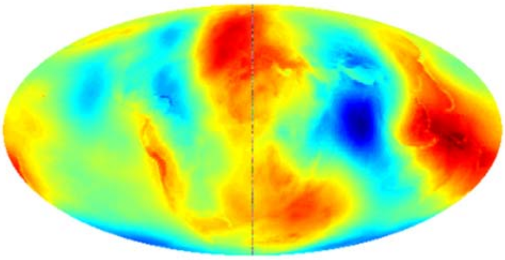

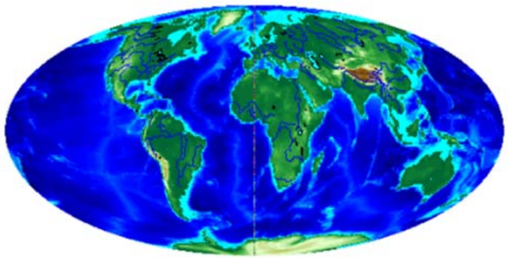

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On the Null Space of a Class of Fredholm Integral Equations of the First Kind

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We investigate the null space of Fredholm integral operators of the first kind with $TD := \int_{\mathcal{B}} D(x)k(x,\cdot)\,\mathrm{d}x$, where $\mathcal{B}$ is a ball, the integral kernel satisfies $k(x,y) = \sum_{n=0}^{\infty} c_n \frac{|x|^{l_n}}{|y|^{n-2+q}} P_n^{(q)}(\frac{x}{|x|} \cdot \frac{y}{|y|})$, and the $P_n^{(q)}$ are Gegenbauer polynomials. We first discuss the case of $\mathcal{B} \subset \mathbb{R}^3$ in detail, where the $P_n^{(3)} = P_n$ are Legendre polynomials, and then derive generalizations for the $\mathbb{R}^q$. The discussed class includes some tomographic inverse problems in the geosciences and in medical imaging. Amongst others, uniqueness constraints are proposed and compared. One result is that information on the radial dependence of D is lost in TD. We are also able to generalize a famous result on the null space of Newton's gravitational potential operator to the $\mathbb{R}^q$. Moreover, we characterize the orthonormal basis of the derived singular value decomposition of T as eigenfunctions of a differential operator and as basis functions of a particular Sobolev space.

Key Words: Fredholm integral equation of the first kind, null space, ill-posed problem, inverse gravimetric problem, inverse MEG, ball, sphere, orthogonal polynomials, spherical harmonics

MSC2010 Classification: primary: 45B05, 45Q05; secondary: 33C45, 33C50, 33C55, 47A42, 86A20, 86A22, 78A30

1 Introduction

A classical inverse problem with many applications is given by the Fredholm integral equation of the first kind, see, for example, [Fredholm, 1903] or [Yosida, 1960]. This particular inverse problem has several applications in geoscience, e.g. the inverse gravimetric problem, see [Stokes, 1867, Stromeyer and Ballani, 1984, Barzaghi and Sansò, 1986, Anger, 1990, Ballani and Stromeyer, 1992], and in medical imaging, for example, the inversion of magnetoencephalography data, see, e.g., [Sarvas, 1987, Fokas et al., 1996, Dassios et al., 2005, Fokas, 2009, Supek and Aine, 2014], and in many other fields.

The inverse gravimetric and the inverse magnetic problem have the non-uniqueness of the solution in common. Hence, in both cases, a study of the null space of the Fredholm operator is necessary. This study is important, since the operator is compact and possesses a singular value decomposition. For this purpose, an appropriate orthonormal basis is needed.

In this article, we consider a general class of Fredholm integral equations of the first kind with a ball-shaped domain, such that the inverse gravimetric and inverse magnetic problem are contained as particular cases. For this general case, we derive a singular value decomposition of the Fredholm operator and we deduce a basis of the null space. We also prove that these basis functions are eigenfunctions of a particular differential operator. In addition, we formulate certain uniqueness constraints for the general setting.

We also consider a generalization to the $\mathbb{R}^q$ and prove that known statements for the inverse gravimetric problem in $\mathbb{R}^3$ can be generalized to the $\mathbb{R}^q$.

Comparing such integral equations has turned out to enable the transfer of insights and solution methods from one problem to the other. This was also done in [Fokas and Michel, 2008] and [Fokas et al., 2012]. With this publication, we hope to foster and support further investigations on this subject.

Our paper is structured as follows. In Section 2, we formulate the Fredholm equation of the first kind and the kind of integral kernels under consideration, and we analyze the corresponding forward problem. In Section 3, we give a characterization of the null space of the Fredholm operator. For this purpose, we construct a tailor-made orthonormal basis system for the problem and calculate the corresponding singular system. In Section 4, we discuss certain uniqueness constraints, for example, the minimum norm solution, a generalization of the harmonicity constraint, and the layer density. In Section 5, we generalize certain results of the null space to higher dimension, i.e. the $\mathbb{R}^q$. In Section 6, we derive a differential operator whose eigenfunctions are the orthonormal basis functions defined in Section 3. In Section 7, we sum up our results.

Notations

In this work, the set of positive integers is denoted by $\mathbb{N}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, $\mathbb{R}$ presents the set of real numbers. Furthermore, the sphere with radius R is denoted by Ω_R and the corresponding closed ball is denoted by $\mathcal{B}$. For $R = 1$, we often use the abbreviation $\Omega := \Omega_1$. A function $F \colon \Omega_R \to \mathbb{R}$ possessing k continuous derivatives on the sphere Ω_R is of class $\mathrm{C}^{(k)}(\Omega_R)$, for $0 \leq k \leq \infty$. $\mathrm{L}^2(\Omega_R)$ is the Hilbert space of (equivalence classes of almost everywhere identical) square-integrable scalar-valued functions $F \colon \Omega_R \to \mathbb{R}$ with the inner product

$$\langle F, G \rangle_{\mathrm{L}^2(\Omega_R)} := \int_{\Omega_R} F(x) G(x) \, \mathrm{d}\omega(x), \qquad F, G \in \mathrm{L}^2(\Omega_R).$$

It is well-known that an orthonormal basis of $\mathrm{L}^2(\Omega)$ can be constructed by some polynomials which are called spherical harmonics and are denoted by $\{Y_{n,j}\}_{n\in\mathbb{N}_0,j=1,\ldots,2n+1}$, where the degree of $Y_{n,j}$ is n. It is easy to verify that $\{\frac{1}{R}Y_{n,j}\left(\frac{\cdot}{R}\right)\}_{n\in\mathbb{N}_0,j=1,\ldots,2n+1}$ is, consequently, an orthonormal basis of $\mathrm{L}^2(\Omega_R)$. For details on spherical harmonics, see e.g. [Müller, 1966] and [Freeden et al., 1998]. Besides this, the volume integral over $\mathcal{B}$ is denoted by $\int_{\mathcal{B}} f(x)\,\mathrm{d}x$, the surface integral over Ω by $\int_\Omega f(x)\,\mathrm{d}\omega(x)$, and a standard integral is denoted by $\int_a^b f(x)\,\mathrm{d}x$. Often, we use the abbreviation $r := |x|$ for the radial part.

2 Inverse Problem

2.1 Formulation of the Problem

Within this paper, we consider a class of inverse problems which are given by a Fredholm integral operator of the first kind

$$T\colon D \mapsto \int_{\mathcal{B}} D(x)k(x,y)\,\mathrm{d}x = V(y) \qquad y \in \mathcal{B}^{\mathrm{ext}}, \tag{1}$$

with a kernel

$$k\colon \mathcal{B}\times\mathcal{B}^{\mathrm{ext}} \to \mathbb{R},$$

a given right-hand side V, a ball $\mathcal{B}$ with radius R, and the outer space $\mathcal{B}^{\mathrm{ext}} := \mathbb{R}^3\backslash\mathcal{B}$. It is the aim to reconstruct D in $\mathcal{B}$ from knowledge of V outside of $\mathcal{B}$. This kind of integral equation arises in many areas, two examples are given below.

Example 2.1. *For the inverse gravimetric problem, the kernel and the integral operator are given by*

$$k^{\mathrm{G}}(x,y) := \frac{\gamma}{|x-y|} = \gamma\sum_{k=0}^{\infty}\frac{|x|^k}{|y|^{k+1}}P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right), \quad x\in\mathcal{B},\, y\in\mathcal{B}^{\mathrm{ext}}, |x|<|y|,$$
$$T^{\mathrm{G}}\colon D\mapsto\int_{\mathcal{B}} D(x)k^{\mathrm{G}}(x,\cdot)\,\mathrm{d}x,$$

where P_k denotes the Legendre polynomial of degree k and γ is the gravitational constant. This problem first occurs in the works of Stokes in 1867 [Stokes, 1867]. It is concerned with the recovery of mass anomalies of the Earth from data of the gravitational potential. It is of particular importance, for example, for the detection of mass transports out of time series of potential models as they are provided by the GRACE mission, see [GRACE, 2002].

Example 2.2. *For the inverse magnetic problem (as we call the problem here), the kernel and the integral operator are given by*

$$k^{\mathrm{M}}(x,y) := \frac{1}{4\pi}\sum_{k=0}^{\infty}\frac{|x|^{k-1}}{|y|^{k+1}(k+1)}P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right), \quad x\in\mathcal{B},\, y\in\mathcal{B}^{\mathrm{ext}}, 0<|x|<|y|, \tag{2}$$

$$T^{\mathrm{M}}\colon D\mapsto\int_{\mathcal{B}} D(x)k^{\mathrm{M}}(x,\cdot)\,\mathrm{d}x. \tag{3}$$

In this case, we want to recover a particular component of the electric current inside $\mathcal{B}$ (which could be the Earth (in particular the outer core) or a human brain). More precisely, the vectorial current j, for example a neuronal current, inside $\mathcal{B}$ can be decomposed via two scalar-valued, up to an additional constant unique, functions F and G and a scalar-valued unique function J^r (see e.g. [Fokas et al., 2012]) as follows:

$$j(r\xi) = \frac{1}{r}\nabla_\xi^* G(r\xi) - \frac{1}{r}\mathrm{L}_\xi^* F(r\xi) + J^r \xi.$$

Here, $\mathcal{B} \setminus \{0\} \ni x = r\xi$ with $\xi \in \Omega$ and $r = |x|$, ∇_ξ^* is the surface gradient and L_ξ^* the surface curl operator on the unit sphere. Due to [Sarvas, 1987] and the above decomposition, the relation between the current and the magnetic potential V in a spherical model can be described by

$$\begin{aligned} V(y) &= \frac{1}{4\pi}\int_{\mathcal{B}} \nabla_x \cdot (j(x) \wedge x) \sum_{k=0}^{\infty} \frac{|x|^k}{|y|^{k+1}(k+1)} P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \mathrm{d}x, \\ &= \frac{1}{4\pi}\int_{\mathcal{B}} \Delta_{\frac{x}{|x|}}^* F(x) \sum_{k=0}^{\infty} \frac{|x|^{k-1}}{|y|^{k+1}(k+1)} P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \mathrm{d}x, \end{aligned}$$

where $\Delta_{\frac{x}{|x|}}^*$ denotes the Beltrami operator and $\wedge$ the vector product. Hence, only the function F and therefore only a tangential component of the current can be reconstructed. With the abbreviation $D(x) := \Delta_{\frac{x}{|x|}}^* F(x)$ we get (3). Note that data from magnetoencephalography (MEG) can be used to obtain data for V in the case where a neuronal current is sought.

