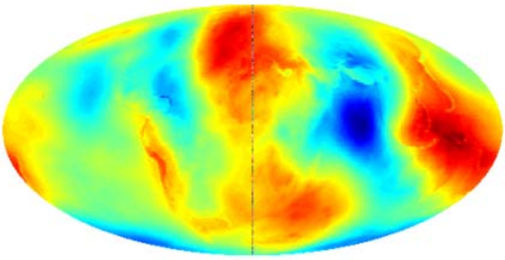
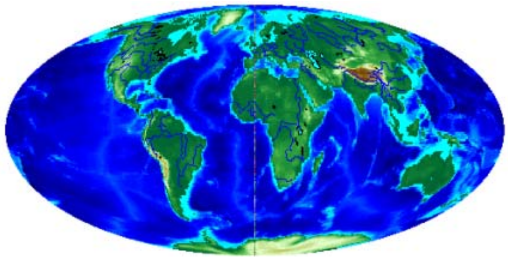




Siegen Preprints on Geomathematics

Reproducing Kernel Based Splines for the Regularization of the Inverse Spheroidal Gravimetric Problem

Nahid Akhtar, Volker Michel

Geomathematics Group,
Department of Mathematics, University of Siegen,
57068 Siegen, Germany.
nahidnariman@yahoo.com,
michel@mathematik.uni-siegen.de

www.geomathematics-siegen.de

4

Reproducing Kernel Based Splines for the Regularization of the Inverse Spheroidal Gravimetric Problem

Nahid Akhtar, Volker Michel
Geomathematics Group, Department of Mathematics,
University of Siegen, 57068 Siegen, Germany.
nahidnariman@yahoo.com, michel@mathematik.uni-siegen.de

March 17, 2011

Abstract

To solve the inverse gravimetric problem, i.e. to reconstruct the Earth's mass density distribution by using the gravitational potential, we introduce a spline interpolation method for the spheroidal Earth model. This problem is ill-posed in the sense of Hadamard as the solution may not exist, it is not unique and it is not stable. Since the anharmonic part (orthogonal complement) of the density function produces a zero potential, we restrict our attention only to reconstruct the harmonic part of the density function by using the gravitational potential. This spline interpolation method gives the existence and uniqueness of the unknown solution. Moreover, this method represents a regularization, i.e. every spline continuously depends on the given gravitational potential. These splines are also combined with a multiresolution concept, i.e. we get closer and closer to the unknown solution by increasing the scale and adding more and more data at each step.

Key Words: inverse gravimetric problem, ill-posed, regularization, scaling function, multiresolution analysis
MSC 2010 Classification: 45Q05, 65R30, 45B05, 41A15, 46E22, 86A22.

1 Introduction

The reconstruction of the Earth's interior is one of the oldest problems in the history of mankind. Fact is that only a few miles of the Earth are drilled down to find the answers about the inner structures of the Earth, which is a very small distance inside the Earth as compared to the diameter of the Earth if we consider it as a ball or to the semi-minor axis in case of a spheroidal Earth model. Anyhow, we have some rough ideas to answer these questions. At present, we are familiar with two methods for the determination of the Earth's mass density distribution. The first method consists of the analysis of seismic data such as travel times of waves and frequency shifts of normal modes. The second method is based on the gravitational potential measurements on and outside the surface of the Earth. Some of the references for the second method (which will be further investigated within this paper) in chronological order are H. Moritz ([27]), W. A. Heiskanen and H. Moritz ([18]), H. Sünkel, C. C. Tscherning ([30]), N. C. Thong and E. W. Grafarend ([31]), L. Ballani, J. Engels and E. W. Grafarend ([6]), E. W. Grafarend and A. A. Ardalan ([17]), V. Michel ([20]), A. A. Ardalan and E. W. Grafarend ([4], [5]), V. Michel ([21]), W. Freeden and V. Michel ([15]), M. J. Fengler, D. Michel and V. Michel ([12]) and the references herein.

According to Newton's law of gravitation, the gravitational potential of the Earth is given by

$$V = k \int_{\text{Earth}} \frac{\rho(x)}{|x - \cdot|} dx \tag{1}$$

where k is Newton's gravitational constant and ρ is the mass density function. If we knew the mass density distribution of the Earth, we could calculate the gravitational potential from Equation (1). However, as we mentioned above, we have very little information about the Earth's interior and so about the mass density distribution. Because of the great importance of the gravitational potential in the geosciences, several methods of measuring the gravitational potential have been developed. Nowadays, high resolution models are available due to satellite missions such as CHAMP, GRACE, and GOCE. So, now the problem is to find the mass density distribution by using the gravitational potential given on and outside the surface of the Earth, which is known as the inverse gravimetric problem. The inverse gravimetric problem is actually the inversion of Equation (1), which is a Fredholm integral equation of the first kind involving Newton's law of gravitation. This inverse problem falls into the category of ill-posed problems because the Fredholm integral equation is not solvable if the right-hand side is non-harmonic. Moreover, even if the right-hand side is harmonic, there may not exist a solution. Also the inversion of the Fredholm integral equation is discontinuous, i.e. small errors in the measurements of the gravitational potential can lead to big changes in the solution. Lastly, we have the problem with the uniqueness of the solution. Since for every arbitrary density distribution there exists an infinite-dimensional set of different density distributions which produce exactly the same potential, we cannot find the solution by a simple inversion of the integral equation given above. Because of this, we need to develop an approximation method to deal such an ill-posed inverse problem.

