The regularizing properties of multistep methods for first kind Volterra integral equations with smooth kernels

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Abstract

In the present paper we consider the regularizing properties of linear multistep methods for the stable solution of perturbed Volterra integral equations of the first kind with smooth kernels. Numerical results are also given.

1 Introduction

In this paper we consider linear Volterra integral equations of the following form,

$$(Au)(x) = \int_0^x k(x,y)u(y)\,dy = f(x) \quad \text{for } 0 \le x \le L, \tag{1}$$

with $L > 0$, and with a sufficiently smooth kernel function $k : \{\,(x,y) \in \mathbb{R}^2 \mid 0 \le y \le x \le L\,\} \to \mathbb{R}$. Moreover, the function $f : [0,L] \to \mathbb{R}$ is supposed to be approximately given, and a function $u : [0,L] \to \mathbb{R}$ satisfying equation (1) is to be determined.

In the sequel we suppose that the kernel function does not vanish on the diagonal $0 \le x = y \le L$, and without loss of generality we may assume that

$$k(x,x) = 1 \quad \text{for } 0 \le x \le L$$

holds.

Composite quadrature methods for the approximate solution of equation (1) are well-investigated if the right-hand side f is exactly given, see e.g., Brunner/van der Houwen [3], Brunner [2], Lamm [12], Linz [13] or Hoog/Weiss [4] and the reference therein. A special class of composite quadrature methods for the approximate solution of (1) is obtained by using in an appropriate manner multistep methods that usually are used to solve initial value problems for first order ordinary differential equations. That class of methods is considered thoroughly in Wolkenfelt ([19], [20]), and see also Holyhead / McKee / Taylor [10], Holyhead/McKee [9] and Taylor [18] for related results. In the present paper, the results and techniques presented in the two papers by Wolkenfelt are extended in order to analyze the regularizing properties of those multistep methods for Volterra integral equations (1) when perturbed right-hand sides are available only. Finally, some numerical illustrations are presented.

2 Numerical integration based on multistep methods

Throughout this section, as a preparation for the numerical solution of Volterra integral equations of the first kind (1) with smooth kernels, we introduce linear multistep methods for solving the associated direct problem. For this purpose we consider equidistant nodes

$$x_n = nh, \qquad n = 1,2,\ldots,N, \quad \text{with } h = \tfrac{L}{N}, \tag{2}$$

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where N is a positive integer. In a first step we consider – for fixed $1 \le n \le N$ – the simple integration

$$\int_0^x \psi(y)\,dy = \varphi(x) \quad \text{for } 0 \le x \le x_n, \tag{3}$$

where $\psi : [0, x_n] \to \mathbb{R}$ is a given continuous function, and the function $\varphi : [0, x_n]$ needs to be determined. This in fact is a direct problem for the associated equation (1) in the special situation $k \equiv 1$. Problem (3) is obviously equivalent to solving the elementary ordinary differential equation

$$\varphi'(x) = \psi(x) \quad \text{for } 0 \le x \le x_n, \qquad \varphi(0) = 0. \tag{4}$$

2.1 Introduction of multistep methods for the direct problem

Next we briefly introduce some basic facts about linear multistep methods to solve initial value problems for ordinary differential equations, with a notation that is adapted to the simple situation considered in (4), or equivalently (3). For a thorough presentation of multistep methods (to solve initial value problems for ordinary differential equations in its general form), see e.g., [15], Hairer / Nørsett / Wanner [7] or Henrici [8].

A linear m-step method with an integer $1 \le m \le n$ is determined by coefficients $a_j \in \mathbb{R}$ and $b_j \in \mathbb{R}$ for $j = 0, 1, \ldots, m$, with $a_m \ne 0$ and $b_j \ne 0$ for some $0 \le j \le m$, all of them being independent of the step size h introduced in (2). When applied to problem (4), or equivalently (3), this scheme is of the form

$$\sum_{j=0}^{m} a_j \varphi_{r+j} = h \sum_{j=0}^{m} b_j \psi_{r+j} \quad \text{for } r = 0, 1, \ldots, n-m. \tag{5}$$

Here, $\psi_s = \psi(x_s), s = 0, 1, \ldots, n$ are given. In addition we have $\varphi_0 = 0$, and the other starting values $\varphi_s \approx \varphi(x_s)$ for $s = 1, 2, \ldots, m-1$ are determined by some procedure not further specified here; an example is considered below (see Example 2.7). The scheme (5) is used then to compute approximations $\varphi_{r+m} \approx \varphi(x_{r+m})$ for $r = 0, 1, \ldots, n-m$. The case $b_m = 0$ is not excluded. In that situation the considered multistep method defines an explicit scheme.

Example 2.1. (a) A class of multistep methods, depending on three parameters τ, μ and m, is obtained by integrating (4) from $x_{r+m-\tau}$ to x_{r+m} and using an interpolatory numerical integration scheme for the resulting integral afterwards, i. e.,

$$\varphi_{r+m} - \varphi_{r+m-\tau} = \int_{x_{r+m-\tau}}^{x_{r+m}} \mathcal{P}_r(x)\,dx, \qquad r = 0, 1, \ldots, n-m. \tag{6}$$

Here $\mathcal{P}_r \in \Pi_{m-\mu}$ satisfies $\mathcal{P}_r(x_s) = \psi_s$ for $s = r, r+1, \ldots, r+m-\mu$. In addition, $1 \le \tau \le m$ and $0 \le \mu \le m$ denote some integers, with τh being the length of the interval used for the local integration, and $m - \mu + 1$ is the number of nodes used for the interpolation. Prominent examples are obtained for $0 \le \mu \le 1$ and $1 \le \tau \le 2$. Some special cases are considered next.

The Adams–Bashfort methods are obtained for $\tau = 1$, $\mu = 1$ and $m \ge 1$; for the special case $m = 1$ this in fact gives the composite forward rectangular rule. The Adams–Moulton methods are obtained for $\tau = 1$, $\mu = 0$ and $m \ge 1$, with the composite trapezoidal rule obtained for the special case $m = 1$. The Nyström methods are given by $\tau = 2$, $\mu = 1$ and $m \ge 2$. For $m = 2$ this gives the repeated midpoint rule. Finally, the Milne–Simpson methods are obtained by $\tau = 2$, $\mu = 0$ and $m \ge 2$, with the repeated Simpson's rule obtained in the case $m = 2$. Each of these methods is in fact of the form (5) and leads to a repeated quadrature method, with adjacent interpolation polynomials $\mathcal{P}_r$ that for $m > \tau$ have overlapping node intervals $[x_r, x_{r+m-\mu}]$.

Note that for $m \ge 2$, the classical usage of composite quadrature methods of the form (6) is to determine approximate integrals $\varphi_{\kappa m}$ for $\kappa = 1, 2, \ldots$, with $\kappa m \le n$ only. This is in contrast to the quadrature scheme (6) used here, with φ_s being computed for each $s = m, m+1, \ldots, n$.

(b) Another class of linear multistep methods are BDF methods (backward differentiation formulas). Here the left-hand side in (4) is replaced by a finite difference scheme. More precisely, for m fixed, approximations $\varphi_{r+m} \approx \varphi(x_{r+m})$ for $r = 0, 1, \ldots, n-m$ are given by $\varphi_{r+m} = \mathcal{P}(x_{r+m})$, where $\mathcal{P} \in \Pi_m$ satisfies $\mathcal{P}(x_s) = \varphi_s$ for $s = r, r+1, \ldots, r+m-1$ and $\mathcal{P}'(x_{r+m}) = \psi_{r+m}$. For $m = 1$ this leads to the composite backward rectangular rule. $\triangle$

2.2 Null stability, order of the method

We next recall some basic notation for multistep methods, adapted to the simple situation considered in (4):

(a) An m-step method is called nullstable, if the first characteristic polynomial

$$\varrho(\xi) = a_m \xi^m + a_{m-1}\xi^{m-1} + \cdots + a_0 \tag{7}$$

of the given m-step method is a simple von Neumann polynomial, i. e.,

$$\text{(i)}\ \ \varrho(\xi) = 0 \text{ implies } |\xi| \le 1, \qquad \text{(ii)}\ \ \varrho(\xi) = 0,\ |\xi| = 1 \text{ implies } \varrho'(\xi) \ne 0. \tag{8}$$

This means that all roots of the characteristic polynomial ϱ belong to the closed unit disk, and each root on the unit circle is simple.

