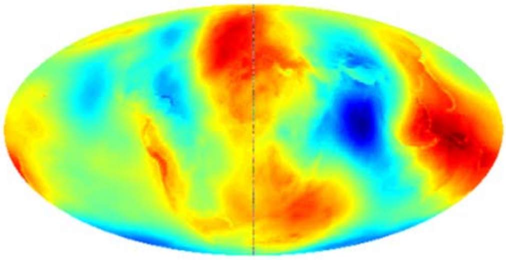 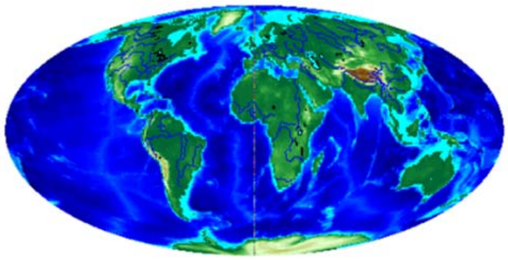

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Electro-Magnetoencephalography for the Spherical Multiple-Shell Model: Novel Integral Operators with Singular-Value Decompositions

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ELECTRO-MAGNETOENCEPHALOGRAPHY FOR THE SPHERICAL MULTIPLE-SHELL MODEL: NOVEL INTEGRAL OPERATORS WITH SINGULAR-VALUE DECOMPOSITIONS

S. LEWEKE[1], V. MICHEL[1], AND A.S. FOKAS[2]

Abstract. The inverse magnetoencephalography and electroencephalography problems for spherical models have been extensively discussed in the literature. Using the spherical multiple-shell model, we derive novel vector-valued and singularity-free integral equations for both problems based on the quasi-static Maxwell's equations. These equations are solved via a Fourier series expansion. We call this procedure the Edmonds approach, since an orthonormal system based on the Edmonds-vector-spherical harmonics is used for the Fourier series. Employing the associated singular-value decomposition, we provide a complete answer to the non-uniqueness question of these two problems: only the harmonic part of the solenoidal component of the neuronal current is visible to the simultaneous use of magnetoencephalography and electroencephalography. The remaining components of the current are invisible to both techniques. We state Picard's condition for the existence of a solution and derive an explicit formula for the best-approximate solution of the neuronal current. In comparison to previous approaches, the Edmonds approach requires the fewest a-priori assumptions on the neuronal current. Finally, we show that the results obtained by means of the Edmonds approach are consistent with results derived earlier via the Helmholtz decomposition.

1. Introduction

Magnetoencephalography (MEG) and electroencephalography (EEG) are used extensively for the non-invasive study of real time brain processes. Foundational work for these imaging techniques can be found in [17, 34]. The neuronal current induces a magnetic field and an electric potential, which are detected by magnetoencephalography and electroencephalography.

The aim of the vectorial inverse MEG problem is the reconstruction of the neuronal current J inside the cerebrum from about 100 measured data points. The magnetic flux density $\nu \cdot B$, which is normal to the sensor surface, is measured outside the head in a magnetically shielded room: superconducting quantum interference device is required for measuring the human brain activity, since its magnetic field is more than a hundred million times smaller than the Earth's magnetic field. An extensive survey of MEG and further references can be found in [19].

The vectorial inverse EEG problem aims at reconstructing the neuronal current J from given values of the voltage difference on the scalp, which is the difference of two electric potentials [19]. A reference level for the potential by means of a reference electrode needs to be chosen in advance. The sensor cap used in clinical medicine contains about 70 sensors that are attached to the scalp non-invasively.

The quasi-static variant of Maxwell's equations is commonly used in medical electromagnetism to describe bioelectric activity. In this context, the changes of the current and of the electric and magnetic fields are

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assumed to be so slow that the quasi-static approximation is valid. In [33], the authors have proven that the errors made by the quasi-static approach, that is the omission of propagation, capacitative and inductive effects, and boundary considerations, are indeed negligible. A second argumentation for the quasi-static approach can be found in [19].

In the case of a continuously distributed current, the non-uniqueness of both problems has been extensively discussed in the literature, for instance, [11] and the references therein. A summary and a comparison to other approaches like the Hodge-decomposition or an expansion by means of Morse-Feshbach-vector-spherical harmonics is given in [4]. An approximation of the brain structure is necessary for the analytic non-uniqueness considerations. In the case of an anatomically correctly shaped conductor model, the magnetic field and the electric potential can only be calculated numerically [19]. Besides homogeneous head models (i.e. constant conductivity in the entire conductor), there exist more realistic models including the three-shell model [3, 12, 13], the multiple-shell model [19, 30], and the elliptic-shell model [2, 6, 7].

Even, for the spherical three-shell and multiple-shell model, there still exist some open questions concerning the ill-posedness of the problem, which we will answer in this article, including the following basic question: When does a solution of the inverse problem exist?

The non-uniqueness question of MEG and EEG for arbitrary geometry was studied in [11]; the particular case of spherical geometry was also analyzed in [11] (see also [12] and [13]). The main difference of the present work with that of [11] is that in the present work we are able to study the non-uniqueness question without introducing the Helmholtz decomposition. This has the important advantage of expressing the 'visible' part of the current directly in terms of appropriate components of the vector J instead of the scalar and vector functions Ψ and A, respectively, appearing in the Helmholtz decomposition. In addition, our approach involves the fewest a-priori assumptions for the rigorous derivation of the relevant results.

In Section 3, we derive a novel vector-valued Fredholm integral equation of the first kind with a regular integral kernel for the inverse MEG problem. This provides a relation between the magnetic potential (and the magnetic field) and the entire vectorial neuronal current. In Section 4, we derive a similar integral equation for the vector-valued inverse EEG problem. Therein, there appears a function H_k depending on the radii of the different shells and their conductivities (this equation holds true on the entire outer shell as opposed only to the outer boundary of the shell). In addition, we analyze the asymptotic behaviour of H_k for $k \to \infty$ in order to guarantee convergence of the series defining the integral kernel. In Section 5, a particular class of integral equations is introduced, which we call the vector Legendre-type integral equation, for solving the two inverse problems simultaneously. We solve the direct problem by means of a Fourier series expansion of the neuronal current. We call this the Edmonds approach since the relevant orthonormal basis utilizes the Edmonds-vector-spherical harmonics for the angular part of the current. In addition, we define an appropriate orthonormal system for the generalized Fourier expansion of the radial part of the neuronal current. Based on the results achieved with the Edmonds approach, in Section 6, we give a novel and full characterization of the MEG and EEG operator null space. We observe that the toroidal component of the neuronal current affects the measurements of the magnetic potential and its field. In addition, only the parts of the solenoidal component of the neuronal current can be reconstructed from joint magnetic and electric potential data. Furthermore, there exist parts of the solenoidal component which are silent with either MEG or EEG. We show that in both cases only the harmonic parts influence the measured quantities. In summary, only the harmonic solenoidal part of the neuronal current can be reconstructed. We also state a singular-value decomposition (SVD) of both compact operators. This enables us to formulate the best-approximate solution of the inverse problem with simultaneous data.

The Edmonds approach has the additional advantage that it only requires $J \in \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$. No further smoothness or boundary conditions on the neuronal current are needed. This is a significant advantage since we do not have such a-priori information about the neuronal current. This is to be contrasted with the approaches using the Hodge [4, 12, 14] or the Helmholtz [4, 8, 11–13] decomposition, where further conditions on the neuronal current are required, such as boundary or smoothness conditions. The further insight into the structure of the inverse problem obtained via our approach, as well as the consistency of our results with the results of [11] are discussed in Section 7. The presented results here are based on parts of the thesis [23].

2. Preliminaries

We denote by $\mathbb{B}_R$ the three-dimensional closed ball with radius $R > 0$ around the origin and by $\mathbb{B}_R^{\mathrm{ext}} := \overline{\mathbb{R}^3 \setminus \mathbb{B}_R}$ its closed exterior. $\mathbb{S}_R$ is the three-dimensional sphere with radius $R > 0$. A spherical shell is defined by

$$\mathbb{S}_{[a,b]} := \left\{ x \in \mathbb{R}^3 \;\middle|\; 0 < a \leq |x| \leq b < \infty \right\}.$$

Here, $r := |x| := \sqrt{x \cdot x}$ denotes the Euclidean norm of x. The Euclidean inner product is denoted by $\cdot$ and the vector product by $\times$. In addition, we will use throughout the paper the abbreviations $x = r\xi$ and $y = s\eta$ with ξ, $\eta \in \mathbb{S} := \mathbb{S}_1$ and $r = |x|$, $s = |y|$. This decomposition is unique for all $x \in \mathbb{R}^3 \setminus \{0\}$. ∇ denotes the gradient whereby the differential operator L^* (independent of the radius r) is defined by

$$L_\xi^* := x \times \nabla_x = \xi \times \nabla_\xi^*. \tag{2.1}$$

In our notation, ∇_ξ^* denotes the part of the gradient containing the tangential derivatives divided by r, [16, Eq. (2.136)]. ∇_ξ^* is often called the surface gradient whereas L_ξ^* is called the surface curl operator. The Beltrami operator is given by $\Delta_\xi^* = \nabla_\xi^* \cdot \nabla_\xi^*$, [16, Eq. (2.140)].

A function of $\mathrm{Harm}_n(\mathbb{S})$ (i.e. the space of all homogeneous, harmonic polynomials of degree n restricted to the unit sphere $\mathbb{S}$, [16, Def. 3.22]) is called a spherical harmonic of degree $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ (the set of all positive integers united with zero). With $\{Y_{n,j}\}_{j=1,\ldots,2n+1}$ we denote an $\mathrm{L}^2(\mathbb{S})$-orthonormal set in $\mathrm{Harm}_n(\mathbb{S})$, [16, Rem. 3.25]. For more details on scalar spherical harmonics and their properties, see for instance [16]. For an introduction to (vector-valued) Lebesgue spaces, see [1].

A set of functions which is closely related to the spherical harmonics is built by the outer spherical harmonics. They are defined for an arbitrary radius $R > 0$ and for all $n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$ by

$$H_{n,j}^{\mathrm{ext}}(R; y) := \frac{1}{R} \left(\frac{R}{s} \right)^{n+1} Y_{n,j}(\eta), \qquad y = s\eta \in \mathbb{R}^3 \setminus \{0\}. \tag{2.2}$$

For further details on inner and outer spherical harmonics, see [16, Ch. 10.3].

By means of the scalar spherical harmonics, we can define a complete, [16, Thm. 5.56], $\mathrm{L}^2(\mathbb{S}, \mathbb{R}^3)$-orthonormal system of vector-valued spherical harmonics which goes back to Edmonds and is defined for example in [16, Eq. (5.309)-(5.311)] by

$$\tilde{y}_{n,j}^{(i)}(\xi) := \left(\tilde{\mu}_n^{(i)} \right)^{-1/2} \tilde{o}_{n,\xi}^{(i)} Y_{n,j}(\xi), \qquad \xi \in \mathbb{S}, \tag{2.3}$$

where

$$\tilde{\mu}_n^{(i)} := \begin{cases} (n+1)(2n+1), & \text{for } i = 1, \\ n(2n+1), & \text{for } i = 2, \\ n(n+1), & \text{for } i = 3, \end{cases} \qquad \tilde{o}_{n,\xi}^{(i)} := \begin{cases} (n+1)\xi - \nabla_\xi^*, & \text{for } i = 1, \\ n\xi + \nabla_\xi^*, & \text{for } i = 2, \\ L_\xi^*, & \text{for } i = 3 \end{cases} \tag{2.4}$$

for all $i = 1, 2, 3$; $n \in \mathbb{N}_{0_i}$, and $j = 1, \ldots, 2n+1$. The set $\mathbb{N}_{0_i}$ is defined as the set of all $n \in \mathbb{N}_0$ with $n \geq 1 - \delta_{1,i}$. Note that these vector-valued spherical harmonics are homogeneous harmonic polynomials, see [16, Lem. 5.12]. The functions belonging to $i = 3$ are toroidal functions, due to (2.1), see [15, Rem. 5.3.2].

These operators provide us with certain useful representations, for example,

$$\nabla_x(r^n Y_{n,j}(\xi)) = \left(\xi\partial_r + \frac{1}{r}\nabla^*_\xi\right)(r^n Y_{n,j}(\xi)) = r^{n-1}\tilde{o}^{(2)}_{n,\xi}Y_{n,j}(\xi), \tag{2.5}$$

$$\nabla_x(r^{-(n+1)} Y_{n,j}(\xi)) = \left(\xi\partial_r + \frac{1}{r}\nabla^*_\xi\right)(r^{-(n+1)} Y_{n,j}(\xi)) = -r^{-(n+2)}\tilde{o}^{(1)}_{n,\xi}Y_{n,j}(\xi). \tag{2.6}$$

The above representations and further information on vector spherical harmonics can be found in [16]. In analogy, we can define, for all $i = 1, 2, 3$ and $n \in \mathbb{N}_{0_i}$, Edmonds-vector-Legendre polynomials by means of the scalar Legendre polynomials P_n, [16, Eq. (3.165)], of order $n \in \mathbb{N}_{0_i}$ as in [16, Lem. 5.63],

$$\tilde{p}^{(i)}_n(\xi,\eta) := \left(\tilde{\mu}^{(i)}_n\right)^{-1/2}\tilde{o}^{(i)}_{n,\xi}P_n(\xi\cdot\eta), \qquad \xi,\eta\in\mathbb{S}. \tag{2.7}$$

Note that scalar Legendre polynomials fulfil, [16, Lem. 3.29], the identity

$$\Delta^*_\xi P_k(\xi\cdot\eta) = -k(k+1)P_k(\xi\cdot\eta), \qquad \xi,\eta\in\mathbb{S}. \tag{2.8}$$

Based on [16, Lem. 5.23], we can easily verify that for all $i = 1, 2, 3$ and all $n \in \mathbb{N}_{0_i}$ the estimate

$$\left|\tilde{p}^{(i)}_n(\xi,\eta)\right|^2 \le \tilde{\mu}^{(i)}_n \tag{2.9}$$

holds true. The Legendre polynomials and their vectorial counterpart satisfy addition theorems, [16, Thm. 3.26, Thm. 5.64]. These imply for all $i = 1, 2, 3$ and all $n \in \mathbb{N}_{0_i}$ the representations

$$\sum_{j=1}^{2n+1} Y_{n,j}(\xi)Y_{n,j}(\eta) = \frac{2n+1}{4\pi}P_n(\xi\cdot\eta), \qquad \sum_{j=1}^{2n+1}\tilde{y}^{(i)}_{n,j}(\xi)Y_{n,j}(\eta) = \frac{2n+1}{4\pi}\tilde{p}^{(i)}_n(\xi,\eta), \qquad \xi,\eta\in\mathbb{S}. \tag{2.10}$$

This immediately leads to the reproducing property of Edmonds-vector-Legendre polynomials for all $i, \iota = 1, 2, 3$, $n \in \mathbb{N}_{0_i}$, $j = 1,\dots,2n+1$, and $k \in \mathbb{N}_{0_\iota}$:

$$\int_{\mathbb{S}}\tilde{y}^{(i)}_{n,j}(\xi)\cdot\tilde{p}^{(\iota)}_k(\xi,\eta)\,\mathrm{d}\omega(\xi) = \frac{4\pi}{2n+1}Y_{n,j}(\eta)\delta_{n,k}\delta_{i,\iota}, \qquad \eta\in\mathbb{S}. \tag{2.11}$$

For the construction of an orthonormal basis of $\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)$, we need appropriate basis functions over the real interval $[0,R]$, which are introduced below.