In this paper, we extend the theory of an existing spline based regularization method to the case of a spheroidal surface of the Earth, since an (oblate) spheroid, i.e. an ellipsoid of revolution, is a better approximation to the surface of the Earth than a sphere (for further details, see, for example, S. Chandrasekhar [10]). For this purpose, spheroidal harmonic spline functions are introduced by the use of reproducing kernel functions (with respect to the spheroidal inner harmonics). The space used in this context is a reproducing kernel Sobolev space. Since spline functions are used to interpolate prescribed data, a spheroidal harmonic spline interpolation problem is explained into which we transform the inverse spheroidal gravimetric problem. Here, the concept of reproducing kernel based spline approximation for a sphere, introduced by W. Freeden in [13] and [14], and for the interior of a ball developed by M. J. Fengler, D. Michel and V. Michel ([12]), V. Michel and K. Wolf ([24]), A. Amirbekyan ([2]), A. Amirbekyan and V. Michel ([3]), P. Berkel and V. Michel ([9]), P. Berkel ([7]), V. Michel ([22]), as well as P. Berkel, D. Fischer and V. Michel ([8]) is extended to the interior of a spheroid. Some interesting properties of spheroidal harmonic splines such as smoothness and best approximation (first and second minimum property) are also discussed. Moreover, the method is equipped with the features of a multiresolution analysis. The proof of a convergence theorem is also given which assures that the interpolating spheroidal harmonic splines converge to the unknown function, which in our case is the harmonic density function.

2 Basic Fundamentals

The set of positive integers, non-negative integers and integers will be represented, respectively, by the letters $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$. The letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}_0$ stand for the set of all real numbers, the set of all positive real numbers and the set of all non-negative real numbers, respectively. $\mathbb{R}^n, n \in \mathbb{N}$ will denote the n-dimensional space of real (column) vectors.

Let $D \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$, $n, m \in \mathbb{N}_0$. Then by $\mathrm{C}^{(k)}(D, W)$, we denote the set of all continuous functions $F : D \longrightarrow W$ such that every derivative of F of order $\leq k$ exists on the interior $\operatorname{int} D$ of D and is continuous. Let $D \subset \mathbb{R}^n$ be an arbitrary measurable set, p with $1 \leq p < \infty$ be a given number and $\omega : D \longrightarrow \mathbb{R}^+$ be a positive measurable weight function then the space $\mathrm{L}^2(D,\omega)$ equipped with the inner product

$$(F,G)_{\mathrm{L}^2(D,\omega)} := \int_D F(x)\, G(x)\, \omega(x)\, dx,$$

is a Hilbert space if almost everywhere identical functions are identified with each other (as it is usual for L^p- spaces).
In terms of the Jacobi ellipsoidal coordinates (u,λ,φ) (following, for example, [4]), a point in space can be located as the intersection of the following family of surfaces:

1. the family of confocal (oblate) ellipsoids of revolution

$$\mathbb{E}^2_{\sqrt{u^2+\varepsilon^2},u} := \left\{ (x,y,z) \in \mathbb{R}^3 \left| \frac{x^2+y^2}{u^2+\varepsilon^2} + \frac{z^2}{u^2} = 1 \right. \right\}, u \in (0,\infty), \varepsilon^2 := a^2 - b^2,$$

2. the family of half planes

$$\mathbb{P}^2_{\cos\lambda,\sin\lambda} := \left\{ (x,y,z) \in \mathbb{R}^3 \left| y = x\tan\lambda \right. \right\}, \lambda \in [0,2\pi],$$

3. the family of confocal half hyperboloids (oblate) ellipsoids of revolution

$$\mathbb{H}^2_{\varepsilon\cos\varphi,\varepsilon\sin\varphi} := \left\{ (x,y,z) \in \mathbb{R}^3 \left| \frac{x^2+y^2}{\varepsilon^2\cos^2\varphi} - \frac{z^2}{\varepsilon^2\sin^2\varphi} = 1 \right. \right\}, \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \varphi \neq 0.$$

The relation between the Cartesian and the (Jacobi) ellipsoidal coordinates is given by

$$x = \sqrt{u^2+\varepsilon^2}\cos\varphi\cos\lambda, \quad y = \sqrt{u^2+\varepsilon^2}\cos\varphi\sin\lambda, \quad z = u\sin\varphi,$$

where $\varphi \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is the reduced latitude, $\lambda \in [0,2\pi]$ is the geocentric longitude, u is the semi-minor axis and $\varepsilon := \sqrt{a^2-b^2}$ is the absolute linear eccentricity of the ellipsoid of revolution E, which will be called a spheroid throughout this paper. We will use $a = 6378136.59$ metres, $b = 6356751.94$ metres and $\varepsilon = \sqrt{a^2-b^2} = 521853.56$ metres for the standard reference ellipsoid used for the Earth model. If $\varepsilon \to 0$ in the above coordinate system then it becomes the spherical coordinate system.
Let us set $\overline{E_{\mathrm{int}}} := E_{\mathrm{int}} \cup E$, where E_{int} and E stand, respectively, for the interior and for the surface of (the Earth) $\overline{E_{\mathrm{int}}}$. In the same fashion, E_{ext} will represent the exterior of $\overline{E_{\mathrm{int}}}$ and $\overline{E_{\mathrm{ext}}} := E_{\mathrm{ext}} \cup E$. Now let us define the mapping $\tau : \mathbb{R}^+_0 \times [0,2\pi] \times \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \mathbb{R}^3$ by $(u,\lambda,\varphi) \longmapsto (x,y,z)^{\mathrm{T}}$. The Jacobian matrix of the transformation from the Jacobi ellipsoidal coordinates into the Cartesian coordinates is given by

$$\begin{aligned}
\tau' &= \begin{pmatrix} x_u & x_\lambda & x_\varphi \\ y_u & y_\lambda & y_\varphi \\ z_u & z_\lambda & z_\varphi \end{pmatrix} \\
&= \begin{pmatrix} \frac{u}{\sqrt{u^2+\varepsilon^2}}\cos\varphi\cos\lambda & -\sqrt{u^2+\varepsilon^2}\cos\varphi\sin\lambda & -\sqrt{u^2+\varepsilon^2}\sin\varphi\cos\lambda \\ \frac{u}{\sqrt{u^2+\varepsilon^2}}\cos\varphi\sin\lambda & \sqrt{u^2+\varepsilon^2}\cos\varphi\cos\lambda & -\sqrt{u^2+\varepsilon^2}\sin\varphi\sin\lambda \\ \sin\varphi & 0 & u\cos\varphi \end{pmatrix}.
\end{aligned}$$

