Minimizing coincidence numbers of maps into projective spaces

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Abstract

In this paper we continue to study ("strong") Nielsen coincidence numbers (which were introduced recently for pairs of maps between manifolds of arbitrary dimensions) and the corresponding minimum numbers of coincidence points and pathcomponents. We explore compatibilities with fibrations and, more specifically, with covering maps, paying special attention to selfcoincidence questions. As a sample application we calculate each of these numbers for all maps from spheres to (real, complex, or quaternionic) projective spaces. Our results turn out to be intimately related to recent work of D. Gonçalves and D. Randall concerning maps which can be deformed away from themselves but not by small deformations; in particular, there are close connections to the Strong Kervaire Invariant One Problem.

1 Introduction and statement of results.

This work is dedicated to the memory of Heiner Zieschang who contributed so substantially to many areas of topology and, in particular, to coincidence theory in codimensions 0 (see e.g. [BGZ 2], [GKZ] and the references given there, as well as the very clear and helpful survey article [BGZ 1]).

In this paper we study coincidences in higher codimensions. As in [K 6] we will use geometric methods which involve (nonstabilized) normal bordism theory and path spaces. It was shown in [K 4] that a similar approach can also be applied fruitfully to certain knotting and linking phenomena – another subject which Heiner Zieschang investigated for many years.

Let $f_1, f_2 : M \to N$ be (continuous) maps between smooth connected manifolds without boundary of arbitrary positive dimensions m and n, M being compact. Our interest centers around coincidence spaces such as

$$(1.1) \qquad C(f_1, f_2) = \{x \in M |\ f_1(x) = f_2(x)\}.$$

We want to determine the **m**inimum number of **c**oincidence points

$$(1.2) \qquad MC(f_1, f_2) \ := \ \min\{\#C(f_1', f_2')|\ f_1' \sim f_1, f_2' \sim f_2\}$$

and the **m**inimum number of **c**oincidence **c**omponents

$$(1.3) \qquad MCC(f_1, f_2) := \min\{\#\pi_0(C(f_1', f_2'))|\ f_1' \sim f_1, f_2' \sim f_2\}$$

(compare [K 6] for the details concerning these and the following definitions).

Generically the map $(f_1, f_2) : M \to N \times N$ is smooth and transverse to the diagonal $\Delta = \{(y, y) \in N \times N|\ y \in N\}$. Then the coincidence locus

$$(1.4) \qquad C = C(f_1, f_2) \ = \ (f_1, f_2)^{-1}(\Delta) \qquad \subset \ M$$

is a closed smooth $(m-n)$–dimensional submanifold of M. Its normal bundle $\nu(C, M)$ is described by a canonically arising vector bundle isomorphism

$$(1.5) \qquad \overline{g}^{\#} \ : \ \nu(C, M) \ \cong \ f_1^*(TN)|C \quad .$$

Moreover the obvious fiber projection from the space

$$(1.6) \qquad E(f_1, f_2) = \{(x, \theta) \in M \times P(N)|\ \theta(0) = f_1(x), \theta(1) = f_2(x)\}$$

(where $P(N)$ denotes the set of continuous paths, equipped with the compact–open topology)) to M allows a canonical section $\widetilde{g}$ over C:

$$(1.7) \qquad \widetilde{g}(x) := (x, \text{ constant path at } f_1(x) = f_2(x)), \quad x \in C \ .$$

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The resulting bordism class

(1.8) $\omega^\#(f_1, f_2) \ = \ [C, \widetilde{g}, \overline{g}^\#] \ \in \ \Omega^\#(f_1, f_2)$

in the bordism set of such triples is our key invariant (see [K 6] for details).

Definition 1.9   (i)   We call a pathcomponent A of the space $E(f_1, f_2)$
strongly essential if the corresponding (partial) bordism class

$$\omega_A^\#(f_1, f_2) \ := \ [C_A := \widetilde{g}^{-1}(A), \ \widetilde{g}|C_A, \ \overline{g}^\#|]$$

is nontrivial.

(ii)   We define the *strong Nielsen number* $N^\#(f_1, f_2)$ of f_1 and f_2 to be
the number of strongly essential pathcomponents $A \in \pi_0(E(f_1, f_2))$.

Theorem 1.10   *(cf. [K 6], 1.11 and 3.5).   The finite number $N^\#(f_1, f_2) =
N^\#(f_2, f_1)$ depends only on the homotopy classes of f_1 and f_2. We have:*

$$0 \ \leq \ N^\#(f_1, f_2) \ \leq \ MCC(f_1, f_2) \leq MC(f_1, f_2) \ \leq \infty \ \ .$$

*If $n \neq 2$, then also $MCC(f_1, f_2) \leq \# \pi_0(E(f_1, f_2))$;   if $(m, n) \neq (2, 2)$,   then
$MC(f_1, f_2) \leq \# \pi_0(E(f_1, f_2))$ or $MC(f_1, f_2) = \infty$.*

(Recall from [K 3], 2.1 that the cardinality $\# \pi_0(E(f_1, f_2))$ can be interpreted
as the cardinality of a wellknown Reidemeister set).

Definition 1.11   We call the pair (f_1, f_2) *loose* if the maps f_1, f_2 can be
deformed away from one another or, equivalently, if $M(C)C(f_1, f_2) = 0$.

We have the obvious implications

(1.12) (f_1, f_2) is loose $\implies$ $\omega^\#(f_1, f_2) = 0$ $\implies$ $N^\#(f_1, f_2) \ = \ 0.$

Question 1.13   Do the converse implications hold? More generally: does the
minimum coincidence number $MCC(f_1, f_2)$ coincide with the strong Nielsen
number $N^\#(f_1, f_2)$?

E.g. if $N^\#(f_1, f_2) = 0$ then there are individual nulbordisms for each C_A (cf.
1.9); but do they fit together to yield a disjointly embedded nulbordism for *all*
of $C = \ II \ C_A$?

The answer to question 1.13 is positive for maps between spheres.

Example 1.14 (cf. [K 6], 1.24). Consider maps $f_1, f_2 : S^m \to S^n$ where $m, n \geq 1$, and let A denote the antipodal involution. Then

$$MCC(f_1, f_2) = N^{\#}(f_1, f_2) = \begin{cases} 0 & \text{if } f_1 \sim A \circ f_2 \text{ ;} \\ \#\pi_0(E(f_1, f_2)) & \text{else} \end{cases}.$$

Remark 1.15 Clearly $N^{\#}, MCC$ and MC coincide for all $f_1, f_2 : M \to N$ whenever $m < n$ or, in case $M = S^m$, whenever $m = 1$ or $n = 1$ (compare e.g. [K 6], 1.12). Thus we will be mainly interested in situations where $m \geq n \geq 2$.

