Consider the time when the last interesting bin is created by packing the second item into it and making it critical. At this time, by Claim 4.13 at least two of the interesting bins are critical, excluding the new one. At most one of the critical bins is matched as $s \leq 1$, thus there is an unmatched critical bin. By rules of FT and using $s \leq 1$ again, in such a case, if a special bin was not created, the number of interesting bins including the new one is at most $\max(3, 4s+1) \leq 5$. ■

Claim 4.15 *We have* $k = 3$.

Proof. By Claim 4.14, there are exactly five interesting bins, so $2k \geq 5$ and $k \geq 3$. By Claim 4.13, there are at least three critical bins, each containing two items with size in $[1/4, 1/2]$. There are $2k - 5$ quick bins, each containing two such items as well. Finally, there is one such item in the special bin and one in D_0. Altogether the number of items with size in $[1/4, 1/2]$ is at least $6 + 2(2k-5) + 2 = 4k - 2$. Using Claim 4.11, we have $4k - 2 \leq 3k + 1$, which yields $k \leq 3$. ■

Thus in the remaining case $\text{OPT} = 3k + 1 = 10$ and FT uses $5k + 2 = 17$ bins. We calculate the total size of the bins to obtain a final contradiction.

We note that the match of the special bin is not the first interesting bin: Claim 4.13 guarantees that one of the critical bins with final level at most $3/4$ is among the at least three interesting bins present when the special bin is created; it is not the first interesting bin. As FT always use the last critical bin as a match, it does not use the first interesting bin in this case.

The first interesting bin has level more than $3/4$ by Claim 4.12. The total size of the special bin and its match is more than $3/2$ by Lemma 2.10. The total size of the remaining three interesting bins is more than 2 by Lemma 2.9. The single quick bin has level at least $3/4$. Finally the total size of the 10 remaining bins, i.e., 9 bins with large items and D_0, is more than 5 by Lemma 2.11 Altogether the total size is more than $3/4 + 3/2 + 2 + 3/4 + 5 = 10 = \text{OPT}$, a contradiction.

Having shown a contradiction in all cases, we conclude the following.

Theorem 4.16 *If every special bin has a large item after all items have been packed, then FT* $\leq 5/3 \cdot \text{OPT}$.

5 There exists a special 1-bin

In this case we again reach a contradiction by comparing the size packed into bins of FT to OPT. This is based on two crucial facts. First, there are many more interesting bins than special bins (Observation 5.1). Second, the large items in special bins have size at least $5/8$ (Lemma 5.3). In general, we may have many special 1-bins and need to account for them. It turns out that it is more advantageous to match them, if possible, to 1-bins with large items rather than to use the matching provided by FT. The matched bins may have average level only about $1/2$, but this is balanced out by the fact that the interesting bins have typically average level at least $2/3$. In fact, the frequency of creating the special bins is sufficiently low so that these two quantities balance out for a large s. These ideas leave out some special cases with a small s that need to be examined separately. Our accounting of sizes is very precise to limit the number of these small cases.

We start by the following fact that follows immediately from the description of FT and Lemma 2.8.

Observation 5.1 *At any moment of the run of FT, if there are $s \geq 1$ special bins, then there are at least $4(s-1)+1 = 4s-3$ interesting bins. If $s=1$, then there are at least 3 interesting bins.*

Let q denote the total number of regular 2^+-bins, including the bins with large items, in the final packing of FT. As these bins include all the interesting bins, Observation 5.1 implies that

$$q = 4s - 3 + k$$

for some integer $k \geq 0$; fix this k.

We now write FT in two forms. First, as all the regular bins are accounted in $q + \delta$, we have FT$= q + s + \delta = 5s - 3 + k + \delta$. Second, fix an integer $x \geq 0$ such that FT$= \frac{5}{3}\textsc{Opt} + 1/3 + x/3$; such an x exists as we assume (3). Now, equating these two forms multiplied by 3, and rearranging we obtain

$$5\textsc{Opt} = 15s - 10 + 3k + 3\delta - x. \tag{10}$$

Let us also denote the number of special 1-bins and 2-bins by s_1 and s_2, respectively. Recall that $\delta = \delta_1 + \delta_0$, where $\delta_0 = 1$ if D_0 and d_0 exist and $\delta_0 = 0$ otherwise. Now we distinguish two cases.