The determinant of the Jacobian matrix is calculated as $\det\tau' = \left(u^2+\varepsilon^2\sin^2\varphi\right)\cos\varphi$. Let an element on the surface E of the spheroid $\overline{E_{\mathrm{int}}}$, with $u = b$ and $a = \sqrt{b^2+\varepsilon^2}$, be given and denoted by

$$\xi := \left(\sqrt{b^2+\varepsilon^2}\cos\varphi\cos\lambda, \sqrt{b^2+\varepsilon^2}\cos\varphi\sin\lambda, b\sin\varphi\right)^{\mathrm{T}}.$$

Then the surface element $d\omega(\xi)$ is given by

$$d\omega(\xi) = \sqrt{KF - G^2}\, d\varphi d\lambda = a\sqrt{b^2 + \varepsilon^2 \sin^2 \varphi}\cos\varphi d\varphi d\lambda \tag{2}$$

where $K := (\xi_\lambda)^{\mathrm{T}} \xi_\lambda$, $F := (\xi_\varphi)^{\mathrm{T}} \xi_\varphi$, $G := (\xi_\lambda)^{\mathrm{T}} \xi_\varphi$.
The references for Definition 1 are, for example, [16], [18], [25] and [26].

Definition 1. *Let $D \subset \mathbb{R}^3$ be a connected set. A function $F \in \mathrm{C}^{(2)}(D)$ is called harmonic iff $\Delta_x F(x) = 0$ for all $x \in \operatorname{int} D$. The set of all harmonic functions in $\mathrm{C}^{(2)}(D)$ is denoted by* $\operatorname{Harm}(D)$.

Note that every harmonic function $H \in \operatorname{Harm}(E_{\mathrm{int}})$ is separable in the coordinate system (u, λ, φ) and the Laplace operator in the Jacobi ellipsoidal coordinates (for a reference, see, for example, [4]) is given by

$$\begin{aligned}\Delta &= \frac{1}{(u^2 + \varepsilon^2 \sin^2 \varphi)} \\ &\quad \times \left(2u\frac{\partial}{\partial u} + (u^2 + \varepsilon^2)\frac{\partial^2}{\partial u^2} - \tan\varphi \frac{\partial}{\partial \varphi} + \frac{\partial^2}{\partial \varphi^2} + \frac{u^2 + \varepsilon^2 \sin^2\varphi}{(u^2+\varepsilon^2)\cos^2\varphi}\frac{\partial^2}{\partial \lambda^2}\right).\end{aligned}$$

For the following investigations, we will need the system of associated Legendre functions. Their definition can be found in [19]. The following theorem is an example of known properties of these functions. For further details, see any standard book on orthogonal polynomials.

Theorem 2. *The associated Legendre functions $\{P_{n,j}\}_{n\in\mathbb{N}_0, j\in\{0,\ldots,n\}}$ given by*

$$P_{n,j}(t) := \frac{1}{2^n n!}\left(1 - t^2\right)^{\frac{j}{2}} \frac{d^{n+j}}{dt^{n+j}}\left(t^2 - 1\right)^n,\ t \in [-1, 1] \tag{3}$$

are not orthogonal in general. Some important relations are given in the following:

$$\int_{-1}^{1} P_{n,j}(t)\, P_{m,j}(t)\, dt = 0,\ n \neq m,\ m \in \mathbb{N}_0,\ j \leq \min(n, m), \tag{4}$$

$$\int_{-1}^{1} (P_{n,j}(t))^2\, dt = \frac{2(n+j)!}{(2n+1)(n-j)!}, \tag{5}$$

and

$$\int_{-1}^{1} P_{n,j}(t)\, P_{n,j+2}(t)\, dt = \frac{-2(n+j)!}{(2n+1)(n-j-2)!}$$

with $n \geq 2$, $j \leq n - 2$.

Moreover, the (real-valued) Legendre functions of the first kind and of the second kind in terms of $\frac{u}{\varepsilon}$ are given, respectively, by

$$\tilde{\mathcal{P}}_{n,j}\left(\frac{u}{\varepsilon}\right) := \frac{(n+j)!}{n!\pi}\int_0^\pi \left(\frac{u}{\varepsilon} + \sqrt{\left(\frac{u}{\varepsilon}\right)^2 + 1}\cos v\right)^n \cos(jv)\, dv \tag{6}$$

with $0 \leq u \leq b$ and

$$\begin{aligned}&\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u}{\varepsilon}\right) \qquad (7)\\ &:= (-1)^{|j|}\frac{2^n (n+|j|)!n!}{(n-|j|)!(2|j|)!}\left(1 + \left(\frac{u}{\varepsilon}\right)^2\right)^{\frac{|j|}{2}} \int_0^\infty \frac{\sinh^{2|j|} v}{\left(\frac{u}{\varepsilon} + \sqrt{\left(\frac{u}{\varepsilon}\right)^2 + 1}\cosh v\right)^{n+|j|+1}}\, dv\end{aligned}$$