Now, we generalize these two problems and analyze the integral kernel

$$\begin{aligned} k(x,y) &:= \sum_{k=0}^{\infty} c_k \frac{|x|^{l_k}}{|y|^{k+1}} P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right), \qquad (x,y) \in \mathrm{dom}(k) \\ \mathrm{dom}(k) &:= \left\{(x,y) \in \mathcal{B} \times \mathcal{B}^{\mathrm{ext}} : |x| < |y|, x \neq 0 \text{ if there exists } n \in \mathbb{N}_0 \text{ with } l_n < 0\right\}, \end{aligned}$$

where $(c_k)_{k\in\mathbb{N}_0}$ is a bounded sequence (i.e. $\sup_{k\in\mathbb{N}_0} |c_k| \le c$) and l_k is an exponent which also depends on k and fulfils $l_k \ge -1$ for all $k \in \mathbb{N}_0$, i.e. $l_k = k$, $c_k = \gamma$ for the inverse gravimetric problem and $l_k = k-1$, $c_k = \frac{1}{4\pi(k+1)}$ for the inverse magnetic problem.

In general, we assume that the integral in (1) exists. We will later see that this is achieved if a summation condition is fulfilled and the integration over the radial part exists.

The problem of the non-uniqueness for the above mentioned problems has been discussed extensively in literature, starting with the paper [Stokes, 1867] for the inverse gravimetric problem. Other references are, for example, [Stromeyer and Ballani, 1984, Barzaghi and Sansò, 1986, Anger, 1990, Ballani and Stromeyer, 1992]. For a survey article on this topic, see [Fokas and Michel, 2008]. For the inverse magnetic problem, see [Sarvas, 1987, Fokas et al., 1996, Fokas et al., 2004, Dassios et al., 2005, Fokas, 2009], and [Dassios and Fokas, 2009, Fokas and Kurylev, 2012, Dassios and Fokas, 2013].

2.2 Forward Problem

In order to derive a relation between the spherical harmonics coefficients of the given and the unknown function, we first consider the forward problem. See, for a similar theorem for the inverse gravimetric problem, [Fokas and Michel, 2008].

Lemma 2.3. *Let the function $D \in \mathrm{L}^2(\mathcal{B})$ be given. We assume that this function is expandable in spherical harmonics*

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} D_{n,j}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right) \tag{4}$$

and that this series converges in $\mathrm{L}^2(\mathcal{B})$. Then the integral

$$V = \int_{\mathcal{B}} D(x) k(x, \cdot)\, \mathrm{d}x$$

with the kernel

$$k(x,y) = \sum_{k=0}^{\infty} c_k \frac{|x|^{l_k}}{|y|^{k+1}} P_k\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \qquad (x,y) \in \mathrm{dom}(k) \tag{5}$$

is given for $|y| > R$ pointwise and for $|y| = R$ in the sense of $\mathrm{L}^2(\Omega_R)$ by

$$V(y) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\int_0^R r^{l_n+2} D_{n,j}(r)\, \mathrm{d}r \right) \frac{4\pi c_n}{2n+1} |y|^{-n-1} Y_{n,j}\left(\frac{y}{|y|}\right), \tag{6}$$

if l_n fulfils the conditions $\sup_{n\in\mathbb{N}_0} R^{l_n - n} < \infty$ (i.e. $\sup_{n\in\mathbb{N}_0}(l_n - n) < \infty$, if $R > 1$, and $\sup_{n\in\mathbb{N}_0}(n - l_n) < \infty$, if $R < 1$) and $\inf_{n\in\mathbb{N}_0} l_n \geq -1$.

Proof. Substituting the representation (4) of D and (5) of the kernel k in (1), we obtain ($y = |y|\,\eta$, $x = r\xi$; η, $\xi \in \Omega$)

$$\begin{aligned} V(y) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{k=0}^{\infty} \int_0^R r^2 D_{n,j}(r) \frac{c_k r^{l_k}}{|y|^{k+1}}\, \mathrm{d}r \int_{\Omega} P_k(\xi \cdot \eta)\, Y_{n,j}(\xi)\, \mathrm{d}\omega(\xi) \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{k=0}^{\infty} \frac{c_k}{|y|^{k+1}} \int_0^R r^{l_k+2} D_{n,j}(r)\, \mathrm{d}r\, \frac{4\pi}{2n+1}\, \delta_{k,n} Y_{n,j}(\eta) \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\int_0^R r^{l_n+2} D_{n,j}(r)\, \mathrm{d}r \right) \frac{4\pi c_n}{(2n+1)\,|y|^{n+1}} Y_{n,j}(\eta), \end{aligned}$$

since $\frac{2n+1}{4\pi} P_n$ is the reproducing kernel for the spherical harmonics of degree n, i.e. for an arbitrary $\eta \in \Omega$ it holds that $\left\langle \frac{2n+1}{4\pi} P_n(\cdot\eta), Y_{n,j} \right\rangle_{\mathrm{L}^2(\Omega)} = Y_{n,j}(\eta)$.

However, we have to check if we were allowed to interchange the limits of both series with the integration over $\mathcal{B}$. The argument concerning the series over n is simple. Due to the strong convergence in $\mathrm{L}^2(\mathcal{B})$ of the series in (4), we obtain the weak convergence

$$\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\left\langle D_{n,j}(|\cdot|)Y_{n,j}\left(\frac{\cdot}{|\cdot|}\right),F\right\rangle_{\mathrm{L}^2(\mathcal{B})}=\langle D,F\rangle_{\mathrm{L}^2(\mathcal{B})}\qquad \text{for all } F\in\mathrm{L}^2(\mathcal{B}).$$

Finally, the kernel $k(\cdot,y)\in\mathrm{L}^2(\mathcal{B})$ satisfies, for each fixed $y\in\mathcal{B}^{\mathrm{ext}}$ and $r:=|x|$,

$$\begin{aligned}
&\int_{\mathcal{B}}\left(\sum_{k=N}^{\infty}c_k\frac{|x|^{l_k}}{|y|^{k+1}}P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\right)^2\mathrm{d}x\\
&\le c^2\int_{\mathcal{B}}\left(\sum_{k=N}^{\infty}\frac{|x|^{l_k}}{|y|^{k+1}}\right)^2\mathrm{d}x=4\pi c^2\int_0^R r^2\left(\sum_{k=N}^{\infty}\frac{r^{l_k}}{|y|^{k+1}}\right)^2\mathrm{d}r\\
&=4\pi c^2\int_0^R\left(\sum_{k=N}^{\infty}\frac{r^{l_k+1}}{|y|^{k+1}}\right)^2\mathrm{d}r\le 4\pi Rc^2\left(\sup_{n\in\mathbb{N}}R^{l_n-n}\right)^2\left(\sum_{k=N}^{\infty}\frac{R^{k+1}}{|y|^{k+1}}\right)^2\to 0\quad (N\to\infty),
\end{aligned}$$

due to the conditions on l_k, the inequality $\frac{R}{|y|}<1$, and the fact that $|P_k(t)|\le 1$ for all $k\in\mathbb{N}$ and all $t\in[-1,1]$. Hence, we have also here a strong convergence. This justifies the interchanging procedure above.

If the conditions on l_n are satisfied, then $V|_{\Omega_R}\in\mathrm{L}^2(\Omega_R)$ due to the Cauchy-Schwarz-inequality and the Parseval identity:

$$\begin{aligned}
\|V|_{\Omega_R}\|^2_{\mathrm{L}^2(\Omega_R)}&=\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\left(\int_0^R r^{l_n+2}D_{n,j}(r)\,\mathrm{d}r\right)^2\left(\frac{4\pi c_n}{(2n+1)R^n}\right)^2\\
&\le\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\left(\int_0^R\frac{r^{2l_n+2}}{R^{2n}}\,\mathrm{d}r\right)\left(\int_0^R r^2(D_{n,j}(r))^2\,\mathrm{d}r\right)\left(\frac{4\pi c}{2n+1}\right)^2\\
&\le 16\pi^2c^2\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\frac{R^{2l_n+3-2n}}{2l_n+3}\left(\int_0^R r^2(D_{n,j}(r))^2\,\mathrm{d}r\right)\\
&\le 16\pi^2c^2R^3\sup_{n\in\mathbb{N}_0}\frac{R^{2l_n-2n}}{2l_n+3}\,\|D\|^2_{\mathrm{L}^2(\mathcal{B})}<\infty.
\end{aligned}$$

The argumentation concerning the pointwise convergence of (6) for $|y|>R$ is based on the estimate (note that $\max_{\xi\in\Omega}|Y_{n,j}(\xi)|\le\sqrt{\frac{2n+1}{4\pi}}$ for all $n\in\mathbb{N}_0$)

$$\begin{aligned}
&\left|\int_0^R r^{l_n+2}D_{n,j}(r)\,\mathrm{d}r\;\frac{4\pi c_n}{2n+1}\;|y|^{-n-1}Y_{n,j}\left(\frac{y}{|y|}\right)\right|\\
&\le\left(\frac{R^{2l_n+3}}{2l_n+3}\int_0^R r^2\left(D_{n,j}(r)\right)^2\,\mathrm{d}r\right)^{1/2}\frac{4\pi c}{2n+1}|y|^{-n-1}\sqrt{\frac{2n+1}{4\pi}}\\
&\le c\left(\frac{R^{2l_n+3}}{R^{2n+2}(2l_n+3)}\int_0^R r^2\left(D_{n,j}(r)\right)^2\,\mathrm{d}r\frac{4\pi}{2n+1}\right)^{1/2}\left(\frac{R}{|y|}\right)^{n+1}.
\end{aligned}$$

The square root is bounded over all n, due to the conditions on l_n and the fact that the Parseval identity of $D \in \mathrm{L}^2(\mathcal{B})$ must converge. Hence, the series (6) is dominated by a geometric series for $|y| > R$. □

Assumption 2.4. *From now on, we assume that* $\sup_{n\in\mathbb{N}_0} R^{l_n - n} < \infty$ *(i.e.* $\sup_{n\in\mathbb{N}_0}(l_n - n) < \infty$*, if* $R > 1$*, and* $\sup_{n\in\mathbb{N}_0}(n - l_n) < \infty$*, if* $R < 1$*) and* $\inf_{n\in\mathbb{N}_0} l_n \geq -1$*.*

For both, the inverse magnetic problem and the inverse gravimetric problem, the conditions of Assumption 2.4 are fulfilled.

We also remark that the existence of the integral in (1) only depends on the existence of the integral of the radial part and the convergence of the series in (6).

Obviously, (6) yields

$$\left\langle V|_{\Omega_R}, \frac{1}{R} Y_{n,j}\left(\frac{\cdot}{R}\right)\right\rangle_{\mathrm{L}^2(\Omega_R)} = \left(\int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r\right) \frac{4\pi c_n}{(2n+1)R^n}.$$

Using the abbreviation

$$V_{n,j} := \left\langle V|_{\Omega_R}, \frac{1}{R} Y_{n,j}\left(\frac{\cdot}{R}\right)\right\rangle_{\mathrm{L}^2(\Omega_R)},$$

we get

$$\frac{(2n+1)R^n}{4\pi c_n} V_{n,j} = \int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r. \tag{7}$$

This relation allows an infinite number of choices for $D_{n,j}$ and, hence, the solution of the inverse problem in (1) is not unique, see also [Fokas and Michel, 2008]. For the inverse gravimetric problem, the last relation is well-known, see for example [Pizzetti, 1910, Rubincam, 1979], or [Moritz, 1990] and for the inverse magnetic problem, see for instance [Fokas et al., 1996].