(b) A linear multistep method is by definition of (consistency) p with an integer $p \ge 1$, if the local discretization error

$$\eta(x,h) := \sum_{j=0}^{m} a_j \varphi(x+jh) - h \sum_{j=0}^{m} b_j \psi(x+jh), \qquad 0 \le x < x_n, \quad 0 < h \le \frac{x_n - x}{m},$$

satisfies, for each $\psi \in C^p[0, x_n]$, the estimate $\eta(x,h) = \mathcal{O}(h^{p+1})$ as $h \to 0$ uniformly for each $0 \le x < x_n$.

Here and in the sequel, at each considered position the symbol $\mathcal{O}(h^\beta)$ with $\beta > 0$ denotes an error term having modulus that is bounded by ch^β, where the constant $c \ge 0$ may be chosen – with a slight abuse of notation – independently of x also.

Note that each multistep method of order $p \ge 1$ is by definition of order $1 \le q \le p$, with reduced smoothness requirements on the involved functions then. In the sequel, occasionally we will make use of this fact in order to consider different smoothness assumptions.

Example 2.2. (a) Each multistep method of the special form (6) is clearly nullstable. In addition, order p here obviously means $\varphi_{r+m} - \varphi_{r+m-\tau} - \int_{x_{r+m-\tau}}^{x_{r+m}} \mathcal{P}_r(x)\, dx = \mathcal{O}(h^p)$ for $r = 0, 1, \ldots, n-m$. The standard error result for polynomial interpolation implies the estimate $\max\{\, |\mathcal{P}_r(x) - \psi(x)| \mid x_{r+m-\tau} \le x \le x_{r+m} \,\} = \mathcal{O}(h^{m-\mu})$ for $\psi \in C^{m-\mu+1}[0, x_n]$. The order of this multistep method is thus at least $m - \mu + 1$.

In some special cases the order is even larger in fact. For example, for $\tau = 2$, $\mu = 0$ and $m = 2$ (the Simpson's rule from the class of Milne–Simpson methods), the order is $p = 4$ for $\psi \in C^4[0, x_n]$ (for those values of τ and μ, the m-step methods coincide for $m = 2$ and $m = 3$ in fact).

(b) The BDF methods are nullstable for $1 \le m \le 6$, with respective order $p = m$.

The basic convergence result in multistep method theory applied to the simple situation (4) is as follows: each nullstable linear multistep method of order p is convergent of order p. The latter means that for each $\psi \in C^p[0, x_n]$ and starting values $\varphi_0 = 0$ and $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}$ of order p, i. e., $|\varphi_r - \varphi(x_r)| = \mathcal{O}(h^p)$ for $r = 1, 2, \ldots, m-1$, one has

$$\max_{r=m,\ldots,n} |\varphi_r - \varphi(x_r)| = \mathcal{O}(h^p) \quad \text{as } h \to 0. \tag{9}$$

Below multistep methods of order p will be applied in an appropriate manner to solve Volterra integral equations of the first kind with smooth kernels (see (1)). It turns out that $\mathcal{O}(h^p)$-error estimates for the associated approximations hold – which is of the same order as for the associated direct problem – but more smoothness on the involved functions must be assumed then. For deriving those estimates, a global error expansion for the associated direct problem is needed. This expansion is considered next.

Lemma 2.3 (Wolkenfelt ([19], [20])). *Consider a nullstable linear multistep method (5) of order $p \geq 1$ for solving the initial value problem (4), with $\psi \in C^{p+1}[0, x_n]$, $n \geq m$. Let the starting values satisfy $\max_{r=1,\ldots,m-1} |\varphi_r - \varphi(x_r)| = \mathcal{O}(h^{p+1})$. Then we have the global error expansion*

$$\varphi_n = \varphi(x_n) - Ch^p \int_0^{x_n} \psi^{(p)}(y)dy + \mathcal{O}(h^{p+1}). \tag{10}$$

Here, $C \in \mathbb{R}$ denotes the error constant of the considered multistep method.

For a definition of the error constant of a multistep method, see e.g., p. 201 in [15], or p. 373 in Hairer / Nørsett / Wanner [7], or p. 223 in Henrici [8]. The proof of the lemma follows easily, e.g., from the procedure described in Henrici [8], pp. 249–255. See also the comments in Wolkenfelt ([19] and [20]).

Remark 2.4. (a) Note that for the starting values considered in Lemma 2.3, the required order is $p + 1$, while in the assumption for (9) the considered order is p only. In addition, more smoothness of the function ψ (and thus of φ) is required in Lemma 2.3. The required order $p + 1$ accuracy of the starting values, however, will be satisfied anyway by the starting procedure used for the inverse problem considered below.

(b) Global error expansions similar to (10) hold also at the other nodes $x_m, x_{m-1}, \ldots, x_{n-1}$. For our purposes it is sufficient to consider that expansion at the endpoint of the considered interval only.

2.3 Reflected coefficients / polynomials

In the next section we present the considered multistep method in a different form which in fact will be useful for the numerical analysis. As a preparation we introduce some more notation. Consider the reflected coefficients of the multistep method under consideration:

$$\alpha_j = a_{m-j}, \quad \beta_j = b_{m-j}, \quad j = 0, 1, \ldots, m, \qquad \alpha_j = \beta_j = 0, \quad j = m, m+1, \ldots . \tag{11}$$

In addition we introduce the second characteristic polynomial

$$\sigma(\xi) := b_m\xi^m + b_{m-1}\xi^{m-1} + \cdots + b_0 \in \Pi_m. \tag{12}$$

The reflected polynomials associated with the two characteristic polynomials ϱ and σ are given by

$$\widetilde{\varrho}(\xi) := \alpha_m\xi^m + \alpha_{m-1}\xi^{m-1} + \cdots + \alpha_0 = \xi^m\varrho(1/\xi) \in \Pi_m, \tag{13}$$

$$\widetilde{\sigma}(\xi) := \beta_m\xi^m + \beta_{m-1}\xi^{m-1} + \cdots + \beta_0 = \xi^m\sigma(1/\xi) \in \Pi_m. \tag{14}$$

In addition, it is convenient to ignore in the sequel the possibly vanishing leading coefficients of the second characteristic polynomial σ. For this purpose let $0 \leq \mu \leq m$ such that

$$b_{m-\mu+1} = \cdots = b_{m-1} = b_m = 0, \quad b_{m-\mu} \neq 0. \tag{15}$$

For the reflected coefficients this in fact means $\beta_0 = \beta_1 = \cdots = \beta_{\mu-1} = 0,\ \beta_\mu \neq 0$. In the case $\mu \geq 1$ this means that the second reflected polynomial $\widetilde{\sigma}(\xi)$ has a root of order μ at $\xi = 0$.