with $b \leq u < \infty$, see [4].
Following for example, [4], [29] and [31], we know that the associated Legendre functions of the second kind are not numerically stable if we calculate them through their recurrence formulae, hence we adopt here the recursive relations for these functions given in [4], where these functions enjoy the numerical stability. These recursive relations are given through the following expressions

$$\begin{aligned}
\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u}{\varepsilon}\right) &= \sum_{k=0}^{k_{\max}} \tilde{\mathcal{Q}}_{n,|j|,k}(u), \quad k_{\max} \in \mathbb{N}_0, \\
\tilde{\mathcal{Q}}_{n,|j|,k}(u) &= \frac{\varepsilon^2 (1-n-|j|-2k)(n+|j|+2k)}{2k(2n+2k+1)u^2} \tilde{\mathcal{Q}}_{n,|j|,k-1}(u) \quad \text{for all } k \geq 1, \\
\tilde{\mathcal{Q}}_{n,|j|,0}(u) &= \left(u^2+\varepsilon^2\right)^{\frac{|j|}{2}} \left(\frac{a}{u}\right)^{n+1}, \quad n \in \mathbb{N}_0,\ j \in \{-n,...,n\}.
\end{aligned}$$

The summation is continued until we get the required accuracy limit σ, i.e.

$$\tilde{\mathcal{Q}}_{n,|j|,k_{\max}}(u) - \tilde{\mathcal{Q}}_{n,|j|,k_{\max}-1}(u) < \sigma.$$

For double precision, σ may be taken equal to 10^{-16}.

3 Construction of the Complete Orthonormal System in $\left(\mathrm{Harm}\left(\overline{\boldsymbol{E}_{\mathrm{int}}}\right), (\cdot,\cdot)_{\mathrm{L}^2\left(\overline{\boldsymbol{E}_{\mathrm{int}}}\right)}\right)$

For the construction of a reproducing kernel, we need an orthonormal system in $\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)$. The details of the construction of such a basis are explained in this section. Note that we treat the non-uniqueness of the problem (see [23] for further details) by using the constraint of a harmonic solution.
Following [18], we describe here the Dirichlet problem or the first boundary-value problem of potential theory in terms of an ellipsoid of revolution E: Given an arbitrary function on the surface E, determine a function H which is harmonic either inside or outside E and which assumes on E the values of the prescribed function.
Let $n \in \mathbb{N}_0$, $j \in \{-n,...,n\}$. Then, following, [18], a solution of the Laplace equation

$$\Delta H(x(u,\lambda,\varphi)) = 0$$

is given by

$$H(x(u,\lambda,\varphi)) = \mathcal{P}_{n,|j|}\left(i\frac{u}{\varepsilon}\right) P_{n,|j|}(\sin\varphi) \begin{cases} \cos j\lambda, & \text{if } 0 \leq j \leq n \\ \sin |j|\lambda, & \text{if } -n \leq j \leq -1, \end{cases} \tag{8}$$

with $0 \leq u \leq b$, and

$$H(x(u,\lambda,\varphi)) = \mathcal{Q}_{n,|j|}\left(i\frac{u}{\varepsilon}\right) P_{n,|j|}(\sin\varphi) \begin{cases} \cos j\lambda, & \text{if } 0 \leq j \leq n \\ \sin |j|\lambda, & \text{if } -n \leq j \leq -1, \end{cases} \tag{9}$$

with $b \leq u < \infty$ where $\mathcal{P}_{n,|j|}$, $\mathcal{Q}_{n,|j|}$ are, respectively, the associated Legendre functions of the first kind and of the second kind of degree n and order $|j|$.
Since, $\mathcal{P}_{n,|j|}\left(i\frac{b}{\varepsilon}\right)$ and $\mathcal{Q}_{n,|j|}\left(i\frac{b}{\varepsilon}\right)$ are non-zero constants (see [4], [18]), the division of Equation (8) by $\mathcal{P}_{n,|j|}\left(i\frac{b}{\varepsilon}\right)$ and of Equation (9) by $\mathcal{Q}_{n,|j|}\left(i\frac{b}{\varepsilon}\right)$ is possible.
If we set

$$H_{\mathrm{i},n,j}(x(u,\lambda,\varphi)) := \frac{\mathcal{P}_{n,|j|}\left(i\frac{u}{\varepsilon}\right)}{\mathcal{P}_{n,|j|}\left(i\frac{b}{\varepsilon}\right)} P_{n,|j|}(\sin\varphi) \begin{cases} \cos j\lambda, & \text{if } 0 \leq j \leq n \\ \sin |j|\lambda, & \text{if } -n \leq j \leq -1, \end{cases} \tag{10}$$

with $0 \le u \le b$, then all $H_{\mathrm{i},n,j}$ are harmonic in $\overline{E_{\mathrm{int}}}$ and are called spheroidal inner harmonics, while

$$H_{\mathrm{e},n,j}\left(x\left(u,\lambda,\varphi\right)\right) := \frac{\mathcal{Q}_{n,|j|}\left(i\frac{u}{\varepsilon}\right)}{\mathcal{Q}_{n,|j|}\left(i\frac{b}{\varepsilon}\right)} P_{n,|j|}\left(\sin\varphi\right) \begin{cases} \cos j\lambda, & \text{if } 0 \le j \le n \\ \sin|j|\lambda, & \text{if } -n \le j \le -1, \end{cases} \tag{11}$$

with $b \le u < \infty$ are harmonic in $\overline{E_{\mathrm{ext}}}$ and are known as spheroidal outer harmonics. Moreover, at $u = b$, Equations (10) and (11) are equal i.e.

$$H_{\mathrm{i},n,j}\left(x\left(b,\lambda,\varphi\right)\right) = H_{\mathrm{e},n,j}\left(x\left(b,\lambda,\varphi\right)\right) = P_{n,|j|}\left(\sin\varphi\right) \begin{cases} \cos j\lambda, & \text{if } 0 \le j \le n \\ \sin|j|\lambda, & \text{if } -n \le j \le -1. \end{cases}$$