Case a: $\delta_1 \leq s_1$

We redefine the matching as follows. See Figure 3. We match arbitrarily all the δ_1 regular 1-bins with large items to some δ_1 special 1-bins, replacing the matches of these special bins. The remaining $s_1 - \delta_1$ special 1-bins keep their match as defined by FT; note that they are matched to interesting bins. We remove all the matches of the special 2-bins. Finally, if D_0 exists, we match it to an arbitrary unmatched interesting bin; such a bin exists by Observation 5.1, as at most s interesting bins are matched.

Let q' be the number of unmatched regular 2^+-bins. Since exactly $(s_1 - \delta_1) + \delta_0$ regular 2^+-bins are matched, we have

$$q' = q + \delta_1 - \delta_0 - s_1 = 4s + k + \delta_1 - \delta_0 - s_1 - 3\,. \tag{11}$$

We also have s_2 unmatched special 2-bins and $2(s_1 + \delta_0)$ bins in $s_1 + \delta_0$ matched pairs. The crucial property of the matching is the following fact.

Lemma 5.2 *Each pair of matched bins has total level more than 1.*

Proof. For each special 1-bin that is matched with a 1-bin with a large item, the large item does not fit together with the special item by Lemma 2.7. For the other special 1-bins, we apply Lemma 2.10, as the pair is matched by FT. Finally, for the pair with D_0 the bound follows by the First Fit property, as both bins are regular. ■

If $q' \geq 3$, we apply Lemma 2.9 for the q' unmatched 2^+-bins. Each special 2-bin has level more than 3/4 by Lemma 2.10; we actually use only a weaker bound of 1/3 in our calculation. Using Lemma 5.2 for the remaining bins and substituting (11), we get

$$\begin{aligned} 3 \cdot \textsc{Opt} \geq 3 \cdot \textsc{size} &> 3\left(\frac{2}{3}q' + \frac{1}{3}s_2 + s_1 + \delta_0\right) \\ &= 2(4s + k + \delta_1 - \delta_0 - s_1 - 3) + 3s_1 + s_2 + 3\delta_0 \\ &= 9s + 2k + 2\delta_1 + \delta_0 - 6\,. \end{aligned}$$

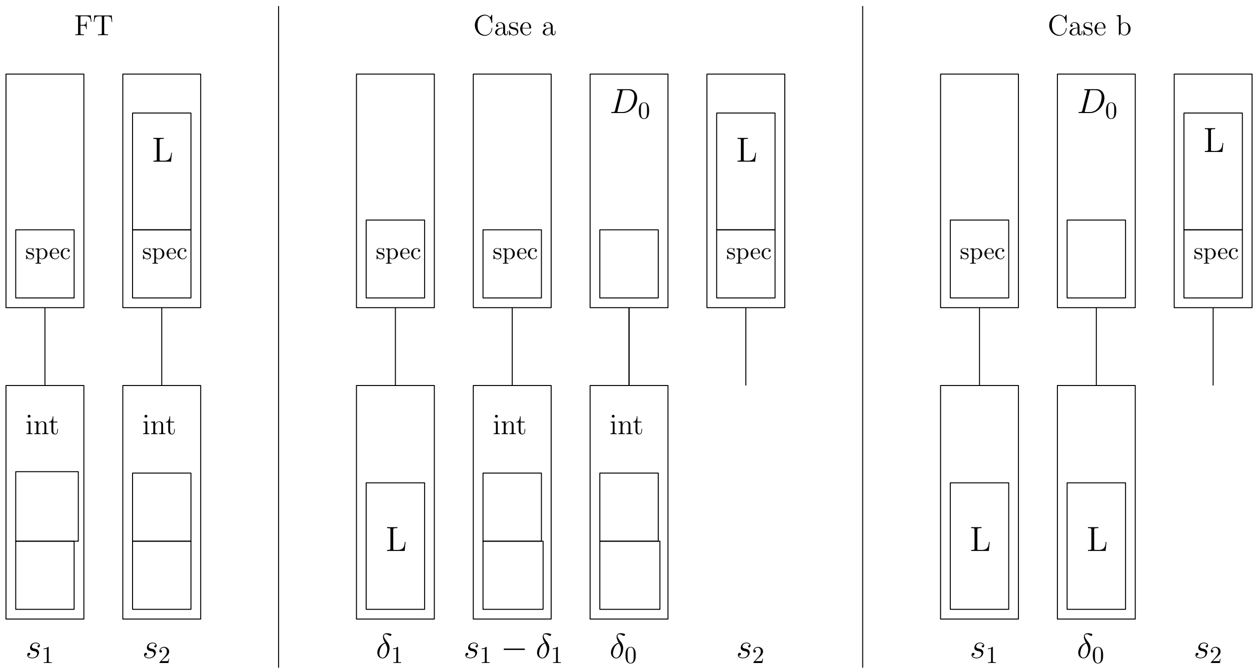

Figure 3: The matchings used in this section compared to the matching of FT. Variables below bins indicate the numbers of these pairs that exist. L refers to large items, spec to special items, int to interesting bins.