3 Characterization of the Null Space

For the characterization of the null space of the Fredholm integral operator of the first kind considered so far, an appropriate basis for $\mathrm{L}^2(\mathcal{B})$ is needed. We use the following orthonormal basis for $\mathrm{L}^2(\mathcal{B})$ which is given by

$$G_{m,n,j}(x) = \gamma_{m,n} P_m^{(0,l_n+1/2)}\left(2\frac{|x|^2}{R^2} - 1\right) \frac{|x|^{l_n}}{R^{l_n}} Y_{n,j}\left(\frac{x}{|x|}\right), \qquad x \in \mathcal{B}\setminus\{0\}, \tag{8}$$

$m, n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$, where $\left\{P_m^{(\alpha,\beta)}\right\}_{m\in\mathbb{N}_0}$ are the Jacobi polynomials (see, for example, [Szegö, 1975]) and $\gamma_{m,n}$ are normalization constants

$$\gamma_{m,n} := \sqrt{\frac{4m + 2l_n + 3}{R^3}}.$$

If the exponents l_n are non-negative, we can extend our functions $G_{m,n,j}$ for all $x \in \mathcal{B}$. Note that the Jacobi polynomials $P_m^{(\alpha,\beta)}$, where $\alpha, \beta > -1$, are uniquely determined by the requirements that

1. each $P_m^{(\alpha,\beta)}$ is a polynomial of degree m,
2. for all m, $n \in \mathbb{N}_0$ with $m \neq n$,
$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\, \mathrm{d}x = 0,$$
and
3. for each $m \in \mathbb{N}_0$, we set $P_m^{(\alpha,\beta)}(1) = \binom{m+\alpha}{m}$.

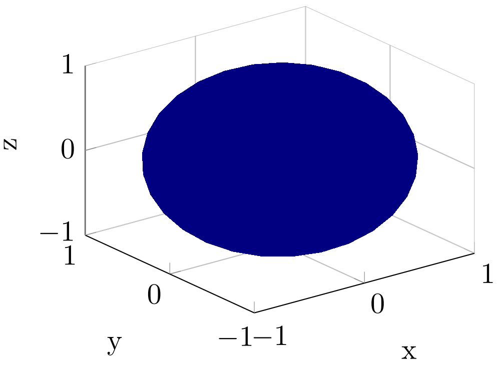
Theorem 3.1. *The set of functions* $\{G_{m,n,j}\}_{m,n\in\mathbb{N}_0, j=1,\ldots,2n+1}$ *given in* (8) *is an orthonormal basis for* $\mathrm{L}^2(\mathcal{B})$.

Proof. First, we need to calculate the inner products. Using the $\mathrm{L}^2(\Omega)$-orthonormality of the spherical harmonics and the substitution $r = R\sqrt{\frac{1+z}{2}}$, we obtain

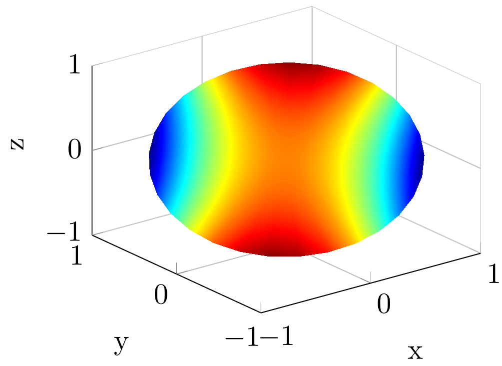
$$\begin{aligned}
&\left\langle G_{m,n,j}, G_{\tilde{m},\tilde{n},\tilde{j}} \right\rangle_{\mathrm{L}^2(\mathcal{B})} \\
&\quad = \gamma_{m,n}\gamma_{\tilde{m},\tilde{n}}\delta_{\tilde{n},n}\delta_{\tilde{j},j} \int_0^R \frac{r^{2l_n+2}}{R^{2l_n}} P_m^{(0,l_n+1/2)}\left(2\frac{r^2}{R^2}-1\right) P_{\tilde{m}}^{(0,l_n+1/2)}\left(2\frac{r^2}{R^2}-1\right) \mathrm{d}r \\
&\quad = \gamma_{m,n}\gamma_{\tilde{m},n}\delta_{\tilde{n},n}\delta_{\tilde{j},j} \frac{R^3}{2^{l_n+5/2}} \int_{-1}^{1} (1+z)^{l_n+1/2} P_m^{(0,l_n+1/2)}(z) P_{\tilde{m}}^{(0,l_n+1/2)}(z)\, \mathrm{d}z \\
&\quad = \gamma_{m,n}\gamma_{\tilde{m},n}\delta_{\tilde{n},n}\delta_{\tilde{j},j} \frac{R^3}{2^{l_n+5/2}} \frac{2^{l_n+3/2}}{2m+l_n+3/2}\delta_{\tilde{m},m} \\
&\quad = \gamma_{m,n}^2 \delta_{\tilde{n},n}\delta_{\tilde{j},j} \frac{R^3}{4m+2l_n+3}\delta_{\tilde{m},m} \\
&\quad = \delta_{\tilde{m},m}\delta_{\tilde{n},n}\delta_{\tilde{j},j},
\end{aligned}$$

where the formula for the integral over the Jacobi polynomials is given in [Nikiforov and Uvarov, 1988]. Thus, the set $\{G_{m,n,j}\}_{m,n\in\mathbb{N}_0, j=1,\ldots,2n+1}$ is $\mathrm{L}^2(\mathcal{B})$-orthonormal. Due to the completeness of the spherical harmonics and the Jacobi polynomials, it is a basis for $\mathrm{L}^2(\mathcal{B})$. □

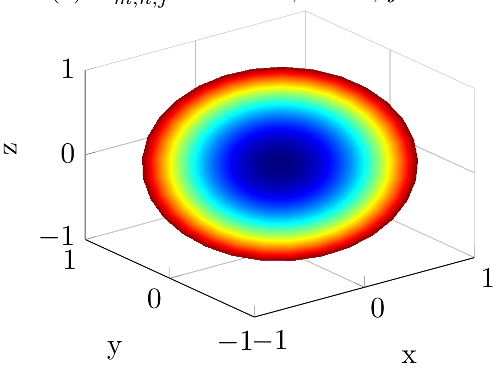
There are many known approaches for the construction of basis systems on the sphere, see e.g. [Dufour, 1977, Ballani et al., 1993, Tscherning, 1996], or [Michel, 2013]. The orthonormal system used here is a generalization of the system which was introduced in [Dufour, 1977] and [Ballani et al., 1993]. It was called $G^{\mathrm{I}}_{m,n,j}$ in [Michel, 2005] and [Michel, 2013] in the case of $l_n = n$ (remember that this setting corresponds to the inverse gravimetric problem). Some of the functions from the system $G^{\mathrm{I}}_{m,n,j}$ are shown in Figure 1. A selection of the functions corresponding to the inverse magnetic problem, where $l_n = n - 1$, is shown in Figure 2.



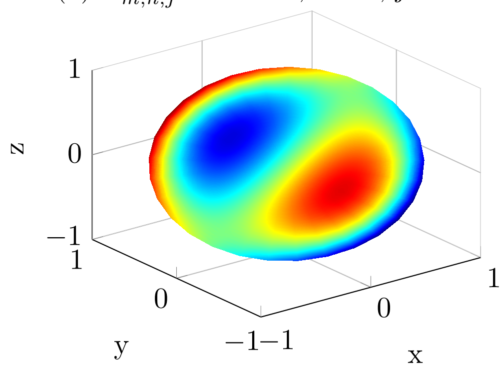
(a) $G^{\mathrm{I}}_{m,n,j}$ for $m = 0$, $n = 0$, $j = 0$



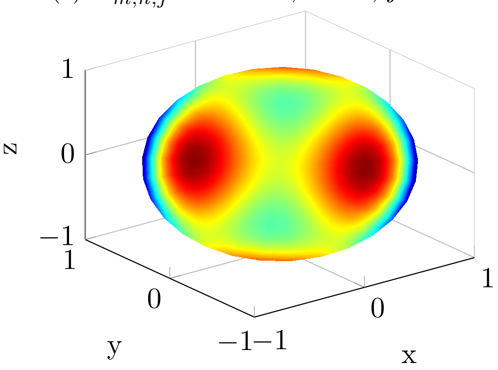
(b) $G^{\mathrm{I}}_{m,n,j}$ for $m = 0$, $n = 2$, $j = 2$



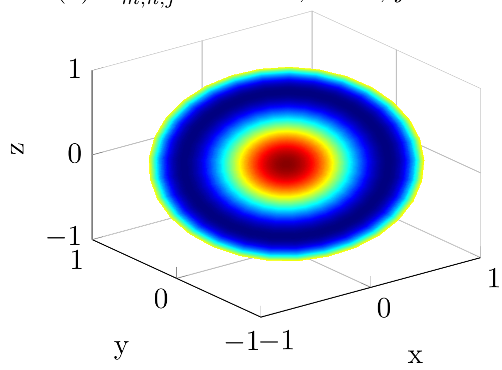
(c) $G^{\mathrm{I}}_{m,n,j}$ for $m = 1$, $n = 0$, $j = 0$



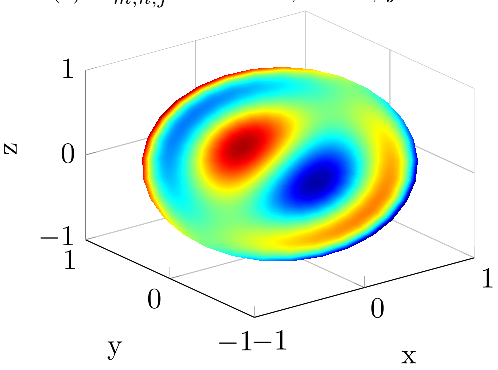
(d) $G^{\mathrm{I}}_{m,n,j}$ for $m = 1$, $n = 1$, $j = 1$



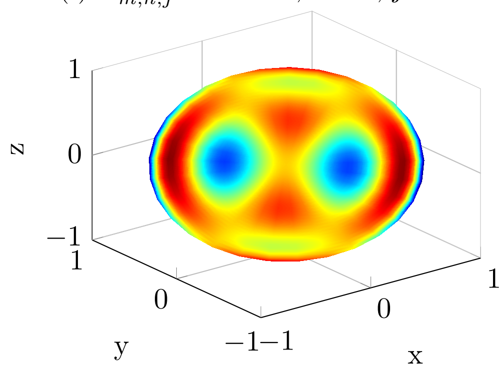
(e) $G^{\mathrm{I}}_{m,n,j}$ for $m = 1$, $n = 2$, $j = 2$