For the computational simplicity, we make $\mathcal{P}_{n,|j|}$ and $\mathcal{Q}_{n,|j|}$ real-valued functions, i.e. we set

$$\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right) := i^{-n}\mathcal{P}_{n,|j|}\left(i\frac{u}{\varepsilon}\right) \text{ and } \tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u}{\varepsilon}\right) := i^{n+1}\mathcal{Q}_{n,|j|}\left(i\frac{u}{\varepsilon}\right) \tag{12}$$

respectively. Let

$$e_{n,j}\left(\xi\left(\lambda,\varphi\right)\right) := P_{n,|j|}\left(\sin\varphi\right) \begin{cases} \cos j\lambda, & \text{if } 0 \le j \le n \\ \sin|j|\lambda, & \text{if } -n \le j \le -1 \end{cases}$$

represent the surface spheroidal harmonics. These harmonics are linearly independent but are not orthogonal in the Hilbert space $\mathrm{L}^2\left(E\right)$. The simple scalar product of the function space $\mathrm{L}^2\left(E\right)$ can be defined by

$$\left(g,h\right)_{\mathrm{L}^2(E)} := \int_E g\left(\xi\right) h\left(\xi\right) d\omega\left(\xi\right)$$

where $d\omega\left(\xi\right)$ is the surface element and following Equation (2), its value is given as follows

$$d\omega\left(\xi\right) = a\sqrt{b^2 + \varepsilon^2 \sin^2\varphi}\cos\varphi d\varphi d\lambda.$$

Let the weight function be denoted and defined by

$$w\left(\xi\left(\lambda,\varphi\right)\right) := \frac{1}{4\pi}\frac{1}{a\sqrt{b^2 + \varepsilon^2\sin^2\varphi}}.$$

Then the inner product of the elements of the weighted Hilbert space $\mathrm{L}^2\left(E,\omega\right)$ is introduced by

$$\begin{aligned} \left(g,h\right)_{\mathrm{L}^2(E,w)} &:= \int_E w\left(\xi\right) g\left(\xi\right) h\left(\xi\right) d\omega\left(\xi\right) \\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\left(\xi\left(\lambda,\varphi\right)\right) h\left(\xi\left(\lambda,\varphi\right)\right)\cos\varphi d\varphi d\lambda. \end{aligned}$$

We know that, if n, m are integers, then we get due to Theorem 2,

$$\begin{aligned} \left(e_{n,j}, e_{m,k}\right)_{\mathrm{L}^2(E,w)} &= \frac{1}{4\pi}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e_{n,j}\left(\xi\left(\lambda,\varphi\right)\right) e_{m,k}\left(\xi\left(\lambda,\varphi\right)\right)\cos\varphi d\varphi d\lambda \\ &= \frac{\left(n+|j|\right)!}{\left(2-\delta_{j,0}\right)\left(2n+1\right)\left(n-|j|\right)!}\delta_{nm}\delta_{jk}. \end{aligned} \tag{13}$$

Now we will normalize the surface spheroidal harmonics. For this purpose, we set

$$Y_{n,j}\left(\xi\right) := \sqrt{\frac{\left(2-\delta_{j0}\right)\left(2n+1\right)\left(n-|j|\right)!}{\left(n+|j|\right)!}} e_{n,j}\left(\xi\right),\ \xi \in E. \tag{14}$$

Using Equations (13) and (14), we obtain the orthonormality of the spheroidal surface harmonics as follows

$$(Y_{n,j}, Y_{m,k})_{\mathrm{L}^2(E,w)} = \int_E w(\xi)\, Y_{n,j}(\xi)\, Y_{m,k}(\xi)\, d\omega(\xi) = \delta_{nm}\delta_{jk}. \tag{15}$$

By setting

$$P^*_{n,j}(\sin\varphi) := \sqrt{\frac{(2-\delta_{j0})(2n+1)(n-|j|)!}{(n+|j|)!}} P_{n,|j|}(\sin\varphi), \quad -n \le j \le n,$$

Equation (14) can be rewritten as

$$Y_{n,j}(\xi(\lambda,\varphi)) = P^*_{n,j}(\sin\varphi) \begin{cases} \cos j\lambda, & \text{if } 0 \le j \le n \\ \sin |j|\lambda, & \text{if } -n \le j \le -1. \end{cases} \tag{16}$$

The spheroidal inner and outer harmonics are now defined and denoted by

$$H_{n,j}(x(u,\lambda,\varphi)) := \frac{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)}{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)} Y_{n,j}(\xi(\lambda,\varphi)) \tag{17}$$

and

$$V_{n,j}(y(u,\lambda,\varphi)) := \frac{\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)}{\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)} Y_{n,j}(\xi(\lambda,\varphi)). \tag{18}$$

For $n, m \in \mathbb{N}_0$, $j \in \{-n, ..., n\}$, $k \in \{-m, ..., m\}$, we have the following observations

$$\int_0^{2\pi} \cos j\lambda \cos k\lambda \, d\lambda = \delta_{jk}(1+\delta_{j0})\pi, \ j,k \in \mathbb{Z}, \tag{19}$$

$$\int_0^{2\pi} \sin j\lambda \sin k\lambda \, d\lambda = \delta_{jk}\pi, \ j,k \in \mathbb{Z}\setminus\{0\}, \tag{20}$$

$$\int_0^{2\pi} \sin j\lambda \cos k\lambda \, d\lambda = 0, \ j,k \in \mathbb{Z}. \tag{21}$$