Using the integrality of all variables and then multiplying by 5 we get

$$15 \cdot \text{OPT} \geq 45s + 10k + 10\delta_1 + 5\delta_0 - 25$$

which together with the first equality (10) gives

$$45s - 30 + 9k + 9\delta - 3x \geq 45s + 10k + 10\delta_1 + 5\delta_0 - 25$$

or equivalently

$$4\delta_0 \geq k + 3x + \delta_1 + 5\,.$$

Since $\delta_0 \leq 1$ and the right hand side is at least 5, we have a contradiction.

Now consider the remaining case $q' \leq 2$. Using (11) and $s_1 \leq s$, we obtain

$$5 + \delta_0 \geq q' + \delta_0 + 3 = 4s + k + \delta_1 - s_1 \geq 3s + \delta_1\,. \tag{12}$$

It follows that $6 \geq 3s$, i.e., $s \leq 2$.

Case a1: $s = 2$. In this case (12) must hold with equality and thus we have $s_1 = s = 2$, $s_2 = 0$, $q' = 2$, $\delta_1 = 0$, and $\delta_0 = 1$. We have $FT = q' + 2(s + \delta_0) = 8$. The total level of the bins in the three matched pairs is more than 3, the total level of the remaining two regular bins is more than 1 by the First Fit property. Thus SIZE > 4 and OPT ≥ 5, a contradiction with $FT > \frac{5}{3}\text{OPT}$.

Case a2: $s = 1$**.** By Observation 5.1 there are at least three interesting bins. Using Lemma 2.9 for these bins we get $\text{SIZE} > 2$ and thus $\text{OPT} \geq 3$. This in turn implies $FT > \frac{5}{3}\text{OPT} \geq 5$ and thus $FT \geq 6$.

On the other hand, $FT = q' + s_2 + 2(s_1 + \delta_0) \leq 6$, as $s_1 \leq s_1 + s_2 = 1$ and $\delta_0 \leq 1$. Thus these bounds all hold with equality and the packing of FT has two pairs of matched bins and $q' = 2$ regular bins. The total level of the bins in the matched pairs is more than 2, the total level of the remaining two regular bins is more than 1 by the First Fit property. Thus $\text{SIZE} > 3$ and $\text{OPT} \geq 4$, a contradiction with $FT > \frac{5}{3}\text{OPT}$.

This completes the proof of Case a, as we have obtained a contradiction in all subcases.

Case b: $\delta_1 \geq s_1 + 1$

In this case we discard the matching defined by FT completely. Instead, we match all special 1-bins and also D_0, if it exists, to regular 1-bins with large items arbitrarily; the case condition guarantees that this is possible. See Figure 3. Each one of the $s_1 + \delta_0$ pairs of matched bins has total level more than 1 by Lemma 2.7, or by the First Fit property in the case of D_0.

To bound the level of the remaining unmatched regular 1-bins with large items we use the following lemma.

Lemma 5.3 *If there exists a special 1-bin after all items have been packed, then every large item in a 1-bin has size more than* 5/8.

Proof. This follows from Lemma 2.7 and Lemma 2.10. ■

As there are at least three interesting bins by Observation 5.1, we have $q \geq 3$ and we apply Lemma 2.9 for these bins. Each special 2-bin has level more than 3/4 by Lemma 2.10. Combining all the bounds, we get

$$\begin{aligned}
24 \cdot \text{OPT} \geq 24 \cdot \text{SIZE} &> 24\left(\frac{2}{3}q + \frac{3}{4}s_2 + \frac{5}{8}(\delta_1 - s_1 - \delta_0) + s_1 + \delta_0\right) \\
&= 24\left(\frac{2}{3}q + \frac{3}{8}s_1 + \frac{3}{4}s_2 + \frac{5}{8}\delta_1 + \frac{3}{8}\delta_0\right) \\
&= 16(4s_1 + 4s_2 - 3 + k) + 9s_1 + 18s_2 + 15\delta_1 + 9\delta_0 \\
&= 73s_1 + 82s_2 + 16k + 15\delta_1 + 9\delta_0 - 48\,.
\end{aligned}$$