In addition we introduce the sequence $\gamma_0, \gamma_1, \ldots$ given by the following discrete convolution equation:

$$\sum_{s=0}^{r} \alpha_{r-s}\gamma_s = \beta_{r+\mu} \quad \text{for } r = 0, 1, \ldots . \tag{16}$$

Notice that $\gamma_0 \neq 0$. It follows from (16) and standard results for difference equations (see e.g., Lemma 5.5 on p. 242 in Henrici [8]) that a nullstable multistep method satisfies

$$\gamma_n = \mathcal{O}(1) \quad \text{as } n \to \infty. \tag{17}$$

We finally note that the coefficients of the power series

$$\gamma(\xi) = \frac{\widetilde{\sigma}(\xi)/\xi^{\mu}}{\widetilde{\varrho}(\xi)} =: \sum_{s=0}^{\infty} \gamma_s \xi^s \tag{18}$$

coincide with the coefficients $\gamma_0, \gamma_1, \ldots$ defined by the recurrence equation (16). This follows easily by considering products of power series.

2.4 Explicit representation of the values φ_r

For the numerical analysis to be considered later on we need to express the values $\varphi_m, \varphi_{m+1}, \ldots, \varphi_n$ generated by the multistep method (5) in terms of the values ψ_s and the starting values $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}$. In Lemma 2.5 below we present some details, and we proceed by introducing weights needed in that lemma.

(a) Consider

$$\omega_{ns} = \gamma_{n-s-\mu} \quad \text{for } 0 \leq s \leq n-\mu, \quad n \geq m. \tag{19}$$

This in particular means that the weights ω_{ns} considered in (19) are of convolution form and uniformly $\mathcal{O}(1)$ (see (17)).

(b) In addition, consider starting weights $\widetilde{\omega}_{nr}$ for $1 \leq r \leq m-1$ and $n \geq m$, which for r fixed are recursively determined as follows,

$$\sum_{t=0}^{\nu} \alpha_{\nu-t} \widetilde{\omega}_{t+m,r} = -\alpha_{\nu+m-r} \quad \text{for } \nu = 0, 1, \ldots . \tag{20}$$

As for (17), it follows from standard results for difference equations that for a nullstable multistep method we have

$$\widetilde{\omega}_{nr} = \mathcal{O}(1) \quad \text{for } 1 \leq r \leq m-1, \quad n \geq m. \tag{21}$$

We are now in a position to represent the multistep method (5) in a different form. Note that the numbers $\psi_0, \psi_1, \ldots, \psi_{n-\mu}$ considered in that lemma does not necessarily coincide with the values of the previously considered function $\psi : [0, x_n] \to \mathbb{R}$ at the considered nodes.

Lemma 2.5. *Consider a multistep method (5), and let the parameter μ be chosen as in (15). Consider the weights ω_{rs} and $\widetilde{\omega}_{rs}$ given by (19) and (20). Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ and $\psi_0, \psi_1, \ldots, \psi_{n-\mu}$ be arbitrary two sequences of real numbers satisfying the multistep method recurrence (5), with $n \geq m$ and $\varphi_0 = 0$. Then the following identity holds:*

$$\varphi_n = h \sum_{s=0}^{n-\mu} \omega_{ns} \psi_s + \sum_{r=1}^{m-1} \widetilde{\omega}_{nr} \varphi_r. \tag{22}$$

PROOF. It follows by induction that a representation of the form (22) with some weights ω_{rs} and $\widetilde{\omega}_{rs}$ exists in general. The special representations of the weights given in (19) and (20) are then obtained by considering canonical bases for $\varphi_1, \varphi_2, \ldots, \varphi_m$ and $\psi_0, \psi_1, \ldots, \psi_{n-\mu}$. Details are omitted. □

We summarize the results of Lemma 2.3 and Lemma 2.5:

Corollary 2.6. *Consider a nullstable linear multistep method (5) of order $p \geq 1$ for solving the initial value problem (4) with $\psi \in C^{p+1}[0, x_n]$, where $n \geq m$ holds. Let $\varphi_0 = 0$, and let the starting values $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}$ be of order $p+1$, i. e., we have $\max_{r=1,\ldots,m-1} |\varphi_r - \varphi(x_r)| = \mathcal{O}(h^{p+1})$. Let the approximations $\varphi_m, \varphi_{m+1}, \ldots, \varphi_n$ be determined by the multistep method (5). Then we have*

$$h \sum_{s=0}^{n-\mu} \omega_{ns} \psi(x_s) + \sum_{r=1}^{m-1} \widetilde{\omega}_{nr} \varphi_r = \varphi(x_n) - Ch^p \int_0^{x_n} \psi^{(p)}(y) dy + \mathcal{O}(h^{p+1}). \tag{23}$$

Here, $C \in \mathbb{R}$ denotes the error constant of the considered multistep method.

2.5 A starting procedure

For multistep methods (5) to solve the initial value problem (4), we next consider, for $m \geq 2$, the determination of starting values $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}$ of order $p+1$. Note that this order is needed in view of Lemma 2.3 and Corollary 2.6. One option for generating starting values of that order is to approximate for each $x = x_r,\ r = 1, 2, \ldots, m-1$, the integral in (3) by using quadrature methods on finer grids. However, this requires evaluations of the function ψ at additional abscissae. Another scheme is considered next, and for this we restrict the considerations to multistep methods of order $p \leq m$. As a matter of fact (see Remark 3.2 below for details), this assumption is no serious restriction when multistep methods are applied to Volterra integral equations of the first kind (1). This scheme is of the form

$$\varphi_r = h \sum_{s=0}^{m-1} \omega_{rs} \psi_s, \quad r = 1, 2, \ldots, m-1, \tag{24}$$

where $\psi_s = \psi(x_s)$ for $s = 0, 1, \ldots, m-1$. Here, $\omega_{rs} \in \mathbb{R}$ for $r = 1, 2, \ldots, m-1$ and $s = 0, 1, \ldots, m-1$, are appropriately chosen starting weights. It is obvious that in (24), each starting value φ_r $(1 \leq r \leq m-1)$ depends on values $\psi(x_s)$ for $s = 1, 2, \ldots, m-1$, in general, i. e., it is affected also by future times.

A standard procedure that results in a scheme of the form (24) is presented in the following example.

Example 2.7. Consider for fixed $r \in \{1, 2, \ldots, m\}$ (the case $r = m$ is not considered in (24) but will be needed below) an interpolatory quadrature method for the integral $\varphi(x_r) = \int_0^{x_r} \psi(x) dx$ using interpolation nodes $x_0, x_1, \ldots, x_{m-1}$. This in fact means that the resulting quadrature scheme $\varphi_r = h \sum_{s=0}^{m-1} \omega_{rs} \psi(x_s) \approx \varphi(x_r)$ is exact for all polynomials ψ of degree $\leq m-1$, with quadrature weights that are given by the following linear system of equations:

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & m-1 \\ 0 & 1 & 4 & \cdots & (m-1)^2 \\ 0 & 1 & 9 & \cdots & (m-1)^3 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 2^{m-1} & \cdots & (m-1)^{m-1} \end{pmatrix}}_{=:\ M} \begin{pmatrix} w_{r0} \\ w_{r1} \\ w_{r2} \\ \vdots \\ w_{r,m-1} \end{pmatrix} = \begin{pmatrix} r \\ r^2/2 \\ r^3/3 \\ \vdots \\ r^m/m \end{pmatrix}. \tag{25}$$

It follows from standard interpolation theory that for each $0 \leq p \leq m$ and each $\psi \in C^p[0, x_m]$, the interpolation error is of order p, i. e., we have $\max\{\, |\mathcal{P}(x) - \psi(x)| \mid 0 \leq x \leq x_m \,\} = \mathcal{O}(h^p)$ for $\mathcal{P} \in \Pi_m$ with $\mathcal{P}(x_s) = \psi(x_s)$ for $s = 0, 1, \ldots, m-1$. From that we immediately obtain $\varphi_r - \varphi(x_r) = \mathcal{O}(h^{p+1})$ for $r = 1, 2, \ldots, m$ which in fact is the order required in Lemma 2.3 and Corollary 2.6. Note that the matrix

$$M = (s^\nu)_{\substack{\nu=0,\ldots,m-1 \\ s=0,\ldots,m-1}} \in \mathbb{R}^{m \times m}$$

