Two for One: Tight approximation of 2D Bin Packing*

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In this paper, we study the two-dimensional geometrical bin packing problem (2DBP): given a list of rectangles, provide a packing of all these into the smallest possible number of unit bins without rotating the rectangles. Beyond its theoretical appeal, this problem has many practical applications, for example in print layout and VLSI chip design.

We present a 2-approximate algorithm, which improves over the previous best known ratio of 3, matches the best results for the problem where rotations are allowed and also matches the known lower bound of approximability. Our approach makes strong use of a PTAS for a related 2D knapsack problem and a new algorithm that can pack instances into two bins if OPT = 1.

Keywords: bin packing; approximation; rectangle packing.

1. Introduction

In recent years, there has been increasing interest in extensions of packing problems such as strip packing [1, 2, 14, 17, 19], knapsack [3, 15, 18] and bin packing [4, 5, 6, 9, 20], to multiple independent criteria (vector packing) or multiple dimensions (geometric packing).

Two-dimensional geometric bin packing, both with and without rotations, is one of the very classical problems in combinatorial optimization and its study has begun several decades ago. This is not only due to its theoretical appeal, but also to a large number of applications, ranging from print and web layout [7] (putting

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all ads and articles onto the minimum number of pages) to office planning (putting a fixed number of office cubicles into a small number of floors), to transportation problems (packing goods into the minimum number of standard-sized containers) and VLSI design [11].

It is easy to see that two-dimensional bin packing without rotations (2DBP) is strongly NP-hard as a generalization of its one-dimensional counterpart, hence the main focus is on algorithms with provable approximation quality.

Consider an algorithm A for 2DBP, and denote for each instance I with $A(I)$ the number of bins A produces and with $\mathrm{OPT}(I)$ the smallest number of bins into which I can be packed. A is an *α-approximation* for 2DBP if $\sup_I\{A(I)/\mathrm{OPT}(I)\} \leq \alpha$ over all instances I, and an *asymptotical α-approximation* if $\limsup_{\mathrm{OPT}(I)\to\infty} A(I)/\mathrm{OPT}(I) \leq \alpha$. A polynomial-time approximation scheme (PTAS) is a family $\{A_\epsilon : \epsilon > 0\}$ of $(1+\epsilon)$-approximation algorithms.

Remark 1. *Assuming $P \neq NP$, 2DBP is $(2-\epsilon)$-inapproximable for all $\epsilon > 0$, since the decision problem "can the n items of the instance be packed into a single bin?" contains the strongly NP-hard problem* 3PARTITION *as a special case (where all n items have the same height $3/n$).*

The best previous result for the problem without rotations was a 3-approximation by Zhang [20]; Harren and van Stee have given another 3-approximation with an improved running time of $O(n \log n)$ [10]. For the case that rotation by 90° is allowed, Harren and van Stee have given a 2-approximation in [9], the same ratio can be achieved using techniques by Jansen and Solis-Oba [14].

As to asymptotical approximation ratios, Bansal and Sviridenko showed in [5] that 2DBP does not admit an asymptotical PTAS. Caprara gave an algorithm with ratio of 1.692 in [6], breaking the important barrier of 2. Bansal, Caprara and Sviridenko improved the rate to 1.526 in [4] for both the problem with and without rotations. Recently, this result was further improved for both problems to an $(1.5+\epsilon)$-approximation with additive constant 69 by Jansen and Prädel [12], for an arbitrary $\epsilon > 0$.

A closely related problem is two-dimensional knapsack: here, every rectangle also has a profit and the objective is to pack a subset of high profit into a constant number (usually one) of target bins. The best currently known results here are a $(2+\epsilon)$-approximation by Jansen and Zhang [16] for the general case, and a PTAS by Jansen and Solis-Oba [13] if all items are squares. For our purposes, the special case that the profit equals the item's area is important. Bansal et al. have shown in [3] that this problem admits a PTAS, and this algorithm is one of the corner stones of the algorithm presented here.

Our contribution We study the geometric two-dimensional bin packing problem without rotations. We are given a list of rectangles (items) $r_1 = (w_1, h_1), \ldots, r_n = (w_n, h_n)$ with all w_i, h_i taken from the interval $(0, 1]$, and the objective is to find a non-overlapping packing of all items into the minimum number of containers (bins) of unit size without rotating the items. The main result of this paper is the following theorem:

Theorem 2. *There is a polynomial-time* 2*-approximation for two-dimensional geometric bin packing.*

This result is achieved using an asymptotic approximation algorithm such as [4, 6] or [12] for large optimal values; smaller (i.e. constant) values are solved by applying the PTAS mentioned above [3], combined with other packing algorithms. If the optimal packing uses only one bin, we conduct a case study, again starting from a packing that covers $1 - \epsilon$ of the bin, and generate a packing into $\mathrm{OPT} + 1 = 2$ bins.

As it turns out, this last case is the most involved one; the following crucial theorem is proven in Sect. 4:

Theorem 3. *There is a polynomial-time algorithm that finds a packing into two bins, provided that a packing into one bin exists.*

2. Definitions

In the following, we consider a bin packing instance specified as a list I of n items $r_1, \dots, r_n$, where each $r_i = (w_i, h_i)$ has height h_i and width w_i taken from the interval $(0, 1]$. A packing into a number k of bins is a mapping

$$p\colon \{r_1, \dots, r_n\} \to \{1, \dots, k\} \times [0, 1) \times [0, 1)$$

that assigns each item's lower left corner a position in one of the bins such that no two items overlap or protude beyond their bin, without rotating the items. For these purposes, we consider an item $r_i = (w_i, h_i)$ at position (x_i, y_i) to be the cartesian product of open-ended intervals $(x_i, x_i + w_i) \times (y_i, y_i + h_i)$.

For this fixed input I, we define the set of *high items* $H := \{r_i : h_i > 1/2\}$ and the set of *wide items* $W := \{r_i : w_i > 1/2\}$. We extend the notion of width and height to sets T of items by setting $w(T) := \sum_{i \in T} w_i$, the *total width of* T and $h(T) := \sum_{i \in T} h_i$, the *total height of* T. The *total area of* T is denoted by $\mathcal{A}(T) = \sum_{i \in T} w_i h_i$.

In many cases, we will pack parts of the instance using the classic 2-approximation for strip packing by Steinberg [19], which we quote without proof:

Theorem 4 (Steinberg) *We can pack a set of items* $\{r_i = (w_i, h_i), i = 1, \dots, n\}$ *into a target area of size* $u \times v$ *if the following conditions hold:*

1. $\max\{w_i : i = 1, \dots, n\} \leq u$,
2. $\max\{h_i : i = 1, \dots, n\} \leq v$,
3. $2 \sum_{i=1}^{n} w_i h_i \leq uv - (2 \max\{w_i : 1 \leq i \leq n\} - u)_+ (2 \max\{h_i : 1 \leq i \leq n\} - v)_+$,

where $(\cdot)_+$ *denotes* $\max\{\cdot, 0\}$.

Corollary 5. *We can pack a set of items* $\{r_i = (w_i, h_i), i = 1, \ldots, n\}$ *into a target area of size* $u \times v$ *if the following conditions hold:*

1. $\max\{w_i : i = 1, \ldots, n\} \leq u$,
2. $\max\{h_i : i = 1, \ldots, n\} \leq v/2$,
3. $2\sum_{i=1}^{n} w_i h_i \leq uv$.

Naturally, this also holds in the symmetrical case of width and height interchanged.

Bansal et al. [3] considered the two-dimensional knapsack problem in which each item $r_i \in I$ has an associated profit p_i and the goal is to maximize the total profit that is packed into a unit-sized bin. Using a technical *Structural Lemma* they derived an algorithm that we call BCJPS algorithm in this paper. They showed the following theorem.

Theorem 6 (Bansal, Caprara, Jansen, Prädel & Sviridenko) *For any fixed* $r \geq 1$ *and* $\delta > 0$, *the BCJPS algorithm returns a packing of value at least* $(1 - \delta)\mathrm{OPT}_{2KP}(I)$ *for instances* I *for which* $p_i/\mathcal{A}(r_i) \in [1, r]$ *for* $r_i \in I$. *The running time of the BCJPS algorithm is polynomial in the number of items.*

Here $\mathrm{OPT}_{2KP}(I)$ denotes the maximal profit that can be packed in a bin of unit size. In the case that $p_i = w_i h_i$ for all items $r_i \in I$ we want to maximize the total packed area. Let $\mathrm{OPT}_{(a,b)}(T)$ denote the maximum area of items from T that can be packed into the rectangle (a, b), where individual items in T do not necessarily fit in (a, b). By appropriately scaling the bin, the items and the accuracy we get the following corollary.

Corollary 7. *For any fixed* $\epsilon > 0$, *the BCJPS algorithm returns a packing of* $I' \subseteq I$ *in a rectangle of width* $a \leq 1$ *and height* $b \leq 1$ *such that* $\mathcal{A}(I') \geq \mathrm{OPT}_{(a,b)}(I) - \epsilon$.

3. Packing instances that have a large optimal value or that fit into a constant number of bins

As mentioned above, Jansen and Prädel have given in [12] an algorithm that for any ϵ and any instance obtains in polynomial time a solution with at most $(1.5 + \epsilon)\mathrm{OPT} + 69$ bins. Let us consider the case that OPT is at least the threshold value $k := 140$. In this case, the algorithm obtains with $\epsilon = 0.005$ a solution that uses at most

$$\begin{aligned}(1.5 + 0.005)\mathrm{OPT} + 69 &= 1.505\mathrm{OPT} + 69k/k = 1.505\mathrm{OPT} + 69k/140 \\ &\leq 1.505\mathrm{OPT} + 0.493k \leq 1.505\mathrm{OPT} + 0.493\mathrm{OPT} < 2\mathrm{OPT}\end{aligned}$$

bins. Hence, we do not need to consider this case explicitly.

In the following we give a brief description of our algorithm that packs the instances I with $2 \leq \mathrm{OPT} < k$ into $2\,\mathrm{OPT}$ bins. Let $\epsilon := 1/(20k^3 + 2)$.

Let $L = \{r_i \mid w_i h_i > \epsilon\}$ be the set of *large* items and let $T = \{r_i \mid w_i h_i \leq \epsilon\}$ be the set of *tiny* items.

Recall the definitions of W and H from Sect. 2. We extend these definitions to the set B of *big items*, which have width and height larger $1/2$ and the set S of *small items*, which have width and height less or equal than $1/2$.

Note that the terms *large* and *tiny* refer to the area of the items whereas *big*, *wide*, *high* and *small* refer to their widths and heights. Also note that, e.g., an item can be tiny and high, or wide and big at the same time.