Definition 2.1. The functions $Q^{(\beta)}_m(R;\cdot)$ of degree $m\in\mathbb{N}_0$ are defined by means of the Jacobi polynomials $P^{(\alpha,\beta)}_m$, $\alpha,\beta > -1$, via

$$Q^{(\beta)}_m(R;r) := \sqrt{\frac{4m+2\beta+2}{R^3}}\left(\frac{r}{R}\right)^{\beta-1/2}P^{(0,\beta)}_m\left(2\frac{r^2}{R^2}-1\right), \qquad r\in(0,R].$$

Theorem 2.2. *The set of functions $Q^{(\beta)}_m(R;\cdot)$ for $m\in\mathbb{N}_0$ builds for each fixed $\beta > -1$ an orthonormal system for $\mathrm{L}^2_w[0,R]$ with the weight function $w(r) := r^2$.*

Proof. The Jacobi polynomials $P^{(\alpha,\beta)}_m$ with $\alpha,\beta > -1$, build a complete orthogonal system for $\mathrm{L}^2_\varpi[-1,1]$ with the weight function $\varpi(r) := (1-r)^\alpha(1+r)^\beta$, [35, Thm. 3.1.5]. We get with $\alpha = 0$ and $z = 2r^2/R^2 - 1$

using [35, Eq. (4.3.3)] that

$$\begin{aligned}\frac{2^{\beta+1}}{2m+\beta+1}\delta_{m,k} &= \int_{-1}^{1}(1+z)^{\beta}P_m^{(0,\beta)}(z)P_k^{(0,\beta)}(z)\,\mathrm{d}z\\ &= \frac{2^{\beta+2}}{R^{2\beta+2}}\int_0^R r^{2\beta+1}P_m^{(0,\beta)}\left(2\frac{r^2}{R^2}-1\right)P_k^{(0,\beta)}\left(2\frac{r^2}{R^2}-1\right)\mathrm{d}r\\ &= \frac{2^{\beta+2}}{\sqrt{4m+2\beta+2}\sqrt{4k+2\beta+2}}\int_0^R Q_m^{(\beta)}(R;r)Q_k^{(\beta)}(R;r)w(r)\,\mathrm{d}r. \qquad \square\end{aligned}$$

Therefore, each function $F\in \mathrm{L}^2_w[0,R]$ can be expanded into a generalized Fourier series by means of these orthonormal basis functions, that is

$$F=\sum_{m=0}^{\infty}\left\langle F, Q_m^{(\beta)}(R;\cdot)\right\rangle_{\mathrm{L}^2_w[0,R]} Q_m^{(\beta)}(R;\cdot).$$

For more information on weighted Lebesgue spaces and generalized Fourier expansions, see [39, Ch. II]. Next, we combine the previous results concerning the Hilbert space over the sphere and the interval $[0,R]$.

Theorem 2.3. *The vector-valued functions $\tilde{g}^{(i)}_{m,n,j}(R;\cdot)$ for $i=1,\ 2,\ 3$, $m\in\mathbb{N}_0$, $n\in\mathbb{N}_{0_i}$, and $j=1,\dots,2n+1$ with the parameter $t_n^{(i)}$ fulfilling $\inf_{n\in\mathbb{N}_0} t_n^{(i)}\geq -\frac{3}{2}$ are defined by*

$$\tilde{g}^{(i)}_{m,n,j}(R;x)\coloneqq Q^{(i)}_{m,n}(r)\tilde{y}^{(i)}_{n,j}(\xi), \qquad x\in\mathbb{B}_R,$$

with $Q^{(i)}_{m,n}\coloneqq Q_m^{(t_n^{(i)}+1/2)}(R;\cdot)$. Then the set of these functions is a complete orthonormal system in $\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)$. Eventually, each $f\in\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)$ has with the abbreviation $f^\wedge(i,m,n,j)\coloneqq\left\langle f,\tilde{g}^{(i)}_{m,n,j}(R;\cdot)\right\rangle_{\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)}$ the representation

$$f=\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1} f^\wedge(i,m,n,j)\tilde{g}^{(i)}_{m,n,j}(R;\cdot), \tag{2.12}$$

which converges strongly in the $\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)$-sense and unconditionally.

Due to the construction of the functions $\tilde{g}^{(i)}_{m,n,j}(R;\cdot)$, it can easily be seen, that for all $m\in\mathbb{N}_0$, $n\in\mathbb{N}$, and $j=1,\dots,2n+1$ the functions $\tilde{g}^{(3)}_{m,n,j}(R;\cdot)$ are toroidal, [15, Rem. 5.3.2].

Next, we analyze some particular basis functions with $t_n^{(2)}\coloneqq n-1$. Then by means of lengthy calculations we can prove that several basis functions are solenoidal, that is

$$\nabla_x\cdot\tilde{g}^{(i)}_{m,n,j}(R;x)=0 \qquad \Leftrightarrow \qquad \begin{cases} i=2,\ m=0,\ n\in\mathbb{N},\ t_n^{(2)}\coloneqq n-1,\ j=1,\dots,2n+1,\\ i=3,\ m\in\mathbb{N}_0,\ n\in\mathbb{N},\ j=1,\dots,2n+1.\end{cases} \tag{2.13}$$

In addition, with $t_n^{(3)}\coloneqq n$, the solenoidal functions $\tilde{g}^{(i)}_{m,n,j}(R;\cdot)$ for $i=2,\ 3$ are harmonic if and only if $m=0$. Now, we come back to the general setting.

Lemma 2.4. *Let $f\in\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)$ be a given function. We define for all $i=1,\ 2,\ 3$, $n\in\mathbb{N}_{0_i}$, and $j=1,\dots,2n+1$ the function*

$$f^{(i)}_{n,j}\coloneqq\sum_{m=0}^{\infty} f^\wedge(i,m,n,j)Q^{(i)}_{m,n}. \tag{2.14}$$

Then, for almost all $r \in [0, R]$, we obtain

$$f_{n,j}^{(i)}(r) = \int_{\mathbb{S}} f(r\xi) \cdot \tilde{y}_{n,j}^{(i)}(\xi) \, \mathrm{d}\omega(\xi).$$

An analogon can be obtained for scalar-valued functions $F \in \mathrm{L}^2(\mathbb{B}_R)$ and almost all $r \in [0, R]$ by

$$F_{n,j}(r) = \int_{\mathbb{S}} F(x) Y_{n,j}(\xi) \, \mathrm{d}\omega(\xi) = \sum_{m=0}^{\infty} F^{\wedge}(m, n, j) Q_m^{(t_n+1/2)}(R; r). \tag{2.15}$$

Using the representation of the gradient in spherical coordinates, (2.5) and lengthy calculations, we obtain the next result.

Theorem 2.5. *Let the vector field $f \in \mathrm{C}^2(\mathbb{B}_R, \mathbb{R}^3)$ be represented by the spherical Helmholtz decomposition, [16, Eq. (5.58)], that is*

$$f(x) = \xi F^{(1)}(x) + \nabla_\xi^* F^{(2)}(x) + L_\xi^* F^{(3)}(x). \tag{2.16}$$

Then the divergence and the curl of f are given by

$$\nabla_x \cdot f(x) = \frac{\partial}{\partial r} F^{(1)}(x) + \frac{1}{r}\left(2F^{(1)}(x) + \Delta_\xi^* F^{(2)}(x)\right),$$
$$\nabla_x \wedge f(x) = \frac{1}{r}\xi \Delta_\xi^* F^{(3)}(x) - \nabla_\xi^* \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) F^{(3)}(x) + L_\xi^* \left(-\frac{1}{r} F^{(1)}(x) + \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) F^{(2)}(x)\right).$$

3. Magnetoencephalography for the multiple-shell model

First, we introduce the multiple shell model, see Figure 1. Therein, the cerebrum is modelled by a closed ball with radius $\varrho_0 > 0$ that is $\mathbb{B}_{\varrho_0}$ and a constant, positive conductivity σ_0. Around the cerebrum, there are $L \geq 2$ spherical shells $\mathbb{S}_{[\varrho_l, \varrho_{l+1}]}$ with $\varrho_l < \varrho_{l+1}$ modelling the various head tissues with constant conductivities σ_{l+1} for all $l = 0, \ldots, L-1$. Consequently, the sphere $\mathbb{S}_{\varrho_L}$ models the head boundary. The permeability is constant everywhere and is equal to μ_0 which is the permeability of the vacuum, see [19]. The continuously distributed neuronal current J is non-vanishing only inside the cerebrum and J is an $\mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$-function. Outside the head, there is assumed to be no conductivity, that is $\sigma_{L+1} = 0$, which is a good approximation of the conductivity of the air in comparison to the conductivities of the various head tissues.

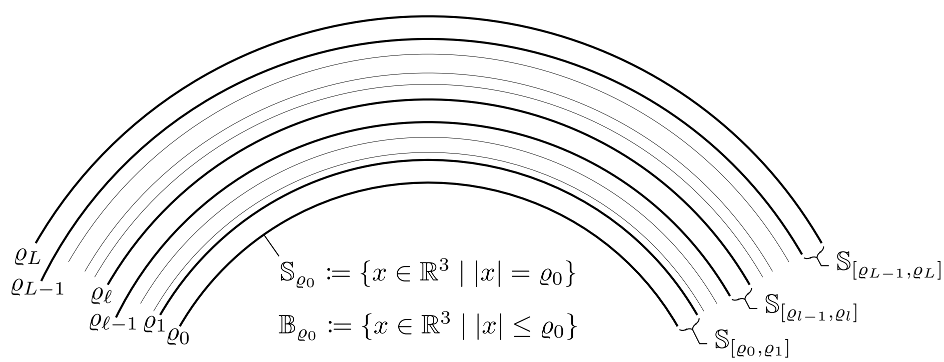

Figure 1. The multiple-shell model of the head with L spherical shells around the cerebrum $\mathbb{B}_{\varrho_0}$

Definition 3.1 (Quasi-static Maxwell's equations [32])**.** Let E be the vectorial electric field, u be the scalar electric potential, and $J^{\mathrm{T}} = J + \sigma E$ be the vectorial total current with the neuronal current J and the Ohmic current σE. Let the function σ be the piecewise constant conductivity. Then

$$E = -\nabla u, \qquad \nabla \cdot B = 0, \qquad \nabla \times B = \mu_0 J^{\mathrm{T}}.$$

Moreover, the electric and magnetic potential are regular at infinity.

Note that the total current has its support inside the cerebrum. Thus, in the exterior of the head (a simply connected subset of $\mathbb{R}^3$) the magnetic field is an irrotational, conservative vector field. Hence, there exists a scalar magnetic potential U such that

$$B = \mu_0 \nabla U \qquad \text{in } \mathbb{B}_{\varrho_L}^{\mathrm{ext}}. \tag{3.1}$$

In [12], the quasi-static Maxwell's equations for the three-shell model are used in order to derive an integral equation for the magnetic potential. This derivation is independent of the number of shells and can, thus, be embedded into our setting.

Lemma 3.2 ([12, Prop. 2.2])**.** *Let the neuronal current $J \in \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$ be a continuously distributed current. Then*

$$U(y) = \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} (J(x) \times x) \cdot \nabla_x \sum_{k=0}^{\infty} \frac{r^k}{s^{k+1}(k+1)} P_k(\xi \cdot \eta)\, \mathrm{d}x, \qquad y \in \mathbb{B}_{\varrho_L}^{\mathrm{ext}}.$$

In order to drop unnecessary differentiability and boundary conditions on J, we avoid using Gauß's theorem for the next step as it was done in [12]. Instead, we use the circular shift of the triple product stated in [22, Sec. 14], that is $(x \times y) \cdot z = x \cdot (y \times z)$ for all $x,\, y,\, z \in \mathbb{R}^3$. This leads to a singularity-free, vector-valued integral equation of the first kind for the entire neuronal current instead of an integral equation for parts or derivatives of J.

Theorem 3.3. *Let the neuronal current $J \in \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$ be a continuously distributed current, then*

$$\left(\mathcal{T}^{\mathrm{M}} J\right)(y) \coloneqq U(y) = \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} J(x) \cdot \left(\sum_{k=1}^{\infty} \sqrt{\frac{k}{k+1}} \frac{r^k}{s^{k+1}} p_k^{(3)}(\xi, \eta) \right) \mathrm{d}x, \qquad y \in \mathbb{B}_{\varrho_L}^{\mathrm{ext}}.$$

Proof. We start with the representation of U from Lemma 3.2 and use the circular shift of the triple product in the first step and (2.1) in the second.

$$\begin{aligned} U(y) &= \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} (J(x) \times x) \cdot \nabla_x \sum_{k=0}^{\infty} \frac{r^k}{s^{k+1}(k+1)} P_k(\xi \cdot \eta)\, \mathrm{d}x \\ &= \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} J(x) \cdot \left(x \times \nabla_x \sum_{k=1}^{\infty} \frac{r^k}{s^{k+1}(k+1)} P_k(\xi \cdot \eta) \right) \mathrm{d}x \\ &= \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} J(x) \cdot \left(L_\xi^* \sum_{k=0}^{\infty} \frac{r^k}{s^{k+1}(k+1)} P_k(\xi \cdot \eta) \right) \mathrm{d}x \\ &= \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} J(x) \cdot \left(\sum_{k=1}^{\infty} \sqrt{\frac{k}{k+1}} \frac{r^k}{s^{k+1}} \tilde{p}_k^{(3)}(\xi, \eta) \right) \mathrm{d}x. \end{aligned} \tag{3.2}$$

In the last step, the differential operator L_ξ^* was interchanged with the series. For this purpose, recall that the scalar, [16], and the Edmonds-vector-Legendre polynomials, (2.9), are uniformly bounded for each $k \in \mathbb{N}$. Thus, the original and the term-by-term differentiated series are uniformly dominated by a convergent power series in $\varrho_0/\varrho_L < 1$ which allows this interchanging. The zeroth summand vanishes as a first order derivative of a constant. □

4. Electroencephalography for the multiple-shell model

Besides the magnetic field, the neuronal current J induces an electric potential u and its field E, respectively. The use of a spherical shell model, where the different radii and conductivities play an important role, is common for the inverse EEG problem, i.e. [3, 11, 30].