(with the notation $0^0 = 1$) is a Vandermonde matrix which does not depend on n. The right-hand side also does not depend on n, and so we finally obtain the estimate

$$w_{rs} = \mathcal{O}(1) \quad \text{for} \quad r = 1, 2, \ldots, m, \quad s = 0, 1, \ldots, m-1. \tag{26}$$

Remark 2.8. If the starting procedure is of the form (24), then the representation (22) can be replaced by a quadrature representation, i. e.,

$$\varphi_n = h \sum_{s=0}^{n-\mu} \omega_{ns} \psi_s. \tag{27}$$

Here, the quadrature weights ω_{ns} for $s \geq m$ are given by (19), and the starting weights ω_{ns} for $s \leq m-1$ are also easy to determine, details are omitted here. A quadrature method (27) generated by a multistep method (5) is called (ϱ, σ)-reducible (see Taylor [18] and Wolkenfelt ([19], [20])).

3 Linear multistep methods for perturbed first kind Volterra integral equations

3.1 Some preparations

We now return to the first kind Volterra integral equation (1). For the numerical approximation we consider this equation at equidistant nodes $x_n = nh$, $n = 1, 2, \ldots, N$ with $h = L/N$, cf. (2). The resulting integrals $\varphi(x) = \int_0^x k(x_n, y) u(y)\, dy$ for $0 \leq x \leq x_n$ are approximated by the multistep method under consideration, respectively, see (5) with $\psi(y) = k(x_n, y) u(y)$ for $0 \leq y \leq x_n$.

In the sequel we suppose that the right-hand side of equation (1) is only approximately given, with

$$|f_n^\delta - f(x_n)| \leq \delta \quad \text{for } n = 1, 2, \ldots, N, \tag{28}$$

where $\delta > 0$ is a known noise level.

For the main convergence results we impose the following conditions.

Assumption 3.1. For the Volterra integral equation (1) of the first kind and a given m-step method with $m \geq 1$ (see (5)) we assume the following:

(a) The considered m-step method with $m \geq 1$ is nullstable and has order $1 \leq p \leq m$.

(b) The second characteristic polynomial σ satisfies a strong root condition:

$$\sigma(\xi) = 0 \Longrightarrow |\xi| < 1 \qquad (\xi \in \mathbb{C}), \tag{29}$$

i. e., all roots of the polynomial σ belong to the open unit disk.

(c) There exists a solution $u : [0, L] \to \mathbb{R}$ to the integral equation (1) with $u \in C^{p+1}[0, L]$.

(d) For some integer $N_0 \geq 1$ and $h_0 = L/N_0$, the kernel function satisfies $k \in C^{p+1}(E)$, where $E \subset \mathbb{R}^2$ is some open set with $E \supset \{\, (x, y) \mid 0 \leq y \leq x \leq L \,\}$ and $E \supset \{\, (x, y) \mid 0 \leq x, y \leq m h_0 \,\}$.

(e) There holds $k(x, x) = 1$ for each $0 \leq x \leq L$.

(f) For a given step size $h = L/N$ with some integer $N \leq N_0$, let $x_1, x_2, \ldots, x_N$ be uniformly distributed nodes given by (2).

(g) The values of the right-hand side of equation (1) are approximately given by (28).

We next present some comments on the strong root condition considered in item (b) of Assumption 3.1.

Remark 3.2. (a) In the stability analysis to be considered, the coefficients of the inverse power series

$$\frac{1}{\gamma(\xi)} = \sum_{n=0}^{\infty} \gamma_n^{(-1)} \xi^n \tag{30}$$

of the generating function $\gamma(\xi) = \sum_{n=0}^{\infty} \gamma_n \xi^n$, with γ_n as in (16) (see also (18)) play a significant role. The strong root condition (29) implies that the power series γ is analytic in an open set of the complex plane that contains a disk $\{ \xi \in \mathbb{C} \mid |\xi| \le R \}$ for some $R > 1$, and Cauchy's theorem then implies that the coefficients $\gamma_n^{(-1)}$ in (30) decay exponentially, i. e.,

$$\gamma_n^{(-1)} = \mathcal{O}(\tau^n) \quad \text{as } n \to \infty \text{ for some } 0 < \tau < 1, \tag{31}$$

with $\tau = 1/R$ in fact.

(b) It is elementary calculus to show that the strong root condition (29) is satisfied, e.g., by the m-step Adams–Bashfort methods with $1 \le m \le 3$, and by the m-step Nyström method with $2 \le m \le 3$ as well. In addition, (29) is obviously satisfied by the BDF methods.

(c) The strong root condition (29) is violated for each multistep method of class (6) with $\mu = 0$ and with order $p > m$. More generally, it is an essential observation mady by Gladwin/Jeltsch [6] that the second characteristic polynomial σ is even not a simple von Neumann polynomial in that situation, with the case $m = \tau = 1$ (the repeated trapezoidal rule) as an exception. In addition, the associated scheme for solving Volterra integral equations of the first kind introduced below is necessarily divergent then, in general. For the mentioned exception $m = \tau = 1$, the associated second characteristic polynomial is obviously a simple von Neumann polynomial but does not satisfy the strong root condition.

As a consequence of the former observations, it therefore makes sense to reduce the considerations to m-step methods of order $1 \le p \le m$ in the sequel (see (a) of Assumption 3.1). Note that the case $p \le m - 1$ becomes relevant for solutions and kernels of the Volterra integral equation of the first kind (1) that have a lower degree of smoothness. $\triangle$

3.2 The numerical scheme

We consider now, under the conditions given in Assumption 3.1, the following scheme for the numerical solution of a Volterra integral equation (1) with a smooth kernel:

Algorithm 3.3. (a) Determine m initial approximations $u_s^\delta \approx u(x_s)$ for $s = 0, 1, \ldots, m-1$ by solving the following linear system of m equations,

$$h \sum_{s=0}^{m-1} \omega_{ns} k(x_n, x_s) u_s^\delta = f_n^\delta, \qquad n = 1, 2, \ldots, m, \tag{32}$$

where the starting weights ω_{ns} are given by (25), with r replaced by n there.

(b) Determine then recursively approximations $u_{n-\mu}^\delta \approx u(x_{n-\mu})$ for $n = m + \mu, \ldots, N$ by the following scheme:

- Set $\psi_s^\delta = k(x_n, x_s) u_s^\delta$ for $s = 0, 1, \ldots, n - \mu - 1$,
- set $\varphi_0^\delta = 0$, and compute (for $m \ge 2$) $\varphi_r^\delta = h \sum_{s=0}^{m-1} \omega_{rs} \psi_s^\delta$ for $r = 1, 2, \ldots, m-1$, cf. (24),
- compute recursively φ_{r+m}^δ for $r = 0, 1, \ldots, n-m-1$ by using on the interval $[0, x_n]$ the perturbed version of the multistep scheme (5):

$$\sum_{j=0}^{m} a_j \varphi_{r+j}^\delta = h \sum_{j=0}^{m-\mu} b_j \psi_{r+j}^\delta \quad \text{for } r = 0, 1, \ldots, n-m-1, \tag{33}$$

- set $\varphi_n^\delta = f_n^\delta$,

- compute $\psi^\delta_{n-\mu}$ by using the identity (33) for $r = n - m$,
- compute $u^\delta_{n-\mu} = \psi^\delta_{n-\mu}/k(x_n, x_{n-\mu})$. $\triangle$

Remark 3.4. (a) Note that due to (e) in Assumption 3.1, for h sufficiently small we have $k(x_n, x_{n-\mu}) \neq 0$ for each n. Thus the numerical procedure considered above can in fact be used for the computation of $u^\delta_{n-\mu}$.