In this paper we give a new interpretation of our Nielsen numbers in terms of (liftings to) covering spaces (see section 3 below). As a sample application we compute Nielsen and minimum coincidence numbers for all pairs of maps from spheres to projective spaces.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ denote the field of real, complex or quaternionic numbers, and let $d = 1, 2$, or 4 be its real dimension. Let $\mathbb{K}P(n')$ and $V_{n'+1,2}(\mathbb{K})$, resp., denote the corresponding space of lines and of orthonormal 2–frames, resp., in $\mathbb{K}^{n'+1}$. The real dimension of $N = \mathbb{K}P(n')$ is $n := d \cdot n'$.

Consider the diagram
(1.16)

$$\ldots \longrightarrow \pi_m(V_{n'+1,2}(\mathbb{K})) \xrightarrow{p_{\mathbb{K}*}} \pi_m(S^{n+d-1}) \xrightarrow{\partial_{\mathbb{K}}} \pi_{m-1}(S^{n-1}) \longrightarrow \ldots$$

$$\downarrow{p_*} \qquad\qquad\qquad \downarrow{E}$$

$$\pi_m(\mathbb{K}P(n')) \qquad\qquad \pi_m(S^n)$$

determined by the canonical fibrations p and $p_{\mathbb{K}}$; E denotes the Freudenthal suspension homomorphism.

In view of example 1.14 and remark 1.15 above (as well as of example 1.24 and the appendix in [K 6]) the following result determines our Nielsen and minimum numbers for all $f_1, f_2 : S^m \to \mathbb{K}P(n')$, $m, n' \geq 1$. (Proofs will be given in section 6 below).

Theorem 1.17 *Assume $m, n' \geq 2$. Given $[f_i] \in \pi_m(\mathbb{K}P(n'))$, there is a unique homotopy class $[\widetilde{f_i}] \in \pi_m(S^{n+d-1})$ such that $p_*([\widetilde{f_i}]) - [f_i]$ lies in the image of $\pi_m(\mathbb{K}P(n'-1))$, $i = 1, 2$. (Since this image is isomorphic to $\pi_{m-1}(S^{d-1})$, we may assume that $\widetilde{f_i}$ is a genuine lifting of f_i when $\mathbb{K} = \mathbb{R}$ or when $m > 2$ and $\mathbb{K} = \mathbb{C}$). Define $[f_i'] := [p \circ \widetilde{f_i}] \in \pi_m(\mathbb{K}P(n'))$.*

Each pair of homotopy classes $[f_1], [f_2] \in \pi_m(\mathbb{K}P(n'))$ satisfies precisely one of the seven conditions which are listed in table 1.18, together with the corresponding Nielsen and minimum numbers.

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Condition	$N^{\#}(f_1,f_2)$	$MCC(f_1,f_2)$	$MC(f_1,f_2)$
1) $f_1' \sim f_2'$, $[\widetilde{f_2}] \in \ker \partial_{\mathbb{K}}$	0	0	0
2) $f_1' \sim f_2'$, $[\widetilde{f_2}] \in \ker E \circ \partial_{\mathbb{K}} - \ker \partial_{\mathbb{K}}$	0	1	1
3) $\mathbb{K} = \mathbb{R}$, $f_1' \sim f_2'$, $\widetilde{f_2} \not\sim A \circ \widetilde{f_2}$	1	1	1
4) $\mathbb{K} = \mathbb{R}$, $f_1' \not\sim f_2'$, $[\widetilde{f_1}] - [\widetilde{f_2}] \in E\,(\pi_{m-1}(S^{n-1}))$	2	2	2
5) $\mathbb{K} = \mathbb{R}$, $[\widetilde{f_1}] - [\widetilde{f_2}] \notin E(\pi_{m-1}(S^{n-1}))$	2	2	∞
6) $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, $[\widetilde{f_1}] = [\widetilde{f_2}] \notin \ker E \circ \partial_{\mathbb{K}}$	1	1	1
7) $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$, $[\widetilde{f_1}] \neq [\widetilde{f_2}]$	1	1	∞

Table 1.18   Nielsen and minimum coincidence numbers of all pairs of maps $f_1, f_2 : S^m \to \mathbb{K}P(n')$, $m, n' \geq 2$: replace each (possibly base point free) homotopy class $[f_i]$ by a base point preserving representative and read off the values of $N^{\#}$ and $M(C)C$. (Here $f_1' \sim f_2'$ means that f_1', f_2' are homotopic in the basepoint free sense; A denotes the antipodal map).

Example 1.19 $: \mathbb{K} = \mathbb{R}$, $m = 11$, $n = 6$.   According to [T] and [P] we have in 1.16

$$\frac{1}{2} H : \pi_{11}(S^6) \overset{\cong}{\longrightarrow} \mathbb{Z}; \quad \pi_{10}(S^5) \cong \mathbb{Z}_2; \quad \pi_{10}(V_{7,2}) = 0$$

where H denotes the Hopf invariant. Thus $\partial_{\mathbb{K}}$ is surjective and $E \equiv 0$ here.

Given maps $f_1, f_2 : S^{11} \to \mathbb{R}P(6)$, the numbers $N^{\#}(f_1, f_2)$ and $MCC(f_1, f_2)$ differ precisely if f_1 and f_2 are homotopic and $H(\widetilde{f_i}) \equiv 2(4)$, $i = 1, 2$. In this case $\omega^{\#}(f_1, f_2) = 0$ but the pair (f_1, f_2) is not loose.   □

In [K 3] we studied the looseness obstruction

(1.20) $\widetilde{\omega}(f_1, f_2) \in \Omega_{m-n}(E(f_1, f_2); \widetilde{\varphi})$

which lies in a manageable normal bordism group (or, equivalently, in a stable homotopy group) and is often accessible to computations. However already when N is a sphere $\widetilde{\omega}$ turned out not to be a *complete* looseness obstruction. In order to remedy this we introduced $\omega^{\#}$ as a "desuspended" version of $\widetilde{\omega}$ which captures also nonstabilized geometric coincidence data. As the counterexamples in 1.19 show we have not quite desuspended far enough: if $[f_1] = [f_2] \in \pi_{11}(\mathbb{R}P(6))$ then the precise looseness obstruction for (f_1, f_2) is

the homotopy class $\partial_{\mathbb{R}}([\widetilde{f_1}])$, but $\omega^\#(f_1, f_2)$ is only as strong as the (once!) suspended value $E(\partial_{\mathbb{R}}([\widetilde{f_1}])$.

Example 1.19 implies that the first part of the first question as well as the second question in 1.13 does not always have a positive answer. However, if $M = S^m$ and N is a projective space then $N^\#$ has the following property (analogous to a norm on a vector space).