From the quasi-static Maxwell's equation, we immediately obtain the equation

$$\sigma \Delta u = \nabla \cdot J.$$

We follow the approach in [3] for an ellipsoidal shell-model and split the differential equation into a Poisson's equation over $\mathbb{B}_{\varrho_0}$ and into L Laplace equations over each spherical shell and the exterior of the head $\mathbb{B}_{\varrho_L}^{\mathrm{ext}}$. Due to [3], the electric potential u_l^{d} for a single dipole with dipole moment Q at a fixed position $x \in \mathbb{B}_{\varrho_0}$ is given for $y \in \mathbb{B}_{\varrho_0}$, $r < s$, by

$$u_0^{\mathrm{d}}(x,y) = Q \cdot \left(\sum_{k=1}^{\infty} \sum_{i=1}^{2k+1} \left(\alpha_k^{(0)} s^k + \frac{1}{\sigma_0(2k+1)} \frac{1}{s^{k+1}} \right) \nabla_x \left(r^k Y_{k,i}(\xi) \right) Y_{k,i}(\eta) \right).$$

The latter summand equals the Legendre series expansion, [16, Eq. (3.212)], of the particular solution of the Poisson's equation, that is $y \mapsto (4\pi\sigma_0)^{-1} Q \cdot \nabla_x |y-x|^{-1}$, see [37, Ch. VI]. According to [3], we obtain for all $l = 1, \ldots, L$, the fixed position $x \in \mathbb{B}_{\varrho_0}$, and $y \in \mathbb{S}_{[\varrho_{l-1},\varrho_l]}$ an expansion in inner and outer spherical harmonics, that is

$$u_l^{\mathrm{d}}(x,y) = Q \cdot \left(\sum_{k=1}^{\infty} \sum_{i=1}^{2k+1} \left(\alpha_k^{(l)} s^k + \beta_k^{(l)} \frac{1}{s^{k+1}} \right) \nabla_x \left(r^k Y_{k,i}(\xi) \right) Y_{k,i}(\eta) \right). \tag{4.1}$$

Note that we are mainly interested in the function u_L^{d}, because it is measured via EEG. The electric potentials over different regions are connected by the transmission conditions, see [3, 7], which require that

$$u_l^{\mathrm{d}}(x,y) = u_{l+1}^{\mathrm{d}}(x,y), \qquad \sigma_l \frac{\partial u_l^{\mathrm{d}}(x,y)}{\partial s} = \sigma_{l+1} \frac{\partial u_{l+1}^{\mathrm{d}}(x,y)}{\partial s}, \qquad y \in \mathbb{S}_{\varrho_l},$$

for all $l = 0, \ldots, L$, where u_{L+1}^{d} is the electric potential in $\mathbb{B}_{\varrho_L}^{\mathrm{ext}}$. These conditions further constrain the coefficients $\alpha_k^{(l)}$ and $\beta_k^{(l)}$. The first condition guarantees the continuity of the potential u^{d} on the transition spheres and the second implies the continuity of the normal component of the Ohmic current on the transition spheres. On the boundary $\mathbb{S}_{\varrho_L}$, we get from the corresponding Neumann condition due to the vanishing conductivity outside the head, the linear dependency of $\alpha_k^{(L)}$ to $\beta_k^{(L)}$:

$$\alpha_k^{(L)} = \frac{k+1}{k} \varrho_L^{-(2k+1)} \beta_k^{(L)}, \qquad k \in \mathbb{N}. \tag{4.2}$$

Eventually, using the abbreviation $\beta_k^{(0)} := (\sigma_0(2k+1))^{-1}$ for all $k \in \mathbb{N}$ and all $l = 0, \ldots, L-1$ we obtain the set of equations

$$\varrho_l^{2k+1} \left(\alpha_k^{(l)} - \alpha_k^{(l+1)} \right) + \beta_k^{(l)} - \beta_k^{(l+1)} = 0, \tag{4.3}$$

$$\varrho_l^{2k+1} k \left(\sigma_l \alpha_k^{(l)} - \sigma_{l+1} \alpha_k^{(l+1)} \right) + (k+1) \left(\sigma_{l+1} \beta_k^{(l+1)} - \sigma_l \beta_k^{(l)} \right) = 0, \tag{4.4}$$

which could be transferred into an adequate system of linear equations. However, the relevant matrix is ill-conditioned [12]. Therefore, we use an ansatz based on the idea of [30, 38]. In contrast to the approach therein, we exploit the fact of knowing a particular solution. For every $l = 0, \dots, L-1$, the transmission conditions can be represented by

$$\begin{pmatrix} \varrho_l^{2k+1} & 1 \\ \varrho_l^{2k+1} k\sigma_l & -(k+1)\sigma_l \end{pmatrix} \begin{pmatrix} \alpha_k^{(l)} \\ \beta_k^{(l)} \end{pmatrix} = \begin{pmatrix} \varrho_l^{2k+1} & 1 \\ \varrho_l^{2k+1} k\sigma_{l+1} & -(k+1)\sigma_{l+1} \end{pmatrix} \begin{pmatrix} \alpha_k^{(l+1)} \\ \beta_k^{(l+1)} \end{pmatrix}.$$

The inversion of the right-hand side matrix is allowed, since $\varrho_l^{2k+1}(2k+1)\sigma_{l+1} \neq 0$. By solving this equation for the coefficients belonging to $(l+1)$ and using the equations recursively, we find

$$\begin{pmatrix} \alpha_k^{(L)} \\ \beta_k^{(L)} \end{pmatrix} = \frac{1}{(2k+1)^L} \left(\coprod_{l=0}^{L-1} \begin{pmatrix} k+1+\frac{\sigma_l}{\sigma_{l+1}}k & (k+1)\left(1-\frac{\sigma_l}{\sigma_{l+1}}\right)\varrho_l^{-(2k+1)} \\ k\left(1-\frac{\sigma_l}{\sigma_{l+1}}\right)\varrho_l^{2k+1} & k+(k+1)\frac{\sigma_l}{\sigma_{l+1}} \end{pmatrix} \right) \begin{pmatrix} \alpha_k^{(0)} \\ \beta_k^{(0)} \end{pmatrix}. \tag{4.5}$$

The product of the matrix $\coprod_{l=0}^{L-1}$ is meant as follows: $\coprod_{l=0}^{L-1} A_l b := A_{L-1} \cdots A_1 A_0 b$ with the matrices A_l, $l = 0, \dots, L-1$ and a vector b. The matrix including the factor $(2k+1)^{-L}$ occurring on the right-hand side of (4.5) will be denoted by $M^{(L)}$ with its entries by $m_{i,j}^{(L)}$ for $i,\ j = 1, 2$. Note that the entries of the matrix $M^{(L)}$ depend on k which is neglected in the notation for the sake of readability.

Theorem 4.1. *Let the neuronal current $J \in \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$ be a continuously distributed current, then*

$$\left(\mathcal{T}^{\mathrm{E}} J\right)(y) := u_L(y) = \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} J(x) \cdot \left(\sum_{k=1}^{\infty} \sqrt{k(2k+1)^3} H_k(s) r^{k-1} \tilde{p}_k^{(2)}(\xi, \eta) \right) \mathrm{d}x, \qquad y \in \mathbb{S}_{[\varrho_{L-1}, \varrho_L]}$$

$$H_k(s) := \left(\frac{k+1}{k} \left(\frac{s}{\varrho_L} \right)^{2k+1} + 1 \right) \beta_k^{(L)} \frac{1}{s^{k+1}}.$$

The occurring coefficients are given for all $k \in \mathbb{N}$ by

$$\beta_k^{(L)} = \frac{k}{\sigma_L(2k+1)\left(k m_{1,1}^{(L)} - (k+1) m_{2,1}^{(L)} \varrho_L^{-(2k+1)}\right)}. \tag{4.6}$$

Proof. We start, again, with a single dipole. We have already found an expansion of the electric potential on the outer shell by means of inner and outer spherical harmonics in (4.1). The coefficients are given by (4.5), which using the introduced abbreviation is equivalent to

$$\alpha_k^{(L)} = m_{1,1}^{(L)} \alpha_k^{(0)} + m_{1,2}^{(L)} \beta_k^{(0)}, \qquad \beta_k^{(L)} = m_{2,1}^{(L)} \alpha_k^{(0)} + m_{2,2}^{(L)} \beta_k^{(0)}.$$

This can be reformulated by means of (4.2) and a straight-forward calculation in the form

$$\beta_k^{(L)} \left(\frac{k+1}{k} \varrho_L^{-(2k+1)} - \frac{m_{1,1}^{(L)}}{m_{2,1}^{(L)}} \right) = -\frac{m_{1,1}^{(L)} m_{2,2}^{(L)} - m_{1,2}^{(L)} m_{2,1}^{(L)}}{m_{2,1}^{(L)}} \beta_k^{(0)}.$$

Solving this equation for $\beta_k^{(L)}$ combined with $\beta_k^{(0)} = (\sigma_0(2k+1))^{-1}$ and the formal identity $\det\left(M^{(L)}\right) = m_{1,1}^{(L)} m_{2,2}^{(L)} - m_{1,2}^{(L)} m_{2,1}^{(L)}$, we obtain, for all $k \in \mathbb{N}$, the relation

$$\beta_k^{(L)} = \frac{k}{\sigma_0(2k+1)\left(k m_{1,1}^{(L)} - (k+1) m_{2,1}^{(L)} \varrho_L^{-(2k+1)}\right)} \det\left(M^{(L)}\right).$$

By means of simplifying the occurring products, we obtain a closed formula for the determinant:

$$\det\left(M^{(L)}\right) = \frac{1}{(2k+1)^{2L}} \prod_{l=0}^{L-1} \left(\left(k+1+\frac{\sigma_l}{\sigma_{l+1}}k\right)\left(k+(k+1)\frac{\sigma_l}{\sigma_{l+1}}\right) - k(k+1)\left(1-\frac{\sigma_l}{\sigma_{l+1}}\right)^2 \right)$$
$$= \frac{1}{(2k+1)^{2L}} \prod_{l=0}^{L-1} (2k+1)^2 \frac{\sigma_l}{\sigma_{l+1}} = \frac{\sigma_0}{\sigma_L}.$$

This yields the stated formula for the coefficients $\beta_k^{(L)}$ for all $k \in \mathbb{N}$. Eventually, for the electric potential on the outer shell generated by a single dipole, by inserting (4.2) in (4.1), we find

$$u_L^{\mathrm{d}}(x,y) = Q \cdot \left(\sum_{k=1}^{\infty} \sum_{i=1}^{2k+1} \left(\frac{k+1}{k} \left(\frac{s}{\varrho_L}\right)^{2k+1} + 1 \right) \frac{\beta_k^{(L)}}{s^{k+1}} \nabla_x \left(r^k Y_{k,i}(\xi)\right) Y_{k,i}(\eta) \right).$$

In order to obtain the electric potential in the case of a continuously distributed neuronal current J, we use the superposition principle and integrate our solution u_L^{d} over the cerebrum $\mathbb{B}_{\varrho_0}$ with respect to the first argument. In this way, we obtain the integral equation for the electric potential on the outer shell:

$$u_L(y) = \int_{\mathbb{B}_{\varrho_0}} J(x) \cdot \left(\sum_{k=1}^{\infty} \sum_{i=1}^{2k+1} H_k(s) \nabla_x \left(r^k Y_{k,i}(\xi)\right) Y_{k,i}(\eta) \right) \mathrm{d}x$$
$$= \int_{\mathbb{B}_{\varrho_0}} J(x) \cdot \left(\sum_{k=1}^{\infty} \sum_{i=1}^{2k+1} H_k(s) r^{k-1} \tilde{o}_{k,\xi}^{(2)} Y_{k,i}(\xi) Y_{k,i}(\eta) \right) \mathrm{d}x,$$

where we have used the representation of the gradient from (2.5). The definition of the Edmonds-vector-spherical harmonics (2.3) and the addition theorem for Edmonds-vector-Legendre polynomials (2.10) yield the stated integral equation. □

An advantage of this approach is that the electric potential u_L is given on the entire outer shell instead of only on the outer sphere as it is stated in [4, 11, 12]. In addition, we will discuss the existence of the series of the integral kernel and the integral which has not been elaborated in the earlier literature. In this connection, we need more information on the coefficients $\beta_k^{(L)}$ and its asymptotic behaviour.

Lemma 4.2. *Let $L \in \mathbb{N}_0$ be the numbers of shells and let $\beta_k^{(L)}$ for all $k \in \mathbb{N}_0$ be defined as in Theorem 4.1, then*

$$\left(k \mapsto \left|\beta_k^{(L)}\right|\right) \in \mathcal{O}\left(k^{-1}\right) \qquad \text{as } k \to \infty,$$

where $\mathcal{O}$ is the usual Landau symbol.

Proof. First, we assume that $L = 0$. We immediately obtain from the particular solution of Poisson's equation the relation $\beta_k^{(0)} = (\sigma_0(2k+1))^{-1} \in \mathcal{O}(k^{-1})$. Now, let $L > 0$. We analyze the matrix entries of $M^{(L)}$ occurring in the denominator of $\beta_k^{(L)}$ inductively. For $L = 1$, we obtain

$$M^{(1)} = \frac{1}{2k+1} \begin{pmatrix} k+1+\frac{\sigma_0}{\sigma_1}k & (k+1)\left(1-\frac{\sigma_0}{\sigma_1}\right)\varrho_0^{-(2k+1)} \\ k\left(1-\frac{\sigma_0}{\sigma_1}\right)\varrho_0^{2k+1} & k+(k+1)\frac{\sigma_0}{\sigma_1} \end{pmatrix}.$$

Obviously, for a finite, real constant $C^{(1)} > 0$, since $\varrho_0 < \varrho_1$ and the conductivities on each shell are non-vanishing, the following limits holds true:

$$\lim_{k\to\infty} \left|m_{1,1}^{(1)}\right| = C^{(1)}, \qquad \lim_{k\to\infty} \left|m_{2,1}^{(1)}\varrho_1^{-(2k+1)}\right| = 0.$$

Now, we assume for an arbitrary integer $L \geq 2$ that the entries in the first column of $M^{(L-1)}$ fulfil the conditions

$$\lim_{k\to\infty} \left| m_{1,1}^{(L-1)} \right| = C^{(L-1)}, \qquad \lim_{k\to\infty} \left| m_{2,1}^{(L-1)} \varrho_{L-1}^{-(2k+1)} \right| = 0$$

with a finite, real constant $C^{(L-1)} > 0$. For $M^{(L)}$, due to its constructions in (4.5) and the matrix multiplication, we obtain the relations

$$\begin{aligned} m_{1,1}^{(L)} &= \frac{1}{2k+1} \left(\left(k + 1 + \frac{\sigma_{L-1}}{\sigma_L} k \right) m_{1,1}^{(L-1)} + (k+1) \left(1 - \frac{\sigma_{L-1}}{\sigma_L} \right) \varrho_{L-1}^{-(2k+1)} m_{2,1}^{(L-1)} \right), \\ m_{2,1}^{(L)} &= \frac{1}{2k+1} \left(k \left(1 - \frac{\sigma_{L-1}}{\sigma_L} \right) \varrho_{L-1}^{2k+1} m_{1,1}^{(L-1)} + \left(k + (k+1) \frac{\sigma_{L-1}}{\sigma_L} \right) m_{2,1}^{(L-1)} \right). \end{aligned}$$