We guess $\ell = \mathrm{OPT} < k$ and open 2ℓ bins that we denote by $B_1, \ldots, B_\ell$ and $C_1, \ldots, C_\ell$. By *guessing* we mean that we iterate over all possible values for ℓ and apply the remainder of this algorithm on every value. As there are only a constant number of values, this is possible in polynomial time. We assume that we know the correct value of ℓ as we eventually consider this value in an iteration. For the ease of presentation, we also denote the sets of items that are associated with the bins by $B_1, \ldots, B_\ell$ and $C_1, \ldots, C_\ell$. We will ensure that the set of items that is associated with a bin is feasible and a packing is known or can be computed in polynomial time. To do this we use the following corollary from Theorem 4 for some of these sets by Jansen and Zhang [16].

Corollary 8 (Jansen & Zhang) *If the total area of a set X of items is at most $1/2$ and there are no wide items (except a possible big item) then the items in X can be packed into a bin.*

Obviously, this corollary also holds for the case that there are no high items (except a possible big item). This corollary is an improvement upon Theorem 4 if there is a big item in T as in this case Theorem 4 would give a worse area bound.

Let I_i^* be the set of items in the i-th bin in an optimal solution. We assume w.l.o.g. that $\mathcal{A}(I_i^*) \geq \mathcal{A}(I_j^*)$ for $i < j$. Then we have

$$\mathcal{A}(I) = \mathcal{A}(I_1^*) + \cdots + \mathcal{A}(I_\ell^*) \leq \ell \cdot \mathcal{A}(I_1^*). \tag{2}$$

In a first step, we guess the assignment of the large items to bins. Using this assignment and the BCJPS algorithm we pack a total area of at least $\mathcal{A}(I_1^*) - \epsilon$ into B_1 and keep C_1 empty. This step has the purpose of providing a good area bound for the first bin and leaving a free bin for later use. We ensure that the large items that are assigned to B_1 are actually packed. For all other bins we reserve B_i for the wide and small items (except the big items) and C_i for the high and big items for $i = 2, \ldots, \ell$. This separation enables us to use Steinberg's algorithm (Corollary 8) to pack up to half of the bins' area. In detail, the first part of the algorithm works as follows.

1. Guess $L_i = I_i^* \cap L$ for $i = 1, \ldots, \ell$.
2. Apply the BCJPS algorithm on $L_1 \cup T$ with $p_i = \mathcal{A}(r_i)(1/\epsilon + 1)$ for $r_i \in L_1$, $p_i = \mathcal{A}(r_i)$ for $r_i \in T$ and an accuracy of $\epsilon^2/(1+\epsilon)$. Assign the output to bin B_1 and keep an empty bin C_1.

3. For $i = 2, \ldots, \ell$, assign the wide and small items of L_i to B_i (omitting big items) and assign the high and big items of L_i to C_i. That is, $B_i = L_i \setminus H$ and $C_i = L_i \cap H$.
4. For $i = 2, \ldots, \ell$, greedily add tiny wide items from $T \cap W$ by non-increasing order of width to B_i as long as $\mathcal{A}(B_i) \leq 1/2$ and greedily add tiny high items from $T \cap H$ by non-increasing order of height to C_i as long as $w(C_i) \leq 1$.

The bins $B_2, \ldots, B_l$ can be packed as follows. If no tiny items have been added in Step 4, a packing of B_i can be derived by an exact algorithm as the number of items in B_i is bounded by $1/\epsilon$ (since all items in B_i are large) and B_i is feasible since $B_i \subseteq L_i \subseteq I_i^*$. Otherwise, $\mathcal{A}(B_i) \leq 1/2$ and thus Corollary 8 can be applied (as there are no high items in B_i). The bins $C_2, \ldots, C_\ell$ can be packed with a simple stack as they contain only high items of total width at most 1. Observe that in Step 4 we only add to a new bin B_i if the previous bins contain items of total area at least $1/2 - \epsilon$ and we only add to a new bin C_i if the previous bins contain items of total width at least $1 - 2\epsilon$ (as the width of the tiny high items is at most 2ϵ) and thus of total area at least $1/2 \cdot (1 - 2\epsilon) = 1/2 - \epsilon$. After the application of this first part of the algorithm, some tiny items $T' \subseteq T$ might remain unpacked. Note that if $\mathcal{A}(B_\ell) < 1/2 - \epsilon$, then there are no wide items in T' and if $\mathcal{A}(C_\ell) < 1/2 - \epsilon$ then there are no high items in T' (as these items would have been packed in Step 4). We distinguish different cases to continue the packing according to the filling of the last bins B_ℓ and C_ℓ.

In the following we show that we actually ensure that the large items that are assigned to B_1 are packed into this bin. First note that Theorem 6 can be applied for $r = 1/\epsilon + 1$ as $p_i/\mathcal{A}(r_i) \in \{1, 1/\epsilon + 1\}$ for all items in $L_1 \cup T$. Now it is easy to see that L_1 is packed since $p_i > 1 + \epsilon$ for $r_i \in L_1$, whereas $p(\widetilde{T}) = \mathcal{A}(\widetilde{T}) \leq 1$ for any feasible set $\widetilde{T} \subseteq T$. Thus $L_1 = I_1^* \cap L = B_1 \cap L$. Furthermore, for the set of packed tiny items $B_1 \cap T$ we have

$$\mathcal{A}(B_1 \cap T) \geq \mathcal{A}(I_1^* \cap T) - \epsilon$$

since $(1/\epsilon + 1)\mathcal{A}(B_1 \cap L) + \mathcal{A}(B_1 \cap T) = p(B_1)$ and

$$\begin{aligned}
p(B_1) &\geq \Big(1 - \frac{\epsilon^2}{1+\epsilon}\Big)\mathrm{OPT}(L_1 \cup T) && \text{by Theorem 6} \\
&\geq \Big(1 - \frac{\epsilon^2}{1+\epsilon}\Big)\Big[\Big(\frac{1}{\epsilon} + 1\Big)\mathcal{A}(I_1^* \cap L) + \mathcal{A}(I_1^* \cap T)\Big] \\
&\geq \Big(\frac{1}{\epsilon} + 1\Big)\mathcal{A}(I_1^* \cap L) + \mathcal{A}(I_1^* \cap T) - \frac{\epsilon^2}{1+\epsilon}\Big(\frac{1}{\epsilon} + 1\Big) \\
&\qquad \text{as } \Big(\frac{\epsilon^2}{1+\epsilon}\Big)\Big[\Big(\frac{1}{\epsilon} + 1\Big)\mathcal{A}(I_1^* \cap L) + \mathcal{A}(I_1^* \cap T)\Big] < \frac{\epsilon^2}{1+\epsilon}\Big(\frac{1}{\epsilon} + 1\Big) \\
&= \Big(\frac{1}{\epsilon} + 1\Big)\mathcal{A}(B_1 \cap L) + \mathcal{A}(I_1^* \cap T) - \epsilon.
\end{aligned}$$

Thus we have

$$\mathcal{A}(B_1) \geq \mathcal{A}(I_1^*) - \epsilon. \tag{3}$$

Now we are ready to start with the case analysis.

Case 1. $\mathcal{A}(B_\ell) < 1/2 - \epsilon$ and $\mathcal{A}(C_\ell) < 1/2 - \epsilon$.

In this case T' does not contain any wide or high items as these items would have been packed to B_ℓ or C_ℓ. Greedily add items from T' into all bins except B_1 as long as the bins contain items of total area at most 1/2. After adding the items from T', either all items are assigned to a bin (and can thus be packed) or each bin contains items of total area at least $1/2 - \epsilon$ and we packed a total area of at least

$$\begin{aligned}
A &\geq \mathcal{A}(B_1) + (2\ell - 1)\Big(\frac{1}{2} - \epsilon\Big) \\
&\geq \mathcal{A}(I_1^*) + \ell - \frac{1}{2} - 2\ell\epsilon && \text{by Inequality 3} \\
&\geq \mathcal{A}(I_1^*) + \ell\mathcal{A}(I_1^*) + (\ell - 1)(1 - \mathcal{A}(I_1^*)) + 1 - \mathcal{A}(I_1^*) - \frac{1}{2} - 2\ell\epsilon \\
&\geq \ell\mathcal{A}(I_1^*) + \frac{1}{2} - 2\ell\epsilon && \text{as } \ell \geq 2 \text{ and } 1 - \mathcal{A}(I_1^*) \geq 0 \\
&> \ell\mathcal{A}(I_1^*) && \text{as } \epsilon < \frac{1}{4\ell}.
\end{aligned}$$

Since this contradicts Inequality 2, all items are packed.

Case 2. $\mathcal{A}(B_\ell) \geq 1/2 - \epsilon$ and $\mathcal{A}(C_\ell) \geq 1/2 - \epsilon$.

In this case T' might contain wide and high items. On the other hand the bin C_1 is still available for packing. We use the area of the items in bin C_ℓ to bound the total area of the packed items and (with a similar calculation as in Case 1) we get a packed area of at least $A \geq \mathcal{A}(B_1) + \mathcal{A}(C_\ell) + (2\ell - 3)(1/2 - \epsilon) \geq \ell\mathcal{A}(I_1^*) + \mathcal{A}(C_\ell) - 1/2 - (2\ell - 2)\epsilon$. As $\mathcal{A}(I) \leq \ell\mathcal{A}(I_1^*)$ (Inequality 2) we get

$$\begin{aligned}
&\mathcal{A}(T') \leq \mathcal{A}(I) - A \leq \frac{1}{2} + (2\ell - 2)\epsilon - \mathcal{A}(C_\ell) \quad \text{and hence} && (4) \\
&\mathcal{A}(T') \leq (2\ell - 1)\epsilon && \text{as } \mathcal{A}(C_\ell) \geq 1/2 - \epsilon \quad (5)
\end{aligned}$$

Assume that $\mathcal{A}(C_\ell) < 1/2 + (2\ell - 2)\epsilon$ as otherwise $T' = \emptyset$ by Inequality 4.

We consider the set $\widehat{H} = \{r_i \in C_\ell \mid h_i \leq 3/4\}$. If $w(\widehat{H}) \geq (4\ell - 3)\epsilon$ then remove $\widehat{H}$ from C_ℓ and pack it in a stack in C_1 instead. As we now have $\mathcal{A}(C_\ell) < 1/2 - (2\ell - 1)\epsilon$ and $\mathcal{A}(T' \setminus W) \leq \mathcal{A}(T') \leq (2\ell - 1)\epsilon$ by Inequality 5, we can pack $T' \setminus W$ together with C_ℓ. The remaining items $T' \cap W$ have total height at most $2(2\ell - 1)\epsilon$ and thus fit above $\widehat{H}$ into C_1.