(b) It immediately follows from Lemma 2.5 that the approximations obtained by Algorithm 3.3 satisfy

$$h \sum_{s=0}^{n-\mu} \omega_{ns} k(x_n, x_s)\, u^\delta_s + \sum_{r=1}^{m-1} \widetilde{\omega}_{nr} \varphi^\delta_r = f^\delta_n, \qquad n = m+\mu, \ldots, N. \tag{34}$$

Here, the weights ω_{ns} and $\widetilde{\omega}_{nr}$ are given by (19) and (20), respectively. Note that the starting values φ^δ_1, $\ldots, \varphi^\delta_{m-1}$ considered in (34) depend also on n. The representation (34) will be used in the proof of the main result, cf. Theorem 3.7.

(c) The scheme considered in Algorithm 3.3 is quite universal. For the backward rectangular rule (which is the 1-step BDF method) considered in part (b) of Example 2.1, an implementation of Algorithm 3.3 without the starting procedure considered in (a) there is possible. This means, however, that no approximation u^δ_0 will be available then. $\triangle$

3.3 Uniqueness, existence and approximation properties of the initial approximations

We now consider uniqueness, existence as well as the approximation properties of the initial approximations $u^\delta_0, u^\delta_1, \ldots, u^\delta_{m-1}$. In a first step we consider in more detail the linear system of equations (32). This system of equations can be written in the form

$$\overbrace{h \begin{pmatrix} \omega_{10}k(x_1,x_0) & \omega_{11}k(x_1,x_1) & \cdots & \omega_{1,m-1}k(x_1,x_{m-1}) \\ \omega_{20}k(x_2,x_0) & \omega_{21}k(x_2,x_1) & \cdots & \omega_{2,m-1}k(x_2,x_{m-1}) \\ \vdots & \vdots & & \vdots \\ \omega_{m0}k(x_m,x_0) & \omega_{m1}k(x_m,x_1) & \cdots & \omega_{m,m-1}k(x_m,x_{m-1}) \end{pmatrix}}^{=:\, S_h} \begin{pmatrix} u^\delta_0 \\ u^\delta_1 \\ \vdots \\ u^\delta_{m-1} \end{pmatrix} = \begin{pmatrix} f^\delta_1 \\ f^\delta_2 \\ \vdots \\ f^\delta_m \end{pmatrix}. \tag{35}$$

Note that the matrix $S_h \in \mathbb{R}^{m\times m}$ introduced in (35) depends on the stepsize h.

Proposition 3.5. *The system matrix S_h in (35) is regular for sufficiently small values of h, and $\|S_h^{-1}\|_\infty = \mathcal{O}(1)$ as $h \to 0$.*

PROOF. We first consider the situation $k \equiv 1$. In a first step we observe that (25) implies the following:

$$\underbrace{M \begin{pmatrix} \omega_{10} & \omega_{11} & \cdots & \omega_{1,m-1} \\ \omega_{20} & \omega_{21} & \cdots & \omega_{2,m-1} \\ \vdots & \vdots & & \vdots \\ \omega_{m0} & \omega_{m1} & \cdots & \omega_{m,m-1} \end{pmatrix}}_{=:\, T} = DB \tag{36}$$

with the matrix $M \in \mathbb{R}^{m\times m}$ from (25), and

$$D = \mathrm{diag}\big(\tfrac{1}{q} \;:\; q = 1, 2, \ldots, m\big) \in \mathbb{R}^{m\times m}, \qquad B = (n^q)_{\substack{q=1,\ldots,m\\ n=1,\ldots,m}} \in \mathbb{R}^{m\times m}.$$

The matrices D, B and M are regular, and hence the matrix $T \in \mathbb{R}^{m\times m}$ introduced in (36) is regular. The latter matrix coincides in the situation $k \equiv 1$ with the matrix S_h.

We now consider the general case for k. We have $k(x, x) = 1$ and $x_n = \mathcal{O}(h)$ for $n = 1, 2, \ldots, m-1$, and thus $k(x_n, x_s) = 1 + \mathcal{O}(h)$ for $n = 1, \ldots, m$ and $s = 0, \ldots, m-1$. This shows $S_h = T + \mathcal{O}(h)$ for $h \to 0$ so that the matrix S_h is regular for sufficiently small values h, with $\|S_h^{-1}\|_\infty$ being bounded as $h \to 0$. This completes the proof of the proposition. □

We next consider the approximation properties of the initial approximations.

Theorem 3.6. *Let the conditions of Assumptions 3.1 be satisfied. Then the initial approximations u_0^δ, $u_1^\delta, \ldots, u_{m-1}^\delta$, determined by (32) for h sufficiently small, satisfy*

$$\max_{n=0,1,\ldots,m-1} |u_n^\delta - u(x_n)| = \mathcal{O}(h^p + \delta/h) \quad \textit{as} \;\; (h, \delta) \to 0.$$

PROOF. It is clear from (35) and Proposition 3.5 that the initial approximations $u_0^\delta, u_0^\delta, \ldots, u_{m-1}^\delta$ exists and are unique for h sufficiently small. We have

$$h \sum_{s=0}^{m-1} \omega_{ns}\, k(x_n, x_s) e_s^\delta = \mathcal{O}(h^{p+1} + \delta) \quad \text{for } \; n = 1, 2, \ldots, m, \tag{37}$$

where

$$e_s^\delta = u_s^\delta - u(x_s), \quad s = 0, 1, \ldots, m-1,$$

denote the approximation errors. This follows from the considerations in Example 2.7, with the notation $r = n$ and for $\psi(y) = k(x_n, y)u(y)$ for $0 \le y \le x_m$. A matrix-vector formulation of (37) yields $hS_h\Delta_h^\delta = \mathcal{O}(h^{p+1} + \delta)$ as $h \to 0$, with $\Delta_h^\delta := (e_0^\delta, e_1^\delta, \ldots, e_{m-1}^\delta)^\top \in \mathbb{R}^m$, and with the matrix S_h from (35). According to Proposition 3.5, this matrix S_h is regular for sufficiently small values of h, and $\|S_h^{-1}\|_\infty = \mathcal{O}(1)$ as $h \to 0$. From this the statement of the theorem follows. □

3.4 The main result

We next present the main result of this paper which extends the results by Wolkenfelt ([19], [20]) to the case of perturbed right-hand sides.

Theorem 3.7. *Let the conditions of Assumption 3.1 be satisfied, and let the approximations $u_0^\delta, u_1^\delta, \ldots, u_{N-\mu}^\delta$ be determined by Algorithm 3.3, for h sufficiently small. Then the following error estimate holds,*

$$\max_{n=0,1,\ldots,N-\mu} |u_n^\delta - u(x_n)| = \mathcal{O}(h^p + \delta/h) \quad \textit{as} \;\; (h, \delta) \to 0. \tag{38}$$

PROOF. The initial approximation errors are already covered by Theorem 3.6, so it remains to estimate the error $u_n^\delta - u(x_n)$ for $n = m, m+1, \ldots, N-\mu$.