(f) $G^{\mathrm{I}}_{m,n,j}$ for $m = 2$, $n = 0$, $j = 0$

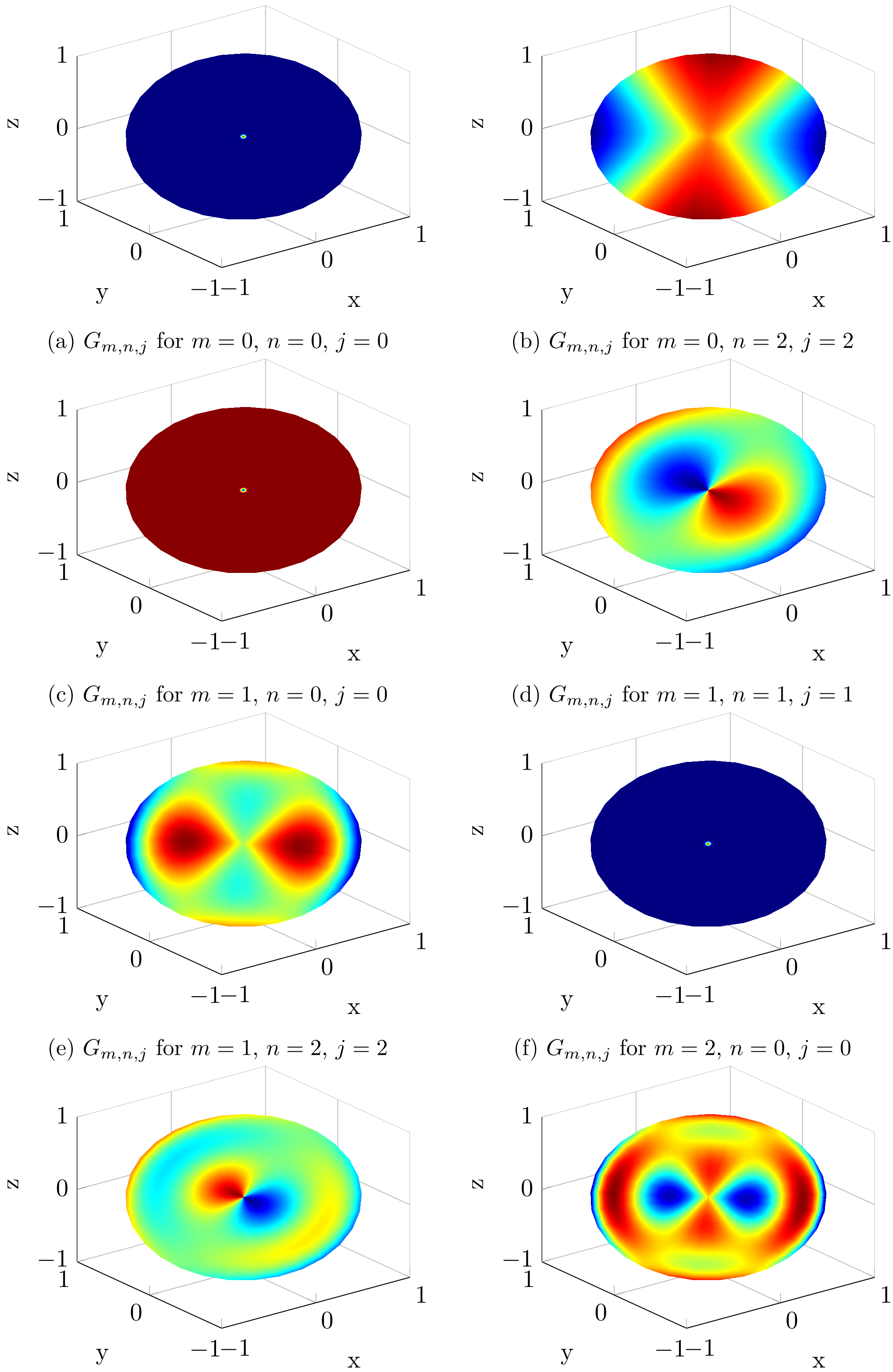


(g) $G^{\mathrm{I}}_{m,n,j}$ for $m = 2$, $n = 1$, $j = 1$



(h) $G^{\mathrm{I}}_{m,n,j}$ for $m = 2$, $n = 2$, $j = 2$

Figure 1: The functions $G_{m,n,j}$ in the case $l_n = n$ (also called $G^{\mathrm{I}}_{m,n,j}$) for different parameters m, n, j are plotted at the plane through the origin with normal vector $(1, 1, -1)^{\mathrm{T}}$. For the particular parameters, see the headline of each plot. The maximum is always red and the minimum is blue.



(a) $G_{m,n,j}$ for $m = 0,\ n = 0,\ j = 0$

(b) $G_{m,n,j}$ for $m = 0,\ n = 2,\ j = 2$

(c) $G_{m,n,j}$ for $m = 1,\ n = 0,\ j = 0$

(d) $G_{m,n,j}$ for $m = 1,\ n = 1,\ j = 1$

(e) $G_{m,n,j}$ for $m = 1,\ n = 2,\ j = 2$

(f) $G_{m,n,j}$ for $m = 2,\ n = 0,\ j = 0$

(g) $G_{m,n,j}$ for $m = 2,\ n = 1,\ j = 1$

(h) $G_{m,n,j}$ for $m = 2,\ n = 2,\ j = 2$

Figure 2: The functions $G_{m,n,j}$ in the case $l_n = n - 1$ for different parameters m, n, j are plotted at the plane through the origin with normal vector $(1, 1, -1)^{\mathrm{T}}$. For the particular parameters, see the headline of each plot. The maximum is always red and the minimum is blue.

With this orthonormal basis, the functions $D_{n,j}$ in (4) have the representation

$$D_{n,j}(r) = \frac{r^{l_n}}{R^{l_n}} \sum_{m=0}^{\infty} d_{m,n,j} \gamma_{m,n} P_m^{(0,l_n+1/2)} \left(2\frac{r^2}{R^2} - 1 \right), \qquad n \in \mathbb{N}_0,\ j = 1, \ldots, 2n+1,$$

where $d_{m,n,j} := \langle D, G_{m,n,j} \rangle_{\mathrm{L}^2(\mathcal{B})}$.

Using this representation in (6), by means of the orthogonality of the Jacobi polynomials, we get $\left(\text{again with the substitution } r = R\sqrt{\frac{1+z}{2}},\ \mathrm{d}r = \frac{R}{4}\left(\frac{2}{1+z}\right)^{1/2} \mathrm{d}z\right)$

$$\begin{aligned}
V(y) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\int_0^R r^{l_n+2} D_{n,j}(r) \,\mathrm{d}r \right) \frac{4\pi c_n}{(2n+1)\,|y|^{n+1}} Y_{n,j}\left(\frac{y}{|y|}\right) \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\int_0^R \frac{r^{2l_n+2}}{R^{l_n}} \sum_{m=0}^{\infty} d_{m,n,j} \gamma_{m,n} P_m^{(0,l_n+1/2)} \left(2\frac{r^2}{R^2} - 1 \right) \mathrm{d}r \right) \\
&\quad \times \frac{4\pi c_n}{(2n+1)\,|y|^{n+1}} Y_{n,j}\left(\frac{y}{|y|}\right) \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{R^{3+l_n}}{2^{l_n+5/2}} \left(\int_{-1}^{1} (1+z)^{l_n+1/2} \sum_{m=0}^{\infty} d_{m,n,j} \gamma_{m,n} P_m^{(0,l_n+1/2)}(z) \,\mathrm{d}z \right) \\
&\quad \times \frac{4\pi c_n}{(2n+1)\,|y|^{n+1}} Y_{n,j}\left(\frac{y}{|y|}\right) \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{R^{3+l_n}}{2^{l_n+5/2}} d_{0,n,j} \gamma_{0,n} \frac{2^{l_n+3/2}}{l_n + 3/2} \frac{4\pi c_n}{(2n+1)\,|y|^{n+1}} Y_{n,j}\left(\frac{y}{|y|}\right) \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} d_{0,n,j} \gamma_{0,n}^{-1} \frac{4\pi c_n R^{l_n}}{(2n+1)\,|y|^{n+1}} Y_{n,j}\left(\frac{y}{|y|}\right).
\end{aligned}$$

This result allows a direct characterization of the null space of the Fredholm integral operator of the first kind

$$\ker T = \overline{\operatorname{span}\left\{ G_{m,n,j} | m \geq 1 \text{ or } c_n = 0 \right\}}^{\|\cdot\|_{\mathrm{L}^2(\mathcal{B})}}.$$

For the inverse gravimetric problem, this leads to the well-known result that the null space is the set of all anharmonic functions, i.e. the orthogonal complement of the set of all harmonic functions.

$$\ker T^{\mathrm{G}} = \overline{\operatorname{span}\left\{ G^{\mathrm{I}}_{m,n,j} \,\middle|\, m \geq 1 \right\}}^{\|\cdot\|_{\mathrm{L}^2(\mathcal{B})}} = \left\{ F \in \mathrm{C}^{(2)}(\mathcal{B}) \,\middle|\, \Delta F = 0 \right\}^{\perp_{\mathrm{L}^2(\mathcal{B})}}.$$

Corollary 3.2 (Singular System)**.** *The singular system of the operator defined by the Fredholm integral operator*

$$T\colon D \mapsto \int_{\mathcal{B}} D(x) \sum_{k=0}^{\infty} \frac{c_k\,|x|^{l_k}}{|\cdot|^{k+1}} P_k\left(\frac{x}{|x|} \cdot \frac{\cdot}{|\cdot|}\right) \mathrm{d}x, \qquad (x, \cdot) \in \operatorname{dom}(k),$$

is given by the $\mathrm{L}^2(\mathcal{B})-$*orthonormal set* $\{G_{0,n,j}\}_{n\in\mathbb{N}_0,c_n\neq 0;j=1,\ldots,2n+1}$ *in the domain, the set of* $\mathrm{L}^2(\Omega_R)$*-orthonormal outer harmonic functions* $\{\frac{R^n}{|\cdot|^{n+1}}Y_{n,j}\left(\frac{\cdot}{R}\right)\}_{n\in\mathbb{N}_0,c_n\neq 0;j=1,\ldots,2n+1}$ *in the range, and the singular values* $\left(\frac{4\pi c_n R^{l_n-n+3/2}}{(2n+1)(2l_n+3)^{1/2}}\right)_{n\in\mathbb{N}_0}$.

Obviously, the sequence of the singular values is a zero sequence, since the sequences $(c_n)_{n\in\mathbb{N}_0}$ and $(R^{l_n-n})_{n\in\mathbb{N}_0}$ are bounded. Due to this condition, the operator T is a compact operator and, hence, the problem in (1) is ill-posed.

In the case $l_n = n$, the functions $G^{\mathrm{I}}_{0,n,j}$ are inner harmonics and, therefore, form a basis for the set of harmonic functions on a ball:

$$G^{\mathrm{I}}_{0,n,j}(x) = \sqrt{\frac{2n+3}{R}}\frac{|x|^n}{R^{n+1}}Y_{n,j}\left(\frac{x}{|x|}\right), \qquad x \in \mathcal{B}.$$

4 Uniqueness Constraints

We discuss here examples of uniqueness constraints as generalizations of the results in [Fokas and Michel, 2008].

4.1 Minimum Norm Solution

Now, for reconstructing the density, we need a relation between the function or Fourier coefficients of the potential V and the density function D. Recall (7) which is repeated below for convenience:

$$\frac{(2n+1)R^n}{4\pi c_n}V_{n,j} = \left(\int_0^R r^{l_n+2}D_{n,j}(r)\,\mathrm{d}r\right).$$

As seen before, we need more assumptions in order to obtain a uniquely determined solution. One approach is the minimum norm solution. The following theorem is a generalization of the minimum norm solution of the inverse gravimetric problem, see [Fokas and Michel, 2008].

Theorem 4.1 (Minimum Norm Solution)**.** *Let* $V\colon \overline{\mathcal{B}^{\mathrm{ext}}} \to \mathbb{R}$ *be an arbitrary function satisfying*

- $V|_{\Omega_R} \in \mathrm{L}^2(\Omega_R)$,
- $\sum_{n=0}^{\infty} \frac{n^2(2l_n+3)R^{2n-2l_n}}{c_n^2}\sum_{j=1}^{2n+1}V_{n,j}^2 < \infty$, *and*
- $\Delta V = 0$ *in* $\mathcal{B}^{\mathrm{ext}}$.