Theorem 1.21 *Given any maps* $f_1, f_2 : S^m \to \mathbb{K}P(n')$ *(where* $m, n' \geq 1$ *and* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$*), we have:*

$$\omega^\#(f_1, f_2) = 0 \quad \text{if and only if} \quad N^\#(f_1, f_2) = 0$$

This follows from the fact that for $m, n' \geq 2$ the Nielsen number vanishes only in the selfintersection setting where $f_1 \sim f_2$ and automatically all pathcomponents of $E(f_1, f_2)$ but one are strongly inessential (cf. 5.1).

Observe also that the weakness of the $\omega^\#$–invariant illustrated by example 1.19 occurs here only in the selfcoincidence setting. In contrast, in the root setting (where f_2 is constant) $\omega^\#(f_1, *)$ is a complete looseness obstruction for all $[f_1] \in \pi_m(\mathbb{K}P(n'))$ (compare [K 6], 6.6).

For $\mathbb{K} = \mathbb{R}$ let us give a more systematic treatment of the case 2 in table 1.18.

Theorem 1.22 *Given a map* $f : S^m \to \mathbb{R}P(n);\ m, n \geq 2,$ *let* $\widetilde{f} : S^m \to S^n$ *be a lifting. Then the following conditions are equivalent:*

(1) $\omega^\#(f, f) = 0,$ *but* (f, f) *is not loose;*

(2) $\partial_{\mathbb{R}}([\widetilde{f}]) \neq 0,$ *but* $E \circ \partial_{\mathbb{R}}([\widetilde{f}]) = 0;$

(3) $(\widetilde{f}, \widetilde{f})$ *is loose, but* (f, f) *is not loose;*

(4) $MC(\widetilde{f}, \widetilde{f}) < MC(f, f);$

(5) $MCC(\widetilde{f}, \widetilde{f}) < MCC(f, f);$

(6) $(\widetilde{f}, \widetilde{f})$ *is loose but not by a small deformation.*

This result settles a question raised in [K 6], 1.16. More importantly, however, it relates the completeness question concerning $\omega^\#$–invariants to fascinating recent work of D. Gonçalves and D. Randall. They produced many maps $\widetilde{f} : S^m \to S^n$ which cannot be deformed away from themselves by small homotopies but only via large deformations which use all the space available in S^n (cf. 1.22 (6)). E.g. all Whitehead products of the form

$$[\widetilde{f}] = [\iota_{4k+2},\ \iota_{4k+2}] \in \pi_{8k+3}(S^{4k+2}), \quad k = 1, 2, \ldots \quad,$$

have this property (cf. [GR 1]). Moreover the existence of such maps in dimension $(m, n) = (4k - 2, 2k)$, $k > 4$, turns out to be equivalent to the Strong Kervaire Invariant One Problem, i.e. the existence of an element of order 2 with Kervaire invariant one in the stable homotopy group π_{2k-2}^S (cf. [GR 2]); examples of such maps exist in dimension $2k = 16, 32, 64$.

2   Covering spaces and the pathcomponents of $E(f_1, f_2)$

Throughout this paper $M, N, \widetilde{N}$, and Q will denote smooth connected manifolds (having the Hausdorff property and countable bases) without boundary, M being compact; $f_1, f_2, f, \overline{f}, \ldots : M \to N$ will be (continuous) maps.

In this section we study the set $\pi_0(E(f_1, f_2))$ of pathcomponents of $E(f_1, f_2)$ (cf. 1.6) with the help of coverings of N. We will need this when we discuss Nielsen decompositions in section 3.

Consider the diagram

$$(2.1) \qquad \begin{array}{c} \widetilde{N} \\ \downarrow {\scriptstyle p} \\ M \xrightarrow{\ f_1, f_2\ } N \end{array}$$

where p is a covering map. Pick base points $x_0 \in M$, $y_{0i} := f_i(x_0) \in N$, $i = 1, 2$, and $\widetilde{y}_{02} \in p^{-1}(\{y_{02}\}) \subset \widetilde{N}$. For any path θ joining y_{01} to y_{02} in N (i.e. $(x_0, \theta) \in E(f_1, f_2)$) define the subset

$$(2.2) \qquad \pi_\theta := \{ [\theta^{-1}(f_1 \circ \sigma)\theta(f_2 \circ \sigma)^{-1}] \in \pi_1(N, y_{02}) \mid [\sigma] \in \pi_1(M, x_0) \}$$

of $\pi_1(N, y_{02})$ (which consists of all homotopy classes of concatenated loops of the form

$$y_{02} \xrightarrow{\ \theta^{-1}\ } y_{01} \xrightarrow{\ f_1 \circ \sigma\ } y_{01} \xrightarrow{\ \theta\ } y_{02} \xrightarrow{\ f_2 \circ \sigma^{-1}\ } y_{02}).$$

Example 2.3   (i)   If $f_{i*}(\pi_1(M)) = 0$, $i = 1, 2$ (e.g. if M is simply connected), then $\pi_\theta = \{0\}$;

(ii)   if $\pi_1(N)$ is abelian and $f_1(x_0) = f_2(x_0)$ then

$$\pi_\theta = (f_{1*} - f_{2*})(\pi_1(M, x_0));$$

(iii)   (root case) if f_2 is constant, then π_θ is conjugated, via $[\theta]$, to $f_{1*}(\pi_1(M, x_0))$; whenever $\pi_1(N)$ is not commutative, here (and in other cases) π_θ may depend strongly on θ.

Now consider the map

$$\psi \ : \ \mathrm{pr}^{-1}(\{x_0\}) = \{(x_0, \theta) \in E(f_1, f_2)\} \ \longrightarrow \ p^{-1}(\{y_{0,1}\}) \ \subset \ \widetilde{N}$$

defined on the fiber of the obvious projection $\mathrm{pr} : E(f_1, f_2) \to M$ (cf. 1.6) as follows: let $\widetilde{\theta} : I \to \widetilde{N}$ be the unique (continuous) lifting of θ such that $\widetilde{\theta}(1) = \widetilde{y}_{02}$ and put $\psi(x_0, \theta) := \widetilde{\theta}(0)$. (The reason for "lifting backwards" will be explained in remark 4.5). There is also the surjective map

$$q : \ \mathrm{pr}_*^{-1}(\{x_0\}) \ \longrightarrow \ \pi_0(E(f_1, f_2))$$

which assigns, to each element (x_0, θ), the corresponding pathcomponent of $E(f_1, f_2)$. When does ψ factor through q?