(1) In a first step we observe that the following system of error equations holds:

$$h \sum_{s=m}^{n-\mu} \gamma_{n-s-\mu} k(x_n, x_s) e_s^\delta = E_h(x_n) + \mathcal{O}(h^{p+1} + \delta) \quad \text{for } \; n = m+\mu, \ldots, N, \tag{39}$$

where

$$e_s^\delta = u_s^\delta - u(x_s), \quad s = m, \dots, N-\mu,$$
$$E_h(x_n) = Ch^p \int_0^{x_n} \psi^{[n,p]}(y)dy, \quad n = m+\mu, \dots, N, \tag{40}$$

with the notation

$$\psi^{[n,t]}(y) = \frac{\partial^t}{\partial y^t}\psi^{[n]}(y), \quad t = 0, 1, \dots, p, \quad 0 \le y \le x_n, \tag{41}$$

with

$$\psi^{[n]}(y) = k(x_n, y)u(y), \quad 0 \le y \le x_n. \tag{42}$$

The error representation (39) follows by considering the difference of the representations (34) and (23), where the latter representation is applied with $\psi = \psi^{[n]}$ on the interval $[0, x_n]$. In addition, the representation (19) of the weights ω_{ns} for $n \ge m$ as well as the boundedness (21) of the starting weights is used here. Note that (37) implies $\varphi_s^\delta - \varphi_s = \mathcal{O}(h^{p+1} + \delta)$ for $s = 0, 1, \dots, m-1$, where φ_s is given by (24) with $\psi = \psi^{[n]}$ there, considered on the interval $0 \le y \le x_n$.

(2) We next consider a matrix-vector formulation of (39). As a preparation we introduce the notation

$$N_1 := N - m - \mu + 1 \tag{43}$$

and consider the system matrix $A_h \in \mathbb{R}^{N_1 \times N_1}$ given by

$$A_h = \begin{pmatrix} \gamma_0 k_{m+\mu,m} & 0 & \cdots & \cdots & 0 \\ \gamma_1 k_{m+\mu+1,m} & \gamma_0 k_{m+\mu+1,m+1} & \ddots & & 0 \\ \vdots & \gamma_1 k_{m+\mu+2,m+1} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \gamma_{N-m-\mu} k_{Nm} & \cdots & \cdots & \gamma_1 k_{N,N-\mu-1} & \gamma_0 k_{N,N-\mu} \end{pmatrix}$$

with the notation

$$k_{ns} \;=\; k(x_n, x_s) \quad \text{for } \; m \le s \le n-\mu, \quad m+\mu \le n \le N.$$

In addition we consider the vectors

$$\Delta_h^\delta = (e_s^\delta)_{s=m,\dots,N-\mu}, \quad R_h = (E_h(x_n))_{n=m+\mu,\dots,N}. \tag{44}$$

Using these notations, the linear system of equations (39) obviously takes the form

$$hA_h\Delta_h^\delta = R_h + F_h^\delta, \quad \text{with some } \; F_h^\delta \in \mathbb{R}^{N_1}, \; \|F_h^\delta\|_\infty = \mathcal{O}(h^{p+1} + \delta), \tag{45}$$

where $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{R}^{N_1}$.

(3) For a further treatment of the identity (45) we next prove that

$$\|D_h\|_\infty = \mathcal{O}(1), \qquad \|(D_hA_h)^{-1}\|_\infty = \mathcal{O}(1), \qquad \|A_h^{-1}\|_\infty = \mathcal{O}(1) \quad \text{as } \; h \to 0, \tag{46}$$

where the matrix $D_h \in \mathbb{R}^{N_1 \times N_1}$ is given by

$$D_h = \begin{pmatrix} \gamma_0^{(-1)} & 0 & \cdots & \cdots & 0 \\ \gamma_1^{(-1)} & \gamma_0^{(-1)} & 0 & & 0 \\ \gamma_2^{(-1)} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \gamma_{N-m-\mu}^{(-1)} & \cdots & \cdots & \gamma_1^{(-1)} & \gamma_0^{(-1)} \end{pmatrix}, \tag{47}$$

and $\|\cdot\|_\infty$ denotes the matrix norm induced by the maximum vector norm on $\mathbb{R}^{N_1}$. In fact, the estimate $\|D_h\|_\infty = \mathcal{O}(1)$ as $h \to 0$ follows immediately from the exponential decay of the coefficients of the inverse of the generating function γ, cf. estimate (31). For the proof of the second statement in (46) we next show that the matrix $D_h A_h$ can be written in the form

$$D_h A_h = I_h + K_h, \tag{48}$$

where $I_h \in \mathbb{R}^{N_1 \times N_1}$ denotes the identity matrix, and $K_h = (\kappa_{nj}(h)) \in \mathbb{R}^{N_1 \times N_1}$ denotes some lower triangular matrix which satisfies $\max_{0 \le j \le n \le N-m-\mu} |\kappa_{nj}(h)| = \mathcal{O}(h)$ as $h \to 0$. This representation shows that the matrix $D_h A_h$ is nonsingular for h small enough, and the discrete version of Gronwall's inequality yields $\|(D_h A_h)^{-1}\|_\infty = \mathcal{O}(1)$ as $h \to 0$ then. The third estimate in (46) follows immediately from the other two estimates considered in (46).

In the sequel it will be shown that the representation (48) is satisfied, and for this we consider the lower triangular matrix

$$D_h A_h = (b_{nj}) \in \mathbb{R}^{N_1 \times N_1}$$

in more detail. In fact, we have for $0 \le j < n \le N - m - \mu$

$$b_{nj} = \sum_{\ell=j}^{n} \gamma_{n-\ell}^{(-1)} \gamma_{\ell-j} k(x_{m+\mu+\ell}, x_{j+m}) = \sum_{\ell=0}^{n-j} \gamma_{n-j-\ell}^{(-1)} \gamma_\ell k(x_{m+\mu+\ell+j}, x_{j+m})$$

$$= k(x_{m+\mu+n}, x_{j+m}) \overbrace{\sum_{\ell=0}^{n-j} \gamma_{n-j-\ell}^{(-1)} \gamma_\ell}^{=\,0} + \sum_{\ell=0}^{n-j} \big[\gamma_{n-j-\ell}^{(-1)} \gamma_\ell \big(k(x_{m+\mu+\ell+j}, x_{j+m}) - k(x_{m+\mu+n}, x_{j+m}) \big) \big]$$

and thus

$$|b_{nj}| = \mathcal{O}\Big(h \underbrace{\sum_{\ell=0}^{n-j} |\gamma_{n-j-\ell}^{(-1)}| |\gamma_\ell| (n-j-\ell)}_{\overset{(*)}{=} \mathcal{O}(1)} \Big) = \mathcal{O}(h) \quad \text{for} \quad 0 \le j < n \le N - m - \mu \tag{49}$$

uniformly with respect to j and n. Here, $(*)$ follows immediately from (17) and (31). Moreover we have

$$b_{nn} = \gamma_0^{(-1)} k(x_{n+m+\mu}, x_{n+m}) \gamma_0 = 1 + \mathcal{O}(h) \quad \text{for} \ \ n = 0, 1, \ldots, N - m - \mu, \tag{50}$$

which in fact follows from the identities $\gamma_0^{(-1)} = 1/\gamma_0$ and $k(x, x) \equiv 1$, cf. (e) in Assumption 3.1. The statements (49) and (50) show that the lower triangular matrix $D_h A_h$ in fact can be written as in (48).