Otherwise, there is no item $r' = (w', h')$ in C_ℓ with $h' \leq 3/4$ and $w' \geq (4\ell - 3)\epsilon$. Let $\widetilde{H} = \{r_i \in C_\ell \cup T' \mid h_i > 3/4\}$. Observe that we have

$$w(\widetilde{H}) \leq \frac{\mathcal{A}(C_\ell \cup T')}{3/4} \leq \frac{4}{3}\Big(\frac{1}{2} + (2\ell - 2)\epsilon + (2\ell - 1)\epsilon\Big) = \frac{2}{3} + \Big(\frac{16}{3}\ell - 4\Big)\epsilon < 1. \tag{6}$$

We take *all* high items from $C_\ell \cup T'$ and order them by non-increasing height. Now pack the items greedily into a stack of width up to 1 and pack this stack into

C_ℓ. We have $w(C_\ell) \geq 1 - (4\ell - 3)\epsilon$ as we bounded the total width of the items from $\widetilde{H}$ in Inequality 6 and thus all further items have width at most $(4\ell - 3)\epsilon$ (as otherwise $\widehat{H} \geq (4\ell - 3)\epsilon$ and we had solved the problem in the previous step). For the remaining items T' we have $h_{\max}(T') \leq 3/4$ and $\mathcal{A}(T') \leq 1/2 - (2\ell - 2)\epsilon - \mathcal{A}(C_\ell) \leq (4\ell - 7/2)\epsilon \leq 1/4$ (by Inequality 4 and as $\mathcal{A}(C_\ell) \geq w(C_\ell)/2 \geq 1/2 - (2\ell - 3/2)\epsilon$ and $\epsilon < 1/(16\ell)$). Thus T' can be packed into bin C_1 using Steinberg's algorithm.

Case 3. $\mathcal{A}(B_\ell) < 1/2 - \epsilon$ and $\mathcal{A}(C_\ell) \geq 1/2 - \epsilon$.
If $w(T' \cap H) \leq 1$ then pack $T' \cap H$ in C_1 and proceed as in Case 1.

The subcase where $w(T' \cap H) > 1$ is the most difficult of all four cases. The challenge that we face is that $w(H)$ can be close to ℓ (which is a natural upper bound) but we can only ensure a packed total width of at least $\ell(1 - 2\epsilon)$ in the bins $C_1, \ldots, C_\ell$. So we have to pack high items into the bins $B_2, \ldots, B_\ell$. We distinguish two further subcases.

1. Assume that there exists $j \in \{2, \ldots, \ell\}$ with $w(L_j \cap H) > 10\ell\epsilon$, i.e., the total width of the items that are large and high and associated with the j-th bin in an optimal packing is large enough such that moving these items away gives sufficient space for the still unpacked high items.

   Go back to Step 3 in the first part of the algorithm and omit separating the items from L_j. Instead we assign the items from L_j to bin B_j and keep C_j free at the moment. Note that L_j admits a packing into a bin as L_j corresponds to the large items in a bin of an optimal solution. Since $|L_j| \leq 1/\epsilon$ we can find such a packing in constant time.

   While greedily adding tiny items in Step 4, we skip B_j for the wide items and we continue packing high items in C_1 after we have filled $C_2, \ldots, C_\ell$. As we moved high items of total width at least $10\ell\epsilon$ to B_j and we can pack high items of total width at least $1 - 2\epsilon$ into each bin, no high items remains after this step. Finally, greedily add remaining tiny items to bins $B_2, \ldots, B_\ell$ except B_j, using the area bound $1/2$.

   Now consider the bins C_1 and C_j. Both contain only tiny items, as we moved the large items from C_j to B_j. We packed the tiny items greedily by height and thus all items in C_j have height greater or equal to any item in C_1. Let h' be greatest height in C_1. Then we have $\mathcal{A}(C_j) \geq h'(1 - 2\epsilon)$. Furthermore, we know that $w(H) > \ell(1 - 2\epsilon)$. Thus we have

   $$\mathcal{A}(H) > (\ell - 1)(1/2 - \epsilon) + h'(1 - 2\epsilon).$$

   If after the modified Step 4 tiny items remain unpacked, then all bins B_i for $i \in \{2, \ldots, \ell\} \setminus \{j\}$ have area $\mathcal{A}(B_i) \geq 1/2 - \epsilon$. By summing up the area of the high items separately we get a total packed area of at least

$$\begin{aligned}
A &\geq \mathcal{A}(B_1) + \overbrace{(\ell-2)\Big(\frac{1}{2}-\epsilon\Big)}^{B_i \text{ for } i\in\{2,\dots,\ell\}\setminus\{j\}} + \mathcal{A}(H) \\
&> \mathcal{A}(B_1) + (\ell-2)\Big(\frac{1}{2}-\epsilon\Big) + (\ell-1)\Big(\frac{1}{2}-\epsilon\Big) + h'(1-2\epsilon) \\
&\geq \mathcal{A}(I_1^*) + \ell - \frac{3}{2} + h' - (2\ell - 2 + 2h')\epsilon && \text{by Inequality 3} \\
&\geq \mathcal{A}(I_1^*) + \ell\mathcal{A}(I_1^*) + (\ell-1)(1-\mathcal{A}(I_1^*)) + 1 - \mathcal{A}(I_1^*) \\
&\quad - \frac{3}{2} + h' - 2\ell\epsilon && \text{as } h' \leq 1 \\
&\geq \ell\mathcal{A}(I_1^*) - \frac{1}{2} + h' - 2\ell\epsilon && \text{as } \ell \geq 2 \text{ and} \\
& && 1 - \mathcal{A}(I_1^*) \geq 0.
\end{aligned}$$

On the other hand we have $A \leq \mathcal{A}(I) \leq \ell\mathcal{A}(I_1^*)$ by Inequality 2. Thus the total area of the remaining items T' is at most $\mathcal{A}(T') \leq 1/2 + 2\ell\epsilon - h'$. If $h' \geq 1/2 + 2\ell\epsilon$ we packed all items.

Otherwise we have $1/2 < h' < 1/2 + 2\ell\epsilon$ and

$$\mathcal{A}(T') \leq 2\ell\epsilon. \tag{7}$$

We will pack T' in C_1 together with the already packed high items. Observe that

$$w(C_1 \cap H) \leq 1 - 8\ell\epsilon \tag{8}$$

as we move high items of total area of at least $10\ell\epsilon$ to B_j and all bins $C_2, \dots, C_\ell$ are filled up to a width of at least $1 - 2\epsilon$.

We pack the remaining items T' into three rectangles $R_1 = (1, 4\ell\epsilon)$, $R_2 = (8\ell\epsilon, 1 - 4\ell\epsilon)$ and $R_3 = (1 - 8\ell\epsilon, 1/2 - 6\ell\epsilon)$ which can be packed in C_1 together with $C_1 \cap H$ as follows—see Fig. 1. Pack the stack of $C_1 \cap H$ in the lower left corner and pack R_3 above this stack. As $h' + h(R_3) \leq 1 - h(R_1)$, R_1 fits in the top of C_1. Finally, pack R_2 in the bottom right corner. This is possible as $h(R_2) \leq 1 - h(R_1)$ and $w(C_1 \cap H) \leq 1 - 8\ell\epsilon = 1 - w(R_3)$ by Inequality 8.

Now pack $T' \cap W$ in a stack in R_1 (which is possible since $h(T' \cap W) \leq 2\mathcal{A}(T') \leq 4\ell\epsilon$ by Inequality 7) and pack all $r_i = (w_i, h_i) \in T'$ with $h_i > 1/2 - 6\ell\epsilon$ in a vertical stack in R_2 (this fits as the total width of items with $h_i > 1/2 - 6\ell\epsilon$ is at most $\mathcal{A}(T')/(1/2 - 6\ell\epsilon) \leq 8\ell\epsilon$ by Inequality 7 and as $6\ell\epsilon \leq 1/4$). Finally, use Steinberg's algorithm to pack the remaining items in R_3. This is possible since $w_{\max} \leq 1/2$, $h_{\max} \leq 1/2 - 6\ell\epsilon$ and

$$\begin{aligned}
2\mathcal{A}(T') \leq 4\ell\epsilon &\leq \frac{1}{2} - 14\ell\epsilon + 96\ell^2\epsilon^2 \\
&= (1 - 8\ell\epsilon)\Big(\frac{1}{2} - 6\ell\epsilon\Big) - (1 - 1 + 8\ell\epsilon)_+\Big(1 - 12\ell\epsilon - \frac{1}{2} + 6\ell\epsilon\Big)_+.
\end{aligned}$$

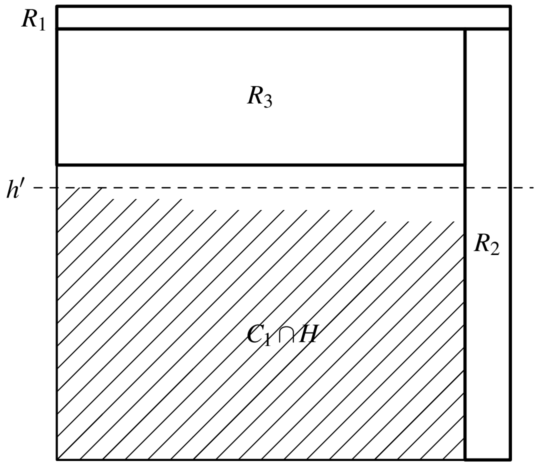


Fig. 1: The three rectangles R_1, R_2 and R_3 for packing T' together with $C_1 \cap H$ in C_1.

This finishes the first case where we assumed that there exists $j \in \{2, \ldots, \ell\}$ with $w(L_j \cap H) > 10\ell\epsilon$.

2. Now assume that we have $w(L_j \cap H) \leq 10\ell\epsilon$ for all $j \in \{2, \ldots, \ell\}$ and in particular $w_i \leq 10\ell\epsilon$ for all items $r_i = (w_i, h_i) \in (L_2 \cup \cdots \cup L_\ell)$. Thus all high items that are not packed in B_1 are thin, i.e., have width at most $10\ell\epsilon$. We use this fact by repacking the high items greedily by non-increasing height in the bins $C_1, \ldots, C_\ell$. Each bin contains high items of total width at least $1 - 10\ell\epsilon$ afterwards. Thus high items of total width at most $10\ell^2\epsilon$ remain unpacked. This is worse than in the previous case but since we repacked all items we can get a nice bound on the height of the unpacked items. Let h' be the smallest height in C_ℓ. Then all items in $C_1, \ldots, C_\ell$ have height at least h' and the remaining items T' have height at most h'. If there is an $i \in \{2, \ldots, \ell\}$ with $\mathcal{A}(B_i) \leq 1 - h' - 10\ell^2\epsilon$ then we can add the remaining items T' to B_i using Steinberg's algorithm. To see this note that $h_{\max}(T' \cup B_i) \leq h'$ and $2\mathcal{A}(T' \cup B_i) \leq 2\mathcal{A}(B_i) + 2(10\ell^2\epsilon)h' \leq 2 - 2h'$ which corresponds to the bound of Theorem 4.