Proposition 2.4 ψ induces a welldefined bijection

$$\Psi : \ \pi_0(E(f_1, f_2)) \ \longleftrightarrow \ p^{-1}(\{y_{0,1}\})$$

if and only if

$$(*) \ \ \pi_\theta \ = \ p_*(\pi_1(\widetilde{N}, \widetilde{y}_{02})) \ \ \text{for all } (x_0, \theta) \in E(f_1, f_2)$$

(This condition holds e.g. if M and $\widetilde{N}$ are simply connected).

Proof Let $(x_0, \theta'), (x_0, \theta)$ be elements of $E(f_1, f_2)$ based at x_0.

First assume that they can be joined by a path (σ, Θ) in $E(f_1, f_2)$. Then Θ yields homotopies from θ' to $(f_1 \circ \sigma)\theta(f_2 \circ \sigma)^{-1}$ and from $\theta^{-1}\theta'$ to $\theta^{-1}(f_1 \circ \sigma)\theta(f_2 \circ \sigma)^{-1}$ which leave the endpoints fixed. Clearly $\psi(\theta') = \psi(\theta)$ precisely if $\theta^{-1}\theta'$ lifts to a closed loop in $\widetilde{N}$ starting and ending in $\widetilde{y}_{02}$. Thus ψ induces a welldefined map on $\pi_0(E(f_1, f_2))$ if and only if $\pi_\theta \subset p_*(\pi_1(\widetilde{N}, \widetilde{y}_{02}))$ for all $(x_0, \theta) \in E(f_1, f_2)$. Surjectivity follows automatically since $\widetilde{N}$ is connected.

Next suppose only that $\psi(\theta') = \psi(\theta)$. If $\pi_\theta \supset p_*(\pi_1(\widetilde{N}, \widetilde{y}_{02}))$, then $[\theta^{-1}\theta'] = [\theta^{-1}(f_1 \circ \sigma)\theta(f_2 \circ \sigma)^{-1}]$ for some $[\sigma] \in \pi_1(M, x_0)$; this yields a path in $E(f_1, f_2)$ joining (x_0, θ') to (x_0, θ). Thus condition $(*)$ in 2.4 implies also the injectivity of Ψ.

On the other hand, given $(x_0, \theta) \in E(f_1, f_2)$ and $[\tau] \in p_*(\pi_1(\widetilde{N}, \widetilde{y}_{02}))$, put $\theta' = \theta\tau$. If Ψ is injective, then there is a path in $E(f_1, f_2)$ from (x_0, θ') to (x_0, θ), and we conclude again that $[\tau] = [\theta^{-1}\theta']$ (and hence all of $p_*(\pi_1(\widetilde{N}, \widetilde{y}_{02}))$) lies in π_θ.

In the special case when $f_{i*}(\pi_1(M)) = 0$, $i = 1, 2$, condition $(*)$ in 2.4 holds if and only if $\widetilde{N}$ is simply connected (cf. 2.2 and 2.3). $\qquad \square$

3 Nielsen numbers and covering spaces

In this section we use liftings to covering spaces in order to give a new description of our Nielsen numbers. As a sample application we discuss maps into spherical space forms.

In the setting of diagram 2.1 we assume that

(i) the group G of covering transformations of the covering space $p : \widetilde{N} \to N$ acts transitively on the fibers of p (or, equivalently, $p_*(\pi_1(\widetilde{N}))$ is a normal subgroup of $\pi_1(N)$); and

(ii) there are liftings $\widetilde{f_i} : M \to \widetilde{N}$ of f_i (or, equivalently, $f_{i*}(\pi_1(M)) \subset p_*(\pi_1(\widetilde{N}))$), $i = 1, 2$.

Any choice of such maps $\widetilde{f_i}$ (satisfying $p \circ \widetilde{f_i} = f_i$) determines a homeomorphism

$$(3.1) \qquad \lambda \; : \; E(f_1, f_2) \; \longleftrightarrow \; \mathrm{II}_{g \in G} \, E(g \circ \widetilde{f_1}, \widetilde{f_2})$$

(cf. 1.6) defined by $\lambda(x, \theta) = (x, \widetilde{\theta})$, $(x, \theta) \in E(f_1, f_2)$, where $\widetilde{\theta}$ is the unique lifting of the path θ such that $\widetilde{\theta}(1) = \widetilde{f_2}(x)$; here the disjoint components $E(g \circ \widetilde{f_1}, \widetilde{f_2})$ on the right hand side correspond to open subsets of $E(f_1, f_2)$.

λ, together with the obvious tangent isomorphism $T\widetilde{N} \cong p^*(TN)$, induces the map

$$(3.2) \qquad \lambda_* \; : \; \Omega^{\#}(f_1, f_2) \; \longrightarrow \; \prod_{g \in G} \Omega^{\#}(g \circ \widetilde{f_1}, \widetilde{f_2}).$$

(cf. 1.8). Clearly

$$(3.3) \qquad \lambda_*(\omega^{\#}(f_1, f_2)) \; = \; (\omega^{\#}(g \circ \widetilde{f_1}, \widetilde{f_2}))_{g \in G}$$

(cf. 1.4 – 1.8) and $\lambda_*([\phi]) = ([\phi])_{g \in G}$. Possibly λ_* is neither onto (e.g. when G is infinite) nor injective (since the disjointness requirements in the definition of $\Omega^{\#}(f_1, f_2)$, concerning e.g. embedded bordisms, may not be preserved). However, if $E(g \circ \widetilde{f_1}, \widetilde{f_2})$ is pathconnected for all $g \in G$, i.e. if the condition $(*)$ in 2.4 holds, then $\lambda_*(\omega^{\#}(f_1, f_2))$ keeps track of strongly essential Nielsen components (compare 1.9). We conclude

Theorem 3.4 *Let $p : \widetilde{N} \to N$ be a covering space such that the group G of covering transformations acts transitively on the fibers and let $\widetilde{f_i} : M \to \widetilde{N}$ be a lifting of f_i, $i = 1, 2$. Assume that condition $(*)$ (cf. proposition 2.4) holds.*

Then $\#\pi_0(E(f_1, f_2)) = \#G$ and the strong Nielsen number of (f_1, f_2) (cf. 1.9) is given by

$$N^{\#}(f_1, f_2) \;=\; \#\{g \in G \mid \omega^{\#}(g \circ \widetilde{f}_1, \widetilde{f}_2) \;\neq\; 0\}.$$

It is easy to see that condition $(*)$ and the lifting condition are independent. E.g. if $\widetilde{f} : M \to \widetilde{N}$ and p are nontrivial coverings and $\pi_1(N)$ is abelian, then $f_1, f_2 := p \circ \widetilde{f}$ have liftings, but $(*)$ fails to hold (cf. 2.3 (ii) and 2.4).