By using Equations (19) $-$ (21), we will prove that the inner harmonics are not even completely orthogonal in the inner product sense as is shown below in Equations (25), (34) and (36). The inner product of inner harmonics is given by

$$\begin{aligned}
&(H_{n,j}, H_{m,k})_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} \\
&= \frac{1}{4\pi}\int_0^b \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)}{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)} \frac{\tilde{\mathcal{P}}_{m,|k|}\left(\frac{u}{\varepsilon}\right)}{\tilde{\mathcal{P}}_{m,|k|}\left(\frac{b}{\varepsilon}\right)} Y_{n,j}(\xi(\lambda,\varphi))\, Y_{m,k}(\xi(\lambda,\varphi)) \\
&\quad \times \left(u^2 + \varepsilon^2 \sin^2\varphi\right)\cos\varphi d\varphi d\lambda du \\
&= \frac{1}{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{P}}_{m,|k|}\left(\frac{b}{\varepsilon}\right)} \Bigg(\int_0^b u^2 \tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)\tilde{\mathcal{P}}_{m,|k|}\left(\frac{u}{\varepsilon}\right) du \\
&\quad \times \frac{1}{4\pi}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Y_{n,j}(\xi(\lambda,\varphi))\, Y_{m,k}(\xi(\lambda,\varphi))\cos\varphi d\varphi d\lambda \\
&\quad + \int_0^b \tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)\tilde{\mathcal{P}}_{m,|k|}\left(\frac{u}{\varepsilon}\right) du \\
&\quad \times \frac{\varepsilon^2}{4\pi}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Y_{n,j}(\xi(\lambda,\varphi))\, Y_{m,k}(\xi(\lambda,\varphi))\sin^2\varphi\cos\varphi d\varphi d\lambda \Bigg).
\end{aligned} \tag{22}$$

Since from Equation (15), we obtain

$$\frac{1}{4\pi}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Y_{n,j}\left(\xi\left(\lambda,\varphi\right)\right)Y_{m,k}\left(\xi\left(\lambda,\varphi\right)\right)\cos\varphi d\varphi d\lambda=\delta_{nm}\delta_{jk} \tag{23}$$

and following Equation (21), we get

$$\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Y_{n,j}\left(\xi\left(\lambda,\varphi\right)\right)Y_{m,k}\left(\xi\left(\lambda,\varphi\right)\right)\sin^2\varphi\cos\varphi d\varphi d\lambda=0 \text{ for all } j\neq k, \tag{24}$$

where $n,m\in\mathbb{N}_0$, $j\in\{-n,...,n\}$, $k\in\{-m,...,m\}$, therefore

$$\left(H_{n,j},H_{m,k}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}=0 \text{ for all } j\neq k. \tag{25}$$

Next we want to calculate the value of the inner products $\left(H_{n,j},H_{m,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}$ if $n\geq m$ and $|j|\leq m$. Thus, first considering $j=k$, $n>m$, $|j|\leq m$ and using Equation (23) in Equation (22), we get

$$\begin{aligned}\left(H_{n,j},H_{m,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} &= \frac{1}{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{P}}_{m,|j|}\left(\frac{b}{\varepsilon}\right)}\left(\int_0^b\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)\tilde{\mathcal{P}}_{m,|j|}\left(\frac{u}{\varepsilon}\right)du\right. \\ &\quad \left.\times\frac{\varepsilon^2}{4\pi}\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}Y_{n,j}\left(\xi\left(\lambda,\varphi\right)\right)Y_{m,j}\left(\xi\left(\lambda,\varphi\right)\right)\sin^2\varphi\cos\varphi d\varphi d\lambda\right).\end{aligned} \tag{26}$$

Let

$$A\left(n,j\right):=\sqrt{\frac{\left(2-\delta_{j0}\right)\left(2n+1\right)\left(n-|j|\right)!}{\left(n+|j|\right)!}} \text{ for all } n\in\mathbb{N}_0,\ j\in\{-n,...,n\} \tag{27}$$

and $\alpha_j:=1+\delta_{j0}$ which gives that

$$\left(2-\delta_{j0}\right)\alpha_j=2 \text{ for all } j\in\mathbb{Z}. \tag{28}$$

Let $t:=\sin\varphi$, $\varphi\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$, where

$$t^2=\frac{2}{3}\left(\frac{1}{2}\left(3t^2-1\right)+\frac{1}{2}\right)=\frac{2}{3}\left(P_{2,0}\left(t\right)+\frac{1}{2}\right)=\frac{2}{3}P_{2,0}\left(t\right)+\frac{1}{3}. \tag{29}$$

For $j=k$, $n\geq m$, $|j|\leq m$, we get, using Equations (19) – (21) in Equation (24),

$$\begin{aligned}&\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}Y_{n,j}\left(\xi\left(\lambda,\varphi\right)\right)Y_{m,j}\left(\xi\left(\lambda,\varphi\right)\right)\sin^2\varphi\cos\varphi d\varphi d\lambda \\ &= \alpha_j\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}P^*_{n,j}\left(\sin\varphi\right)P^*_{m,j}\left(\sin\varphi\right)\sin^2\varphi\cos\varphi d\varphi \\ &= A\left(n,j\right)A\left(m,j\right)\alpha_j\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}P_{n,|j|}\left(\sin\varphi\right)P_{m,|j|}\left(\sin\varphi\right)\sin^2\varphi\cos\varphi d\varphi \\ &\overset{t=\sin\varphi}{=} A\left(n,j\right)A\left(m,j\right)\alpha_j\pi\int_{-1}^{1}P_{n,|j|}\left(t\right)P_{m,|j|}\left(t\right)t^2dt \\ &\overset{(29)}{=} A\left(n,j\right)A\left(m,j\right)\alpha_j\pi\int_{-1}^{1}P_{n,|j|}\left(t\right)P_{m,|j|}\left(t\right)\left(\frac{2}{3}P_{2,0}\left(t\right)+\frac{1}{3}\right)dt \\ &= \frac{2}{3}A\left(n,j\right)A\left(m,j\right)\alpha_j\pi\int_{-1}^{1}P_{n,|j|}\left(t\right)P_{m,|j|}\left(t\right)P_{2,0}\left(t\right)dt \\ &\quad +\frac{1}{3}A\left(n,j\right)A\left(m,j\right)\alpha_j\pi\int_{-1}^{1}P_{n,|j|}\left(t\right)P_{m,|j|}\left(t\right)dt.\end{aligned} \tag{30}$$