We use the existence of the limits concerning the matrix entries of $M^{(L-1)}$ and hence we find

$$\lim_{k\to\infty} \left| m_{1,1}^{(L)} \right| = \frac{1}{2} \left(1 + \frac{\sigma_{L-1}}{\sigma_L} \right) C^{(L-1)} =: C^{(L)}.$$

Since the limits of all summands exist, and are finite, positive constants, it follows that the limit exists. In analogy, we get with $\varrho_{L-1} < \varrho_L$

$$\begin{aligned} \lim_{k\to\infty} \left| m_{2,1}^{(L)} \varrho_L^{-(2k+1)} \right| = \lim_{k\to\infty} \Bigg| & \left(\frac{k}{2k+1} \left(1 - \frac{\sigma_{L-1}}{\sigma_L} \right) \left(\frac{\varrho_{L-1}}{\varrho_L} \right)^{2k+1} m_{1,1}^{(L-1)} \right. \\ & \left. + \left(\frac{k}{2k+1} + \frac{k+1}{2k+1} \frac{\sigma_{L-1}}{\sigma_L} \right) m_{2,1}^{(L-1)} \varrho_{L-1}^{-(2k+1)} \left(\frac{\varrho_{L-1}}{\varrho_L} \right)^{2k+1} \right) \Bigg| = 0. \end{aligned}$$

Eventually, using (4.6) we obtain the following equation:

$$\limsup_{k\to\infty} \left| \beta_k^{(L)} k \right| = \lim_{k\to\infty} \left| \frac{k^2}{\sigma_L (2k+1) \left(k m_{1,1}^{(L)} - (k+1) m_{2,1}^{(L)} \varrho_L^{-(2k+1)} \right)} \right| = \frac{1}{2\sigma_L C^{(L)}} < \infty,$$

which implies the stated asymptotic behaviour of $k \mapsto \beta_k^{(L)}$ as $k \to \infty$. □

In addition to the asymptotic behaviour, we need to know if $\beta_k^{(L)}$ vanishes for certain $k \in \mathbb{N}$.

Lemma 4.3. *Let $L \in \mathbb{N}_0$ be the numbers of shells and let $\beta_k^{(L)}$ be defined as in Theorem 4.1 for all $k \in \mathbb{N}$, then $\beta_k^{(L)} \neq 0$ for all $k \in \mathbb{N}$.*

Proof. For $L = 0$, the above statement is clear due to the existence of a particular solution. The statement for $L \geq 1$ remains to be proven. Let us assume that $\beta_k^{(L)} = 0$, for an arbitrary $k \in \mathbb{N}$. Then by means of (4.2), we obtain $\alpha_k^{(L)} = 0$. Inserting this relation in (4.3) and (4.4) for $l = L - 1$, we obtain two equations for the unknown coefficients $\beta_k^{(L-1)}$ and $\alpha_k^{(L-1)}$, thus

$$\begin{aligned} \varrho_{L-1}^{2k+1} \alpha_k^{(L-1)} + \beta_k^{(L-1)} &= 0, \\ \varrho_{L-1}^{2k+1} k \sigma_{L-1} \alpha_k^{(L-1)} - (k+1) \sigma_{L-1} \beta_k^{(L-1)} &= 0. \end{aligned}$$

Inserting the first equation into the second, we directly obtain $\alpha_k^{(L-1)} = 0$ and thus, $\beta_k^{(L-1)} = 0$. Accordingly, this can be done inductively for $l = L - 2, \ldots, 1$, and hence we obtain $\alpha_k^{(l)} = \beta_k^{(l)} = 0$ for all $l = 1, \ldots, L$.

Then, (4.3) and (4.4) yield for $l = 0$ the equations

$$\varrho_0^{2k+1}\alpha_k^{(0)} = \frac{-1}{(2k+1)\sigma_0}, \qquad \varrho_0^{2k+1}k\sigma_0\alpha_k^{(0)} = \frac{k+1}{2k+1}.$$

Both equations cannot be true simultaneously. Hence, this leads to a contradiction and we get $\beta_k^{(L)} \neq 0$ for all $k \in \mathbb{N}$. □

5. Vector Legendre-type integral equation

In order to solve the magnetoencephalography and the electroencephalography problems simultaneously, we introduce a particular class of integral equations.

Definition 5.1. Let $\mathbb{B}_R \subset \mathbb{R}^3$ be a ball with radius $R > 0$ and $\mathbb{G} \subset \mathbb{B}_R^{\mathrm{ext}}$ be a measurable (unbounded) outer region with $\inf_{y\in\mathbb{G}} s > R$. We define the vector Legendre-type operator $\mathcal{T}$ with its vector Legendre-type kernel $k^{(\iota)}$ for an arbitrary $\iota \in \{1,2,3\}$ and all $f \in \mathrm{L}^2(\mathbb{B}_R, \mathbb{R}^3)$ by

$$(\mathcal{T}f) \coloneqq \int_{\mathbb{B}_R} f(x)\cdot k^{(\iota)}(x,\cdot)\,\mathrm{d}x,$$

$$k^{(\iota)}(x,y) \coloneqq \sum_{k=1}^{\infty} \gamma_k(s) r^{t_k^{(\iota)}} \tilde{p}_k^{(\iota)}(\xi,\eta), \qquad (x,y) \in \mathbb{B}_R \times \mathbb{G}.$$

Let the occurring quantities fulfil the following assumptions:

(1) The exponents $t_k^{(\iota)}$ are bounded from below by zero, that is $\inf_{k\in\mathbb{N}} t_k^{(\iota)} \geq 0$.
(2) The exponents fulfil the growth condition $\sup_{k\in\mathbb{N}} R^{t_k^{(\iota)}-k} < \infty$.
(3) Let the non-negative constants $\{\Gamma_k\}_{k\in\mathbb{N}}$ fulfil for a fixed $M \in \mathbb{N}_0$ the growth condition $(k \mapsto \Gamma_k) \in \mathcal{O}\left(k^M\right)$ as $k \to \infty$. Each scalar-valued function $\gamma_k\colon [0,R] \to \mathbb{R}$ is continuous and satisfies for all $k \in \mathbb{N}$ the estimate

$$|\gamma_k(s)| \leq \frac{\Gamma_k}{s^{k+1}}.$$

We call the function f the density and the resulting function $\mathcal{T}f$ the potential.

Remark 5.2. We see directly that this integral equation covers the representation of the magnetic potential (i.e. $R \coloneqq \varrho_0$, $\mathbb{G} \coloneqq \mathbb{B}_{\varrho_L}^{\mathrm{ext}}$, $\iota = 3$, $t_k^{(\iota)} \coloneqq k$, and $4\pi\gamma_k(s) \coloneqq \sqrt{k/(k+1)}s^{-(k+1)}$ for all $s \geq \varrho_L$) as well as the electric potential (i.e. $R \coloneqq \varrho_0$, $\mathbb{G} \coloneqq \mathbb{S}_{[\varrho_{L-1},\varrho_L]}$, $\iota = 2$, $t_k^{(\iota)} \coloneqq k-1$, and $4\pi\gamma_k(s) \coloneqq \sqrt{k(2k+1)^3}H_k(s)$ for all $s \in [\varrho_{L-1}, \varrho_L]$). Note that the condition $\inf_{y\in\mathbb{G}} s > R$ is only fulfilled if the number of shells L is at least $L \geq 1$ in the case of the MEG problem and $L \geq 2$ in the case of the EEG problem. In the case of the electric potential the condition (3) is fulfilled due to the representation of H_k from Theorem 4.1 and Lemma 4.2.

Now, we summarize certain properties of the vector Legendre-type equation.

Theorem 5.3. *Let all conditions of Definition 5.1 be fulfilled. Then*

(1) the integral equation is well-defined and the relevant series converges absolutely and uniformly,
(2) the operator $\mathcal{T}\colon \mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3) \to \mathrm{L}^2(\mathbb{G})$ is bounded, and
(3) the potential is continuous (i.e. $\mathcal{T}f \in \mathrm{C}(\mathbb{G})$ for all $f \in \mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)$).

Proof. Due to the construction of $\mathbb{G}$, there exists an $\varepsilon > 0$ such that $s \geq R + \varepsilon$ for all $y \in \mathbb{G}$.

(1) First, we prove that the series of the vector Legendre-type kernel converges absolutely and uniformly. By means of (2.9) we obtain the estimate

$$\left|\sum_{k=K+1}^{\infty} \gamma_k(s) r^{t_k^{(\iota)}} \tilde{p}_k^{(\iota)}(\xi,\eta)\right| \leq \left(\sup_{k\in\mathbb{N}} R^{t_k^{(\iota)}-k}\right) \sum_{k=K+1}^{\infty} \Gamma_k \sqrt{\tilde{\mu}_k^{(\iota)}} \frac{R^k}{(R+\varepsilon)^{k+1}}. \tag{5.1}$$

The series on the right-hand side converges to 0 as $K \to \infty$, due to the convergence properties of a power series. Thus, the function $k^{(\iota)}$ from Definition 5.1 is well-defined. If the integral kernel is an $\mathrm{L}^2(\mathbb{B}_R \times \mathbb{G}, \mathbb{R}^3)$-function, the integral is well-defined due to Cauchy-Schwarz' inequality. This property of the integral kernel is proven in the next item.

(2) We start the proof of this statement by verifying that $k^{(\iota)} \in \mathrm{L}^2(\mathbb{B}_R \times \mathbb{G}, \mathbb{R}^3)$. Due to Tonelli's theorem, [1, Thm. 23.6], we concentrate our calculations on the following:

$$\begin{aligned}
\int_{\mathbb{G}} \int_{\mathbb{B}_R} \left(k^{(\iota)}(x,y)\right)^2 \mathrm{d}x\,\mathrm{d}y &\leq \int_{\mathbb{G}} \left(\sum_{k=1}^{\infty} |\gamma_k(s)|\, R^{t_k^{(\iota)}} \sqrt{\tilde{\mu}_k^{(\iota)}}\right)^2 \mathrm{vol}(\mathbb{B}_R)\,\mathrm{d}y \\
&\leq \mathrm{vol}(\mathbb{B}_R) \int_{\mathbb{S}} \int_{R+\varepsilon}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\Gamma_k}{s^{k+1}} R^{t_k^{(\iota)}} \sqrt{\tilde{\mu}_k^{(\iota)}}\right)^2 s^2\,\mathrm{d}s\,\mathrm{d}\omega(\eta) \\
&= 4\pi\,\mathrm{vol}(\mathbb{B}_R) \int_{R+\varepsilon}^{\infty} \left(\sum_{k=1}^{\infty} \frac{\Gamma_k}{s^{k}} R^{t_k^{(\iota)}} \sqrt{\tilde{\mu}_k^{(\iota)}}\right)^2 \mathrm{d}s \\
&= 4\pi\,\mathrm{vol}(\mathbb{B}_R) \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\Gamma_k \Gamma_l}{(k+l-1)} \sqrt{\tilde{\mu}_k^{(\iota)}\tilde{\mu}_l^{(\iota)}} \frac{R^{t_k^{(\iota)}+t_l^{(\iota)}}}{(R+\varepsilon)^{k+l-1}} \\
&\leq 4\pi\,\mathrm{vol}(\mathbb{B}_R) \left(\sup_{k\in\mathbb{N}} R^{t_k^{(\iota)}-k}\right)^2 \left(\sum_{k=1}^{\infty} \Gamma_k \sqrt{\tilde{\mu}_k^{(\iota)}} \frac{R^k}{(R+\varepsilon)^{k-1/2}}\right)^2 < \infty.
\end{aligned}$$

Note that we have interchanged the integration and the summation which is allowed due to the monotone convergence theorem for series, [1, Thm. 11.4]. The latter series is finite, due to the assumptions of Definition 5.1 and the convergence properties of the power series. Thus, we indeed have an operator $\mathcal{T}\colon \mathrm{L}^2(\mathbb{B}_R, \mathbb{R}^3) \to \mathrm{L}^2(\mathbb{G})$. In addition, it is a well-known fact that an integral operator $\mathcal{T}$ with an $\mathrm{L}^2(\mathbb{B}_R, \mathbb{G})$-kernel is linear and bounded, see [9].

(3) Let $f \in \mathrm{L}^2(\mathbb{B}_R, \mathbb{R}^3)$ be fixed. It is sufficient to verify that $\lim_{n\to\infty}(\mathcal{T}f)(y_n) = (\mathcal{T}f)(y)$ for all sequences $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{G}$ with $\lim_{n\to\infty} y_n = y \in \mathbb{G}$. For this purpose, we apply the dominated convergence theorem, see [1, Thm. 15.6], to the function $g_n(x) \coloneqq f(x) \cdot k^{(\iota)}(x, y_n)$. Due to the proof of the previous item, $k^{(\iota)} \in \mathrm{L}^2(\mathbb{B}_R \times \mathbb{G}, \mathbb{R}^3)$ and we get via Hölders' inequality $g_n \in \mathrm{L}^1(\mathbb{B}_R)$ for all $n \in \mathbb{N}$. Each summand of the integral kernel $k^{(\iota)}$ is a composition of continuous functions, thus it is continuous, and the series converges uniformly due to the estimate in item (1). Thus, $k^{(\iota)}$ is continuous in both arguments. Hence, $\lim_{n\to\infty} g_n = f \cdot k^{(\iota)}(\cdot, y)$ almost everywhere. In addition, we obtain

$$|g_n(x)| \leq |f(x)| \sup_{(x,y_n)\in\mathbb{B}_R\times\mathbb{G}} \left|k^{(\iota)}(x,y_n)\right| =: g(x).$$

The absolute value of the integral kernel can be estimated uniformly by (5.1). Hence, $g \in \mathrm{L}^2(\mathbb{B}_R) \subset \mathrm{L}^1(\mathbb{B}_R)$ since $f \in \mathrm{L}^2(\mathbb{B}_R, \mathbb{R}^3)$. Thus, the dominated convergence theorem allows to drag the limit into the integral. Then the continuity of $k^{(\iota)}$ in the second argument implies the continuity of $\mathcal{T}f$. □

After verifying that the vector Legendre-type integral equation is well-defined, we can solve the direct problem via an adequate Fourier series.