Otherwise for all $i \in \{2, \ldots, \ell\}$ we have $\mathcal{A}(B_i) \geq 1 - h' - 10\ell^2\epsilon$. Then we packed a total area of at least

$$\begin{aligned}
A &\geq \mathcal{A}(B_1) + \mathcal{A}(C_1 \cup \cdots \cup C_\ell) + \mathcal{A}(B_2 \cup \cdots \cup B_\ell) \\
&\geq \mathcal{A}(B_1) + h'\ell(1 - 10\ell\epsilon) + (\ell - 1)(1 - h' - 10\ell^2\epsilon) \\
&\geq \mathcal{A}(I_1^*) + \ell - 1 + h' - (10\ell^3 + 1)\epsilon \\
&> \mathcal{A}(I_1^*) + \ell\mathcal{A}(I_1^*) + \ell(1 - \mathcal{A}(I_1^*)) - 1 + \frac{1}{2} - (10\ell^3 + 1)\epsilon \quad \text{as } h' > \frac{1}{2} \\
&\geq \ell\mathcal{A}(I_1^*) + \frac{1}{2} - (10\ell^3 + 1)\epsilon \quad \text{as } \ell \geq 1 \text{ and} \\
&\qquad 1 - \mathcal{A}(I_1^*) \geq 0.
\end{aligned}$$

And since $1/2 - (10\ell^3 + 1)\epsilon \geq 0$ and $A \leq \mathcal{A}(I) \leq \ell\mathcal{A}(I_1^*)$ by Inequality 2, no item remains unpacked.

Thus in both subcases we are able to derive a feasible packing.

Case 4. $\mathcal{A}(B_\ell) \geq 1/2 - \epsilon$ and $\mathcal{A}(C_\ell) < 1/2 - \epsilon$.
In this case T' contains no high items. If there are also no wide items remaining in T', apply the methods of Case 1. Otherwise we use the following process to free some space in the bins for wide and small items, i.e., $B_2, \ldots, B_\ell$. The idea of the process is to move small items from bins B_i to bins C_i and thereby move the tiny high items $T' \cap H$ further in direction C_ℓ. To do this, let $S_i = L_i \cap S$ be the set of small items in B_i.

Remove the tiny items from $C_2, \ldots, C_\ell$. If there exists an item $r \in S_i \cap B_i$ for some $i \in \{2, \ldots, \ell\}$ then remove r from B_i and add it to C_i, otherwise stop. Adding r to C_i is possible as C_i is a subset of $L_i = I_i^*$ and thus feasible. Add wide items from $W \cap T'$ to B_i until $\mathcal{A}(B_i) \geq 1/2 - \epsilon$ again or $W \cap T' = \emptyset$. Finally, add the high items from $H \cap T'$ to $C_2, \ldots, C_\ell$ in a greedy manner analogously to Step 4 of the first part of the algorithm but using the area bound $\mathcal{A}(C_i) \leq 1/2$. This ensures that all sets C_i can be packed with Steinberg's algorithm (we use Corollary 8 here as there might be big items in C_i). Repeat this process until $S_i \cap B_i = \emptyset$ for all $i \in \{2, \ldots, \ell\}$ or T' contains a high item at the end of an iteration.

There are two ways in which this process can stop. First if we moved all items from S_i to C_i, and second if in the next step a high item would remain in T' after the process. In the first case we have reached a situation as in Case 2 or Case 3, i.e., the roles of the wide and the high items are interchanged and $\mathcal{A}(B_\ell) \geq 1/2 - \epsilon$. Thus by rotating all items and the packing derived so far, we can solve this case analogously to Case 2 or Case 3, depending on $\mathcal{A}(C_\ell)$.

In the second case, let r^* be the item that stopped the process, i.e., if r^* is moved from B_i to C_i for some $i \in \{2, \ldots, \ell\}$, at least one high item would remain in T'. Then, instead of moving r^* to C_i we move r^* to C_1 and add items from T' to C_1 and C_i as long as $\mathcal{A}(C_1) \leq 1/2$ and $\mathcal{A}(C_i) \leq 1/2$. The resulting sets can be packed with Steinberg's algorithm as no item has width greater than 1/2. If after this step still items remain unpacked then a calculation similar to Case 1 gives a total packed area of

$$
\begin{aligned}
A &\geq \mathcal{A}(B_1) + \overbrace{(2\ell-2)\Big(\frac{1}{2}-\epsilon\Big)}^{\text{all bins except } B_1, B_i} + \overbrace{\frac{1}{2}-\epsilon-\mathcal{A}(r^*)}^{\text{bin } B_i} \\
&\geq \ell\mathcal{A}(I_1^*) + \frac{1}{2} - 2\ell\epsilon - \mathcal{A}(r^*) > \ell\mathcal{A}(I_1^*) \qquad \text{since } \mathcal{A}(r^*) \leq 1/4 \text{ and } \epsilon < 1/(8\ell).
\end{aligned}
$$

As we have a contradiction to $A \leq \mathcal{A}(I) \leq \ell\mathcal{A}(I_1^*)$ by Inequality 2, all items are packed.

We showed the following lemma which concludes our presentation of the 2-approximation algorithm for two-dimensional bin packing.

Lemma 9. *There exists a polynomial-time algorithm that, given an instance I with $1 < \mathrm{OPT}(I) < k$, returns a packing in $2\,\mathrm{OPT}(I)$ bins.*

4. Solving with 2 bins for OPT = 1

In this section, we consider the remaining case that there exists a packing of all items into a single bin. We will start off by some general statements that can be shown for packings into a single bin before showing that each instance falls into one of four cases, each of which we consider separately.

Let us first make easy observations about the presence of high and wide items in an instance that admits a packing into one bin. As everywhere in this paper, these results still hold for wide items instead of high items by rotating the construction by 90 degrees.

Remark 10. *We can always fit all high items into a single bin by packing them next to each other in non-increasing order of height, since no two of them fit on top of each other in the optimum.*

Remark 11. *If we can pack a set of items which includes all high items into one bin such that the total area of the packed items is at least $1/2$, we can pack the remainder of the instance into the second bin using Steinberg's algorithm.*

Lemma 12. *Consider some $\gamma \in [0, 1/2)$ and let w the total width of all items of height at least $1-\gamma$. Then, the total height of items of width larger than $\max\{1/2, 1-w\}$ and height less than $1-\gamma$ is at most 2γ.*

Proof. Consider a horizontal line $y = y_0$ in any feasible packing, for any $y_0 \in (\gamma, 1-\gamma)$. Such a line clearly must intersect all items of height at least $1-\gamma$, cf. Fig. 2, which take up total width w. In particular, it cannot intersect any other item of width more than $1-w$, so all these items must be located in the outermost γ of the bin. Since the items are also wide, no two of them could be next to one another, so the total height can be at most 2γ. □

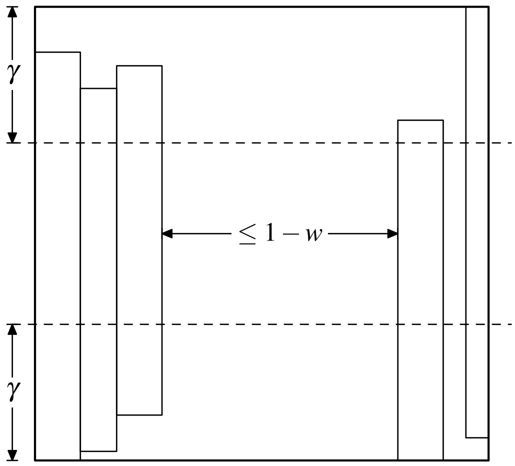


Fig. 2: Items of height $> 1 - \gamma$ limit items of width $> 1 - w$.

Two parameters will appear in the following analysis, a width limit for high items δ and the accuracy ϵ used for the knapsack PTAS Corollary 7. We set

$$\delta := 1/25; \quad \epsilon := \min\{\delta/144, 1/308.4\} = 1/3600\,, \tag{9}$$

but note that the same arguments can be carried out for a slightly larger δ at the expense of a more involved analysis in Subsect. 4.1.

4.1. *Many high or many wide items*

In this section, we consider the case that the subset of high items, i.e. those of height more than 1/2, is comparatively large. (Symmetrically, this also solves the case that many items are wide; again, we only consider high items explicitly.) To be more specific, we want to prove that if the total width of high items $w(H)$ is at least $1 - \delta$, we can pack all items into two bins.

We will show this in the following way: we first pack all high items by Remark 10, and then try to pack additional items so that the total area covered is at least 1/2, at which point we can invoke Remark 11 on the remainder. Note that the high items alone already cover at least $(1 - \delta)/2$, so we need only $\delta/2$ in extra items. We will try this in five different ways corresponding to five classes of items, which in total cover all non-high items. If none of these five succeeds, we know that the area of each class is bounded by a small term in ϵ. In particular, the total unpacked area will then be bounded by 1/2 so we can still use Steinberg's algorithm on the second bin and the unpacked items.

To see this, we first show some technical results:

Observation 13. *Each item r_i satisfies at least one of the following conditions:*

1. $h_i > 1/3$,

2. $h_i \cdot w_i \geq \delta/2$,
3. $h_i \leq \sqrt{\delta/2}$ *and* $w_i \leq 1/2$,
4. $h_i \leq 1/3$ *and* $w_i \leq \sqrt{\delta/2}$,
5. $h_i \leq 1/2$ *and* $w_i > 1/2$.