For $n \geq m$, $|j| \leq m$, we get, following Equations (4) $-$ (5) and (28),

$$A(n,j)\,A(m,j)\,\alpha_j \int_{-1}^{1} P_{n,|j|}(t)\,P_{m,|j|}(t)\,dt = 4\delta_{n,m}. \tag{31}$$

For $n > m$, $|j| \leq m$ and following Equation (31), Equation (30) reduces to

$$\begin{aligned}
&\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} Y_{n,j}(\xi(\lambda,\varphi))\,Y_{m,j}(\xi(\lambda,\varphi))\sin^2\varphi\cos\varphi\, d\varphi\, d\lambda \\
&= \frac{2}{3} A(n,j)\,A(m,j)\,\alpha_j \pi \int_{-1}^{1} P_{n,|j|}(t)\,P_{m,|j|}(t)\,P_{2,0}(t)\,dt.
\end{aligned} \tag{32}$$

Following [1], we know that

$$\int_{-1}^{1} P_{n,|j|}(t)\,P_{m,|j|}(t)\,P_{2,0}(t)\,dt = \begin{cases} \frac{12(m+2)!(2m)!(m+2+|j|)!}{(m)!(5+2m)!(m-|j|)!}, & \text{if} \quad n-m=2 \\ \frac{-8(m+|j|)!(m+1)!}{(2m+3)(2m+2)(2m+1)(2m)} \\ \times \frac{\left(3|j|^2-m(m+1)\right)}{(m-|j|)!(m-1)!(2m-1)}, & \text{if} \quad n=m\neq 0 \\ 0, & \text{else} \end{cases} \tag{33}$$

Taking into account Equations (32) and (33), Equation (26) has the value

$$(H_{n,j}, H_{m,j})_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} = 0, \text{ for } n > m,\ n-m \neq 2,\ |j| \leq m. \tag{34}$$

Now, for $n > m$ with $n - m = 2$ and $|j| \leq m$, following Equations (32) and (33), Equation (26) reduces to

$$\begin{aligned}
(H_{m+2,j}, H_{m,j})_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} &= 2\alpha_j\varepsilon^2 A(m,j)\,A(m+2,j)\,\frac{(m+2)!\,(2m)!\,(m+2+|j|)!}{m!\,(5+2m)!\,(m-|j|)!} \\
&\times \frac{1}{\tilde{\mathcal{P}}_{m,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{P}}_{m+2,|j|}\left(\frac{b}{\varepsilon}\right)} \int_0^b \tilde{\mathcal{P}}_{m,|j|}\left(\frac{u}{\varepsilon}\right)\tilde{\mathcal{P}}_{m+2,|j|}\left(\frac{u}{\varepsilon}\right) du,
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
&2\alpha_j\varepsilon^2 A(m,j)\,A(m+2,j)\,\frac{(m+2)!\,(2m)!\,(m+2+|j|)!}{m!\,(5+2m)!\,(m-|j|)!} \\
&\overset{(27)}{=} \quad 2\alpha_j\varepsilon^2\sqrt{\frac{(2-\delta_{j0})(2m+1)(m-|j|)!}{(m+|j|)!}} \\
&\qquad \times \sqrt{\frac{(2-\delta_{j0})(2m+5)(m+2-|j|)!}{(m+2+|j|)!}}\,\frac{(m+2)!\,(2m)!\,(m+2+|j|)!}{m!\,(5+2m)!\,(m-|j|)!} \\
&\overset{(28)}{=} \quad 4\varepsilon^2\frac{(m+2)!\,(2m)!}{m!\,(5+2m)!}\sqrt{\frac{(2m+1)(2m+5)(m+2+|j|)!\,(m+2-|j|)!}{(m+|j|)!\,(m-|j|)!}} \\
&= \quad \varepsilon^2\sqrt{\frac{\left((m+2)^2-|j|^2\right)\left((m+1)^2-|j|^2\right)}{(2m+5)(2m+3)^2(2m+1)}}.
\end{aligned}$$

Thus, Equation (35) with $|j| \leq m$, $n > m$, $n - m = 2$, has the value

$$\begin{aligned}
(H_{m+2,j}, H_{m,j})_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} &= \varepsilon^2\sqrt{\frac{\left((m+2)^2-|j|^2\right)\left((m+1)^2-|j|^2\right)}{(2m+5)(2m+3)^2(2m+1)}} \\
&\times \frac{1}{\tilde{\mathcal{P}}_{m,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{P}}_{m+2,|j|}\left(\frac{b}{\varepsilon}\right)} \int_0^b \tilde{\mathcal{P}}_{m,|j|}\left(\frac{u}{\varepsilon}\right)\tilde{\mathcal{P}}_{m+2,|j|}\left(\frac{u}{\varepsilon}\right) du.
\end{aligned} \tag{36}$$

Hence, not all pairs of functions in the system $\{H_{m,j}\}_{m\in\mathbb{N}_0,j\in\{-m,...,m\}}$ are orthogonal in $\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)$. For this reason, we construct a new system $\{H'_{m,j}\}_{m\in\mathbb{N}_0,j\in\{-m,...,m\}}$ by setting

$$H'_{m,j} := H_{m,j} \text{ if } |j| = m, m-1 \tag{37}$$

and

$$H'_{m,j} := H'_{m-2,j} - \frac{\left\|H'_{m-2,j}\right\|^2_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}{\left(H'_{m-2,j}, H_{m,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}} H_{m,j} \tag{38}$$

for $m \geq 2$, $|j| \leq m-2$.