Theorem 5.4. *Let $f \in \mathrm{L}^2(\mathbb{B}_R, \mathbb{R}^3)$ and let the condition of Definition 5.1 be fulfilled. Then the potential is given by*

$$\begin{aligned}(\mathcal{T}f)(y) &= 4\pi \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{2n+1} \left(\int_0^R f_{n,j}^{(\iota)}(r) r^{t_n^{(\iota)}+2} \,\mathrm{d}r \right) \gamma_n(s) Y_{n,j}(\eta) \\ &= 4\pi \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{R^{t_n^{(\iota)}+3/2}}{(2n+1)\sqrt{2t_n^{(\iota)}+3}} f^\wedge(\iota, 0, n, j) \gamma_n(s) Y_{n,j}(\eta),\end{aligned}$$

which converges in the $\mathrm{L}^2(\mathbb{G})$*-sense, absolutely, and uniformly.*

Proof. Since $f \in \mathrm{L}^2(\mathbb{B}_R, \mathbb{R}^3)$, its Fourier series with respect to the orthonormal basis $\tilde{g}_{m,n,j}^{(i)}(R;\cdot)$ for $i = 1$, 2, 3; $m \in \mathbb{N}$, $n \in \mathbb{N}_{0_i}$, $j = 1, \ldots, 2n+1$ is given by (2.12). The strong convergence of the Fourier series of f, [37, Ch. V], immediately implies the uniform convergence of the series with respect to k. Hence, (2.11) yields

$$\begin{aligned}(\mathcal{T}f)(y) &= \int_{\mathbb{B}_R} f(x) \cdot k^{(\iota)}(x,y) \,\mathrm{d}x \\ &= \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} \sum_{k=1}^{\infty} \left(\gamma_k(s) \int_0^R f_{n,j}^{(i)}(r) r^{t_k^{(\iota)}+2} \,\mathrm{d}r \int_{\mathbb{S}} \tilde{y}_{n,j}^{(i)}(\xi) \cdot \tilde{p}_k^{(\iota)}(\xi,\eta) \,\mathrm{d}\omega(\xi) \right) \\ &= 4\pi \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{2n+1} \left(\int_0^R f_{n,j}^{(\iota)}(r) r^{t_n^{(\iota)}+2} \,\mathrm{d}r \right) \gamma_n(s) Y_{n,j}(\eta).\end{aligned}$$

Thus, for all $n \in \mathbb{N}$, $j = 1, \ldots, 2n+1$ the radial integral remains to be calculated. This is done by means of the auxiliary function $h_n(r) := r^{t_n^{(\iota)}} = R^{t_n^{(\iota)}+3/2} \left(2t_n^{(\iota)}+3\right)^{-1/2} Q_{0,n}^{(\iota)}(r)$ see Definition 2.1, the generalized Fourier expansion for $f_{n,j}^{(\iota)}$ from (2.14), and Parseval's identity, [37, Thm. III.4.2]:

$$\begin{aligned}\int_0^R f_{n,j}^{(\iota)}(r) r^{t_n^{(\iota)}+2} \,\mathrm{d}r &= \left\langle f_{n,j}^{(\iota)}, h_n \right\rangle_{\mathrm{L}^2_w[0,R]} \\ &= \sum_{m=0}^{\infty} \left\langle f_{n,j}^{(\iota)}, Q_{m,n}^{(\iota)} \right\rangle_{\mathrm{L}^2_w[0,R]} \left\langle h_n, Q_{m,n}^{(\iota)} \right\rangle_{\mathrm{L}^2_w[0,R]} \\ &= \sum_{m=0}^{\infty} f^\wedge(\iota, m, n, j) \sqrt{\frac{R^{2t_n^{(\iota)}+3}}{2t_n^{(\iota)}+3}} \left\langle Q_{0,n}^{(\iota)}, Q_{m,n}^{(\iota)} \right\rangle_{\mathrm{L}^2_w[0,R]} \\ &= f^\wedge(\iota, 0, n, j) \sqrt{\frac{R^{2t_n^{(\iota)}+3}}{2t_n^{(\iota)}+3}}.\end{aligned}$$

Finally, the stated convergence needs to be verified. For the uniform convergence, we use Cauchy-Schwarz' inequality for series and Parseval's identity:

$$\begin{aligned}
&\left|\sum_{n=N}^{\infty}\sum_{j=1}^{2n+1}\frac{R^{t_n^{(\iota)}+3/2}}{(2n+1)\sqrt{2t_n^{(\iota)}+3}}f^\wedge(\iota,0,n,j)\gamma_n(s)Y_{n,j}(\eta)\right|^2\\
\leq&\left(\sum_{n=N}^{\infty}\sum_{j=1}^{2n+1}\frac{R^{2t_n^{(\iota)}+3}}{(2n+1)^2\left(2t_n^{(\iota)}+3\right)}\left(\gamma_n(s)\right)^2\left(Y_{n,j}(\eta)\right)^2\right)\left(\sum_{n=N}^{\infty}\sum_{j=1}^{2n+1}f^\wedge(\iota,0,n,j)^2\right)\\
\leq&\frac{1}{4\pi}\left(\sum_{n=N}^{\infty}\frac{R^{2t_n^{(\iota)}+3}}{(2n+1)\left(2t_n^{(\iota)}+3\right)}\frac{\Gamma_n^2}{s^{2n+2}}\right)\|f\|^2_{\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)}.
\end{aligned}$$

The remaining series can be estimated in analogy to the proof of Theorem 5.3. Hence, the right-hand side of the previous equation converges to zero as $N\to\infty$. For the $\mathrm{L}^2(\mathbb{G})$-convergence, we find

$$\begin{aligned}
&\int_{\mathbb{G}}\left(\sum_{n=N}^{\infty}\sum_{j=1}^{2n+1}\frac{R^{t_n^{(\iota)}+3/2}}{(2n+1)\sqrt{2t_n^{(\iota)}+3}}f^\wedge(\iota,0,n,j)\gamma_n(s)Y_{n,j}(\eta)\right)^2\mathrm{d}y\\
&\quad\leq\|f\|^2_{\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)}\int_{\mathbb{G}}\left(\sum_{n=N}^{\infty}\sum_{j=1}^{2n+1}\frac{R^{2t_n^{(\iota)}+3}}{(2n+1)^2\left(2t_n^{(\iota)}+3\right)}\left(\gamma_n(s)Y_{n,j}(\eta)\right)^2\right)\mathrm{d}y.
\end{aligned}$$

The right-hand side converges also to zero as $N\to\infty$ which can be proven with the same estimates as in the proof of Theorem 5.3. □

Remark 5.5. Note that with the above setting scalar-valued problems like the inverse gravimetric problem, [26], can also be solved. Let F be a scalar-valued $\mathrm{L}^2(\mathbb{B}_R)$-function and the operator $\mathcal{S}\colon\mathrm{L}^2(\mathbb{B}_R)\to\mathrm{C}(\mathbb{G})$ be given by

$$(\mathcal{S}F)(y):=\int_{\mathbb{B}_R}F(x)K(x,y)\,\mathrm{d}x,\qquad K(x,y):=\sum_{k=1}^{\infty}\gamma_k(s)r^{t_k}P_k(\xi\cdot\eta),\quad(x,y)\in\mathbb{B}_R\times\mathbb{G},$$

where the occurring quantities fulfil the conditions from Definition 5.1. Then with $f(x):=\xi F(x)$, (2.4), (2.7), Theorem 5.4, and the linearity of the integral, we obtain the following:

$$\begin{aligned}
(\mathcal{S}F)(y)&=\int_{\mathbb{B}_R}f(x)\cdot\left(\sum_{k=1}^{\infty}\gamma_k(s)r^{t_k}\xi P_k(\xi\cdot\eta)\right)\mathrm{d}x\\
&=\int_{\mathbb{B}_R}f(x)\cdot\left(\sum_{k=1}^{\infty}\frac{\gamma_k(s)}{2k+1}r^{t_k}\left(\tilde{o}^{(1)}_{k,\xi}+\tilde{o}^{(2)}_{k,\xi}\right)P_k(\xi\cdot\eta)\right)\mathrm{d}x\\
&=\int_{\mathbb{B}_R}f(x)\cdot\left(\sum_{k=1}^{\infty}\frac{\gamma_k(s)}{2k+1}r^{t_k}\left(\left(\tilde{\mu}^{(1)}_k\right)^{1/2}\tilde{p}^{(1)}_k(\xi,\eta)+\left(\tilde{\mu}^{(2)}_k\right)^{1/2}\tilde{p}^{(2)}_k(\xi,\eta)\right)\right)\mathrm{d}x\\
&=4\pi\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1}\frac{1}{(2n+1)^2}\left(\int_0^R\left(\left(\tilde{\mu}^{(1)}_n\right)^{1/2}f^{(1)}_{n,j}(r)+\left(\tilde{\mu}^{(2)}_n\right)^{1/2}f^{(2)}_{n,j}(r)\right)r^{t_n+2}\,\mathrm{d}r\right)\gamma_n(s)Y_{n,j}(\eta).
\end{aligned}$$

In analogy with the above calculations, we get with (2.15) for almost all $r \in [0, R]$ the identity

$$\begin{aligned} F_{n,j}(r) &= \int_{\mathbb{S}} F(r\xi) Y_{n,j}(\xi)\,\mathrm{d}\omega(\xi) \\ &= \frac{1}{2n+1}\left(\left(\tilde{\mu}_n^{(1)}\right)^{1/2}\int_{\mathbb{S}} f(r\xi)\cdot\tilde{y}_{n,j}^{(1)}(\xi)\mathrm{d}\omega(\xi) + \left(\tilde{\mu}_n^{(2)}\right)^{1/2}\int_{\mathbb{S}} f(r\xi)\cdot\tilde{y}_{n,j}^{(2)}(\xi)\mathrm{d}\omega(\xi)\right) \\ &= \frac{1}{2n+1}\left(\left(\tilde{\mu}_n^{(1)}\right)^{1/2} f_{n,j}^{(1)}(r) + \left(\tilde{\mu}_n^{(2)}\right)^{1/2} f_{n,j}^{(2)}(r)\right). \end{aligned}$$

Combining the above calculations, we eventually obtain

$$(\mathcal{S}F)(y) = 4\pi\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1}\frac{1}{2n+1}\left(\int_0^R F_{n,j}(r) r^{t_n+2}\,\mathrm{d}r\right)\gamma_n(s) Y_{n,j}(\eta).$$

For the inverse gravimetric problem, where $t_n = n$ for all $n \in \mathbb{N}$, this is a well-known expansion of $\mathcal{S}F$, see for example [26] and the references therein.

In most applications, one is interested in solving the inverse problem instead of the direct problem. For this purpose, the compactness of the integral operator plays a major role.

Theorem 5.6. *Let $f \in \mathrm{L}^2(\mathbb{B}_R, \mathbb{R}^3)$ and let the condition of Definition 5.1 be fulfilled. Then the operator $\mathcal{T}\colon \mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3) \to \mathrm{L}^2(\mathbb{G})$ is nuclear and compact.*

Proof. Due to [37, Ch. X.2], each nuclear operator is compact. Thus, it remains to prove that $\mathcal{T}$ is nuclear. Since $\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)$ is a Hilbert space, the operator $\mathcal{T}$ has the desired $\mathrm{L}^2(\mathbb{G})$-convergent series representation, see Theorem 5.4. The relevant quantities have to fulfil the following condition:

$$\sup_{n\in\mathbb{N}_{0_\iota}, j=1,\ldots,2n+1}\left\|\tilde{g}_{0,n,j}^{(\iota)}\right\|_{\mathrm{L}^2(\mathbb{B}_R,\mathbb{R}^3)} < \infty, \qquad \sup_{n\in\mathbb{N}_{0_\iota}, j=1,\ldots,2n+1}\left\|(n+1)^2 R^n \gamma_n Y_{n,j}\right\|_{\mathrm{L}^2(\mathbb{G})} < \infty,$$

and a summability condition for the coefficients

$$\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1}\left|\frac{R^{t_n^{(\iota)}-n+3/2}}{(n+1)^3\sqrt{2t_n^{(\iota)}+3}}\right| < \infty.$$

The first condition is immediately fulfilled, due to the normalization property of the basis. The second condition is implied by the asymptotic behaviour of the functions γ_n, $n \in \mathbb{N}$ and the fact that $\|Y_{n,j}\|_\infty \le \sqrt{(2n+1)/4\pi}$, see [16, Lem. 3.31]. The last condition is fulfilled due to the assumptions on the exponents $t_n^{(\iota)}$ for all $n \in \mathbb{N}$. □

6. Inverse electro-magnetoencephalography problem

By means of the forward solution of the vector Legendre-type equation, we achieve immediately solutions of the direct MEG problem as well as the direct EEG problem. However, we are interested in the inverse problem, since the magnetic field ($B = \mu_0\nabla U$, (3.1)) and the electric potential are measured and a reconstruction of the neuronal current inside the brain is desired. We have already seen in Theorem 5.6 that the operators $\mathcal{T}^{\mathrm{M}}$ and $\mathcal{T}^{\mathrm{E}}$, which are particular cases of the vector Legendre-type operator, are compact. Compact operators with infinite dimensional range cannot have a bounded inverse, [9, Pro. 2.7]. Thus, the corresponding inverse problem is ill-posed according to Hadamard. For more details on (ill-posed) inverse problems and their regularization, see for example [9]. Furthermore, compact operators have a singular-value decomposition (SVD), [9, Ch. 2.2]. For the particular operators $\mathcal{T}^{\mathrm{M}}$ and $\mathcal{T}^{\mathrm{E}}$, we state below their singular-value decompositions.

Theorem 6.1 (SVD for MEG). *Let $J \in \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$ be the neuronal current and the MEG forward operator $\mathcal{T}^{\mathrm{M}} \colon \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3) \to \mathrm{L}^2\left(\mathbb{B}^{\mathrm{ext}}_{\varrho_L}\right)$ be defined as in Theorem 3.3 with $L \geq 1$. Then*

(1) the forward operator has the representation

$$\left(\mathcal{T}^{\mathrm{M}} J\right)(y) = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{2n+1} \sqrt{\frac{n}{n+1}} \left(\int_0^{\varrho_0} J^{(3)}_{n,j}(r) r^{n+2} \, \mathrm{d}r \right) \frac{1}{s^{n+1}} Y_{n,j}(\eta);$$

(2) the null space of $\mathcal{T}^{\mathrm{M}}$ is characterized by

$$\left(\ker \mathcal{T}^{\mathrm{M}}\right)^{\perp} = \overline{\operatorname{span}\left\{ \tilde{g}^{(3)}_{0,n,j}(\varrho_0; \cdot) \;\middle|\; n \in \mathbb{N}, j = 1, \ldots, 2n+1 \right\}}^{\mathrm{L}^2(\mathbb{B}_{\varrho_0})}.$$

Thus, only the harmonic part of the toroidal component of the neuronal current is not in the operator null space;

(3) the range of the operator is a subset of $\operatorname{Harm}\left(\mathbb{B}^{\mathrm{ext}}_{\varrho_L}\right) \subset \mathrm{L}^2\left(\mathbb{B}^{\mathrm{ext}}_{\varrho_L}\right)$ (i.e. the Hilbert space of all harmonic functions in the exterior of the head which have a bounded $\mathrm{L}^2\left(\mathbb{B}^{\mathrm{ext}}_{\varrho_L}\right)$-norm equipped with the $\mathrm{L}^2\left(\mathbb{B}^{\mathrm{ext}}_{\varrho_L}\right)$-inner product, [16, Ch. 10.8]);

(4) a complete orthonormal basis of $\operatorname{Harm}\left(\mathbb{B}^{\mathrm{ext}}_{\varrho_L}\right)$ is given by means of the outer harmonics from (2.2), $H^{\mathrm{ext}}_{n,j}(\varrho_L; x)$, by

$$S_{n,j}(\varrho_L; x) \coloneqq \sqrt{\frac{2n-1}{\varrho_L}} H^{\mathrm{ext}}_{n,j}(\varrho_L; x), \tag{6.1}$$

for all $n \in \mathbb{N}$ and $j = 1, \ldots, 2n+1$.