Proof. Suppose by contradiction, that there is an item r_i that does not satisfy any of the above conditions. It follows that $h_i \leq 1/3$, since otherwise it would satisfy Condition 1. Condition 4 and 5 restricts the width to $w_i \in (\sqrt{\delta/2}, 1/2]$. We have $w_i \leq 1/2$, thus the Condition 3 bounds the height by $h_i \in (\sqrt{\delta/2}, 1/3]$. Therefore, $h_i > \sqrt{\delta/2}$ and $w_i > \sqrt{\delta/2}$ and we have $h_i \cdot w_i \geq \delta/2$. This is a contradiction, since r_i satisfies Condition 2, cf. Fig. 3. □

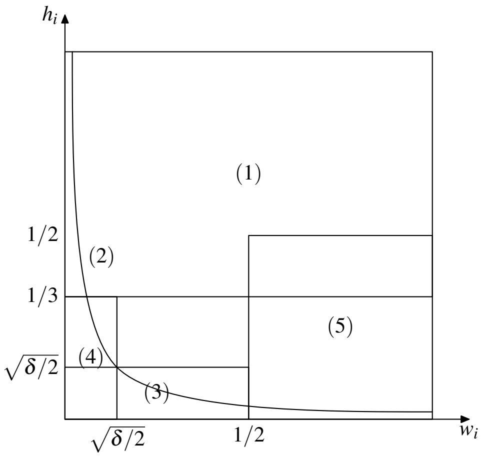


Fig. 3: Schematic representation of the different conditions

Lemma 14. *Given a list of rectangles* $q_1 = (w_1, h_1), \ldots, q_m = (w_m, h_m)$ *of total width at most* 1 *and one extra rectangle* $q' = (w', h')$ *with* $h' \leq 1/2$*, we can either pack these items into one bin or the set* $\{q'\} \cup \{q_i : h_i > 1 - h'\}$ *cannot be packed into a single bin at all.*

Note in particular that we do not require the q_i *to be a subset of the input instance.*

Proof. By reindexing, we assume w.l.o.g. $h_1 \geq h_2 \geq \cdots \geq h_m$. We pack the items at the bottom of the bin in this order, cf. Fig. 4. This is feasible since their total width is at most 1. Assume that placing q' in the top-right corner creates an overlap with a certain q_i. Since $h' < 1/2$, we have $h_1 \geq \cdots \geq h_i > 1 - h' \geq 1/2 \geq h'$, so no two of these could be on top of one another in any feasible packing. However, we

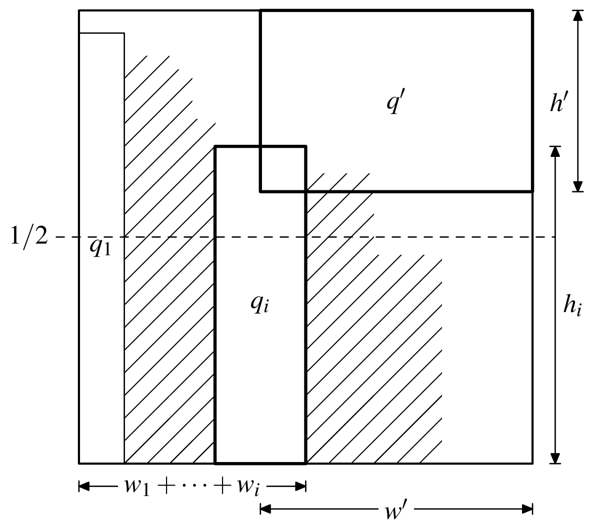


Fig. 4: A high item intersecting the extra item of Lemma 14

have $w_1 + \cdots + w_i > 1 - w'$, so a bin of width 1 does not admit a packing of all these items next to each other either. □

From Remark 11, Remark 10 and Lemma 14, we obtain:

Corollary 15. *If the total area of high items is (at least) $1/2 - \delta/2$, then all other items have individual area at most $\delta/2$, or else we can pack the instance into two bins.*

Note that for purposes of proving Lemma 22, this means that we can restrict ourselves to the case that Case 2 of Observation 13 does not hold true for any other item, i.e. all other items have (individual) area of less than $\delta/2$, and in particular they are bounded in at least one direction by $\sqrt{\delta/2}$.

Similar to Lemma 14, we can show:

Lemma 16. *If the total width of high items $w(H)$ is at least $1 - \delta$, we can pack all high items and leave an empty area sized $(1 - \delta/(1 - 2h)) \times h$ in the top right corner for any desired $0 < h < 1/2$, or we can directly pack the instance into two bins.*

Proof. As before, we order the high items by non-increasing height and pack them from left to right. Note that there is a total area of at least $(1 - \delta)/2$ covered by high items below the line $y = 1/2$. If the area $(1 - \delta/(1 - 2h)) \times h$ intersects the high items, then in particular the point $(\delta/(1 - 2h); 1 - h)$ is within some high item, cf. Fig. 5. This means that there is covered area above the line $y = 1/2$ of at least $\delta/(1 - 2h) \cdot (1/2 - h) = \delta/2$. Thus, the total area of high items would be at least $1/2$, and we can pack the instance by Remark 11. □

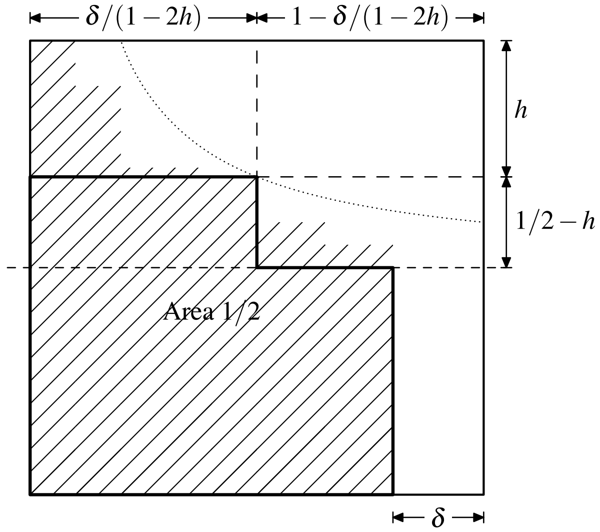


Fig. 5: Free space available in the top right corner.

We will now use this bound of the area for two classes of items, those that satisfy either Case 3 or 4 in Observation 13. Let us first consider Case 3, the set of items that have height at most $\sqrt{\delta/2}$ and width at most $1/2$. Assume that the total area of these items is at least $\delta/2$ and greedily select a subset S of total area in the interval $[\delta/2, \delta)$. This is possible, since every individual item can be assumed to have area at most $\delta/2$ by Corollary 15.

We define a container of size $1/2 \times 2\sqrt{\delta}$ and note that its area is $\sqrt{\delta} = 1/5 \geq 2\delta$ and its height is more than $2\sqrt{\delta/2}$. In particular, we can pack S into this container with Steinberg's algorithm by Corollary 5. It remains to verify that the container itself can be packed by Lemma 16, and indeed its height is $2\sqrt{\delta} < 1/2$ since $\delta = 1/25$ and the allowed width of a container of this height is

$$1 - \frac{\delta}{1 - 4\sqrt{\delta}} = 1 - \frac{1/25}{1 - 4/5} = 4/5 > 1/2\,, \tag{10}$$

so the container fits. This shows:

Lemma 17. *If $w(H) \geq 1 - \delta$ and the total area of items with $h_i \leq \sqrt{\delta/2}$ and $w_i \leq 1/2$ is at least $\delta/2$, we can pack all items into two bins.*

We now turn to the items of Case 4, having height at most $1/3$ and width at most $\sqrt{\delta/2}$. (Note this set might not be disjoint from the previous set.) Again, if these items have total area at least $\delta/2$, we can select a subset S with area in the interval $[\delta/2, \delta)$ and pack this subset into a container sized $2\sqrt{\delta} \times 1/3$ which has area $2\sqrt{\delta}/3 = 2/15 > 2\delta$. Again by Lemma 16, this container will fit into the bin since its height is less than 1/2 and the allowed width is

$$1 - \delta/(1 - 2/3) = 1 - 3\delta = 22/25 > 2/5 = 2\sqrt{\delta}\,. \tag{11}$$

This shows:

Lemma 18. *If $w(H) \geq 1 - \delta$ and the total area of items with $h_i \leq 1/3$ and $w_i \leq \sqrt{\delta/2}$ is at least $\delta/2$, we can pack all items into two bins.*

Next, we consider Case 5 of Observation 13, the items of width more than 1/2 which are not already packed. (There could be one wide item that is also high.) Their individual height is automatically less than δ by Corollary 15. We can pack a specific subset of these by using the following lemma:

Lemma 19. *Let $r_1, \ldots, r_m$ be the items of $W \setminus H$, i.e. all wide items apart from up to one item which is high as well, ordered by non-decreasing width, and let $k \leq m$ such that $\sum_{i=1}^{k} h_i \leq h(W \setminus H)/2$. We can then pack H and $\{r_1, \ldots, r_k\}$ into a single bin.*

Proof. Pack the high items from left to right ordered by non-increasing height at the bottom of the bin and stack the wide items from the top right corner downwards ordered by non-increasing width as shown in Fig. 6, and assume that there is an overlap. Choose $j \leq k$ maximal such that r_j intersects a high item r_ℓ. Clearly, the total width of items at least as high as r_ℓ is larger than $1 - w_j$, otherwise, the overlap would not have occurred. By Lemma 12, setting $\gamma = 1 - h_\ell < 1/2$, the total height of wide non-high items of width at least w_j is then at most $2(1 - h_\ell)$, however, it is also at least

$$\sum_{i=j}^{m} h_i = \sum_{i=j}^{k} h_i + \sum_{i=k+1}^{m} h_i \geq \sum_{i=j}^{k} h_i + h(W \setminus H)/2 \geq 2 \sum_{i=j}^{k} h_i > 2(1 - h_\ell)\,, \tag{12}$$

which contradicts the assumption of overlap. □

Assume that their total area is at least 4δ, then their total height is also at least 4δ. In particular, by greedily selecting the narrowest wide items of total height at least δ and at most 2δ, this shows

Corollary 20. *If $w(H) \geq 1 - \delta$ and the total area of items with $w_i > 1/2$ and $h_i \leq 1/2$ is at least 4δ, we can pack all items into two bins.*

Finally, we consider Case 1, items of height larger than 1/3, but at most 1/2. Each item's width is then bounded by $3\delta/2$ by Corollary 15. We then succeed by the following lemma:

Lemma 21. *We can pack all but one item of height larger than 1/3 into one bin, and the unpacked item has height at most 1/2.*

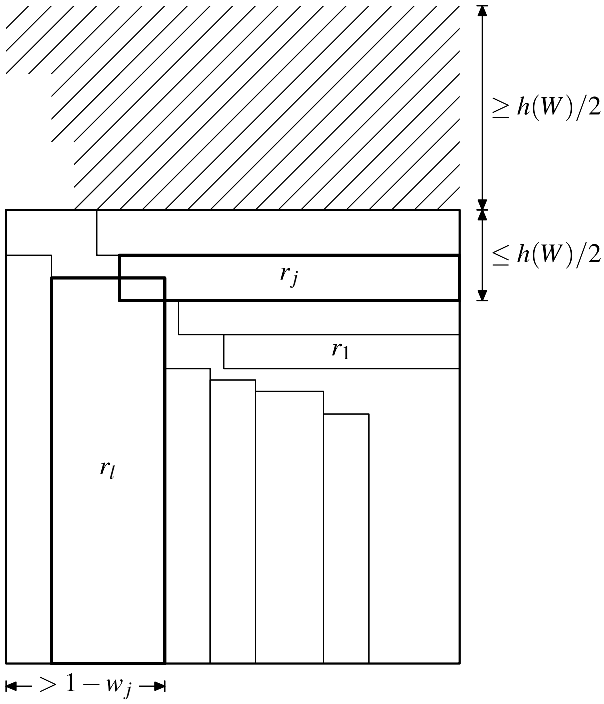


Fig. 6: High and wide items in a single bin.