By the integrality of all the variables we obtain

$$24 \cdot \text{OPT} \geq 73s_1 + 82s_2 + 16k + 15\delta_1 + 9\delta_0 - 47.$$

Together with (10), this gives

$$\begin{aligned}
120 \cdot \text{OPT} &= 24(15s - 10 + 3k + 3\delta - x) \\
&= 360s_1 + 360s_2 + 72k + 72\delta_1 + 72\delta_0 - 240 - 24x \\
&\geq 5(73s_1 + 82s_2 + 16k + 15\delta_1 + 9\delta_0 - 47) \\
&= 365s_1 + 410s_2 + 80k + 75\delta_1 + 45\delta_0 - 235\,.
\end{aligned}$$

After rearranging we obtain

$$27\delta_0 \geq 5s_1 + 50s_2 + 5 + 8k + 3\delta_1 + 24x\,.$$

This implies that we have $\delta_0 = 1$ and $x = s_2 = 0$. After substituting, the inequality is

$$22 \geq 5s_1 + 8k + 3\delta_1 = 5(s_1 + k) + 3(k + \delta_1)\,. \tag{13}$$

Examining (10), using $x = 0$ and integrality of all variables, it follows that $k + \delta$ is divisible by five. Since $\delta_0 = 1$, we have $k + \delta > 0$. Also, $k + \delta \geq 10$ would imply $k + \delta_1 \geq 9$ contradicting (13). Thus the only possibility is $k + \delta = 5$ and thus $k + \delta_1 = 4$. Substituting back into (13), we get that

$$2 \geq s_1 + k,. \tag{14}$$

For $s = s_1 \leq 1$ we would get $q = 4s + k - 3 = 3s - 3 + (s + k) \leq 3 - 3 + 2 = 2$ contradicting $q \geq 3$ from Observation 5.1. Thus (14) implies $s_1 = s = 2$ and $k = 0$.

The only remaining case is therefore $s_2 = x = k = 0$, $s_1 = 2$, $\delta_0 = 1$, $\delta_1 = 4$, and $q = 4s+k-3 = 5$. Thus $FT = q + s + \delta = 12$. From (10) we obtain $\text{Opt} = 7$.

Since we have at least five interesting bins when the second special bin is created, all five 2^+-bins are interesting. In particular there is no 2^+-bin with a large item. Thus all the large items are in 1-bins and there are four of them, as $\delta_1 = 4$.

Denote the first and second special items c_1 and c_2. As there is no special 2-bin, both special bins are created in Step 4, where instead of creating a critical bin, FT packs the new item in a new bin, and the 1-bin with a smaller item becomes special while the other 1-bin remains regular. Let p_1 resp. p_2 be the item in the regular 1-bin in Step 4 when c_1 resp. c_2 becomes special.

Observe that after c_1 becomes special, we either have exactly three interesting bins, or we have exactly one critical bin (among the more than three interesting bins) that becomes matched. This holds, as otherwise a special item would have been created earlier. Furthermore, all the interesting bins precede the bins of c_1 and p_1 by Lemma 2.8. When c_2 becomes special, we have five interesting bins and one of them is an unmatched critical bin. Thus another bin must become interesting between the times when c_1 and c_2 become special. We claim that the first such bin is the bin of p_1. No large item is packed with p_1, as all the large items are in 1-bins. So the bin p_1 remains the only regular 1-bin with a small item until another small item is packed into it. Thus no other bin can newly become interesting, and the bin of p_1 becomes interesting as all 2^+-bins are interesting in the end. This shows that at the time when c_2 becomes special, the bin of p_1 is a 2^+-bin, and in particular c_1, p_1, c_2, and p_2 are four distinct items.

We claim that none of c_1, p_1, c_2, and p_2 fit into a 1-bin with a large item. This is true for c_1 and c_2 by Lemma 2.7. However, then it follows also for p_1 and p_2, as they are larger than c_1 and c_2, respectively. Thus c_1, p_1, c_2, p_2 and the four large items have total size larger than 4. The bin of p_1 is not the first interesting bin, as we observed that three other interesting bins precede it. There is another item in the bin of p_1 which does not fit in the first interesting bin by the First Fit property, accounting for volume at least 1. The remaining three interesting bins have total size more than 2 by Lemma 2.9. Thus $\text{size} > 4 + 1 + 2 = 7$, contradicting $\text{Opt} = 7$.

In all the subcases of Case b we have obtained a contradiction as well. Thus we have shown the following theorem.

Theorem 5.4 *If there exists a special 1-bin after all items have arrived, then* $FT \leq 5/3 \cdot \text{Opt}$.

Combining Theorems 3.1, 4.16 and 5.4 immediately leads to our main result.

Theorem 5.5 *The algorithm FT has absolute competitive ratio* $5/3$.

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