(4) The statement of the theorem now follows easily from the error representation (39) and its matrix version (44), (45), and from the stability estimates in (46). We only need to take a closer look at the global error representation (40) of the quadrature error. For this purpose we observe that by Assumption 3.1 we have $\psi^{[n,p]} \in C^1[0, x_n]$ (for the definition of this function see (41)). Since each m-step method of order $p \geq 1$ is in particular of order 1, the integral in the global error expansion (40) can be written as

$$\int_0^{x_n} \psi^{[n,p]}(y)dy = h \sum_{s=0}^{n-\mu} \omega_{ns} \psi^{[n,p]}(x_s) + \mathcal{O}(h) = h \sum_{s=m}^{n-\mu} \omega_{ns} \psi^{[n,p]}(x_s) + \mathcal{O}(h) \tag{51}$$

for $n = m+\mu, \ldots, N$. The first identity in (51) follows from Lemma 2.5, applied with $\psi_s = \psi^{[n,p]}(x_s)$ for $s = 0, 1, \ldots, n-\mu$ and with $\varphi_1 = \cdots = \varphi_{m-1} = 0$, and from (9) with $p = 1$. In the second identity of (51), the boundedness of the starting weights (see (26)) is taken into account. From (51) and the Leibniz rule for derivatives, we obtain the following for $E_h(x_n)$ defined by (40):

$$E_h(x_n) = Ch^{p+1} \sum_{q=0}^{p} \tbinom{p}{q} \sum_{s=m}^{n-\mu} \gamma_{n-s-\mu} g^{[n,p-q]}(x_s) u^{(q)}(x_s) + \mathcal{O}(h^{p+1})$$

uniformly for $n = m + \mu, \ldots, N$, where

$$g^{[n,t]}(y) = \tfrac{\partial^t}{\partial y^t} k(x_n, y) \quad \text{for } t = 0, 1, \ldots, p.$$

From that it follows that $R_h \in \mathbb{R}^{N_1}$ considered in (44) can be written as follows,

$$R_h = Ch^{p+1} \sum_{q=0}^{p} \tbinom{p}{q} B_h^{(p-q)} U_h^{(q)} + \mathcal{E}_h, \tag{52}$$

where $\mathcal{E}_h \in \mathbb{R}^{N_1}$ denotes some vector with $\|\mathcal{E}_h\|_\infty = \mathcal{O}(h^{p+1})$, and

$$B_h^{(t)} = \begin{pmatrix} \gamma_0 g^{[m+\mu,t]}(x_m) & 0 & \cdots & \cdots & 0 \\ \gamma_1 g^{[m+\mu+1,t]}(x_m) & \gamma_0 g^{[m+\mu+1,t]}(x_{m+1}) & \ddots & & 0 \\ \vdots & \gamma_1 g^{[m+\mu+2,t]}(x_{m+1}) & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \gamma_{N-m-\mu} g^{[N,t]}(x_m) & \cdots & \cdots & \gamma_1 g^{[N,t]}(x_{N-\mu-1}) & \gamma_0 g^{[N,t]}(x_{N-\mu}) \end{pmatrix},$$

$$U_h^{(t)} = \begin{pmatrix} u^{(t)}(x_m) \\ u^{(t)}(x_{m+1}) \\ \vdots \\ u^{(t)}(x_{N-\mu}) \end{pmatrix} \quad \text{for } t = 0, 1, \ldots, p.$$

The representations (45) and (52) give

$$hA_h\Delta_h^\delta = Ch^{p+1}\sum_{q=0}^{p}\binom{p}{p}B_h^{(p-q)}U_h^{(q)} + \mathcal{E}_h + F_h^\delta,$$

and thus

$$D_hA_h\Delta_h^\delta = Ch^p\sum_{q=0}^{p}\binom{p}{q}D_hB_h^{(p-q)}U_h^{(q)} + \tfrac{1}{h}D_h\mathcal{E}_h + \tfrac{1}{h}D_hF_h^\delta. \tag{53}$$

For $t = 0, 1, \ldots, p$, the lower triangular matrices $D_hB_h^{(t)}$ can be written as follows, $D_hB_h^{(t)} = M_h^{(t)} + C_h^{(t)}$ with the diagonal matrix $M_h = \mathrm{diag}(g^{[m+\mu+n,t]}(x_n) : n = 0, 1, \ldots, N-m-\mu)$ and some lower triangular matrix $C_h^{(t)} = (c_{nj}^{(t)}(h)) \in \mathbb{R}^{N_1\times N_1}$ with $\max_{0\le j\le n\le N-m-\mu}|c_{nj}^{(t)}(h)| = \mathcal{O}(h)$ as $h \to 0$ (see the third part of this proof for similar results for the matrix D_hA_h; details are omitted here). This in particular means

$$\|D_hB_h^{(t)}\|_\infty = \mathcal{O}(1) \quad \text{as } h \to 0 \quad (t = 0, 1, \ldots, p). \tag{54}$$

This completes the proof of the theorem. □

Remark 3.8. The stability analysis presented in the third part of the proof of Theorem 3.7 uses techniques similar to those used in Eggermont [5]; see also Lubich [14] as well as [16] and [17].

In the sequel, for step sizes $h = h(\delta) = L/N$, with a slight abuse of notation we write $h \sim \delta^\beta$ as $\delta \to 0$, if there exist real constants $c_2 \ge c_1 > 0$ such that $c_1h \le \delta^\beta \le c_2h$ holds for $\delta \to 0$. As an immediate consequence of Theorem 3.7 we obtain the following main result of this paper.

Corollary 3.9. *Let Assumption 3.1 be satisfied. For $h = h(\delta) \sim \delta^{1/(p+1)}$ we have*

$$\max_{n=0,1,\ldots,N-\mu}|u_n^\delta - u(x_n)| = \mathcal{O}(\delta^{p/(p+1)}) \quad \textit{as } \delta \to 0.$$

Each m-step method of order $p \ge 1$ is also of order $1 \le q \le p$, with accordingly reduced smoothness assumptions on the involved functions in the considered Volterra integral equation of the first kind (1) then. This fact allows to consider lower degrees of smoothness there.

Corollary 3.10. *Let Assumption 3.1 be satisfied, with the smoothness assumptions in items (c) and (d) there replaced by $u \in C^{q+1}[0, L]$ and $k \in C^{q+1}(E)$, respectively, with some $1 \le q \le p$. Let the approximations $u_0^\delta, u_1^\delta, \ldots, u_{N-\mu}^\delta$ be determined by the scheme considered in Algorithm 3.3. Then the following holds:*

(a) We have

$$\max_{n=0,1,\ldots,N-\mu}|u_n^\delta - u(x_n)| = \mathcal{O}(h^q + \delta/h) \quad \textit{as } (h, \delta) \to 0.$$

(b) For $h = h(\delta) \sim \delta^{1/(q+1)}$ we have

$$\max_{n=0,1,\ldots,N-\mu}|u_n^\delta - u(x_n)| = \mathcal{O}(\delta^{q/(q+1)}) \quad \textit{as } \delta \to 0.$$

We conclude this section with some remarks.

Remark 3.11. (a) The result of part (a) of the latter corollary follows in the case $1 \le q \le p-1$ more easily than for $q = p$. In the former case $1 \le q \le p-1$, in the proof of Theorem 3.7 it would be in fact sufficient to make use of the global error estimate (9) (with p replaced by q there), instead of using the global error expansion considered in Lemma 2.3 (with p replaced by q there).

(b) Part (b) of Corollary 3.10 is of limited use only. In that case it is useful to apply an explicit m-step method for $m = q$ in order to keep computational complexity as small as possible.

Remark 3.12. For results on the regularization properties of the composite midpoint rule, see e. g. Apartsin [1] or Kaltenbacher [11]. For other special regularization methods for the approximate solution of Volterra integral equations of the first kind with smooth kernels and perturbed right-hand sides, see e.g., Lamm [12].