Proof. The idea of this proof is a generalization of a result implicit in Graham's proof of the performance of the Longest Processing Time scheduling heuristic [8], i.e. that this heuristic is optimal as long as there are at most two jobs per machine. We sort all items by non-increasing height (assume by reindexing $h_1 \geq h_2 \dots$) and start packing them at the bottom of the bin until the total width is at least 1. If the width is strictly larger, the last item, r_k, protrudes beyond the bin and we split it. (This is the one item that we are allowed to not pack at the end.) By Remark 10, the split item cannot have height larger than 1/2. The rest of the split item and all further items are packed from right to left at the top of the bin into a 'reverse shelf', cf. Fig. 7.

Assume now that there is a collision of items. Clearly, this can only happen if one of the items involved, say r_i at position $(x_i, 0)$ in the lower shelf, has height larger than 1/2, and the other one, r_j at $(x_j, 1 - h_j)$ in the upper shelf, has height at most 1/2. In particular, we can conclude that the total width of items of height at least h_j, $w_1 + \dots + w_j$, is at least $2 - x_j$, and the total width of items of height at least $1 - h_j$ is at least $w_1 + \dots + w_i > x_j$.

However, in *any* feasible packing of the items $r_1, \dots, r_j$, there are vertical strips of total width $w_1 + \dots + w_i > x_j$ that can contain only $r_1, \dots, r_i$. In the remaining width, $1 - w_1 - \dots - w_i$, at most two rectangles can be on top of one another at any

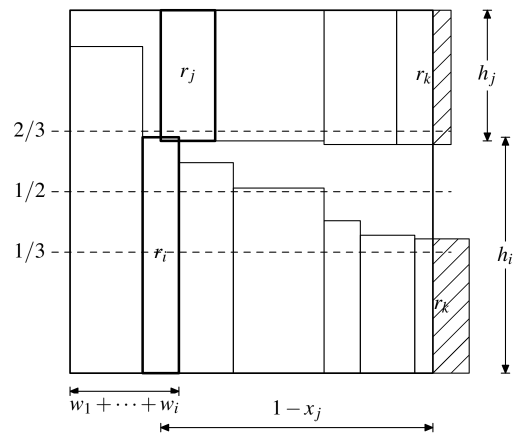


Fig. 7: Items of height larger than 1/3 in a single bin

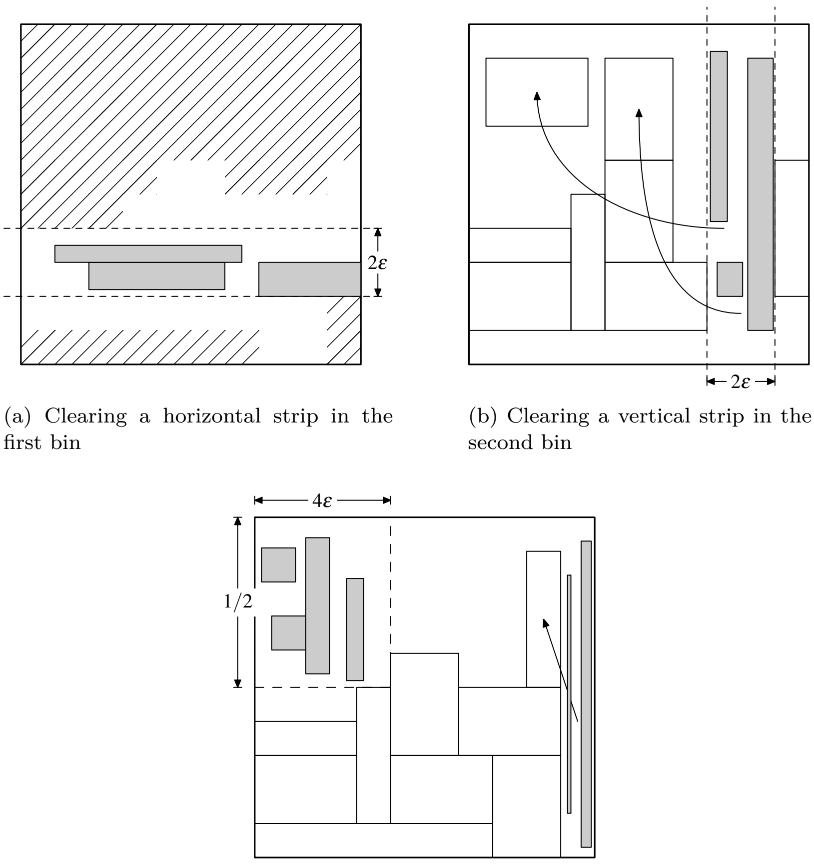
place, so the total width of those items there is at most $2 - 2(w_1 + \cdots + w_i)$. Hence, the entire width of items at least as high as r_j is at most $2 - w_1 - \cdots - w_i < 2 - x_j$, which contradicts that it is also at least $2 - x_j$ by assumption of overlap.

Hence, our packing is feasible apart from the fact that at most one item is split. We discard this item. □

Assume the total area of items of height larger than 1/3 and at most 1/2 is at least δ. The previous lemma then packs all but one item, but the area of the discarded item is at most $\delta/2$ by Corollary 15, so we have still packed at least $\delta/2$ additional area.

Lemma 22. *If the total width of high items $w(H)$ is at least $1 - \delta$, we can pack all items into two bins.*

Proof. We pack all high items by Remark 10. The total area of them is at least $(1 - \delta)/2$. Afterwards we add some items of a total area at least $\delta/2$ to this bin, so that we are able to invoke Remark 11 on the remainder. We do this by considering the five conditions of Observation 13. If the total area of the remaining unpacked items that satisfy Condition 1, i.e. the items that have height larger than 1/3 but at most 1/2, is larger than δ then we invoke Lemma 21. If there is at least one item that satisfies Condition 2, i.e. it has an area of at least $\delta/2$, we are able to adopt Corollary 15. If there are items of the total area δ that satisfy Condition 3, i.e. the items that have height at most $\sqrt{\delta/2}$ and width at most 1/2, then we make use of Lemma 17. The same holds with Lemma 18 if there are items of the total area δ that satisfy Condition 4, i.e. the items that have height at most 1/3 and width at most $\sqrt{\delta/2}$. Corollary 20 is used when there are items of the total area 4δ that satisfy Condition 5, i.e. the items that have height at most 1/2 and width larger than 1/2. If none of these attempts solves the problem, we can hence bound the



(a) Clearing a horizontal strip in the first bin

(b) Clearing a vertical strip in the second bin

(c) Finding two areas in the second bin

Fig. 8: General approach

total area of non-high items by $\delta + 0 + \delta/2 + \delta/2 + 4\delta = 6\delta \leq 1/2$, so we can pack all non-high items in the second bin using Steinberg's algorithm. $\square$

In the following, we therefore always assume that $w(H) \leq 1 - \delta$ and, by symmetry, $h(W) \leq 1 - \delta$.

In the remaining cases, we will always pursue the same angle of attack: starting off with a packing of area $(\sum_{i=1}^{n} w_i h_i) - \epsilon^2$ into one bin generated with the BCJPS-algorithm (Corollary 7) and precision ϵ^2, we will identify a suitable strip of size 2ϵ and move all items that properly intersect the strip into the second bin, cf. Fig. 8a. For convenience, we always consider horizontal strips, but all results still hold with 'horizontal' and 'vertical' interchanged.

All unpacked items that are bounded in height by ϵ can then be packed into the empty strip sized $1 \times 2\epsilon$ in the first bin using Steinberg's algorithm by Corollary 5.

We will then re-arrange the moved items in the second bin in such a way that the second bin also accomodates the other unpacked items of area at most ϵ^2 (each of which is bounded in width by ϵ) in one of two ways: we either clear a full-height area of size $2\epsilon \times 1$, Fig. 8b, into which they can be packed by Steinberg's algorithm again, or we will argue that in specific cases, a certain subset of high unpacked items can be packed 'manually' so that the rest can fit into a free area of height less than 1 but width larger than 2ϵ as in Fig. 8c.

The following lemma will prove useful for rearranging items of height at most 1/2:

Lemma 23. *Given a set $\{a_1 \geq \ldots \geq a_m\}$ of numbers, a total width $S \geq \sum_{i=1}^m a_i$ and a desired target value T such that $S \geq 2T + a_1$, we can find in linear time a subset $P \subseteq \{1, \ldots, m\}$ such that $\sum_{i \in P} a_i \leq S - T$ and $\sum_{i \notin P} a_i \leq S - T$.*

Proof. If $\sum_{i=1}^m a_i \leq S - T$, $P := \emptyset$ is a trivial solution. Otherwise, we find $k < m$ such that $\sum_{i=1}^k a_i \leq S - T < \sum_{i=1}^{k+1} a_i$. Then, we also have

$$\sum_{i=k+1}^{m} a_i \leq S - \sum_{i=1}^{k+1} a_i + a_{k+1} < T + a_{k+1} \leq T + a_1 \leq S - T\,, \tag{13}$$

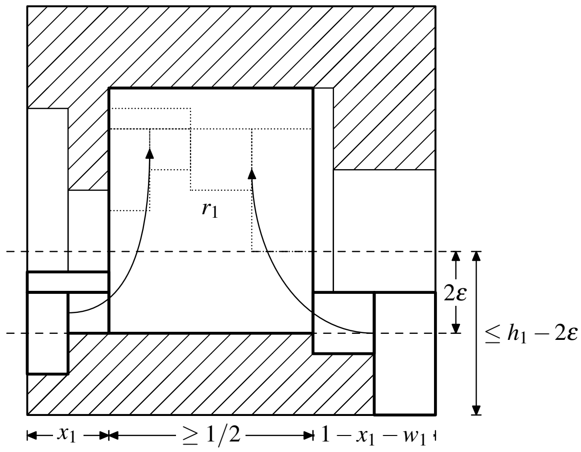
so $P = \{1, \ldots, k\}$ is the desired set. □

4.2. *One big item*

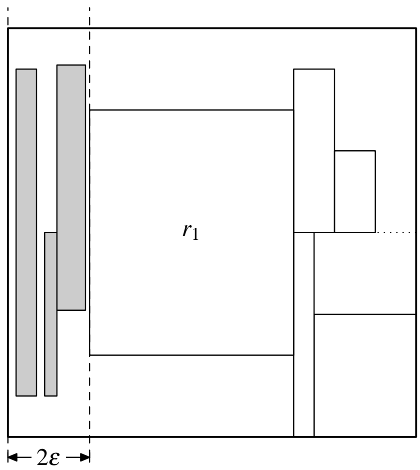
In this section, we will consider the case that there exists one item in the packing generated by the BCJPS-algorithm (Corollary 7), say r_1, such that $w_1, h_1 > 1/2$. By Lemma 22, we also may assume that $w_1, h_1 \leq 1 - \delta$. Let (x_1, y_1) the coordinates of r_1's lower left corner. Without loss of generality, we assume the bottom edge of r_1 is closer to the bottom of the bin than the top edge to the top, i.e. $y_1 \leq 1 - h_1 - y_1$, i.e. $y_1 \leq (1 - h_1)/2$, otherwise, we would flip the packing upside down. We consider the strip defined by $y \in (y_1, y_1 + 2\epsilon)$ and denote with S the set of items that intersect $y = y_1 + 2\epsilon$ (in particular, r_1). We move all items in S to the second bin. Note that all items that intersect the strip and do not intersect $y = y_1 + 2\epsilon$ are already packed in two areas sized $x_1 \times (y_1 + 2\epsilon)$ and $(1 - w_1 - x_1) \times (y_1 + 2\epsilon)$, because they were either to the left or to the right of r_1. Since $x_1 + (1 - w_1 - x_1) = 1 - w_1 \leq 1/2 \leq w_1$ and $y_1 + 2\epsilon \leq (1 - h_1)/2 + 2\epsilon \leq 1/4 + 2\epsilon \leq h_1 - 2\epsilon$, we can pack these areas into the empty space freed by r_1 without obstructing the horizontal strip at the bottom, cf. Fig. 9a.