Then, among all $D \in \mathrm{L}^2(\mathcal{B})$ *with* $V = \int_{\mathcal{B}} D(x)k(x,\cdot)\,\mathrm{d}x$ *there is a unique minimizer of the functional*

$$\mathcal{F}(D) := \int_{\mathcal{B}} (D(x))^2\,\mathrm{d}x.$$

This minimum norm solution is given in $\mathrm{L}^2(\mathcal{B})$ *by*

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (2l_n + 3) \frac{2n+1}{4\pi c_n} R^{n-l_n-3} V_{n,j} \frac{|x|^{l_n}}{R^{l_n}} Y_{n,j}\left(\frac{x}{|x|}\right), \qquad x \in \mathcal{B}, \tag{9}$$

provided that this series converges with respect to $\mathrm{L}^2(\mathcal{B})$*.*

Proof. The solvability of the inverse problem is guaranteed by the conditions on V, where the second one is the Picard condition.

According to the calculations above, any solution solving the inverse problem can be represented by the following series, which converges in $\mathrm{L}^2(\mathcal{B})$:

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} D_{n,j}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right),$$

where

$$\frac{(2n+1)R^n}{4\pi c_n} V_{n,j} = \left(\int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r\right),$$

with $V_{n,j} := \left\langle V|_{\Omega_R}, \frac{1}{R} Y_{n,j}\right\rangle$, such that

$$V(y) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} V_{n,j} \frac{R^n}{|y|^{n+1}} Y_{n,j}\left(\frac{y}{|y|}\right), \qquad y \in \mathcal{B}^{\mathrm{ext}}.$$

Since

$$\|D\|^2_{\mathrm{L}^2(\mathcal{B})} = \int_{\mathcal{B}} (D(x))^2\,\mathrm{d}x = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_0^R r^2 (D_{n,j}(r))^2\,\mathrm{d}r,$$

we now have the following minimization problem for each $n \in \mathbb{N}_0$ and $j = 1, \ldots, 2n+1$:

$$\begin{aligned} &\text{minimize} && \int_0^R r^2 (D_{n,j}(r))^2\,\mathrm{d}r, \\ &\text{subject to} && \int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r = \frac{2n+1}{4\pi c_n} R^n V_{n,j}. \end{aligned}$$

Setting $F_{n,j}(r) := r D_{n,j}(r)$, the problem reduces to

$$\begin{aligned} &\text{minimize} && \int_0^R (F_{n,j}(r))^2\,\mathrm{d}r, \\ &\text{subject to} && \int_0^R r^{l_n+1} F_{n,j}(r)\,\mathrm{d}r = \frac{2n+1}{4\pi c_n} R^n V_{n,j}. \end{aligned}$$

We decompose $F_{n,j}$ by $F_{n,j}(r) = \alpha_{n,j} r^{l_n+1} + H_{n,j}(r)$, where $H_{n,j}$ is $\mathrm{L}^2[0,R]$-orthogonal to $r \mapsto r^{l_n+1}$. Then we have to solve

$$\begin{aligned} &\text{minimize} && \alpha_{n,j}^2 \int_0^R r^{2l_n+2}\,\mathrm{d}r + \|H_{n,j}\|^2_{\mathrm{L}^2[0,R]}, \\ &\text{subject to} && \alpha_{n,j} \int_0^R r^{2l_n+2}\,\mathrm{d}r = \frac{2n+1}{4\pi c_n} R^n V_{n,j}. \end{aligned}$$

Obviously, $H_{n,j} \equiv 0$ and

$$\alpha_{n,j} = (2l_n+3)\frac{2n+1}{4\pi c_n}\frac{R^n}{R^{2l_n+3}} V_{n,j}.$$

□

In terms of the orthonormal basis functions, this yields

$$D(x) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \sqrt{\frac{2l_n+3}{R^3}}\frac{2n+1}{4\pi c_n} R^{n-l_n} V_{n,j} G_{0,n,j}(x) \tag{10}$$

in the sense of $\mathrm{L}^2(\mathcal{B})$ for the minimum norm solution.

In the case of $l_n = n$, it can be shown that the minimum norm solution is equivalent to the harmonic solution, see [Fokas and Michel, 2008] i.e., if $l_n = n$ and $c_n = 1$ for all $n \in \mathbb{N}_0$, then

$$D(x) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \sqrt{\frac{2n+3}{R^3}}\frac{2n+1}{4\pi} V_{n,j} G^{\mathrm{I}}_{0,n,j}(x)$$

represents the minimum norm solution of the inverse gravimetric problem.

It is wise to analyze the convergence of the solution in (9).

Theorem 4.2. *Under Assumption 2.4, the series in (9) converges in $\mathrm{L}^2(\mathcal{B})$ if and only if*

$$\sum_{n=0}^{\infty} \frac{n^2(2l_n+3)}{c_n^2 R^{2(l_n-n)}} \sum_{j=1}^{2n+1} V_{n,j}^2 < \infty. \tag{11}$$

Proof. We need to calculate the $\mathrm{L}^2(\mathcal{B})$-norm of D. Using the orthonormality of the $G_{m,n,j}$, we obtain from (10) that

$$\|D\|^2_{\mathrm{L}^2(\mathcal{B})} = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{2l_n+3}{R^3}\left(\frac{2n+1}{4\pi c_n}\right)^2 R^{2n-2l_n} V_{n,j}^2.$$

Obviously, this series converges if and only if (11) holds true. □

Note that (11) is the Picard condition imposed in Theorem 4.1. Again, for the inverse gravimetric problem, i.e. $l_n = n$, $c_n = 1$ for all $n \in \mathbb{N}_0$, this convergence condition is satisfied, since the (empirical) Kaula's rule of thumb holds:

$$\sum_{j=1}^{2n+1} \langle V|_\Omega, Y_{n,j}\rangle^2_{\mathrm{L}^2(\Omega)} = \mathcal{O}(\vartheta^{n+1} n^{-3}), \qquad n \to \infty,$$

for a constant $\vartheta \in]0,1[$, see, for example, [Kaula, 1966] or [Sansò and Rummel, 1997].

4.2 A Generalization of the Harmonicity Constraint

An alternative approach to the minimum norm solution is to search for a solution D in a subspace $U \subset \mathrm{L}^2(\mathcal{B})$ with a given basis $\left\{ \frac{|\cdot|^{k_n}}{R^{k_n+1}} Y_{n,j}\left(\frac{\cdot}{|\cdot|}\right) \right\}_{n\in\mathbb{N}_0, j=1,\ldots,2n+1}$ and sequence $(k_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$. In the case $k_n = n$, the subspace U is the set of all harmonic functions. In addition, we now need the condition $2k_n + 3 > 0$ added to the condition $l_n \geq -1$ for all $n \in \mathbb{N}_0$ for the existence of the integral over $\mathcal{B}$. A short calculation shows that the given basis functions are $\mathrm{L}^2(\mathcal{B})$-orthogonal. Setting $B_{n,j}(x) := \frac{|x|^{k_n}}{R^{k_n+1}} Y_{n,j}\left(\frac{x}{|x|}\right)$ and $r = |x|$, we have

$$\begin{aligned}
\left\langle B_{n,j}, B_{\tilde{n},\tilde{j}} \right\rangle_{\mathrm{L}^2(\mathcal{B})} &= \int_{\mathcal{B}} \frac{|x|^{k_n}}{R^{k_n+1}} Y_{n,j}\left(\frac{x}{|x|}\right) \frac{|x|^{k_{\tilde{n}}}}{R^{k_{\tilde{n}}+1}} Y_{\tilde{n},\tilde{j}}\left(\frac{x}{|x|}\right) \mathrm{d}x \\
&= \int_0^R \frac{r^{2k_n+2}}{R^{2k_n+2}} \mathrm{d}r \; \delta_{n,\tilde{n}} \delta_{j,\tilde{j}} = \frac{R^{2k_n+3}}{(2k_n+3)R^{2k_n+2}} \; \delta_{n,\tilde{n}} \delta_{j,\tilde{j}} \\
&= \frac{R}{2k_n+3} \; \delta_{n,\tilde{n}} \delta_{j,\tilde{j}}.
\end{aligned}$$

Hence, the density D can be represented by

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} d_{n,j} \sqrt{\frac{2k_n+3}{R}} \frac{|x|^{k_n}}{R^{k_n+1}} Y_{n,j}\left(\frac{x}{|x|}\right), \qquad x \in \mathcal{B}, \tag{12}$$

in the sense of $\mathrm{L}^2(\mathcal{B})$. In analogy to above, we set

$$D_{n,j}(r) = d_{n,j} \sqrt{\frac{2k_n+3}{R}} \frac{r^{k_n}}{R^{k_n+1}}.$$

Thus, (7) becomes

$$\begin{aligned}
\frac{(2n+1)R^n}{4\pi c_n} V_{n,j} &= \int_0^R d_{n,j} \sqrt{\frac{2k_n+3}{R}} \frac{r^{l_n+2+k_n}}{R^{k_n+1}} \mathrm{d}r \\
&= d_{n,j} \sqrt{\frac{2k_n+3}{R}} \frac{R^{l_n+k_n+3}}{(l_n+k_n+3)R^{k_n+1}} \\
&= d_{n,j} \sqrt{\frac{2k_n+3}{R}} \frac{R^{l_n+2}}{(l_n+k_n+3)}.
\end{aligned} \tag{13}$$

We can formulate the following theorem.

Theorem 4.3. *Let V satisfy the following conditions*

- $V|_{\Omega_R} \in \mathrm{L}^2(\Omega_R)$,
- $\sum_{n=0}^{\infty} \frac{n^2(l_n+k_n+3)^2 R^{2n-2l_n}}{c_n^2(2k_n+3)} \sum_{j=1}^{2n+1} V_{n,j}^2 < \infty$, *and*

- $\Delta V = 0$ *in* $\mathcal{B}^{\text{ext}}$.

Let U *be the subspace of* $\mathrm{L}^2(\mathcal{B})$ *with the basis*

$$B_{n,j}(x) := \frac{|x|^{k_n}}{R^{k_n+1}} Y_{n,j}\left(\frac{x}{|x|}\right), \qquad n \in \mathbb{N}_0, j = 1, \ldots, 2n+1,$$

where $(k_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$. *Then the unique solution* $D \in U$ *of*

$$\int_{\mathcal{B}} D(x) k(x,y) \,\mathrm{d}x = V(y) \qquad \text{in } \overline{\mathbb{R}^3 \setminus \mathcal{B}},$$

with $(x,y) \in \operatorname{dom}(x,y)$ *is given by*

$$D(x) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi c_n} (l_n + k_n + 3) R^{n-l_n} \frac{|x|^{k_n}}{R^{k_n+3}} \sum_{j=1}^{2n+1} V_{n,j} Y_{n,j}\left(\frac{x}{|x|}\right),$$

in the sense of $\mathrm{L}^2(\mathcal{B})$.

Proof. Again, the function D can be represented by

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} D_{n,j}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right).$$

Using (12) and (13), we obtain

$$\begin{aligned} D(x) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} d_{n,j} \sqrt{\frac{2k_n+3}{R}} \frac{|x|^{k_n}}{R^{k_n+1}} Y_{n,j}\left(\frac{x}{|x|}\right) \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi c_n} (l_n + k_n + 3) R^{n-l_n} \frac{|x|^{k_n}}{R^{k_n+3}} \sum_{j=1}^{2n+1} V_{n,j} Y_{n,j}\left(\frac{x}{|x|}\right) \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi c_n} (l_n + k_n + 3) R^{n-l_n-2} \sum_{j=1}^{2n+1} V_{n,j} B_{n,j}(x). \end{aligned}$$

The convergence of the series is guaranteed by the conditions on V. □

In the case $l_n = n$ and $k_n = n$, this approach yields the harmonic solution of the problem in (1) and in the case $l_n = n$ and $k_n = n + p$, $p \in \mathbb{R}_0^+$ the quasi-harmonic solution is obtained.

