

COINCIDENCE THEORY IN ARBITRARY CODIMENSIONS: THE MINIMIZING PROBLEM

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Let $f_1, f_2 : M \longrightarrow N$ be two (continuous) maps between smooth connected manifolds M and N without boundary, of strictly positive dimensions m and n , resp., M being compact. We are interested in making the coincidence locus

$$C(f_1, f_2) := \{x \in M \mid f_1(x) = f_2(x)\}$$

as small (or simple in some sense) as possible after possibly deforming f_1 and f_2 by a homotopy.

Question. How large is the **minimum number of coincidence components**

$$MCC(f_1, f_2) := \min\{\#\pi_0(C(f'_1, f'_2)) \mid f'_1 \sim f_1, f'_2 \sim f_2\} ?$$

In particular, when does this number vanish, i.e. when can f_1 and f_2 be deformed away from one another?

This is a very natural generalization of one of the central problems of classical fixed point theory (where $M = N$ and $f_2 = \text{identity map}$): determine the minimum number of fixed points among all maps in a given homotopy class (see [Br] and [BGZ], proposition 1.5). Note, however, that in higher codimensions $m - n > 0$ the coincidence locus is generically a closed $(m - n)$ -manifold so that it makes more sense to count *pathcomponents* rather than points. Also the methods of (first order, singular) (co)homology will no longer be strong enough to capture the subtle geometry of coincidence manifolds.

In this lecture I will use the language of normal bordism theory (and a nonstabilized version thereof) to define and study lower bounds $N(f_1, f_2)$ (and $N^\#(f_1, f_2)$) for $MCC(f_1, f_2)$.

After performing an approximation we may assume that the map $(f_1, f_2) : M \rightarrow N \times N$ is smooth and transverse to the diagonal $\Delta = \{(y, y) \in N \times N \mid y \in N\}$. Then the coincidence locus

$$C = C(f_1, f_2) = (f_1, f_2)^{-1}(\Delta)$$

is a closed smooth $(m - n)$ -dimensional manifold, equipped with

i) maps

$$\begin{array}{ccc} & & E(f_1, f_2) := \left\{ (x, \theta) \in M \times N^I \mid \begin{array}{l} \theta(0) = f_1(x); \\ \theta(1) = f_2(x) \end{array} \right\} \\ & \nearrow \tilde{g} & \downarrow \text{pr} \\ C & \xrightarrow{g = \text{incl}} & M \end{array}$$

where \tilde{g} is the natural lifting which adds the constant path at $f_1(x) = f_2(x)$ to $g(x) = x \in C$; and

ii) a stable vector bundle isomorphism

$$\bar{g} : TC \oplus g^*(f_1^*(TN)) \cong g^*(TM)$$

deduced from the isomorphism

$$\bar{g}^\# : \nu(C, M) \cong (f_1, f_2)^*(\nu(\Delta, N \times N)) \cong f_1^*(TN) | C$$

of (nonstable) normal bundles.

The triple (C, \tilde{g}, \bar{g}) gives rise to a welldefined bordism class

$$\tilde{\omega}(f_1, f_2) := [C, \tilde{g}, \bar{g}] \in \Omega_{m-n}(E(f_1, f_2); \tilde{\varphi})$$

in the normal bordism group of such triples (here the virtual coefficient bundle is defined by

$$\tilde{\varphi} := pr^*(f_1^*(TN) - TM) ;$$

e.g. if M and N are stably parallelized, then $\tilde{\varphi}$ is trivial and we are dealing with (stably) framed bordism).

Keeping track also of the fact that C is a smooth submanifold of M with (non-stabilized) normal bundle described by $\bar{g}^\#$, we obtain a sharper invariant

$$\omega^\#(f_1, f_2) \in \Omega^\#(f_1, f_2)$$

which, however, lies in general only in a suitable bordism *set* (not group).

A crucial ingredient of both the $\tilde{\omega}$ - and the $\omega^\#$ -invariant is the map \tilde{g} . Indeed, the path space $E(f_1, f_2)$ has a very rich topology. Already its set $\pi_0(E(f_1, f_2))$ of pathcomponents can be huge – it corresponds bijectively to the so called Reidemeister set $R(f_1, f_2)$, a well-studied set-theoretic quotient of the fundamental group $\pi_1(N)$. This leads to a natural decomposition

$$C(f_1, f_2) = \coprod_{A \in \pi_0(E(f_1, f_2))} \tilde{g}^{-1}(A) .$$

Let $N(f_1, f_2)$, and $N^\#(f_1, f_2)$, resp., denote the corresponding number of nontrivial contributions by the various pathcomponents A of $E(f_1, f_2)$ to $\tilde{\omega}(f_1, f_2)$ and $\omega^\#(f_1, f_2)$, resp.

Theorem 1. (i) *The integers $N(f_1, f_2)$ and $N^\#(f_1, f_2)$ depend only on the homotopy classes of f_1 and f_2 ;*

(ii) *$N(f_1, f_2) = N(f_2, f_1)$ and $N^\#(f_1, f_2) = N^\#(f_2, f_1)$;*

(iii) *$0 \leq N(f_1, f_2) \leq N^\#(f_1, f_2) \leq MCC(f_1, f_2) < \infty$;*

(iv) *if $m = n$ then $N(f_1, f_2) = N^\#(f_1, f_2)$ coincides with the classical Nielsen number (which has a standard definition at least if both M and N are orientable or if f_2 is the identity map).*

Recall the decisive progress made by *J. Nielsen* on the classical minimizing problem when he decomposed fixed point sets into equivalence classes. In our interpretation this is just the decomposition of a 0-dimensional bordism class according to the pathcomponents of its target space. In higher (co)dimensions $m - n$ the map \tilde{g} into $E(f_1, f_2)$ and the twisted framing $\bar{g}^\#$ contain much more information. E.g. if $M = S^m$ and $n \geq 2$ then $\Omega^\#(f_1, f_2)$ can be identified with the homotopy group $\pi_m(S^n \wedge \Omega(N)^+)$, and $\omega^\#(f_1, f_2)$ is closely related to a Hopf-Ganea invariant. This allows us to reduce many aspects of our problem to questions in standard homotopy theory.

Details of definitions, proofs, and applications will be given elsewhere (compare e.g. [K 3] and [K 2]). Here we present just one sample result.

Theorem 2. *Let N be an odd-dimensional spherical space form (i.e. the quotient of S^n by a free action of a finite group). Then we have for all $f_1, f_2 : S^m \rightarrow N$:*

$$MCC(f_1, f_2) = N^\#(f_1, f_2) = \begin{cases} 0 & \text{if } f_1 \sim f_2 \text{ or } m < n; \\ \#\pi_1(N) & \text{if } f_1 \not\sim f_2 \text{ and } m > 1; \\ |d^0(f_1) - d^0(f_2)| & \text{if } m = n = 1. \end{cases}$$

(Here $d^0(f_i) \in \mathbb{Z}$ denotes the usual degree).

Finally note that our approach applies also to over- and undercrossings of link maps into a manifold of the form $N \times \mathbb{R}$. This yields unlinking obstructions which often settle unlinking questions and which, in addition, turn out to distinguish a great number of different link homotopy classes. In certain cases they even allow a complete link homotopy classification. Moreover, our approach leads also to the notion of Nielsen numbers of link maps (cf. [K 4]).

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