Let us now order the items in the second bin by non-increasing height, and note in particular that by Lemma 22, we may assume that the total width of high items, $w_H := w(S \cap H)$, is at most $1 - \delta$.

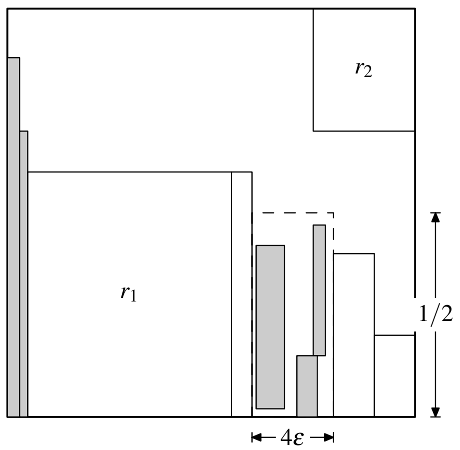
We consider two cases now: either there is a non-high item, say r_2, of width at least $1 - w_H - 4\epsilon$ or not. If there is no such item, we can apply Lemma 23 on the



(a) Re-packing items in the first bin



(b) Clearing a strip if many items are narrow enough



(c) Clearing areas if a sufficiently wide item exists

Fig. 9: Re-packing if a big item r_1 exists

widths of non-high items with total available width S at least $\max\{\delta, 1 - w_H\}$ and target size $T := 2\epsilon$. The two sets can be packed on two shelves atop each other, and this frees a vertical strip of width at least 2ϵ as shown in Fig. 9b.

If, however, such r_2 exists, it has width at least $1 - w_H - 4\epsilon \geq \delta - 4\epsilon \geq 6\epsilon$. In particular, we can apply Lemma 14 on the following items: (1) $S \setminus \{r_2\}$, (2) all unpacked high items (of total width at most 2ϵ) (3) a container sized $4\epsilon \times 1/2$, into which we can pack all remaining unpacked items by Steinberg's algorithm, and use r_2 as the extra item, as depicted in Fig. 9c. Note that r_2 and the container will not intersect since both their heights are bounded by $1/2$.

4.3. *One medium item*

In this section, we will consider the case that there exists one item in the packing, say r_1, such that $w_1, h_1 \geq 12\epsilon$, and this item's lower left corner is at (x_1, y_1). In light of the previous section, we can assume $\min\{w_1, h_1\} \leq 1/2$, say $h_1 \leq 1/2$. We also assume that $y_1 \leq 1 - y_1 - h_1$, i.e. $y_1 \leq (1 - h_1)/2$, otherwise we flip the packing upside-down.

We now set $y_0 := \max\{2\epsilon, y_1\}$ and consider three consecutive horizontal strips: Strip I is defined by $y \in (y_0, y_0 + 2\epsilon)$, Strip II by $y \in (y_0 + 2\epsilon, y_0 + 4\epsilon)$ and Strip III by $y \in (y_0 + 4\epsilon, y_0 + 6\epsilon)$, see Fig. 11a. Since $y_0 + 6\epsilon \leq y_1 + 2\epsilon + 6\epsilon < y_1 + h_1$, all three strips are entirely bisected by r_1.

We claim the following properties hold:

$$y_0 + 6\epsilon \leq 1/2\,, \tag{14}$$

$$y_0 + 4\epsilon \leq 1 - h_1\,. \tag{15}$$

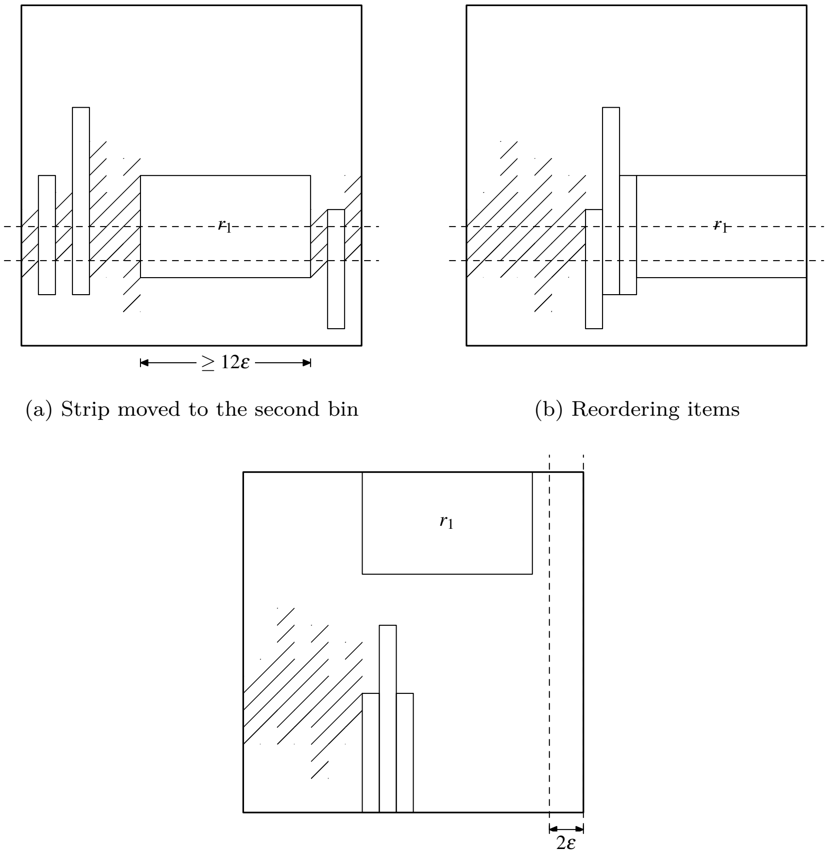
Eq. (14) is trivial for $y_0 = 2\epsilon$ since $\epsilon \leq 1/16$, and for $y_0 = y_1$ we have $y_0 + 6\epsilon \leq y_1 + h_1/2 \leq (1 - h_1)/2 + h_1/2 = 1/2$. As to (15), it is now sufficient to note that $y_0 + 4\epsilon < y_0 + 6\epsilon \leq 1/2 \leq 1 - h_1$.

We are now interested in the sets of items that intersect the strips, which we will denote by S_I, S_{II} and S_{III}, respectively.

Remark 24. *If one of Strips I, II, III contains items other than r_1 of height at most $1 - h_1$ and total width at least 2ϵ that totally bisect the strip, we can pack the instance into two bins.*

Proof. Move the strip in question and the corresponding items to the second bin, maintaining their packing, as shown in Fig. 10a. We modify the packing as follows: By reordering, we can assume that all bisecting items are adjacent to r_1 and r_1 is at the right side of the bin, as shown in Fig. 10b. r_1 is shifted up to the top of the bin, all other items that bisect the strip are shifted down to the bottom of the bin. It is then possible to shift r_1 to the left by at least 2ϵ, which frees a vertical strip of width 2ϵ, cf. Fig. 10c. □

If this does not apply, we will move S_{II} to the second bin and rearrange it to accomodate the remaining items. In more detail, we create the following packing (cf. Fig. 11b): the item r_1 is packed in the top right corner of the bin. Below it, there are two containers, C_1 sized $4\epsilon \times (1 - h_1)$ and C_2 sized $6\epsilon \times (1 - h_1)$. The first holds all unpacked items of height at most $1 - h_1$, packed with Steinberg's algorithm. This is feasible by Corollary 5 since each unpacked item's width is bounded by $\epsilon \leq 4\epsilon/2$ and their total area is at most $\epsilon^2 \leq \epsilon \leq (4\epsilon)/4 \leq (4\epsilon) \cdot (1 - h_1)/2$. The container C_2 contains all items of S_{II} with height at most $1 - h_1$ that bisected at least one of Strip I, II or III entirely, which means they can be packed next to each other since by Remark 24, their total width is at most 6ϵ. (These are marked in a darker shade



(a) Strip moved to the second bin

(b) Reordering items

(c) Shifting items

Fig. 10: Packing items if enough items exist in Remark 24

in both Fig. 11a and Fig. 11b.) Note in particular that all items with height in the interval $[4\epsilon, 1 - h_1]$ end up in C_2.

The remaining items in $S_{II} \setminus S_{III}$, shaded darkest, can be shifted into a container C_3 sized $(1 - w') \times 4\epsilon$, where w' is the total width of all items in $S_{II} \cap S_{III}$ that bisect Strip II. It is immediate that this width is sufficient, because all items in $S_{II} \setminus S_{III}$ do not intersect Strip III and no item in S_{II} is below an item that bisects Strip II. As to the height, note that all these items are bounded in height by $1 - h_1$ by (15), and all those that bisected an entire strip were already removed to C_2 above. This means that all remaining items in $S_{II} \setminus S_{III}$ do not cross the line $y = y_0$ nor $y = y_0 + 4\epsilon$. We position C_3 at the bottom of the bin, next to C_1 and C_2, and note that it is shifted at least 2ϵ under r_1. Since its width is at most $1 - w_1$, the combined width of C_1, C_2, C_3 is less than $1 - 2\epsilon$.

Now, the following items are still remaining: unpacked items of height more than

$1 - h_1$, and packed items of height either larger than $1 - h_1$ or smaller than 4ϵ from the set $S_{II} \cap S_{III} \setminus \{r_1\}$, i.e. they all intersected the line $y = y_0 + 4\epsilon$. Note that the total width of all these is at most $\epsilon/(1 - h_1) + (1 - w_1) \leq 2\epsilon + 1 - w_1 \leq 1 - 10\epsilon$.

We sort these items by decreasing height and pack them left-to-right, starting at position $(0, 0)$, and continuing on top of C_3. The total width available is hence $1 - 10\epsilon$, so the items will not intersect C_2, but conceivably intersect r_1 or extend beyond the top of the bin.