(5) the singular values of $\mathcal{T}^{\mathrm{M}}$ are given by

$$\lambda^{\mathrm{M}}_n \coloneqq \sqrt{\frac{n \varrho_0^3 \varrho_L}{(n+1)(2n+3)(2n-1)}} \frac{1}{2n+1} \left(\frac{\varrho_0}{\varrho_L}\right)^n.$$

Proof. (1) Theorem 5.4 yields the desired result for $R \coloneqq \varrho_0$, $\mathbb{G} \coloneqq \mathbb{B}^{\mathrm{ext}}_{\varrho_L}$, $\iota = 3$, $t^{(\iota)}_k \coloneqq k$, and $4\pi\gamma_k(s) \coloneqq \sqrt{k/(k+1)} s^{-(k+1)}$, see Remark 5.2.

(2) The characterization of the null space is an immediate consequence of the representation of $\mathcal{T}^{\mathrm{M}}$ in Theorem 5.4, since $\gamma_n \neq 0$ for all $n \in \mathbb{N}$. In addition, the particular occurring orthonormal basis functions build a basis for the space of all harmonic $\mathbf{L}^2(\mathbb{B}_{\varrho_0})$-function.

(3) From the quasi-static Maxwell's equation, we immediately obtain that the magnetic potential U is a harmonic function. Thus, $\mathcal{T}^{\mathrm{M}} J \in \operatorname{Harm}\left(\mathbb{B}^{\mathrm{ext}}_{\varrho_L}\right)$.

(4) In [16, Ch. 10.8], it is shown that the outer harmonics build a basis for $\operatorname{Harm}\left(\mathbb{B}^{\mathrm{ext}}_{\varrho_L}\right)$. It can be calculated brute force that they are also orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathrm{L}^2\left(\mathbb{B}^{\mathrm{ext}}_{\varrho_L}\right)}$. Moreover, the calculation of their normalization factor is straight-forward.

(5) From Theorem 5.4, we directly obtain with Remark 5.2

$$\begin{aligned}
\left(\mathcal{T}^{\mathrm{M}} J\right)(y) &= \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} J^{\wedge}(3,0,n,j) \sqrt{\frac{n\varrho_0^3}{(n+1)(2n+1)^2(2n+3)}} \frac{\varrho_0^n}{s^{n+1}} Y_{n,j}(\eta) \\
&= \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \sqrt{\frac{n\varrho_0^3 \varrho_L}{(n+1)(2n+1)^2(2n+3)(2n-1)}} J^{\wedge}(3,0,n,j) \left(\frac{\varrho_0}{\varrho_L}\right)^n S_{n,j}(\varrho_L; y). \qquad (6.2)
\end{aligned}$$

From this Fourier expansion we can extract the singular values. □

Before we state an SVD of the EEG related operator $\mathcal{T}^{\mathrm{E}}$, we need to define a Hilbert space containing the range of the operator.

Definition 6.2. We define a system of functions for $n \in \mathbb{N}$, $j = 1, \dots, 2n+1$, and all $y \in \mathbb{S}_{[\varrho_{L-1},\varrho_L]}$ by

$$G_{n,j}(y) \coloneqq \left(\frac{n+1}{2n+1} \left(\frac{s}{\varrho_L} \right)^{2n+1} + \frac{n}{2n+1} \right) \left(\frac{\varrho_L}{s} \right)^n \frac{1}{s} Y_{n,j}(\eta).$$

Then $\{G_{n,j}\}_{n\in\mathbb{N}, j=1,\dots,2n+1}$ is a complete orthonormal system for the Hilbert space

$$\mathrm{Z} \coloneqq \mathrm{Z}(\mathbb{S}_{[\varrho_{L-1},\varrho_L]}) \coloneqq \overline{\operatorname{span}\left\{ G_{n,j} \mid n \in \mathbb{N}, j = 1, \dots, 2n+1 \right\}}^{\mathrm{Z}}$$

equipped with the inner product $\langle F, H \rangle_{\mathrm{Z}} \coloneqq \langle F|_{\mathbb{S}_{\varrho_L}}, G|_{\mathbb{S}_{\varrho_L}} \rangle_{\mathrm{L}^2(\mathbb{S}_{\varrho_L})}$.

Obviously, $\langle \cdot, \cdot \rangle_{\mathrm{Z}}$ is a symmetric bilinear form over $\mathrm{Z}(\mathbb{S}_{[\varrho_{L-1},\varrho_L]})$ and $\langle F, F \rangle_{\mathrm{Z}} \geq 0$. The bilinear form $\langle \cdot, \cdot \rangle_{\mathrm{Z}}$ is positive definite if $0 = \langle F, F \rangle_{\mathrm{Z}}$ implies $F \equiv 0$. For this purpose, let $F \in \mathrm{Z}(\mathbb{S}_{[\varrho_{L-1},\varrho_L]})$. Due to the construction of Z, we get $F = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} c_{n,j} G_{n,j}$. From $0 = \langle F, F \rangle_{\mathrm{Z}}$ we immediately obtain $0 \equiv F|_{\mathbb{S}_{\varrho_L}} = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} c_{n,j} \varrho_L^{-1} Y_{n,j}$. Since the spherical harmonics are linearly independent, we get $c_{n,j} = 0$ for all $n \in \mathbb{N}$ and $j = 1, \dots, 2n+1$. Note that $\mathrm{Z}(\mathbb{S}_{[\varrho_{L-1},\varrho_L]})$ is a (with respect to the $\|\cdot\|_{\mathrm{Z}}$-norm) closed subspace of all harmonic functions over $\mathbb{S}_{[\varrho_{L-1},\varrho_L]}$.

Theorem 6.3 (SVD EEG). *Let $J \in \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$ be the neuronal current and the operator $\mathcal{T}^{\mathrm{E}} \colon \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3) \to \mathrm{Z}(\mathbb{S}_{[\varrho_{L-1},\varrho_L]})$ be defined as in Theorem 4.1 with $L \geq 2$. Then*

(1) the forward operator has the representation

$$\left(\mathcal{T}^{\mathrm{E}} J\right)(y) = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{\sqrt{n(n+1)^3}}{2n+1} \left(\int_0^{\varrho_0} J_{n,j}^{(2)}(r) r^{n+1} \, \mathrm{d}r \right) H_n(s) Y_{n,j}(\eta);$$

(2) the null space of $\mathcal{T}^{\mathrm{E}}$ is characterized by

$$\left(\ker \mathcal{T}^{\mathrm{E}}\right)^{\perp} = \overline{\operatorname{span}\left\{ \tilde{g}_{0,n,j}^{(2)}(\varrho_0; \cdot) \;\middle|\; n \in \mathbb{N}, j = 1, \dots, 2n+1 \right\}}^{\mathrm{L}^2(\mathbb{B}_{\varrho_0})}.$$

Thus, only the harmonic part of direction related to inner harmonics, see (2.5)*, of the neuronal current is not in the operator null space;*

(3) the range of the operator is a subset of $\mathrm{Z}(\mathbb{S}_{[\varrho_{L-1},\varrho_L]})$;

(4) the non-zero (Lemma 4.3) singular values of $\mathcal{T}^{\mathrm{E}}$ are given by

$$\lambda_n^{\mathrm{E}} \coloneqq (2n+1) \sqrt{\frac{\varrho_0}{n}} \beta_n^{(L)} \left(\frac{\varrho_0}{\varrho_L} \right)^n.$$

Proof. (1) Theorem 5.4 yields the desired result with $R \coloneqq \varrho_0$, $\mathbb{G} \coloneqq \mathbb{S}_{[\varrho_{L-1},\varrho_L]}$, $\iota = 2$, $t_k^{(\iota)} \coloneqq k-1$, $4\pi\gamma_k(s) \coloneqq \sqrt{k(2k+1)^3} H_k(s)$.
(2) The characterization of the null space is an immediate consequence of the representation of $\mathcal{T}^{\mathrm{E}}$ in Theorem 5.4, since $\gamma_n \neq 0$ for all $n \in \mathbb{N}$ due to Lemma 4.3. In addition, the harmonicity of the occurring basis functions can be verified easily.
(3) Due to the definition of the functions H_n, we get via Theorem 5.4 the representation

$$\begin{aligned} \left(\mathcal{T}^{\mathrm{E}} J\right)(y) &= \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \sqrt{n} \varrho_0^{n+1/2} J^\wedge(\iota, 0, n, j) H_n(s) Y_{n,j}(\eta) \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} (2n+1) \sqrt{\frac{\varrho_0}{n}} \beta_n^{(L)} \left(\frac{\varrho_0}{\varrho_L} \right)^n J^\wedge(\iota, 0, n, j) G_{n,j}(y). \end{aligned}$$

In addition, due to Lemma 4.2 and properties of the power series, we get for all $J \in \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$

$$\begin{aligned}\left\|\mathcal{T}^{\mathrm{E}} J\right\|_{\mathrm{Z}}^2 &= \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} (2n+1)^2 \frac{\varrho_0}{n} \left(\beta_n^{(L)}\right)^2 \left(\frac{\varrho_0}{\varrho_L}\right)^{2n} \left(J^\wedge(\iota, 0, n, j)\right)^2 \\ &\leq \sup_{n\in\mathbb{N}, j=1,\ldots,2n+1} \left(J^\wedge(\iota, 0, n, j)\right)^2 \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} (2n+1)^2 \frac{\varrho_0}{n} \left(\beta_n^{(L)}\right)^2 \left(\frac{\varrho_0}{\varrho_L}\right)^{2n} < \infty.\end{aligned}$$

Thus, $\operatorname{ran} \mathcal{T}^{\mathrm{E}} \subset \mathrm{Z}(\mathbb{S}_{[\varrho_{L-1}, \varrho_L]})$.
(4) The singular values can be extracted from the latter Fourier expansion of $\mathcal{T}^{\mathrm{E}} J$. □

The SVD provides us with a bundle of properties of the operators $\mathcal{T}^{\mathrm{M}}$ and $\mathcal{T}^{\mathrm{E}}$. First, an SVD of the adjoint operators can be stated which leads immediately to

$$\overline{\operatorname{ran} \mathcal{T}^{\mathrm{M}}} = \left(\ker\left(\left(\mathcal{T}^{\mathrm{M}}\right)^*\right)\right)^{\perp} = \operatorname{Harm}\left(\mathbb{B}_{\varrho_L}^{\mathrm{ext}}\right), \qquad \overline{\operatorname{ran} \mathcal{T}^{\mathrm{E}}} = \left(\ker\left(\left(\mathcal{T}^{\mathrm{E}}\right)^*\right)\right)^{\perp} = \mathrm{Z}(\mathbb{S}_{[\varrho_{L-1}, \varrho_L]}).$$

Second, we achieve the exponentially fast decrease of the singular values λ_n^{M} and λ_n^{E} to zero for n tending to infinity. Therefore, the inverse MEG and inverse EEG problem are severely ill-posed. The best-approximate solution can now be given by means of the general result for compact operators, [9, Thm 2.8].

Theorem 6.4. *Let the magnetic potential* $U \in \operatorname{Harm}\left(\mathbb{B}_{\varrho_L}^{\mathrm{ext}}\right)$ *and the electric potential* $u_L \in \mathrm{Z}(\mathbb{S}_{[\varrho_{L-1}, \varrho_L]})$ *fulfil Picard's conditions,*

$$\sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \left(\lambda_n^M\right)^{-2} \left\langle U, S_{n,j}(\varrho_L; \cdot)\right\rangle_{\mathrm{L}^2\left(\mathbb{B}_{\varrho_L}^{\mathrm{ext}}\right)}^2 < \infty, \qquad \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \left(\lambda_n^E\right)^{-2} \left\langle u_L, G_{n,j}\right\rangle_{\mathrm{Z}}^2 < \infty.$$

Then using the auxiliary operator $\mathcal{T} \coloneqq (\mathcal{T}^{\mathrm{M}}, \mathcal{T}^{\mathrm{E}})$ *with an appropriate product space in the range, the best-approximate solution of the neuronal current is given by*

$$\begin{aligned} J^+(x) &= \left(\mathcal{T}^+(U, u_{\mathrm{L}})\right)(x) \\ &\overset{\mathrm{L}^2(\mathbb{B}_{\varrho_L})}{=} \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \lambda_n^{\mathrm{M}} \left\langle U, S_{n,j}(\varrho_L; \cdot)\right\rangle_{\mathrm{L}^2\left(\mathbb{B}_{\varrho_L}^{\mathrm{ext}}\right)} \tilde{g}_{0,n,j}^{(3)}(\varrho_0; x) + \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \lambda_n^{\mathrm{E}} \left\langle u_L, G_{n,j}\right\rangle_{\mathrm{Z}} \tilde{g}_{0,n,j}^{(2)}(\varrho_0; x).\end{aligned}$$

A discussion of these results will be found in the conclusions.