On the other hand consider the case where $\pi_1(\widetilde{N}) = 0$ and $f_1(x_0) = f_2(x_0) =: y_0$. Here G acts transitively; moreover $(*)$ is satisfied if and only if $f_{1*} = f_{2*}$ maps $\pi_1(M, x_0)$ into the center of $\pi_1(N, y_0)$; thus if e.g. $\pi_1(N, x_0)$ is abelian and $f_1 = f_2$ is a non-universal covering map, then no liftings $\widetilde{f}_i$ exist, but condition $(*)$ holds.

In any case all the assumptions in theorem 3.4 are satisfied whenever both M and $\widetilde{N}$ are simply connected.

Corollary 3.5 (Spherical space forms) *Given a free smooth action of a nontrivial finite group G on S^n, $n \geq 1$, let $N = S^n/G$ be the quotient manifold. Consider maps $f_1, f_2 : M \to S^n/G$ where (i) $M = S^m$, $m \geq 2$; or (ii) M is simply connected, having dimension $m < 2n - 2$.*

Then f_1 and f_2 are homotopic whenever $N^{\#}(f_1, f_2) \neq \#G$.

In particular, if n is odd then

$$N^{\#}(f_1, f_2) = MCC(f_1, f_2) = \begin{cases} \#G & \text{if } f_1 \not\sim f_2 \ ; \\ 0 & \text{if } f_1 \sim f_2 \ . \end{cases}$$

Proof Theorem 3.4 offers a new and more powerful approach when compared to the treatment in [K 6] (see statement 1.25 there and its proof (preceding 6.23) which works only if f_1 or f_2 is not coincidence producing, cf. 5.8 (iii) below).

Our claim holds trivially when $n = 1$ since then f_1 and f_2 are nulhomotopic. Thus we may assume that $\pi_1(S^n) = 0$. Suppose $N^{\#}(f_1, f_2) < \#G$. Then there exist liftings $\widetilde{f}_1, \widetilde{f}_2 : M \to S^n$ such that $\omega^{\#}(\widetilde{f}_1, \widetilde{f}_2) = 0$ (cf. 3.4). Therefore $(\widetilde{f}_1, \widetilde{f}_2)$ is loose; this follows from 1.14 in case (i) and from [K 6], 1.11 (iii) and [K 3], 1.10 in case (ii). After a homotopy $\widetilde{f}_1, \widetilde{f}_2$ are coincidence free. Thus $\widetilde{f}_1$ is homotopic to $A \circ \widetilde{f}_2$, and so is $g \circ \widetilde{f}_2$ for every nontrivial element $g \in G$ (cf. [DG], 2.10, or the beginning of section 8 in [K 3]; here A denotes the antipodal map). We conclude that

$$f_1 \;=\; p \circ \widetilde{f}_1 \;\sim\; p \circ g \circ \widetilde{f}_2 \;=\; p \circ \widetilde{f}_2 \;=\; f_2.$$

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If n is odd, f_1 can be pushed away from itself along a nowhere zero vector field. In view of theorem 1.10 this completes the proof. $\square$

4 Fibrations and the $\omega^{\#}$–invariants

In this section we generalize λ_* (cf. 3.2 and 3.3). We use this to explore compatibilities of our invariants with fibrations, e.g. with the natural projections from spheres to complex or quaternionic projective spaces.

Consider commuting diagrams

(4.1)

$$
\begin{array}{ccc}
 & & Q \\
 & \overset{\widetilde{f_i}}{\nearrow} & \downarrow{\scriptstyle p} \\
M & \xrightarrow{\ f_i\ } & N
\end{array}
\qquad ,
$$

$i = 1, 2$, where p is a smooth locally trivial fibration. We want to compare the $\omega^{\#}$–invariants of (f_1, f_2) and of the pair $(\widetilde{f}_1, \widetilde{f}_2)$ of liftings.

Given a bordism class

$$
c \;=\; [\, C \subset M,\ \widetilde{g} : C \to E(f_1, f_2),\ \overline{g}^{\#}\,] \;\in\; \Omega^{\#}(f_1, f_2)
$$

(compare 1.4 – 1.8), let $H : C \times I \to N$ be the homotopy from $f_1|C$ to $f_2|C$ which is adjoint to $\widetilde{g}$, and let $\widetilde{H} : C \times I \to Q$ be a lifting of H which *ends* at $\widetilde{H}(\ ,1) = \widetilde{f}_2|C$. Then $\widetilde{f}_1|C$ and $\widetilde{H}(\ ,0)$ determine sections of the pulled back fibration $(f_1|C)^*(p)$. Consider their transverse intersection locus $C' \subset C \subset M$. The normal bundle $\nu(C', C)$ is canonically isomorphic to $(\widetilde{f}_1|C')^*(TF)$, the pullback of the tangent bundle along the fibers of p; together with $\overline{g}^{\#} : \nu(C, M) \cong f_1^*(TN)|C$ this yields a vector bundle isomorphism

(4.2)
$$
\widetilde{\overline{g}}^{\#} \;:\; \nu(C', M) \;\cong\; \widetilde{f}_1^*(TQ)|C'
$$

(compare 1.5). Moreover the adjoint of $\widetilde{H}|(C' \times I)$ determines a section $\widetilde{\widetilde{g}}$ over C' of the fibration $E(\widetilde{f}_1, \widetilde{f}_2) \to M$ (compare 1.6). We obtain the bordism class

(4.3)
$$
\lambda_{e*}([c]) \;=\; [\, C' \subset M,\ \widetilde{\widetilde{g}},\ \widetilde{\overline{g}}^{\#}\,] \;\in\; \Omega^{\#}(\widetilde{f}_1, \widetilde{f}_2) \,.
$$

Proposition 4.4 *This construction yields a welldefined map*

$$
\lambda_{e*} \;:\; \Omega^{\#}(f_1, f_2) \;\longrightarrow\; \Omega^{\#}(\widetilde{f}_1, \widetilde{f}_2)
$$

which takes $\omega^{\#}(f_1, f_2)$ *to* $\omega^{\#}(\widetilde{f}_1, \widetilde{f}_2)$ *and* $[\phi]$ *to* $[\phi]$. *In particular, if* $\omega^{\#}(f_1, f_2)$ *is trivial then so is* $\omega^{\#}(\widetilde{f}_1, \widetilde{f}_2)$.

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In the special case where p is a covering map as in 3.1 and 3.2 λ_{e*} is the component map of λ_* corresponding to the unit e of the group G of covering transformations.

The proof is fairly straight forward.