4 Numerical experiments

As an illustration of the main result considered in Corollary 3.9, we next present the results of numerical experiments for three examples of Volterra integral equations of the first kind with smooth kernels of the form (1), treated by different multistep methods, respectively. Here are some remarks on the numerical tests:

- Numerical experiments with step sizes $h = 1/2^\nu$ for $\nu = 5, 6, \ldots, 12$ are employed, respectively, with the exception of the order 4 BDF method. In the latter method, the influence of rounding errors becomes clearly visible for $\nu \ge 10$.
- For each considered step size h and each considered p-order multistep method, the noise level $\delta = h^{1/(p+1)}$ is considered.
- In the numerical experiments, the perturbations are of the form $f_n^\delta = f(x_n) + \Delta_n$ with uniformly distributed random values Δ_n with $|\Delta_n| \le \delta$.

First we consider the repeated midpoint rule which in fact coincides with the 2-step Nyström method (see Example 2.1). In the formulation (5), this method reads as follows, $\varphi_{r+2} - \varphi_r = 2h\psi_{r+1}$ for $r = 0$, $1, \ldots, n-2$. This method is applied to the following linear Volterra integral equation of the first kind,

$$\int_0^x \cos(x-y)u(y)\,dy = \sin x =: f(x) \quad \text{for } 0 \le x \le 1, \tag{55}$$

with exact solution $u(y) = 1$ for $0 \le y \le 1$. The conditions of Assumption 3.1 are satisfied with $m = 2$ and $p = 2$. The numerical results of are shown in Table 1. There, $\|f\|_\infty$ denotes the maximum norm of the function f. All numerical experiments are employed using the program system OCTAVE (http://www.octave.org).

N	δ	$100 \cdot \delta / \|f\|_\infty$	$\max_n \lvert u_n^\delta - u(x_n)\rvert$	$\max_n \lvert u_n^\delta - u(x_n)\rvert \,/\delta^{2/3}$
32	$3.1 \cdot 10^{-5}$	$3.70 \cdot 10^{-3}$	$1.05 \cdot 10^{-3}$	1.07
64	$3.8 \cdot 10^{-6}$	$4.58 \cdot 10^{-4}$	$3.09 \cdot 10^{-4}$	1.27
128	$4.8 \cdot 10^{-7}$	$5.70 \cdot 10^{-5}$	$6.56 \cdot 10^{-5}$	1.08
256	$6.0 \cdot 10^{-8}$	$7.10 \cdot 10^{-6}$	$1.69 \cdot 10^{-5}$	1.11
512	$7.5 \cdot 10^{-9}$	$8.87 \cdot 10^{-7}$	$7.25 \cdot 10^{-6}$	1.90
1024	$9.3 \cdot 10^{-10}$	$1.11 \cdot 10^{-7}$	$1.09 \cdot 10^{-6}$	1.14
2048	$1.2 \cdot 10^{-10}$	$1.38 \cdot 10^{-8}$	$2.71 \cdot 10^{-7}$	1.14
4096	$1.5 \cdot 10^{-11}$	$1.73 \cdot 10^{-9}$	$6.71 \cdot 10^{-8}$	1.13

Table 1: Numerical results of the repeated midpoint rule applied to equation (55)

Next we present some numerical results of the order 4 BDF method which in the formulation (5) reads as follows, $\frac{1}{12}(25\varphi_{r+4} - 48\varphi_{r+3} + 36\varphi_{r+2} - 16\varphi_{r+1} + 3\varphi_r) = h\psi_{r+4}$ for $r = 0, 1, \ldots, n-4$. This

method is applied to the same operator as for the first numerical experiment but with another right-hand side:

$$\int_0^x \cos(x-y)u(y)\,dy = \underbrace{1-\cos x}_{=:\,f(x)} \quad \text{for } 0 \le x \le 1, \tag{56}$$

with exact solution $u(y) = y$ for $0 \le y \le 1$. The conditions of Assumption 3.1 are satisfied with $m = 4$ and $p = 4$. Step sizes, noise levels, initial approximations and starting values are chosen similar to the example considered above. The results are shown in Table 2.

N	δ	$100 \cdot \delta / \|f\|_\infty$	$\max_n \lvert u_n^\delta - u(x_n)\rvert$	$\max_n \lvert u_n^\delta - u(x_n)\rvert \,/\delta^{4/5}$
32	$3.0 \cdot 10^{-8}$	$6.48 \cdot 10^{-6}$	$7.14 \cdot 10^{-6}$	7.48
64	$9.3 \cdot 10^{-10}$	$2.03 \cdot 10^{-7}$	$4.85 \cdot 10^{-7}$	8.14
128	$2.9 \cdot 10^{-11}$	$6.33 \cdot 10^{-9}$	$2.85 \cdot 10^{-8}$	7.65
256	$9.1 \cdot 10^{-13}$	$1.98 \cdot 10^{-10}$	$2.11 \cdot 10^{-9}$	9.07
512	$2.8 \cdot 10^{-14}$	$6.18 \cdot 10^{-12}$	$1.28 \cdot 10^{-10}$	8.83
1024	$8.9 \cdot 10^{-16}$	$1.93 \cdot 10^{-13}$	$2.32 \cdot 10^{-11}$	25.50

Table 2: Numerical results of the 4th order BDF method applied to equation (56)

Finally we present the results of numerical experiments with the second order Adams–Bashfort method $\varphi_{r+2} - \varphi_{r+1} = \frac{h}{2}(3\psi_{r+1} - \psi_r)$ for $r = 0, 1, \ldots, n-2$, applied to the following test problem:

$$\int_0^x (1+x-y)u(y)\,dy = \underbrace{x-1+e^{-x}}_{=:\,f(x)} \quad \text{for } 0 \le x \le 1, \tag{57}$$

with exact solution $u(y) = ye^{-y}$ for $0 \le y \le 1$. The conditions of Assumption 3.1 are satisfied with $m = 2$ and $p = 2$. Step sizes, noise levels, initial approximations and starting values are chosen similar to the example considered above. The results are shown in Table 3.

N	δ	$100 \cdot \delta / \|f\|_\infty$	$\max_n \lvert u_n^\delta - u(x_n)\rvert$	$\max_n \lvert u_n^\delta - u(x_n)\rvert \,/\delta^{2/3}$
32	$3.1 \cdot 10^{-5}$	$8.76 \cdot 10^{-3}$	$1.93 \cdot 10^{-3}$	1.98
64	$3.8 \cdot 10^{-6}$	$1.07 \cdot 10^{-3}$	$5.21 \cdot 10^{-4}$	2.13
128	$4.8 \cdot 10^{-7}$	$1.31 \cdot 10^{-4}$	$1.29 \cdot 10^{-4}$	2.11
256	$6.0 \cdot 10^{-8}$	$1.63 \cdot 10^{-5}$	$3.84 \cdot 10^{-5}$	2.52
512	$7.5 \cdot 10^{-9}$	$2.03 \cdot 10^{-6}$	$8.99 \cdot 10^{-6}$	2.36
1024	$9.3 \cdot 10^{-10}$	$2.54 \cdot 10^{-7}$	$2.36 \cdot 10^{-6}$	2.47
2048	$1.2 \cdot 10^{-10}$	$3.17 \cdot 10^{-8}$	$5.95 \cdot 10^{-7}$	2.50
4096	$1.5 \cdot 10^{-11}$	$3.96 \cdot 10^{-9}$	$1.60 \cdot 10^{-7}$	2.68

Table 3: Numerical results of the 2nd order Adams–Bashfort method applied to equation (57)

Note that the relative errors in the right-hand side presented in the third column (of all three tables in fact) are rather small, respectively.

5 Conclusions

In the present paper we present a comprehensive analysis for multistep methods for the regularization of Volterra integral equations of the first kind with smooth kernels and perturbed given right-hand sides. The applied techniques are closely related to those used in Wolkenfelt ([19], [20]). The results presented here (which include some numerical experiments) have useful applications for the stable solution of inverse problems.

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