If the basis of U is given by the sum of two or more radial parts, i.e. $\left\{\left(\sum_{i=1}^{K} |\cdot|^{k_{i,n}}\right) Y_{n,j}\left(\frac{\cdot}{|\cdot|}\right)\right\}_{n\in\mathbb{N}_0, j=1,\ldots,2n+1}$ for a fixed $K \in \mathbb{N}$, then a unique solution cannot be obtained. In this case, $K-1$ additional pieces of information are needed. An example of this situation is given in [Fokas and Michel, 2008, Theorem 6.1], where the biharmonic solution for $l_n = n$ is discussed.

4.3 Layer Density

In this section, a particular case of a non-radially dependent density is discussed. Let the sought-after solution be given on a shell $\Omega^{\tau}_{\tau+\varepsilon}$ with given $\tau > 0$, $\varepsilon > 0$:

$$\Omega^{\tau}_{\tau+\varepsilon} := \left\{ x \in \mathbb{R}^3 : \tau \leq |x| \leq \tau + \varepsilon \leq R \right\}.$$

Theorem 4.4. *Let $\Omega^{\tau}_{\tau+\varepsilon}$ be a given spherical shell. If $D \in \mathrm{L}^2(\mathcal{B})$ has the form*

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} D_{n,j}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right),$$
$$D_{n,j}(|x|) := \kappa d_{n,j} \chi_{[\tau,\tau+\varepsilon]}(|x|),$$

where the normalization constant κ is given by

$$\kappa = \sqrt{\frac{3}{(\tau+\varepsilon)^3 - \tau^3}},$$

then the unique solution of (1) is given by

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{(2n+1)(l_n+3)}{4\pi c_n} \frac{R^n}{(\tau+\varepsilon)^{l_n+3} - \tau^{l_n+3}} V_{n,j} \chi_{[\tau,\tau+\varepsilon]}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right), x \in \mathcal{B},$$

in the sense of $\mathrm{L}^2(\mathcal{B})$.

Proof. Using (7), we have

$$\begin{aligned}
\frac{(2n+1)R^n}{4\pi c_n} V_{n,j} &= \int_0^R r^{l_n+2} D_{n,j}(r)\, \mathrm{d}r \\
&= \kappa \int_0^R r^{l_n+2} d_{n,j} \chi_{[\tau,\tau+\varepsilon]}(r)\, \mathrm{d}r \\
&= \kappa d_{n,j} \frac{(\tau+\varepsilon)^{l_n+3} - \tau^{l_n+3}}{l_n+3}.
\end{aligned}$$

As a consequence,

$$\kappa d_{n,j} = \frac{(2n+1)R^n}{4\pi c_n} V_{n,j} \frac{l_n+3}{(\tau+\varepsilon)^{l_n+3} - \tau^{l_n+3}}.$$

And hence,

$$\begin{aligned}
D(x) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} d_{n,j} \kappa \chi_{[\tau,\tau+\varepsilon]}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right), \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{(2n+1)(l_n+3)}{4\pi c_n} \frac{R^n}{(\tau+\varepsilon)^{l_n+3} - \tau^{l_n+3}} V_{n,j} \chi_{[\tau,\tau+\varepsilon]}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right). \qquad \square
\end{aligned}$$

Under the conditions of the previous theorem, the corresponding potential V has the representation

$$V(y) = \kappa \sum_{n=0}^{\infty} \frac{4\pi c_n}{(2n+1)(l_n+3)} \left((\tau+\varepsilon)^{l_n+3} - \tau^{l_n+3}\right) |y|^{-n-1} \sum_{j=1}^{2n+1} d_{n,j} Y_{n,j}\left(\frac{y}{|y|}\right).$$

This series converges pointwise for $|y| > R$, because $(c_n)_{n\in\mathbb{N}_0}$ is bounded and $(l_n)_{n\in\mathbb{N}_0}$ satisfies Assumption 2.4. Due to the square-integrability of $D_{n,j}$, this can be extended to Ω_R in the sense that we get a function in $\mathrm{L}^2(\Omega_R)$:

$$\begin{aligned}
\|V|_{\Omega_R}\|^2_{\mathrm{L}^2(\Omega_R)} &= \kappa^2 \sum_{n=0}^{\infty} \left(\frac{4\pi c_n}{(2n+1)(l_n+3)}\left((\tau+\varepsilon)^{l_n+3} - \tau^{l_n+3}\right)\right)^2 R^{-2n} \sum_{j=1}^{2n+1} d_{n,j}^2 \\
&\leq 16\pi^2 c^2 \kappa^2 \sum_{n=0}^{\infty} \frac{\left(R^{l_n+3} + R^{l_n+3}\right)^2}{(2n+1)^2(2l_n+3)^3 R^{2n}} \sum_{j=1}^{2n+1} d_{n,j}^2 \\
&\leq 64\pi^2 c^2 \kappa^2 \sup_{n\in\mathbb{N}_0} \left(\frac{R^{2l_n+6-2n}}{(2n+1)^2(2l_n+3)^2}\right) \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} d_{n,j}^2 < \infty,
\end{aligned}$$

where the supremum is finite because of Assumption 2.4.

This kind of density is for example used in [Fokas et al., 2012] for the inverse MEG problem.

5 Generalization to Higher Dimension

The results derived above can be further generalized to higher dimensions, if we consider $\mathcal{B}$ to be the ball of radius R with centre 0 in $\mathbb{R}^q$ and use the spherical harmonics $Y_{n,j}^{(q)}$ on $\mathbb{S}^{q-1}$, i.e. the unit sphere in $\mathbb{R}^q$. Note that, in our previous notation, $\mathcal{B} = \mathcal{B}_3$ and $\Omega = \mathbb{S}^2$. From [Müller, 1966], we have the addition theorem

$$\sum_{j=1}^{N(q,n)} Y_{n,j}^{(q)}(\xi) Y_{n,j}^{(q)}(\eta) = \frac{N(q,n)}{\omega_q} P_n^{(q)}(\xi\cdot\eta), \qquad \xi,\eta \in \mathbb{S}^{q-1}$$

where $\{Y_{n,j}^{(q)}\}_{j=1,\ldots,N(q,n)}$ is an orthonormal basis of spherical harmonics of degree n on $\mathbb{S}^{q-1}$, $\omega_q = \int_{\mathbb{S}^{q-1}} 1\,\mathrm{d}\omega$, and the Gegenbauer polynomials $\{P_n^{(q)}\}_{n\in\mathbb{N}_0}$ are defined by

1. $P_n^{(q)}$ is a polynomial of degree n.
2. For $n \neq m$,

$$\int_{-1}^{1} P_n^{(q)}(t) P_m^{(q)}(t) (1-t^2)^{(q-3)/2}\,\mathrm{d}t = 0.$$

3. $P_n^{(q)}(1) = 1$.

Note that we use a slightly different notation here than in [Müller, 1966]. From [Müller, 1966, Lemma 19], we also know that

$$|x-y|^{2-q} = \sum_{n=0}^{\infty} \gamma_n(q) \frac{|x|^n}{|y|^{n-2+q}} P_n^{(q)}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)$$

for all $x, y \in \mathbb{R}^q$ with $|x| < |y|$, where

$$\gamma_n(q) = \frac{\Gamma(n+q-2)}{n!\,\Gamma(q-2)}$$

with Γ being the Gamma function. Accordingly, we can generalize the integral operator in (1) to

$$T_q\colon D \mapsto \int_{\mathcal{B}_q} D(x)k(x,y)\,\mathrm{d}x = V(y), \qquad y \in \mathbb{R}^q \setminus \mathcal{B}_q =: \mathcal{B}_q^{\text{ext}},$$

with

$$\begin{aligned} k(x,y) &:= \sum_{n=0}^{\infty} c_n \frac{|x|^{l_n}}{|y|^{n-2+q}} P_n^{(q)}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \\ &= \sum_{n=0}^{\infty} c_n \frac{|x|^{l_n}}{|y|^{n-2+q}} \frac{\omega_q}{N(q,n)} \sum_{j=1}^{N(q,n)} Y_{n,j}^{(q)}\left(\frac{x}{|x|}\right) Y_{n,j}^{(q)}\left(\frac{y}{|y|}\right) \end{aligned}$$

and $|x| \neq 0$ if $l_n < 0$ for a $n \in \mathbb{N}_0$. Since the Laplace operator in $\mathbb{R}^q$ is representable as (see [Müller, 1966, p. 38])

$$\Delta_q = \frac{\partial^2}{\partial r^2} + (q-1)\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_q^*, \tag{14}$$

where

$$\Delta_q^* Y_{n,j}^{(q)} = -n(n+q-2)Y_{n,j}^{(q)}, \tag{15}$$

we observe that

$$\begin{aligned} &\Delta_q\left(|y|^{-n+2-q} Y_{n,j}^{(q)}\left(\frac{y}{|y|}\right)\right) \\ &= \left[(-n+2-q)(-n+1-q) + (q-1)(-n+2-q) - n(n+q-2)\right] |y|^{-n-q} Y_{n,j}^{(q)}\left(\frac{y}{|y|}\right) \\ &= \left(n^2 - 3n + 2qn + 2 - 3q + q^2 - qn + 3q - q^2 + n - 2 - n^2 - nq + 2n\right) \\ &\qquad \times |y|^{-n-q} Y_{n,j}^{(q)}\left(\frac{y}{|y|}\right) = 0. \end{aligned}$$

As a consequence, the generalized operator T_q, again, produces functions $T_q D$ which are harmonic in $\mathcal{B}_q^{\text{ext}}$. Furthermore, the ansatz

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(q,n)} D_{n,j}(|x|) Y_{n,j}^{(q)}\left(\frac{x}{|x|}\right), \qquad x \in \mathcal{B}_q,$$

yields (in analogy to (6))

$$(TD)(y) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(q,n)} c_n \int_0^R r^{l_n+q-1} D_{n,j}(r) \mathrm{d}r \frac{\omega_q}{N(q,n)} \frac{1}{|y|^{n-2+q}} Y_{n,j}^{(q)} \left(\frac{y}{|y|} \right)$$

for $y \in \mathcal{B}_q^{\mathrm{ext}}$ under appropriate conditions for the convergence. Using the procedure in Theorem 4.1, we find here that the minimum norm solution is characterized by

$$\begin{aligned} &\text{minimize} && \int_0^R r^{q-1} (D_{n,j}(r))^2 \, \mathrm{d}r, \\ &\text{subject to given values} && \int_0^R r^{l_n+q-1} D_{n,j}(r) \, \mathrm{d}r. \end{aligned}$$

With $F_{n,j}(r) \coloneqq r^{(q-1)/2} D_{n,j}(r)$, the task becomes

$$\begin{aligned} &\text{minimize} && \int_0^R (F_{n,j}(r))^2 \, \mathrm{d}r, \\ &\text{subject to given values} && \int_0^R r^{l_n+\frac{q-1}{2}} F_{n,j}(r) \, \mathrm{d}r. \end{aligned}$$

We then see, with the decomposition $F_{n,j}(r) = \alpha_{n,j} r^{l_n+\frac{q-1}{2}} + I_{n,j}(r)$, where $I_{n,j}(r)$ is $\mathrm{L}^2[0,R]$-orthogonal to $r \mapsto r^{l_n+\frac{q-1}{2}}$, that the minimum norm solution has the form

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(q,n)} \alpha_{n,j} |x|^{l_n} Y_{n,j}^{(q)} \left(\frac{x}{|x|} \right), \qquad x \in \mathcal{B}_q,$$

in the sense of $\mathrm{L}^2(\mathcal{B}_q)$, where those $\alpha_{n,j}$ for which $c_n \neq 0$ can be obtained uniquely via the constraint.