Remark 4.5 In the definitions of ψ (cf. 2.4), λ (cf. 3.1), and λ_{e*} (cf. 4.3) we lifted paths and homotopies "backwards", starting at the end. This makes our constructions more easily compatible with our convention to describe normal bundles such as $\nu(C, M)$ in terms of $f_1^*(TN)$ (and not of $f_2^*(TN)$, cf. 1.5) and helps us to avoid reframings as the ones necessitated e.g. in the definitions of α and β in [K 6], 6.7 - 6.9.

The following sample application of 4.4 will be needed in the proof of theorem 1.17.

Corollary 4.6 Let $p : S^{dn'+d-1} \to \mathbb{K}P(n')$, $n' \geq 1$, be the canonical fibration where $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ (with real dimension $d = 2$ or 4). Assume that $M = S^m$ or that M has dimension $m < 2dn' - 2$. If the maps $f_i : M \to \mathbb{K}P(n')$ allow genuine liftings $\widetilde{f_i}$ to $S^{dn'+d-1}$ (i.e. $p \circ \widetilde{f_i} = f_i$), $i = 1, 2$, and if $\omega^{\#}(f_1, f_2) = 0$, then f_1 and f_2 are homotopic.

Proof If $\omega^{\#}(\widetilde{f_1}, \widetilde{f_2}) = 0$ (in view of 4.4) then the pair $(\widetilde{f_1}, \widetilde{f_2})$ is loose and $\widetilde{f_1} \sim A \circ \widetilde{f_2}$ (cf. 1.12, 1.14, [K 6], 1.11 (iii), [K 3], 1.10, and [DG], 2.10). Therefore $f_1 \sim p \circ A \circ \widetilde{f_2} = p \circ \widetilde{f_2} = f_2$. $\qquad\square$

5 Removing selfcoincidences

In view of results such as corollaries 3.5 and 4.6 a closer look at the selfcoincidence setting seems to be in order. Here we are dealing essentially with pairs of the form (f, f). Only the pathcomponent of $E(f, f)$ which contains all elements of the form $(x, \text{constant path})$ (cf. 1.6) can possibly be strongly essential (cf. 1.9). As a consequence our invariants $\omega^{\#}$ and $N^{\#}$ turn out to be not as powerful here as in the root case (where f_1 or f_2 is constant, compare e.g. [K 6], 6.6 b) (iv)). Furthermore the second part of the first question in 1.13 is easily answered.

Proposition 5.1 Let $f_1, f_2 : M \to N$ be homotopic. Then $MCC(f_1, f_2) \leq 1$; moreover $\omega^{\#}(f_1, f_2)$ is trivial if and only if $N^{\#}(f_1, f_2) = 0$. In the special case where M is a sphere $MC(f_1, f_2)$ equals $MCC(f_1, f_2)$.

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Proof The coincidence space $C(f_1, f_1) = M$ is pathconnected by assumption. If (f_1, f_2) is a generic pair of sufficiently close maps then C_A (cf. 1.9 (i)) is empty for all but possibly one pathcomponent A of $E(f_1, f_2)$; the corresponding partial bordism class, i.e. $\omega^\#(f_1, f_2)$, vanishes precisely when $N^\#(f_1, f_2) = 0$. If f_1 and f_2 are related by an arbitrary (possibly "large") homotopy, apply [K 6], 2.3.

If $M = S^m$ the vector bundle $f_1^*(TN)$ allows a section with at most one zero. This yields a map f_2' such that $\#C(f_1, f_2') \leq 1$ and $f_2' \sim f_2$. $\square$

Let $\xi \subset TN$ be a sub–vectorbundle and let ξ' denote the image of its total space under the composed diffeomorphism

(5.2)
$$TN \xleftarrow[p_{1*}]{\cong} \nu(\Delta, N \times N) \cong U \quad \subset \quad N \times N$$

where U is a tubular neighbourhood of the diagonal Δ in $N \times N$ and p_{1*} denotes the vector bundle isomorphism induced by the first projection $p_1 : N \times N \to N$.

Definition 5.3 Given a map $f : M \to N$, we say (f, f) is *loose by a small ξ–deformation* if for every metric on N and for every $\epsilon > 0$ there exists an ϵ–approximation $\overline{f} : M \to N$ of f such that $(f, \overline{f})(M) \subset \xi' - \Delta$. If this holds for $\xi = TN$ we simply say that (f, f) is *loose by a small deformation* (compare [DG] or [GR 2], § 1).

A homotopy lifting argument shows that (f, f) is loose by a small ξ– deformation precisely if the pulled back vector bundle $f^*(\xi)$ has a nowhere vanishing section over M (compare [DG]).

If $M = S^m$ we can approach such phenomena using standard tools of homotopy theory.

Assume $m, n \geq 2$ and consider the commuting diagram

(5.4)

$$
\begin{array}{ccccccc}
 & & & & \pi_m(S^n) & & \\
 & & & \nearrow^{\omega^\#} & \uparrow{\scriptstyle E} & & \\
\ldots \pi_m(STN) & \longrightarrow & \pi_m(N) & \xrightarrow{\ \partial\ } & \pi_{m-1}(S^{n-1}) & \xrightarrow{\ \mathrm{incl}_*\ } & \pi_{m-1}(STN) \\
\downarrow & & \| & & \downarrow{\scriptstyle j_*} & & \downarrow \\
\ldots \pi_m\left(\widetilde{C}_2(N)\right) & \longrightarrow & \pi_m(N) & \xrightarrow{\ \partial'\ } & \pi_{m-1}(N - \{x_0\}) & \xrightarrow{\ \mathrm{incl}'_*\ } & \pi_{m-1}\left(\widetilde{C}_2(N)\right)
\end{array}
$$

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where the fibrations of the space STN of unit tangent vectors and of the configuration space

$$(5.5) \qquad \widetilde{C}_2(N) \ = \ \{(y_1, y_2) \in N \times N |\ y_1 \neq y_2\} \ = \ N \times N - \Delta$$

over N yield the exact horizontal sequences; incl, incl$'$ (and E, resp.) denote fiber inclusions (and the Freudenthal suspension., resp.). The downward pointing vertical arrows are induced by a diffeomorphism as described in 5.2; in particular, j denotes the inclusion of the boundary sphere of an n–ball in N around the basepoint x_0.