It should be noted that MEG measures certain components of the magnetic field, as opposed to components of the magnetic potential. Thus, we are interested in transferring the results from the potential to the field using relation (3.1). With the same technique as in Section 5, we can indeed prove that the potential $\mathcal{T} f$ is continuously differentiable. We can also interchange the gradient with the integral and obtain

$$\nabla_y (\mathcal{T} f)(y) = \int_{\mathbb{B}_{\varrho_0}} \left(\operatorname{jac}_y \left(k^{(\iota)}(x, y)\right)\right)^{\mathrm{T}} f(x) \, \mathrm{d}x.$$

However, for the sake of brevity we skip the details. In analogy to the scalar outer harmonics, we use the vector outer harmonics, [16, Eq. (10.486)], (2.6)

$$h_{n,j}^{(1)}(\varrho_L; y) \coloneqq \frac{1}{\varrho_L} \left(\frac{\varrho_L}{s}\right)^{n+2} \tilde{y}_{n,j}^{(1)}(\eta) = \frac{-\varrho_L}{\sqrt{(2n+1)(n+1)}} \nabla_y H_{n,j}^{\mathrm{ext}}(\varrho_L, y), \qquad y \in \mathbb{B}_{\varrho_L}^{\mathrm{ext}}, \tag{6.3}$$

and the Hilbert space, [31, Ch. 7.2.1],

$$\mathrm{H}\left(\mathbb{B}_{\varrho_L}^{\mathrm{ext}}\right) \coloneqq \overline{\operatorname{span}_{n\in\mathbb{N}, j=1,\ldots,2n+1} \left\{h_{n,j}^{(1)}(\varrho_L; \cdot)\right\}}^{\mathrm{L}^2\left(\mathbb{B}_{\varrho_L}^{\mathrm{ext}}, \mathbb{R}^3\right)},$$

with the orthonormal basis

$$s_{n,j}(\varrho_L;\cdot) \coloneqq \sqrt{\frac{2n+1}{\varrho_L}} h^{(1)}_{n,j}(\varrho_L;\cdot), \tag{6.4}$$

for all $n \in \mathbb{N}$, $j = 1, \dots, 2n+1$. Then, with (6.1), (6.2), (6.3), and (6.4) the continuous magnetic field has the uniformly convergent SVD given by the following formulae:

$$\begin{aligned} B(y) &= \mu_0 \nabla_y \left(\mathcal{T}^{\mathrm{M}} J\right)(y) \\ &= \mu_0 \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \sqrt{\frac{n\varrho_0^3 \varrho_L}{(n+1)(2n+1)^2(2n+3)(2n-1)}} J^\wedge(3,0,n,j) \left(\frac{\varrho_0}{\varrho_L}\right)^n \nabla_y S_{n,j}(\varrho_L; y) \\ &= -\mu_0 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \sqrt{\frac{n\varrho_0\varrho_L}{(2n+1)^2(2n+3)}} \left(\frac{\varrho_0}{\varrho_L}\right)^{n+1} J^\wedge(3,0,n,j) s_{n,j}(\varrho_L; y). \end{aligned} \tag{6.5}$$

7. Comparison with previous results

There exist several approaches in the literature for solving the inverse MEG and inverse EEG problem. They all based upon certain decompositions of the neuronal current. Now, we discuss them in comparison to the novel approach presented in this article. Recall that this approach is called the Edmonds approach since it is based on a Fourier expansion using Edmonds-vector-spherical harmonics for the angular part.

7.1. **Morse-Feshbach approach for MEG.** We start with the inverse MEG problem and consider an ansatz which is similar to the Edmonds approach and is stated, for example, in [4, 5]. Therein, the angular part of the neuronal current is also decomposed by means of (complex) vector-valued spherical harmonics which are based on the Morse-Feshbach-vector-spherical harmonics, [16, Ch. 5.2]. Note that in [5] a non-normalized version of the spherical harmonics is erroneously used. However, the radial part of the neuronal current is not analyzed in [4, 5]. We can construct an orthonormal basis system on the ball similar to the one from Theorem 2.3 with the Morse-Feshbach-vector-spherical harmonics instead of the Edmonds-vector-spherical harmonics as the angular component. Then, we call the analysis of the problem based on a Fourier expansion with the orthonormal system related to the Morse-Feshbach vector spherical harmonics the Morse-Feshbach approach. Since type $i = 3$ of the Morse-Feshbach and the Edmonds-vector-spherical harmonics coincide, both approaches yield the same result of the angular part of the neuronal current.

7.2. **Hodge decomposition for MEG.** The neuronal current can also be decomposed by means of the Hodge decomposition, sometimes called the Hansen decomposition, and used in [4, 12, 14]:

$$J(x) = J^r(x)\xi + \frac{1}{r}\left(\nabla^*_\xi G(x) + L^*_\xi F(x)\right), \qquad x \in \mathrm{B}_{\varrho_0} \setminus \{0\}. \tag{7.1}$$

Note that for this decomposition the scalar-valued functions need to fulfil the conditions $J^r \in \mathrm{L}^2(\mathbb{B}_{\varrho_0})$ and G, $F \in \mathrm{C}^1(\mathbb{B}_{\varrho_0})$, which implies a higher smoothness of J than the smoothness required for the Edmonds approach. In addition, this decomposition is unique if $F|_{\mathbb{S}_r}$, $G|_{\mathbb{S}_r}$ have no constant parts for all $r \in [0, R]$, [16, Thm. 5.4]. In certain cases, the division by r could lead to an involuntary singularity at the origin.

Theorem 7.1. *If the neuronal current is decomposed by the Hodge decomposition with $J^r \in \mathrm{L}^2(\mathbb{B}_{\varrho_0})$ and G, $F \in \mathrm{C}^1(\mathbb{B}_{\varrho_0})$, then, for all $y \in \mathbb{B}^{\mathrm{ext}}_{\varrho_L}$, the relation*

$$U(y) = -\frac{1}{4\pi}\int_{\mathbb{B}_{\varrho_L}} F(x)\frac{\partial}{\partial r}\frac{1}{|x-y|}\,\mathrm{d}x = -\frac{1}{4\pi}\int_{\mathbb{B}_{\varrho_L}} F(x)\sum_{k=1}^{\infty}\frac{kr^{k-1}}{s^{k+1}}P_k(\xi\cdot\eta)\,\mathrm{d}x$$

holds true. This leads to the solution of the direct problem

$$U(y) = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{n}{2n+1} \left(\int_0^{\varrho_0} F_{n,j}(r) r^{n+1} \, \mathrm{d}r \right) \frac{1}{s^{n+1}} Y_{n,j}(\eta).$$

Proof. We insert the Hodge decomposition in the integral equation from Theorem 3.3. Then with the orthogonality of the occurring functions we get

$$U(y) = \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} L_\xi^* F(x) \cdot \left(\sum_{k=1}^{\infty} \sqrt{\frac{k}{k+1}} \frac{r^{-1}}{s^{k+1}} p_k^{(3)}(\xi, \eta) \right) \mathrm{d}x. \tag{7.2}$$

With $f(x) \coloneqq L_\xi^* F(x)$ and $t_n^{(3)} = n-1$ we obtain with the notation from Lemma 2.4 the equation $f_{n,j}^{(3)} = \sqrt{n(n+1)} F_{n,j}(r)$. Combined with $\gamma_n(s) = 1/4\pi \sqrt{k/(k+1)} s^{-(k+1)}$ and Theorem 5.4, we get the solution of the direct problem.

Applying a corollary of the surface Green's formula, [16, Eq. (2.160)], to (7.2), we achieve with $L_\xi^* \cdot L_\xi^* = \Delta^*$, [16, Eq. (2.141)], the identity

$$U(y) = -\frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} F(x) \Delta_\xi^* \left(\sum_{k=1}^{\infty} \frac{1}{k+1} \frac{r^{k-1}}{s^{k+1}} P_k(\xi, \eta) \right) \mathrm{d}x.$$

With (2.8), and the Legendre series in [16, Eq. (3.212)] we get the desired scalar integral equations. □

Another scalar integral equation involving the function $F \in \mathrm{C}^2(\mathbb{B}_{\varrho_0})$ from the Hodge decomposition is derived via the Green's formula in [12, Thm. 2.1], that is

$$U(y) = -\frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} \Delta_\xi^* F(x) \sum_{k=1}^{\infty} \frac{r^{k-1}}{(k+1) s^{k+1}} P_k(\xi \cdot \eta) \, \mathrm{d}x.$$

The corresponding forward solution, stated in [12, Eq. (20)], equals the one from Theorem 7.1. In addition, due to the construction of the Edmonds-vector-spherical harmonics, we get directly

$$\tilde{J}_{n,j}^{(3)}(r) = \frac{\sqrt{n(n+1)}}{r} F_{n,j}(r) \qquad r \in (0, \varrho_0].$$

Thus, the two results obtained via the Hodge decomposition are consistent to the result from the Edmonds approach in Theorem 6.1. However, the Edmonds approach appears to be advantageous since it uses less a-priori assumptions and does not require further calculations in order to obtain J.

7.3. **Helmholtz decomposition for MEG.** The Helmholtz decomposition (note that this is sometimes also called the Hodge-Morrey decomposition) can also be used, [4, 11–13]:

$$J = \nabla \Psi + \nabla \times a$$

with the scalar potential $\Psi \in \mathrm{C}^2(\mathbb{B}_{\varrho_L})$ and the vector potential $a \in \mathrm{C}^2(\mathbb{B}_{\varrho_L}, \mathbb{R}^3)$. Note that this decomposition is not unique without an additional gauge.

The vector potential a is a $\mathrm{C}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$-quantity, thus, we can decompose it by means of the spherical Helmholtz decomposition from (2.16). Due to Theorem 2.5, we get a further representation of the neuronal current

$$\begin{aligned} J(x) = {} & \xi \left(\frac{\partial}{\partial r} \Psi(x) + \frac{1}{r} \Delta_\xi^* A^{(3)}(x) \right) + \nabla_\xi^* \left(\frac{1}{r} \Psi(x) - \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) A^{(3)}(x) \right) \\ & + L_\xi^* \left(\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) A^{(2)}(x) - \frac{1}{r} A^{(1)}(x) \right). \end{aligned} \tag{7.3}$$

We can embed this directly into the vector-valued integral equation from Theorem 5.4. Then the magnetic potential depends on two components of the vector potential which is of disadvantage for a unique reconstruction. However, this can be fixed with an appropriate a-priori gauge. First, we reduce the vector-valued problem to a scalar-valued one.

Theorem 7.2. *If the neuronal current is decomposed by the Helmholtz decomposition with* $\Psi \in \mathrm{C}^2(\mathbb{B}_{\varrho_L})$ *and* $a \in \mathrm{C}^2(\mathbb{B}_{\varrho_L}, \mathbb{R}^3)$ *without additional gauge, then*

$$U(y) = \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} \left(\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) A^{(2)}(x) - \frac{1}{r} A^{(1)}(x) \right) \left(\sum_{k=1}^{\infty} \frac{k r^k}{s^{k+1}} P_k(\xi \cdot \eta) \right) \mathrm{d}x, \qquad y \in \mathbb{B}_{\varrho_L}^{\mathrm{ext}}.$$

Proof. We start with the representation of the magnetic potential from (3.2). Inserting (7.3) and using orthogonality properties of the surface curl and surface gradient, [16, Ch. 2.6], we obtain for U the representation

$$U(y) = \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} L_\xi^* \left(\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) A^{(2)}(x) - \frac{1}{r} A^{(1)}(x) \right) \cdot \left(L_\xi^* \sum_{k=1}^{\infty} \frac{r^k}{s^{k+1}(k+1)} P_k(\xi \cdot \eta) \right) \mathrm{d}x.$$

With an analogon of Green's surface identities for the surface curl operator, see [16, Eq. (2.163)], we get for the previous integral

$$U(y) = -\frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} \left(\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) A^{(2)}(x) - \frac{1}{r} A^{(1)}(x) \right) \left(\Delta_\xi^* \sum_{k=1}^{\infty} \frac{r^k}{s^{k+1}(k+1)} P_k(\xi \cdot \eta) \right) \mathrm{d}x. \tag{7.4}$$

Again, with (2.8), we obtain the stated integral equation. □

Lemma 7.3. *Under the assumptions of Theorem 7.2 and the Poincaré gauge (i.e. for all* $x \in \mathbb{B}_{\varrho_0}$ *the relation* $x \cdot a(x) = 0$ *implies* $A^{(1)} \equiv 0$*), the magnetic potential is given by*

$$\begin{aligned} U(y) &= \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) A^{(2)}(x) \left(\sum_{k=1}^{\infty} \frac{k r^k}{s^{k+1}} P_k(\xi \cdot \eta) \right) \mathrm{d}x \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{n}{2n+1} \left(\int_0^{\varrho_0} \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r} \right) A_{n,j}^{(2)}(r) r^{n+2} \, \mathrm{d}r \right) \frac{1}{s^{n+1}} Y_{n,j}(\eta). \end{aligned}$$

Under the Coulomb gauge (i.e. $\nabla \cdot a = 0$*, used in e.g. [11, Prop. 4.1]) we get*

$$\begin{aligned} U(y) &= \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} \Delta_x \left(r A^{(1)}(x) \right) \left(\sum_{k=1}^{\infty} \frac{r^k}{s^{k+1}(k+1)} P_k(\xi \cdot \eta) \right) \mathrm{d}x \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{(2n+1)(n+1)} \left(\frac{\mathrm{d}A_{n,j}^{(1)}}{\mathrm{d}x}(\varrho_0)\varrho_0 - (n-1) A_{n,j}^{(1)}(\varrho_0) \right) \frac{{\varrho_0}^{n+2}}{s^{n+1}} Y_{n,j}(\eta). \end{aligned}$$

Proof. The first integral equation is an immediate consequence of Theorem 7.2. By using the result of Remark 5.5, we obtain the solution of the forward problem. For the Coloumb gauge, we start with the integral representation of U from (7.4) and apply the second Green's surface identity [16, Eq. (2.158)]. Thus,

$$U(y) = -\frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} \Delta_\xi^* \left(\left(\frac{\partial}{\partial r} + \frac{1}{r} \right) A^{(2)}(x) - \frac{1}{r} A^{(1)}(x) \right) \left(\sum_{k=1}^{\infty} \frac{r^k}{s^{k+1}(k+1)} P_k(\xi \cdot \eta) \right) \mathrm{d}x.$$

By means of the representation of the divergence of a from Theorem 2.5 a relation between $A^{(1)}$ and the Beltrami operator of $A^{(2)}$ is given by the Coloumb gauge, which yields

$$U(y) = \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} \left(\frac{1}{r} \Delta^*_\xi + \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \left(r \frac{\partial}{\partial r} + 2 \right) \right) A^{(1)}(x) \left(\sum_{k=1}^{\infty} \frac{r^k}{s^{k+1}(k+1)} P_k(\xi \cdot \eta) \right) \mathrm{d}x.$$

Combined with the product rule, we get after lengthy calculations the desired result. The latter integral-free identity is stated in [11, Eq. (4.21)]. □

Thus, the reconstruction of the neuronal current depends on the chosen gauge, which is a disadvantage since there exists no medical justification for a preferred gauge (although perhaps a gauge could be fixed via appropriate boundary conditions). Without a chosen gauge, we obtain the relation due to (7.3) as follows:

$$J^{(3)}_{n,j}(r) = \sqrt{n(n+1)} \left(\frac{1}{r} \left(A^{(2)}_{n,j}(r) - A^{(1)}_{n,j}(r) \right) + \frac{\mathrm{d}}{\mathrm{d}r} A^{(2)}_{n,j}(r) \right), \qquad r \in (0, \varrho_0],$$

which implies the consistency of our approach to the Helmholtz decomposition (cf. [4, 11–13]). Thus, the visible part of the neuronal current depends on two functions of the vector potential, which does not allow a unique attribution.