Assume that some item r_i collides with r_1. This cannot be an item of height at most 4ϵ since $y_i + h_i \leq 4\epsilon + 4\epsilon \leq 1/2 \leq 1 - h_1$, hence its height is larger than $1 - h_1$, and the collision would then contradict Lemma 14, since all such items must have fit next to r_1 in an optimal packing.

Finally consider that some item r_i might protude beyond the top of the bin (whether or not it collides with r_1). Such an item must have $h_i > 1 - 4\epsilon$ and be positioned atop C_3. However, since $1/2 > y_0 \geq 2\epsilon$, this means that r_i either completely bisected both S_{II} and S_{III} or was unpacked. The total width of such items (other than r_1) is at most $(w' - w_1) + \epsilon/(1 - 4\epsilon) < w' - w_1 + 2\epsilon$. The width of the area next to C_1, C_2, C_3 is $1 - (1 - w') - 10\epsilon = w' - 10\epsilon \geq w' - w_1 + 2\epsilon$ since $w_1 \geq 12\epsilon$, so all items of height more than $1 - 4\epsilon$ were successfully packed there by the algorithm.

4.4. *All small and elongated items*

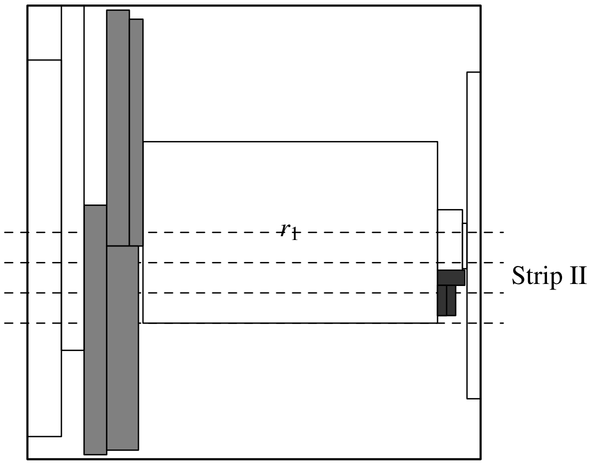
In this section, we consider the remaining case that every packed item is bounded in at least one direction by 12ϵ. Note that all unpacked items are even bounded by ϵ in one direction. First of all, we want to show that the difficult subcase here is if there are few items which have one 'medium' sidelength. (We show the claim for items of medium height, but the same argument works for medium width.)

Lemma 25. *If the total area of packed items of height at least 12ϵ and at most $1/2$ and width at most 12ϵ ('tallish items') is at least 19.2ϵ, we can pack all items into two bins.*

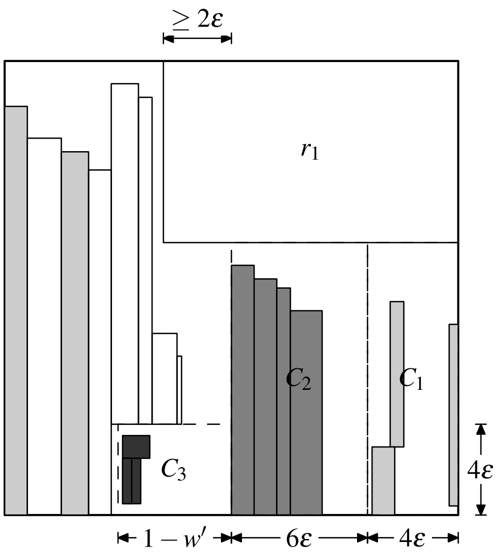
Proof. Suppose for illustration that there is a strip $y \in (y_0, y_0 + 2\epsilon)$ that is entirely bisected by some tallish items. If the total width of these items is at least 16ϵ, we can move this strip to the second bin and apply Lemma 23 with $T := 2\epsilon$ to clear a vertical strip of width 2ϵ in the second bin.

To formalize this notion and show that such a strip must exist, we define for every tallish item r_i packed at location (x_i, y_i) the function

$$\chi_i(y) := \begin{cases} w_i, & y \in [y_i, y_i + h_i - 2\epsilon) \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$



(a) A suitable strip, before reordering



(b) after reordering

Fig. 11: Reordering items that intersect Strip II.

Note that r_i will completely bisect the strip $(y, y + 2\epsilon)$ iff $\chi_i(y) = w_i$. (See also Fig. 12, where the missing 2ϵ are hatched.) We have that

$$\int_0^1 \chi_i(y)dy = w_i \cdot (h_i - 2\epsilon) \geq w_i \cdot h_i \cdot 10/12\,, \tag{17}$$

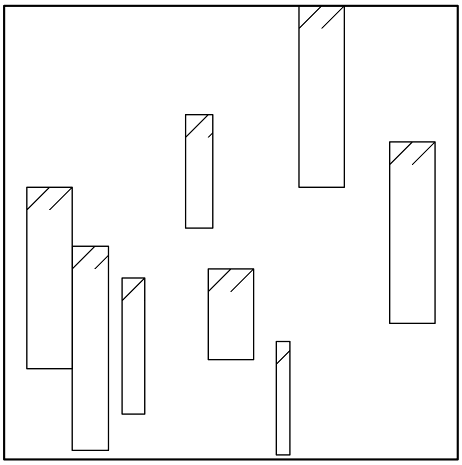


Fig. 12: Areas of tallish items that "do not count towards the 16ϵ".

since $h_i \geq 12\epsilon$. Summing over all tallish items, we obtain that

$$\int_0^1 \sum_{r_i \text{ tallish}} \chi_i(y)dy \geq \sum_{r_i \text{ tallish}} 10w_ih_i/12 = 10/12 \cdot \sum_{r_i \text{ tallish}} w_ih_i \geq 10/12 \cdot 19.2\epsilon = 16\epsilon\,. \quad (18)$$

In particular, there exists some y such that $\sum\{\chi_i(y) : r_i \text{ tallish}\} \geq 16\epsilon$, which identifies a suitable strip for re-packing in the second bin.

We can also find such a strip in polynomial time. To do this, note that when sweeping the horizontal line from the bottom of the bin upwards, the amount of 'counting' tallish items $\sum\{\chi_i(y) : r_i \text{ tallish}\}$ only increases if $y = y_i$ for some tallish item r_i. In particular, the maximum value, which is at least 16ϵ, is attained in one of the at most n elements of $\{y_i : r_i \text{ packed and tallish}\}$. □

If the previous lemma does not give us a solution, we know that most of the area of the instance is in items that are either high or wide or very small in both directions. (We have $2 \cdot 19.2\epsilon$ in tallish and widish items and ϵ^2 in unpacked items that we have not reasoned about yet.) In this case, we will construct a packing from scratch as shown in Fig. 13. Beforehand, we would like to recall a folklore lemma concerning Next Fit Decreasing Height (NFDH) when applied to small items:

Lemma 26. *Given a set of items which are bounded in width and height by* 12ϵ *and a target area sized* $a \times b$ *(for* $a, b \geq 12\epsilon$*), NFDH packs all the items or covers an area of at least* $(a - 12\epsilon)(b - 24\epsilon)$.

In the following, we denote with $\mathcal{A}(H)$ the total area of all high items, with $\mathcal{A}(W)$ the total area of wide items and with $\mathcal{A}(S)$ the total area of items which are bounded by 12ϵ in both directions. Without loss of generality, we assume $\mathcal{A}(H) \geq \mathcal{A}(W)$. We have already shown in Lemma 19 that we can arrange all of the high items and

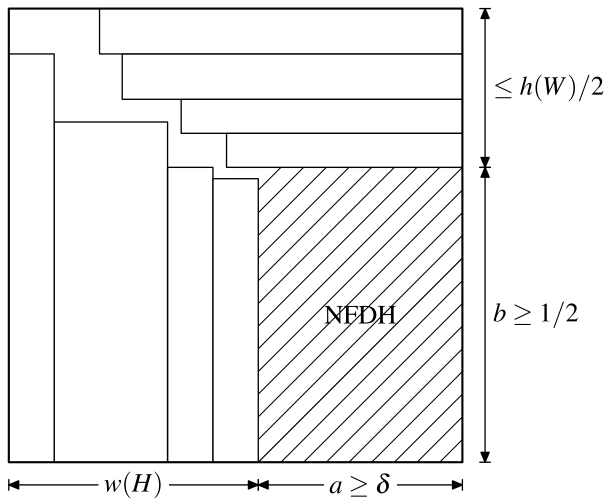


Fig. 13: Packing items if few tallish and widish items exist

approximately half (in terms of total height) of the wide items as shown in Fig. 13. The area covered by these items is at least $\mathcal{A}(H)+\mathcal{A}(W)/3-12\epsilon$, since all packed wide items might have width close to $1/2$ while all unpacked wide items might have width 1, and one item of individual area at most 12ϵ might be split. Denote with a and b the width and height of the area to be filled with NFDH. Following Lemma 22, we may assume that $a \geq \delta$ and $b \geq 1-(1-\delta)/2 > 1/2$. (Bear in mind we have not packed at least half of the stack of wide items in Lemma 19.) In particular, by Lemma 26 we either pack all small items there or cover an area of

$$(a-12\epsilon)(b-24\epsilon) > ab-12\epsilon(2a+b) \geq ab-36\epsilon \geq ab-\delta/4 \geq ab/2\,, \tag{19}$$

where we use that $\epsilon \leq \delta/144$ and $ab \geq \delta/2$. In this case, we have filled the entire bin at least halfway: the area sized $(1-a)\times 1$ at the left side of the bin is covered at least halfway by the high items, the (not disjoint) area sized $1\times(1-b)$ at the top of the bin is covered at least halfway by wide items, and as we have just seen, the NFDH region is also covered with at least $ab/2$.

Even if we run out of small items, the area remaining for the second bin is small: it is bounded by $(2\mathcal{A}(W)/3+12\epsilon)+2\cdot 19.2\epsilon+\epsilon^2$ for wide, widish and tallish and unpacked (non-high non-wide) items, respectively. Since $\mathcal{A}(W) \leq \mathcal{A}(H)$, we have $\mathcal{A}(W) \leq 1/2$, so the above sum is bounded by $1/3+51.4\epsilon$, which is at most $1/2$ since $\epsilon \leq 1/308.4$. Hence, in either case, the second bin can be packed using Steinberg's algorithm.

5. Conclusion

We have presented an algorithm that generates 2-approximate solutions for two-dimensional geometric bin packing, which matches the rate known for the problem with rotations. Since both the problem with and without rotations are not approximable to any $2 - \epsilon$ unless $P = NP$, this settles the question of absolute approximability of these problems. For practical applications, it would be interesting to find faster algorithms: our algorithm relies heavily on the knapsack PTAS in [3] and techniques in [14] with a doubly-exponential dependency on ϵ, in particular when compared to the running time $O(n \log n)$ of Harren and van Stee's 3-approximation in [10].

Another important open problem is the gap in asymptotic behaviour between the non-existence of an APTAS and the best known algorithm with asymptotic quality of 1.525.

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