Given a homotopy class $[f] \in \pi_m(N)$, how does its image under the boundary operator ∂ in 5.4 compare to our looseness obstruction $\omega^{\#}(f, f)$? Recall that the path space approach yields no added information in the selfcoincidence setting; hence $\omega^{\#}(f, f)$ is just as strong as the invariant

$$(5.6) \qquad\qquad \underline{\omega}^{\#}(f, f) \ := \ \mathrm{pr}_*(\omega^{\#}(f, f)) \ \in \pi_m(S^n)$$

defined by the framed bordism class of the zero set Z of a generic section s of $f^*(TN)$ (cf. [K 6], $5.1 - 5.3$ and section 6). We may assume that Z lies in the interior of a small ball $B \subset S^m$ (over which $f^*(TN)$ is trivialized) and that s maps the boundary $\partial B \cong S^{m-1}$ to unit vectors (and thus defines a map $s| : S^{m-1} \to S^{n-1}$). Then $\partial([f]) = [s|\ ]$ and Z is framed bordant to $s|^{-1}(\{*\})$, $*$ a regular value of $s|$. We conclude:

Proposition 5.7 *For all* $[f] \in \pi_m(N)$, *we have*

$$\underline{\omega}^{\#}(f, f) \ = \ \pm E \circ \partial([f]) \ .$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Given $[f] \in \pi_m(N)$, consider the following conditions:

(5.8)

  (i) (f, f) is loose by a small deformation; equivalently, $\partial([f]) = 0$;

  (ii) (f, f) is loose (by any deformation);

  (iii) there exists any map $\overline{f} : S^m \to N$ such that $(f, \overline{f})$ is loose (we say that f is *not coincidence producing*, cf. [BS]); equivalently, $\partial'([f]) = 0$;

(iii') $\omega^{\#}(f, f) = 0$.

Clearly (i) implies (ii). In turn, (ii) implies both (iii) and (iii').

Proposition 5.9   *The conditions* (i) *and* (iii) *are equivalent for all* $[f] \in \pi_m(N)$ *if and only if the homomorphism*

$$(j_*, \mathrm{incl}_*) \; : \; \pi_{m-1}(S^{n-1}) \; \longrightarrow \; \pi_{m-1}(N - \{x_0\}) \oplus \pi_{m-1}(STN)$$

is injective (where incl *denotes the fiber inclusion).*

Indeed, this is the precise condition for the kernels of ∂ and $\partial' = j_* \circ \partial$ in 5.4 to agree. In view of 5.6 and 5.7 we obtain similarly

Proposition 5.10   *The conditions* (i) *and* (iii') *are equivalent for all* $[f] \in \pi_m(N)$ *if and only if the homomorphism*

$$(E, \mathrm{incl}_*) \; : \; \pi_{m-1}(S^{n-1}) \; \longrightarrow \; \pi_m(S^n) \oplus \pi_{m-1}(STN)$$

is injective.

In particular, if j_* is injective on $\mathrm{im}\,\partial = \ker \mathrm{incl}_*$ but E is not, then there is a map $f : S^m \to N$ such that $\omega^{\#}(f, f) = 0$ but (f, f) is not loose by any deformation. In the next section we will study such examples systematically in case N is a projective space. They show that even the nonstabilized "desuspended" invariant $\omega^{\#}$ (compare [K 3]) can be (at least one desuspension) short of yielding complete looseness criteria.

6   Maps into projective spaces

In this section we apply the previous discussion to the case $M = S^m$, $N = \mathbb{K}P(n')$ (where $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$), exploiting natural compatibilities with the canonical fiber map

(6.1) $$p \; : \; S^{n+d-1} \; \longrightarrow \; \mathbb{K}P(n') \; .$$

This leads to a proof of theorems 1.17, 1.21, and 1.22.

First we show that the inequalities of theorem 1.10 hold here in full generality, without an exception at the dimension $n = 2$ which is so critical in classical fixed point and coincidence theory (cf. [Ji 1], [Ji 2] and also [BGZ 1], [BGZ 2], and [GKZ]).

Proposition 6.2   *For all maps* $f_1, f_2 : S^m \to \mathbb{K}P(n')$ *(where* $m, n' \geq 1$ *and* $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$*) we have:*

(i) $MCC(f_1, f_2) \; \leq \; \#\pi_0(E(f_1, f_2));$ and

(ii) $\quad MC(f_1, f_2) \leq \#\pi_0(E(f_1, f_2))$ or $MC(f_1, f_2) = \infty$.

Proof In view of 1.10, 1.14, 1.15, and [Je], 4.0 (see also [K 6], 6.24) we need to consider only the case when $m > 2$, $N = \mathbb{R}P(2)$. Then $\pi_{m-1}(S^{n-1}) = 0$ and hence (f_2, f_2) is loose (cf. 5.4 and 5.8).

Thus $MCC(f_1, f_2) = MCC(f, *)$ where $[f] = [f_1] - [f_2]$ (by [K 6], 6.2). Given liftings $\widetilde{f}, \widetilde{*} : S^m \to S^2$ of f and of the constant map, $C(\widetilde{f}, \widetilde{*}) = \widetilde{f}^{-1}(\{*\})$ is generically a framed submanifold which we may make connected by a suitable surgery. Taking also the inverse image of a nearby (and, after an isotopy, antipodal point) we see that $f^{-1}(\{*\})$ consists of two "parallel" connected components. Hence $MCC(f, *) \leq 2 = \#\pi_0(E(f_1, f_2))$ (cf. [K 3], 2.1). $\quad\square$

For the remainder of this section we assume that $m, n' \geq 2$. Comparing the exact homotopy sequences of the fibrations p (cf. 6.1) and $p| : S^{n-1} \to \mathbb{K}P(n' - 1)$ we see that $\pi_m(\mathbb{K}P(n'))$ is the direct sum of $p_*(\pi_m(S^{n+d-1}))$ and of the image of $\pi_m(\mathbb{K}P(n' - 1))$; moreover p_* is injective. Thus

(6.3) $$[f_i] \quad = \quad [p \circ \widetilde{f_i}] \ + \ [f_i'']$$

where $(f_i'', *)$ is loose (for $* \notin \mathbb{K}P(n' - 1)$), $i = 1, 2$. Clearly the pairs (f_1, f_2) and $(f_1', f_2') := (p \circ \widetilde{f_1}, p \circ \widetilde{f_2})$ have identical $\omega^\#$–invariants as well as Nielsen and minimum numbers (cf. [K 6], 6.1 and 6.2). Moreover since f_i'' factors through $\mathbb{K}P(n' - 1)$ the corresponding pullback of the sphere bundle $ST(\mathbb{K}P(n'))$ allows a section; thus $\partial([f_i'']) = 0$ and

(6.4) $$\partial([f_i]) \quad = \quad \partial([p \circ \widetilde{f_i}]) \ .$$

This reduces the proof of theorem 1.17 to the case where $f_i = p \circ \widetilde{f_i} = f_i'$, $i = 1, 2$.