7.4. **Morse-Feshbach approach for EEG.** Now, we discuss the existing results for the inverse EEG problem. In [5], the non-normalized complex Morse-Feshbach-vector-spherical harmonics are also used for the EEG problem, but only for the homogeneous case. Due to the connection of the Edmonds and Morse-Feshbach-vector-spherical harmonics, [16, Ch. 5.13], we immediately obtain that the measurable part of the neuronal current depends on two Morse-Feshbach-vector-spherical harmonics related orthonormal basis functions. Thus, in [5] the null space of the EEG operator could not be characterized precisely. Nevertheless, it was possible to conclude that the part of the neuronal current which is responsible for the generation of the electric potential lives in the orthogonal complement of the part of the neuronal current generating the magnetic potential, [5, p. 2547]. The approach based on the Edmonds-vector-spherical harmonics presented here enables a precise association of the visible neuronal parts. Furthermore, it yields a characterization of the radial part of the neuronal current which is omitted in [5].

7.5. **Hodge decomposition for EEG.** To the knowledge of the authors, the Hodge decomposition has not been used so far for solving the direct or inverse EEG problem. For the sake of completeness, we state our results here.

Theorem 7.4. *If the neuronal current is decomposed via the Hodge decomposition with $J^r \in \mathrm{L}^2(\mathbb{B}_{\varrho_0})$ and G, $F \in \mathrm{C}^1(\mathbb{B}_{\varrho_0})$, then, for all $y \in \mathbb{S}^{\mathrm{ext}}_{[\varrho_{L-1}, \varrho_L]}$, we obtain*

$$\begin{aligned} u_L(y) &= \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} J^r(x) \left(\sum_{k=1}^{\infty} (2k+1) k H_k(s) r^{k-1} P_k(\xi \cdot \eta) \right) \mathrm{d}x \\ &\quad + \frac{1}{4\pi} \int_{\mathbb{B}_{\varrho_0}} G(x) \left(\sum_{k=1}^{\infty} (2k+1) k(k+1) H_k(s) r^{k-2} P_k(\xi \cdot \eta) \right) \mathrm{d}x \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \left(\int_0^{\varrho_0} \left(r J^r_{n,j}(r) + (n+1) G_{n,j}(r) \right) r^n \mathrm{d}r \right) n H_n(s) Y_{n,j}(\eta). \end{aligned}$$

Proof. We insert the Hodge decomposition (7.1) in the integral equation from Theorem 4.1. Then, combining (2.5) with (2.3) and the addition theorems (2.10), we get $\nabla_x(r^k P_k(\xi)) = \left(\tilde{\mu}_n^{(2)}\right)^{1/2} r^{k-1}\tilde{p}_k^{(2)}(\xi,\eta)$. Hence,

$$\begin{aligned} u_L(y) &= \frac{1}{4\pi}\int_{\mathbb{B}_{\varrho_0}} J^r(x)\frac{\partial}{\partial r}\left(\sum_{k=1}^{\infty}(2k+1)H_k(s)r^k P_k(\xi\cdot\eta)\right)\mathrm{d}x \\ &\quad + \frac{1}{4\pi}\int_{\mathbb{B}_{\varrho_0}} \nabla_\xi^* G(x)\cdot\nabla_\xi^*\left(\sum_{k=1}^{\infty}(2k+1)H_k(s)r^{k-2}P_k(\xi\cdot\eta)\right)\mathrm{d}x \\ &= \frac{1}{4\pi}\int_{\mathbb{B}_{\varrho_0}} J^r(x)\left(\sum_{k=1}^{\infty}(2k+1)kH_k(s)r^{k-1}P_k(\xi\cdot\eta)\right)\mathrm{d}x \\ &\quad + \frac{1}{4\pi}\int_{\mathbb{B}_{\varrho_0}} G(x)\left(\sum_{k=1}^{\infty}(2k+1)k(k+1)H_k(s)r^{k-2}P_k(\xi\cdot\eta)\right)\mathrm{d}x. \end{aligned}$$

In the last step, the second Green's surface identity, [16, Eq. (2.158)], is used. The latter proposition of this theorem is given by Remark 5.5. □

In addition, we find

$$\tilde{J}_{n,j}^{(2)}(r) = \sqrt{\frac{n}{2n+1}}\left(J_{n,j}^r(r) + \frac{n+1}{r}G_{n,j}(r)\right), \qquad r\in(0,\varrho_0].$$

Since the neuronal current to which the electric potential is sensitive depends on two functions of the Hodge decomposition, this decomposition is not as suitable for the inversion as the Edmonds decomposition.

7.6. **Helmholtz decomposition for EEG.** In the literature, the Helmholtz decomposition is the first choice decomposition for the inversion of the continuous EEG problem, see [4, 8, 11–13]. The advantage of this approach is that the corresponding integral equation only depends on the scalar potential Ψ.

Theorem 7.5. *If the neuronal current is decomposed by the Helmholtz decomposition with $\Psi\in \mathrm{C}^2(\mathbb{B}_{\varrho_L})$ and $a\in\mathrm{C}^2(\mathbb{B}_{\varrho_L},\mathbb{R}^3)$ without additional gauge but with the given boundary values $(x\cdot J(x))|_{\mathbb{S}_{\varrho_0}}=0$, then, for all $y\in\mathbb{S}_{[\varrho_{L-1},\varrho_L]}$, we get*

$$\begin{aligned} u_L(y) &= -\frac{1}{4\pi}\int_{\mathbb{B}_{\varrho_0}}(\Delta_x\Psi(x))\sum_{k=1}^{\infty}(2k+1)H_k(s)r^kP_k(\xi\cdot\eta)\mathrm{d}x \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{2n+1}\left(n\Psi_{n,j}(\varrho_0) - \frac{\mathrm{d}\Psi_{n,j}}{\mathrm{d}x}(\varrho_0)\varrho_0\right){\varrho_0}^{n+1}H_n(s)Y_{n,j}(\eta). \end{aligned}$$

Note that in [8] the additional boundary condition was also avoided.

Proof. We start with the integral equation of u_L from Theorem 4.1. We use, again, $\nabla_x(r^k P_k(\xi)) = \left(\tilde{\mu}_n^{(2)}\right)^{1/2} r^{k-1}\tilde{p}_k^{(2)}(\xi,\eta)$ and apply Gauß's theorem to the integral equation, that is

$$\begin{aligned} u_L(y) &= \frac{1}{4\pi}\int_{\mathbb{B}_{\varrho_0}} J(x)\cdot\nabla_x\left(\sum_{k=1}^{\infty}\sqrt{\frac{k(2k+1)^3}{k(2k+1)}}H_k(s)r^kP_k(\xi)\right)\mathrm{d}x \\ &= -\frac{1}{4\pi}\int_{\mathbb{B}_{\varrho_0}}(\nabla_x\cdot J(x))\left(\sum_{k=1}^{\infty}(2k+1)H_k(s)r^kP_k(\xi)\right)\mathrm{d}x \\ &\quad+\frac{1}{4\pi}\int_{\mathbb{S}_{\varrho_0}}(\xi\cdot J(x))\sum_{k=1}^{\infty}H_k(s)(2k+1)r^kP_k(\xi\cdot\eta)\mathrm{d}\omega(\xi). \end{aligned}$$

Due to our assumption, the boundary term vanishes. Then via $\nabla\cdot(\nabla\Psi)=\Delta\Psi$, we obtain the first stated equation. A similar derivation can be found in [11, Prop. 4.1] for the restriction of the electric potential u_L onto the sphere $\mathbb{S}|_{\varrho_L}$. Therein, the latter statement is proven for the particular case of $u_L|_{\mathbb{S}_{\varrho_L}}$. However, the result is still valid in the general case considered here. Note that the sequence $(s_n)_n$ occurring in [11, Prop. 4.1] is closely related to the functions $(H_n)_n$ via

$$s_n = (2n+1)H_n(\varrho_L), \qquad n\in\mathbb{N}. \qquad \square$$

By means of (7.3), we get without the additional boundary conditions for all $r\in(0,\varrho_0]$ the relation

$$\tilde{J}_{n,j}^{(2)}(r) = \sqrt{\frac{n}{2n+1}}\left(\frac{n+1}{r}+\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\Psi_{n,j}(r)-(n+1)A_{n,j}^{(3)}(r)\right).$$

In order to prove that the Helmholtz decomposition is consistent with the Edmonds approach, we insert this identity in the radial integral from Theorem 6.3. Thus,

$$\begin{aligned} \int_0^{\varrho_0}\tilde{J}_{n,j}^{(2)}(r)r^{n+1}\mathrm{d}r &= \sqrt{\frac{1}{n(2n+1)}}\int_0^{\varrho_0} n\frac{\mathrm{d}}{\mathrm{d}r}\left(\Psi_{n,j}(r)r^{n+1}\right)-n(n+1)\frac{\mathrm{d}}{\mathrm{d}r}\left(A_{n,j}^{(3)}(r)r^{n+1}\right)\mathrm{d}r \\ &= \sqrt{\frac{1}{n(2n+1)}}\left(n\Psi_{n,j}(\varrho_0)-n(n+1)A_{n,j}^{(3)}(\varrho_0)\right){\varrho_0}^{n+1}. \end{aligned}$$

In the last step, the fundamental theorem of calculus is used. If the condition $(x\cdot J(x))|_{\mathbb{S}_{\varrho_0}}=0$, and (7.3) which yields $\Psi_{n,j}'(\varrho_0)=\frac{1}{\varrho_0}n(n+1)A_{n,j}^{(3)}(\varrho_0)$ are used, then the result of the Edmonds approach from Theorem 6.3 is consistent to the result from Theorem 7.5.

In this case, without the boundary condition the measurable part depends on two functions of the Helmholtz decomposition. In summary, with the additional boundary condition that the current vanishes on the boundary, the Helmholtz decomposition yields a scalar-valued integral equation for the electric potential. We were able to transfer the existing results to the entire shell $\mathbb{S}_{[\varrho_{L-1},\varrho_L]}$. In addition, an integral-free relation for the electric potential can be obtained. A lack of this approach is the missing characterization of the angular part of the neuronal current. Thus, the Helmholtz decomposition is an excellent approach for the inverse EEG problem, if the sufficiently smooth current vanishes at the boundary of the cerebrum. Nevertheless, the results presented in this paper suggest that the Edmonds approach could be a valuable alternative which should be further pursued in the future.

8. Conclusions

In this article, we gave an alternative general setup for the direct and inverse electro-magnetoencephalography problem in the case of the multiple-shell model. Our ansatz has several advantages over previous ones. Moreover, these previous approaches can be derived as particular cases of our novel modelling.

In Section 3, we derived a novel integral operator mapping the continuously distributed neuronal current onto the magnetic potential based on the quasi-static Maxwell's equations. An advantage of this approach is that as few as possible assumptions on the current are required. The approach only uses the assumption $J \in \mathrm{L}^2(\mathbb{B}_{\varrho_0}, \mathbb{R}^3)$. There are no additional differentiability or boundary conditions needed. In addition, the corresponding integral kernel represented by an Edmonds-vector-Legendre series is singularity-free inside the cerebrum.

In Section 4, we also derived a new integral operator mapping the current onto the electric potential with as few assumptions as in the MEG case. In addition, we enable the evaluation of the electric potential on the entire outer shell (scalp), which is relevant for numerical implementation. A theoretical result is the convergence of the corresponding integral kernel. For this purpose, we analyzed the asymptotic behaviour of the non-vanishing coefficients $\beta_k^{(L)}$ as $k \to \infty$ and found a dominating, convergent power series for the EEG integral kernel.

In order to solve both vector-valued integral equations simultaneously, we defined a new class of these equations in Section 5, which we called the vector Legendre-type equation. We proved that the corresponding integral operator is compact, even if $\mathbb{G}$ is not compact (which is often required for Fredholm integral equations in the literature).

Accordingly, we were able to transfer these results to our particular applications in Section 6. Further, we stated singular-value decompositions for both integral operators and enlarged this result for the magnetic field as the gradient of the magnetic potential by means of a Fourier expansion of the density. Based on the SVD, we see that the reconstruction of the neuronal current from magnetic and electric data is severely ill-posed due to several reasons.

(1) Not all directions of the neuronal current can be reconstructed. In (2.13), it is stated which of the used orthonormal basis functions are solenoidal. By means of Theorem 6.1 and Theorem 6.3, we deduce that only the solenoidal directions ($i = 2$, 3) of the neuronal current are not silent to either the magnetoencephalography or the electroencephalography measurements. In addition, from Theorem 6.1, it is shown that only the direction belonging to the toroidal part of the neuronal current (Edmonds type $i = 3$) is not silent for MEG. The association of solenoidal and toroidal directions of the neuronal current to types of particular vector-valued basis functions is new for the continuously distributed MEG and EEG problem.

(2) Not all parts of the solenoidal direction are effecting the measured quantities. We have seen that the radial part of the retraceable directions can be expanded by a generalized Fourier expansion. Thus, in both applications only the degree $m = 0$ is in the orthogonal complement of the null space, which coincides with the harmonic parts of these directions. Thus, further information is needed in order to obtain a unique reconstruction. This can be, for example, the minimum-norm condition, which leads directly to the best-approximate solution of the neuronal current stated in Theorem 6.4. The minimum-norm condition yields the same result as the assumption that the neuronal current is a harmonic and solenoidal function. Alternatively, further additional radial uniqueness constraints can be added. For this purpose, one should consider the integrand

$$\int_0^R f_{n,j}^{(\iota)}(r) r^{t_n^{(\iota)}+2}\, \mathrm{d}r.$$

Note that we are able to choose a different additional uniqueness constraint for each direction of the neuronal current. A variety of possible conditions for $f_{n,j}^{(\iota)}$ is stated by some of the authors in [24, 27]. Even though only scalar-valued integral equations are considered therein, the results can be transferred one-to-one to the setting of this article.

(3) The inversion of either the magnetic data or the electric data contains a downward continuation (i.e. the factor $(\varrho_0/\varrho_L)^n$ in the SVD), which makes them severely unstable.
(4) In the real data situation, only a few (at most 170) measurements of the magnetic field and the electric potential are available per time step. Thus, the lack of data needs to be handled.
(5) The obtained data is noisy. Besides the noise level of MEG and EEG, the data is influenced by certain effects like head, eye, jaw, or neck movements, magnetic field generated by the heart, particles attached or implanted to the patient's body, and non-human outer fields and currents, see [18].

Summing up, a regularization method is needed in order to solve these two problems numerically. A Tikhonov-based regularization method, the so-called regularized functional matching pursuit algorithm [10, 25, 29], and its enhancement, the regularized orthogonal matching pursuit algorithm [28, 36], will be used in a forthcoming work to solve these two problems numerically. Based on these methods, there exists a recently developed algorithm called the regularized weak functional matching pursuit algorithm which could improve and accelerate the numerical calculations of the inverse MEG and inverse EEG problem, see [20, 21].

In addition, in Section 7 we discussed that the presented approach should be the preferred ansatz in order to solve the (joint) inverse MEG and EEG problem. The reason is that it requires only a minimum of assumptions for the neuronal current and yields the most precise characterization of the operator null spaces. We listed formulae in order to convert the results obtained by the other approaches like the use of the Morse-Feshbach basis, the Hodge and the Helmholtz decomposition to the approach presented here. These formulae prove the consistency of our approach to previous approaches.

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