Theorem 5.1. *If all $c_n \neq 0$, then the orthogonal complement of the null space of T_q is spanned by the basis*

$$H_{n,j}(x) \coloneqq \frac{|x|^{l_n}}{R^{l_n}} Y_{n,j}^{(q)} \left(\frac{x}{|x|} \right), \qquad x \in \mathcal{B}_q.$$

Results on the uniqueness constraints can analogously be derived for the case of q dimensions. In particular, since a function

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(q,n)} h_{n,j} H_{n,j}(x), \qquad x \in \mathcal{B}_q,$$

is uniquely determined by its surface values

$$D(R\xi) = \sum_{n=0}^{\infty} \sum_{j=1}^{N(q,n)} h_{n,j} Y_{n,j}^{(q)}(\xi), \qquad \xi \in \mathbb{S}^{q-1},$$

provided again that an appropriate convergence is given in both cases, we conclude that, in all considered cases, the information of a radial dependence of the solution D of $T_q D = V$ is lost. In other words: given V, then the minimum norm solution D of $T_q D = V$ is uniquely determined by its restriction $D|_{\mathbb{S}_R^{q-1}}$ to the surface $\mathbb{S}_R^{q-1} := \{x \in \mathbb{R}^q | |x| = R\}$.

Furthermore, in the case of the (q-dimensional) inverse gravimetric problem (where $l_n = n$ and $c_n = \gamma_n(q) \neq 0$ for all $n \in \mathbb{N}_0$), the basis in Theorem 5.1 is harmonic since

$$\begin{aligned}\Delta_q \left(|x|^n Y_{n,j}^{(q)} \left(\frac{x}{|x|} \right) \right) &= [n(n-1) + (q-1)n - n(n+q-2)] \, |x|^{n-2} Y_{n,j}^{(q)} \left(\frac{x}{|x|} \right) \\ &= \left(n^2 - n + qn - n - n^2 - nq + 2n \right) |x|^{n-2} Y_{n,j}^{(q)} \left(\frac{x}{|x|} \right) = 0.\end{aligned}$$

Theorem 5.2. *The null space of the inverse gravimetric problem in* $\mathbb{R}^q$, $q \geq 3$, *is given by the orthogonal complement of the space of all harmonic functions, i.e.,*

$$\ker T_q^{\mathrm{G}} = \left\{ F \in \mathrm{C}^{(2)}(\mathcal{B}_q) \middle| \Delta_q F = 0 \right\}^{\perp_{\mathrm{L}^2(\mathcal{B}_q)}},$$

where

$$T_q^{\mathrm{G}} D := \int_{\mathcal{B}_q} \frac{D(x)}{|x - \cdot|^{q-2}} \, \mathrm{d}x, \qquad D \in \mathrm{L}^2(\mathcal{B}_q).$$

We call the elements of $\ker T_q^{\mathrm{G}}$ *the anharmonic functions on* $\mathcal{B}_q$ *in analogy to the nomenclature in* $\mathbb{R}^3$ *(see [Ballani et al., 1993]).*

Proof. Since the inner Dirichlet problem for the Laplace equation is uniquely solvable and the $Y_{n,j}^{(q)}$ represent a basis on the sphere $\mathbb{S}^{q-1}$, the functions $H_{n,j}$ in Theorem 5.1 with $l_n = n$ for all $n \in \mathbb{N}_0$ represent a basis for the harmonic functions on $\mathcal{B}_q$. Furthermore, the space of all harmonic functions on $\mathcal{B}_q$ is closed in $\mathrm{L}^2(\mathcal{B}_q)$ (see [Mikhlin, 1970, Theorem 11.9.2]). Finally, Theorem 5.1 yields the desired result. □

6 A Differential Operator for the Orthonormal System on the Ball

In order to characterize the orthonormal system on the ball $\{G_{m,n,j}\}_{m,n \in \mathbb{N}_0, j=1,\ldots,2n+1}$ which was introduced in Section 3 in further detail, it is useful to find a differential operator whose eigenfunctions are the functions $G_{m,n,j}$. In the case of the inverse gravimetric problem this was done in [Akram et al., 2011]. Now, we generalize these results for the orthonormal system given in Equation (8) for $m, n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$

$$G_{m,n,j}(r\xi) = \gamma_{m,n} P_m^{(0, l_n + 1/2)} \left(2\frac{r^2}{R^2} - 1 \right) \frac{r^{l_n}}{R^{l_n}} Y_{n,j}(\xi).$$

6.1 Derivation of a Differential Operator

Analogously to [Akram et al., 2011], we start with the Jacobi polynomials $P_m^{(0,l_n+1/2)}$. It is well known (see [Szegö, 1975]) that they solve the following differential equation

$$0 = (1-u^2)\frac{\mathrm{d}^2}{\mathrm{d}u^2}P_m^{(0,l_n+1/2)}(u) + \left(l_n + \frac{1}{2} - \left(l_n + \frac{5}{2}\right)u\right)\frac{\mathrm{d}}{\mathrm{d}u}P_m^{(0,l_n+1/2)}(u)$$
$$+ m\left(m + l_n + \frac{3}{2}\right)P_m^{(0,l_n+1/2)}(u).$$

Substituting $u = \frac{2r^2}{R^2} - 1$, i.e. $r = R\sqrt{\frac{u+1}{2}}$, and using the chain rule for differentiation, the differential equation becomes

$$(R^2 - r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2}P_m^{(0,l_n+1/2)}\left(\frac{2r^2}{R^2}-1\right) + 2\left(l_n\left(1-\frac{r^2}{R^2}\right) - 2\frac{r^2}{R^2} + 1\right)\frac{R^2}{r}\times$$
$$\frac{\mathrm{d}}{\mathrm{d}r}P_m^{(0,l_n+1/2)}\left(\frac{2r^2}{R^2}-1\right) + 4m\left(m + l_n + \frac{3}{2}\right)P_m^{(0,l_n+1/2)}\left(\frac{2r^2}{R^2}-1\right) = 0$$
$$\Leftrightarrow \left((R^2-r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2} + 2\left(l_n\left(1-\frac{r^2}{R^2}\right) - 2\frac{r^2}{R^2} + 1\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)P_m^{(0,l_n+1/2)}\left(\frac{2r^2}{R^2}-1\right)$$
$$= -4m\left(m + l_n + \frac{3}{2}\right)P_m^{(0,l_n+1/2)}\left(\frac{2r^2}{R^2}-1\right). \tag{16}$$

Now we define a new function $Y(r) := P_m^{(0,l_n+1/2)}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^{l_n}$, in order to derive a differential equation for the radius part of the function $G_{m,n,j}$. In addition, we use the abbreviation $P(r) := P_m^{(0,l_n+1/2)}\left(\frac{2r^2}{R^2}-1\right)$. We get

$$Y(r) = P(r)\frac{r^{l_n}}{R^{l_n}}, \qquad Y'(r) = P'(r)\frac{r^{l_n}}{R^{l_n}} + \frac{l_n}{r}Y(r),$$
$$Y''(r) = P''(r)\frac{r^{l_n}}{R^{l_n}} + 2l_n\frac{r^{l_n-1}}{R^{l_n}}P'(r) + l_n(l_n-1)\frac{r^{l_n-2}}{R^{l_n}}P(r).$$

Using the same calculations as in [Akram et al., 2011], we can see that Y fulfils the following differential equation

$$(R^2-r^2)Y'' = (R^2-r^2)P''\frac{r^{l_n}}{R^{l_n}} + (R^2-r^2)\frac{2l_n}{r}Y' - (R^2-r^2)\frac{l_n(l_n+1)}{r^2}Y. \tag{17}$$

Inserting the value of $(R^2-r^2)P''$ from (16) in (17) we obtain

$$(R^2-r^2)Y'' = -2\left(l_n\left(1-\frac{r^2}{R^2}\right) - 2\frac{r^2}{R^2} + 1\right)\frac{R^2}{r}\frac{r^{l_n}}{R^{l_n}}P' - 4m\left(m + l_n + \frac{3}{2}\right)\frac{r^{l_n}}{R^{l_n}}P$$
$$+ (R^2-r^2)\frac{2l_n}{r}Y' - (R^2-r^2)\frac{l_n(l_n+1)}{r^2}Y.$$

Using the identities $Y = P\frac{r^{l_n}}{R^{l_n}}$ and $Y' = \frac{r^{l_n}}{R^{l_n}}P' + \frac{l_n}{r}Y$ we can simplify the above equation

$$
\begin{aligned}
(R^2-r^2)Y'' =& -2\left(l_n\left(1-\frac{r^2}{R^2}\right)-2\frac{r^2}{R^2}+1\right)\frac{R^2}{r}\left(Y'-\frac{l_n}{r}Y\right)-4m\left(m+l_n+\frac{3}{2}\right)Y\\
&+(R^2-r^2)\frac{2l_n}{r}Y'-(R^2-r^2)\frac{l_n(l_n+1)}{r^2}Y\\
=& -2\left(\frac{l_n}{r}\left(R^2-r^2\right)-2r+\frac{R^2}{r}-(R^2-r^2)\frac{l_n}{r}\right)Y'-(R^2-r^2)\frac{l_n(l_n+1)}{r^2}Y\\
&+\left(2\left(\frac{l_n^2}{r^2}\left(R^2-r^2\right)-2l_n+\frac{l_nR^2}{r^2}\right)-4m\left(m+l_n+\frac{3}{2}\right)\right)Y\\
=& \left(\frac{l_n(l_n-1)}{r^2}\left(R^2-r^2\right)-4l_n+\frac{2l_nR^2}{r^2}-4m\left(m+l_n+\frac{3}{2}\right)\right)Y\\
&+\left(4r-\frac{2R^2}{r}\right)Y'\\
=& \left(4r-\frac{2R^2}{r}\right)Y'+\left(l_n(l_n+1)\frac{R^2}{r^2}-l_n(l_n+3)-4m\left(m+l_n+\frac{3}{2}\right)\right)Y.
\end{aligned}
$$

It yields

$$
\begin{aligned}
&(R^2-r^2)Y''-\left(4r-\frac{2R^2}{r}\right)Y'-l_n(l_n+1)\frac{R^2}{r^2}Y\\
&\qquad\qquad=-\left(l_n(l_n+3)+4m\left(m+l_n+\frac{3}{2}\right)\right)Y\\
\Leftrightarrow\ &\left((R^2-r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2}+2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-l_n(l_n+1)\frac{R^2}{r^2}\right)Y\\
&\qquad\qquad=-\left(l_n(l_n+3)+4m\left(m+l_n+\frac{3}{2}\right)\right)Y. \qquad (18)
\end{aligned}
$$

This proves that the differential operator D_r defined by

$$
D_r \coloneqq (R^2-r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2}+2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-l_n(l_n+1)\frac{R^2}{r^2}
$$

has the eigenfunctions $Y(r) = P_m^{(0,l_n+1/2)}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^{l_n}$, $r \in [0,R]$ for all $m,n \in \mathbb{N}_0$ and the corresponding eigenvalues $-\left(l_n(l_n+3)+4m\left(m+l_n+\frac{3}{2}\right)\right)$ for all $m,n \in \mathbb{N}_0$. Note that the differential operator D_r depends on the exponent l_n of r, but for a better readability, we omit this in our notation.