Now apply the discussion of the diagram 5.4 to the case $N = \mathbb{K}P(n')$. Clearly j_* is injective here since j agrees with $p|S^{n-1}$ up to the homotopy equivalence $\mathbb{K}P(n') - \{x_0\} \sim \mathbb{K}P(n' - 1)$. Thus the three looseness conditions (i), (ii), and (iii) in 5.8 are equivalent for all $[f] \in \pi_m(\mathbb{K}P(n')$ (cf. 5.9). They hold precisely if $\partial([f]) = 0$ and, in case $[f] = p_*([\widetilde{f}])$, precisely if $\partial_{\mathbb{K}}([\widetilde{f}]) = 0$. Indeed, the exact horizontal sequences in the diagrams 1.16 and 5.4 are closely related via maps induced by p; in particular,

(6.5) $$\partial_{\mathbb{K}} \quad = \quad \partial \circ p_* \quad .$$

Here we observe that the vector bundle $p^*(T\mathbb{K}P(n'))$ is isomorphic to the orthogonal complement

(6.6) $\quad \xi_{\mathbb{K}} := \{(x, v) \in S^{n+d-1} \times \mathbb{K}^{n'+1} \mid v \perp \mathbb{K} \cdot x \} \quad \subset \ TS^{n+d-1}$

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of the $\mathbb{K}$–line bundle over S^{n+d-1} which is spanned by the locus vectors; e.g. $\xi_{\mathbb{R}} = TS^n$. The corresponding space $S(\xi_{\mathbb{K}})$ of unit vectors is the Stiefel manifold $V_{n'+1,2}(\mathbb{K})$.

In view of proposition 5.1 the following result implies theorem 1.22.

Theorem 6.7 *Assume* $m, n' \geq 2$ *and* $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$. *Given* $[f] \in \pi_m(\mathbb{K}P(n'))$, *let* $[\widetilde{f}] \in \pi_m(S^{n+d-1})$ *be the corresponding component in the decomposition 6.3. Then the following conditions are equivalent:*

(i) (f, f) *is loose by a small deformation;*

(ii) (f, f) *is loose by any deformation;*

(iii) f *is not coincidence producing (cf. 5.8, iii);*

(iv) $(\widetilde{f}, \widetilde{f})$ *is loose by a small $\xi_{\mathbb{K}}$–deformation (cf. 5.3);*

(v) $\partial_{\mathbb{K}}([\widetilde{f}]) = 0$.

(*Note that "loose by a small $\xi_{\mathbb{R}}$–deformation" is the same as "loose by a small deformation").*

Furthermore, consider also the following (possibly weaker) conditions:

(iii') $\omega^{\#}(f, f) = 0$;

(iii") $\omega^{\#}(\widetilde{f}, \widetilde{f}) = 0$;

(iv') $(\widetilde{f}, \widetilde{f})$ *is loose;*   *and*

(v') $E(\partial_{\mathbb{K}}([\widetilde{f}])) = 0$.

If $\mathbb{K} = \mathbb{R}$*, then all four conditions (iii') – (v') are equivalent. If* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{H}$*, we still have the following implications:*

$$(\text{v'}) \iff (\text{iii'}) \implies (\text{iii"}) \iff (\text{iv'}) .$$

Proof   The first claim follows from the previous discussion and is based essentially on 5.9, 1.16, and the remark following 5.3.

If $\mathbb{K} = \mathbb{R}$ then $\widetilde{f}$ is a genuine lifting of f to a double cover. Clearly $\underline{\omega}^{\#}(f, f) = \underline{\omega}^{\#}(\widetilde{f}, \widetilde{f})$ (cf. 5.6). But these invariants contain just as much information as $\omega^{\#}(f, f)$ and $\omega^{\#}(\widetilde{f}, \widetilde{f})$. In view of 1.14, 4.4, 5.7, and 6.5 this completes the proof.   □

Proof of theorems 1.17 and 1.21.   In view of the discussion of 6.3 we may assume that $f_i = f_i' = p \circ \widetilde{f_i}$, $i = 1, 2$. Also note that

$$\#\pi_0(E(f_1, f_2)) = \#\pi_1(\mathbb{K}P(n')) = \begin{cases} 2 & \text{if } \mathbb{K} = \mathbb{R} \quad, \\ 1 & \text{if } \mathbb{K} = \mathbb{C} \text{ or } \mathbb{H} \quad, \end{cases}$$

since $\pi_1(S^m) = 0$ (cf. [K 3], 2.1).

First consider the case where $N^{\#}(f_1, f_2) < \#\pi_0(E(f_1, f_2))$. Then $f_1 \sim f_2$ and $MCC(f_1, f_2) = MC(f_1, f_2) \leq 1$ (cf. 3.5, 4.6, and 5.1). According to theorem 1.10 (and 5.1; 3.4, 1.14) we can distinguish (and characterize) the three special cases:

(1) $N^{\#}(f_1, f_2) = MCC(f_1, f_2) = 0$ (i.e. $f_1 \sim f_2$ and (f_2, f_2) is loose);

(2) $N^{\#}(f_1, f_2) < MCC(f_1, f_2) = 1$ (or, equivalently, $f_1 \sim f_2$ and $\omega^{\#}(f_2, f_2) = 0$ but (f_2, f_2) is not loose);

(3) $N^{\#}(f_1, f_2) = MCC(f_1, f_2) = 1$ (or, equivalently, $\mathbb{K} = \mathbb{R}$, $f_1 \sim f_2$ and $\widetilde{f_2}$ is not homotopic to $A \circ \widetilde{f_2}$).

In view of theorem 6.7 (and of the injectivity of p_*) these are just the first three cases in table 1.18.

In the remaining cases $N^{\#}(f_1, f_2)$ is equal to $\#\pi_0(E(f_1, f_2))$ and hence also to $MCC(f_1, f_2)$ and to $MC(f_1, f_2)$ (unless $MC(f_1, f_2) = \infty$, cf. 1.10 and 6.2). If $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ and f_1, f_2 have only finitely many coincidence points we may deform $\widetilde{f_1}$ away from $\widetilde{f_2}$ along the fibers of p; hence $\widetilde{f_1} \sim A \circ \widetilde{f_2}$, $f_1 \sim f_2$ and $MC(f_1, f_2) \leq 1$ (cf. 5.1 and [DG], 2.10). Finally assume $\mathbb{K} = \mathbb{R}$. According to 5.1 and [K 6], 6.2, $MC(f_1, f_2) < \infty$ if and only if $MC(f_1 - f_2, *)$ (and hence $MC(\widetilde{f_1} - \widetilde{f_2}, \widetilde{*})$) is finite. Then $\widetilde{f_1} - \widetilde{f_2}$ is a suspended map (cf. e.g. [K 6], 1.24 or 6.18). Conversely any suspended map (in the unreduced sense) maps only the two suspension poles in S^m to the suspension poles in S^n, i.e. to the fiber of the corresponding point in $\mathbb{R}P(n)$. This establishes table 1.18.   □

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