The angular part of our orthonormal function $G_{m,n,j}$ is given by spherical harmonics of degree $n \in \mathbb{N}_0$ and order $j = 1,\ldots,2n+1$. Recall that the spherical harmonics are eigenfunctions of the Beltrami operator Δ^* with the corresponding eigenvalues $-n(n+1)$ for $n \in \mathbb{N}_0$, $j = 1,\ldots,2n+1$ (for more details see e.g. [Freeden et al., 1998]). We sum up this result in the following theorem.

Theorem 6.1. *The orthonormal basis functions $G_{m,n,j}$ for $m,n \in \mathbb{N}_0$ are eigenfunctions of the differential operator $^{*}\Delta_x$ defined by*

$$^{*}\Delta_x \coloneqq D_{|x|} \circ \Delta^{*}_{\frac{x}{|x|}}.$$

with the corresponding eigenvalues

$$^{*}\Delta^{\wedge}(m,n) \coloneqq \left(l_n(l_n+3) + 4m\left(m + l_n + \frac{3}{2}\right)\right) n(n+1),$$

i.e.,

$$\begin{aligned} ^{*}\Delta_x G_{m,n,j}(x) &= \left(D_{|x|} \circ \Delta^{*}_{\frac{x}{|x|}}\right) G_{m,n,j}(x) \\ &= \left(l_n(l_n+3) + 4m\left(m + l_n + \frac{3}{2}\right)\right) n(n+1) G_{m,n,j}(x). \end{aligned}$$

6.2 Properties of the Differential Operator $^{**}\Delta$

If we have a closer look at the eigenvalues of the differential operator $^{**}\Delta$ which is defined by

$$^{**}\Delta_x \coloneqq \left(-D_{|x|} + \frac{9}{4}\right) \circ \left(-\Delta^{*}_{\frac{x}{|x|}} + \frac{1}{4}\right),$$

we observe that $^{**}\Delta^{\wedge}(m,n)$ is equal to zero if $n = 0$, $m = -\frac{l_n}{2}$, or $m = -\frac{1}{2}(l_n+3)$. Hence, the differential operator is not invertible. But, with the help of the calculations from the previous section, we are able to construct a similar operator which is invertible. For this purpose, we use the same method as in [Akram et al., 2011]. Note that there exist similar results for spherical differential operators, see, for example, [Freeden et al., 1998].

Theorem 6.2. *The differential operator $^{**}\Delta_x$ and its iterates $^{**}\Delta_x^s$, $s \in \mathbb{N}$ are invertible. The corresponding eigenvalues have the representation*

$$\left(^{**}\Delta^s\right)^{\wedge}(m,n) = \left(\left(n + \frac{1}{2}\right)\left(2m + l_n + \frac{3}{2}\right)\right)^{2s}.$$

Proof. The proof is analogous to the one in [Akram et al., 2011] and it suffices to prove

$$\left(-D_{|x|} + \frac{9}{4}\right)\left(P_m^{(0,l_n+1/2)}\left(2\frac{r^2}{R^2} - 1\right)\frac{r^{l_n}}{R^{l_n}}\right) = \left(2m + l_n + \frac{3}{2}\right)^2 P_m^{(0,l_n+1/2)}\left(2\frac{r^2}{R^2} - 1\right)\frac{r^{l_n}}{R^{l_n}},$$

since it is known from [Freeden et al., 1998] that $(-\Delta^{*}_{\frac{x}{|x|}} + \frac{1}{4})Y_{n,j}(\frac{x}{|x|}) = \left(n + \frac{1}{2}\right)^2 Y_{n,j}(\frac{x}{|x|})$ for all $n \in \mathbb{N}_0$ and $j = 1, \ldots, 2n+1$. Using (18) one gets, again with the abbreviation $Y(r) = P_m^{(0,l_n+1/2)}\left(2\frac{r^2}{R^2} - 1\right)\frac{r^{l_n}}{R^{l_n}}$,

$$\left(-D_r + \frac{9}{4}\right) Y(r) = \left(l_n(l_n+3) + 4m\left(m + l_n + \frac{3}{2}\right) + \frac{9}{4}\right) Y(r)$$

and, completing the square, we get

$$\left(l_n(l_n+3)+4m\left(m+l_n+\frac{3}{2}\right)+\frac{9}{4}\right)=\left(2m+l_n+\frac{3}{2}\right)^2.$$

Due to Assumption 2.4, the inequality $\inf_{n\in\mathbb{N}_0} l_n \geq -1$ holds true and hence, for all $m,n\in\mathbb{N}_0$ it yields $\left(2m+l_n+\frac{3}{2}\right)^2 \neq 0$, and the operator $^{**}\Delta_x$ is invertible. In summary, we obtain

$$\left(^{**}\Delta_x^s\right)^\wedge(m,n)=\left(\left(2m+l_n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)\right)^{2s}. \qquad \square$$

In order to find a relation between the domain and the range of the differential operator $^{**}\Delta$, we can construct particular Sobolev spaces. We define a Sobolev space corresponding to the sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ and our orthonormal basis functions $G_{m,n,j}$ for m, $n\in\mathbb{N}_0$, $j=1,\ldots,2n+1$ as follows:

Definition 6.3. *Let $(A_{m,n})_{m,n\in\mathbb{N}_0}$ be a given real sequence. Then the space $\mathcal{E}((A_{m,n}),\mathcal{B})$ consists of all $F\in\mathrm{C}^{(\infty)}(\mathcal{B})$ such that*

$$\langle F,G_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})}=0 \qquad \text{for all } (m,n,j) \text{ with } A_{m,n}=0$$

and

$$\sum_{m,n=0}^{\infty}\sum_{j=1}^{2n+1}A_{m,n}^2\langle F,G_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})}^2<\infty.$$

The space is equipped with the inner product

$$\langle F_1,F_2\rangle_{\mathcal{H}}:=\sum_{m,n=0}^{\infty}\sum_{j=1}^{2n+1}A_{m,n}^2\langle F_1,G_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})}\langle F_2,G_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})},$$

for F_1, $F_2\in\mathcal{E}((A_{m,n}),\mathcal{B})$. $\mathcal{H}((A_{m,n}),\mathcal{B})$ is defined as the completion of $\mathcal{E}((A_{m,n}),\mathcal{B})$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and is called the Sobolev space on $\mathcal{B}$ with respect to $(A_{m,n})_{m,n\in\mathbb{N}_0}$

This approach for a Sobolev space on the ball is also used in, e.g., [Akram et al., 2011] and [Michel, 2013]. In order to classify some reproducing kernel Hilbert spaces on the ball, we define the summability condition in this case.

Definition 6.4. *A sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ satisfies the summability condition if*

$$\sum_{\substack{m,n=0\\A_{m,n}\neq 0}}^{\infty}A_{m,n}^{-2}n(2m+l_n)p_{m,n}<\infty,$$

with

$$p_{m,n}=\begin{cases}\frac{\left(m+l_n+\frac{1}{2}\right)^{2m}}{(m!)^2}, & l_n>-\frac{1}{2}\\ 1, & l_n\leq-\frac{1}{2}.\end{cases}$$

We derive this definition in analogy to the summability condition in [Michel, 2013, p. 267] in the case of the inverse gravimetric problem. Note that the parameter $p_{m,n}$ comes from the supremum norm of the Jacobi polynomials.

Roughly speaking, we say that the sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ is summable. By means of an approach from [Freeden et al., 1998] for constructing reproducing kernels and reproducing kernel Hilbert spaces on the sphere, we can construct similar structures on the ball. In the particular case for $l_n = n$, this was also done in [Akram et al., 2011].

Theorem 6.5. *The spaces $\mathcal{H}((A_{m,n}),\mathcal{B})$, which are defined in Definition 6.3, are reproducing kernel Hilbert spaces with the unique reproducing kernel*

$$K_{\mathcal{H}}(x,y) := \sum_{\substack{m,n=0\\ A_{m,n}\neq 0}}^{\infty} \sum_{j=1}^{2n+1} A_{m,n}^{-2} G_{m,n,j}(x) G_{m,n,j}(y), \qquad x,y\in\mathcal{B},$$

if the sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ is summable.

Based on these Sobolev spaces, we can specify the domain and the range of our differential operator $^{**}\Delta$.

Definition 6.6. *For any $s\in\mathbb{R}_0^+$, we define the Sobolev spaces*

$$\mathcal{H}_s(\mathcal{B}) := \mathcal{H}\left(\left(\left(2m+l_n+\frac{3}{2}\right)^s \left(n+\frac{1}{2}\right)^s\right),\mathcal{B}\right).$$

Obviously, $\mathcal{H}_{s_1}(\mathcal{B}) \subset \mathcal{H}_{s_2}(\mathcal{B})$ for $s_1 \geq s_2$. In addition, $\mathcal{H}_0(\mathcal{B}) = \mathrm{L}^2(\mathcal{B})$. Now, we can redefine the operator in Theorem 6.1.

Definition 6.7. *Let $s,\ t\in\mathbb{R}_0^+$ with $s>2t$. Then the operator*

$$(^{**}\Delta)^t \colon \mathcal{H}_s(\mathcal{B}) \to \mathcal{H}_{s-2t}(\mathcal{B})$$

can be defined by

$$(^{**}\Delta)^t F := \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} \left(2m+l_n+\frac{3}{2}\right)^{2t} \left(n+\frac{1}{2}\right)^{2t} \langle F, G_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})} G_{m,n,j}$$

for $F\in\mathcal{H}_s(\mathcal{B})$.

Using the norm induced by the inner product of the Sobolev space $\mathcal{H}_s(\mathcal{B})$, we can easily see that the following holds true.

Corollary 6.8. *Let $s,\ t\in\mathbb{R}_0^+$ with $s>2t$. If $F\in\mathcal{H}_s(\mathcal{B})$, then*

$$\left\| (^{**}\Delta)^t F \right\|_{\mathcal{H}_{s-2t}(\mathcal{B})} = \|F\|_{\mathcal{H}_s(\mathcal{B})} < \infty.$$

7 Conclusions

In our paper, we considered a generalization of the inverse gravimetric and inverse magnetic problem. By analyzing the forward problem, we found a relation between the Fourier coefficients of the given data and parts of the solution. In order to understand this relation in further detail, we constructed a particular orthonormal system on the ball. With this system, we characterized the null space of the Fredholm integral operator and calculated the singular system. One interesting result is that, in all cases, most of the radial information gets lost. In the particular case of the inverse gravimetric problem, our results coincide with the corresponding well-known results. Moreover, they also approve some existing results for the inverse magnetic problem. Since the null space of the operator is infinite-dimensional, the solution of the inverse problem is (in a severe sense) not unique. However, we showed that, under additional assumptions, uniqueness of the solution can be obtained. We generalized known results of the inverse gravimetric problem, for example, regarding the harmonicity constraint. With this approach, a large class of uniqueness constraints can be analyzed. We also discussed the case of a minimum norm constraint and generalized this ansatz for the inverse gravimetric problem in the $\mathbb{R}^q$, which yielded a generalization of a famous result for Newton's gravitational potential operator. Finally, a representation of a differential operator having the orthonormal basis functions as eigenfunctions was derived including an investigation of associated reproducing kernel Hilbert spaces.

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