Theorem 3. *The system $\{H'_{n,j}\}_{n\in\mathbb{N}_0,\, j\in\{-n,...,n\}}$ forms an orthogonal system in the inner product space $\left(\mathrm{Harm}\left(\overline{E_{\mathrm{int}}}\right), (\cdot,\cdot)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}\right)$.*

Proof. The proof requires lengthy but elementary calculations and the distinction of several cases. For further details, see [1]. □

For $n \in \mathbb{N}_0$, $j \in \{-n, ..., n\}$, we set

$$U_{n,j} := \frac{H'_{n,j}}{\left\|H'_{n,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}. \tag{39}$$

Corollary 4. *The system $\{U_{n,j}\}_{n\in\mathbb{N}_0,\, j\in\{-n,...,n\}}$, defined in (39), is orthonormal in the inner product space $\left(\mathrm{Harm}\left(\overline{E_{\mathrm{int}}}\right), (\cdot,\cdot)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}\right)$.*

Proof. This corollary is an immediate consequence of Theorem 3. □

4 Calculation of the Kernel Function

We investigate now the integral kernel of the Fredholm integral equation in (1). For $n \in \mathbb{N}_0$, $j \in \{-n, ..., n\}$, let us first set

$$c_{n,j} := \begin{cases} -\frac{\left(H'_{n-2,j}, H_{n,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} \left\|H'_{n,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}{\left\|H'_{n-2,j}\right\|^2_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}, & \text{if } n \geq 2,\ |j| \leq n-2 \\ \left\|H'_{n,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}, & \text{if } |j| = n-1, n \\ 0, & \text{if } |j| > n \end{cases} \tag{40}$$

and

$$d_{n,j} := \begin{cases} \frac{\left(H'_{n-2,j}, H_{n,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}{\left\|H'_{n-2,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}, & \text{if } n \geq 2,\ |j| \leq n-2 \\ 0, & \text{else.} \end{cases} \tag{41}$$

Following Equation (38), we know that for $n \geq 2$, $|j| \leq n-2$,

$$H'_{n,j}\left(x\left(u,\lambda,\varphi\right)\right) = H'_{n-2,j}\left(x\left(u,\lambda,\varphi\right)\right) - \frac{\left\|H'_{n-2,j}\right\|^2_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}{\left(H'_{n-2,j}, H_{n,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}} H_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)$$

which can be rewritten as

$$\begin{aligned}
&H_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)\\
&= \frac{\left(H'_{n-2,j},H_{n,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}{\left\|H'_{n-2,j}\right\|^2_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}\left(-H'_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)+H'_{n-2,j}\left(x\left(u,\lambda,\varphi\right)\right)\right)\\
&\overset{(39)}{=} \frac{\left(H'_{n-2,j},H_{n,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}{\left\|H'_{n-2,j}\right\|^2_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}\\
&\quad\times\left(-\left\|H'_{n,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}U_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)+\left\|H'_{n-2,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}U_{n-2,j}\left(x\left(u,\lambda,\varphi\right)\right)\right)\\
&\overset{(40),(41)}{=} c_{n,j}U_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)+d_{n,j}U_{n-2,j}\left(x\left(u,\lambda,\varphi\right)\right). \qquad (42)
\end{aligned}$$

For $n \geq 0, |j| = n-1, n$ and using Equations $(37), (39), (40)$ and (41), we obtain

$$\begin{aligned}
H_{n,j}\left(x\left(u,\lambda,\varphi\right)\right) &= H'_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)\\
&= \left\|H'_{n,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}U_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)\\
&= c_{n,j}U_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)+d_{n,j}U_{n-2,j}\left(x\left(u,\lambda,\varphi\right)\right) \qquad (43)
\end{aligned}$$

with the following assumption: $U_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)=0$, if $|j|>n$.
Thus, following Equations (42) and (43), we obtain

$$H_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)=c_{n,j}U_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)+d_{n,j}U_{n-2,j}\left(x\left(u,\lambda,\varphi\right)\right) \qquad (44)$$

for all $n \in \mathbb{N}_0$, $j \in \{-n,...,n\}$. Next, let $x := x\left(u,\lambda,\varphi\right), y := y\left(u',\lambda',\varphi'\right)$ be arbitrary but fixed inner and outer points of the spheroid E, respectively. Let $L := |x-y|$. Then, for example, from [30], we know that

$$\begin{aligned}
\frac{\varepsilon}{L} &= i\sum_{n=0}^{\infty}(2n+1)\left[\mathcal{Q}_{n,0}\left(i\frac{u'}{\varepsilon}\right)\mathcal{P}_{n,0}\left(i\frac{u}{\varepsilon}\right)P_{n,0}\left(\sin\varphi'\right)P_{n,0}\left(\sin\varphi\right)\right.\\
&\quad+2\sum_{j=1}^{n}\left(-1\right)^j\left(\frac{(n-j)!}{(n+j)!}\right)^2\mathcal{Q}_{n,j}\left(i\frac{u'}{\varepsilon}\right)\mathcal{P}_{n,j}\left(i\frac{u}{\varepsilon}\right)\\
&\quad\left.\times P_{n,j}\left(\sin\varphi'\right)P_{n,j}\left(\sin\varphi\right)\cos\left[j\left(\lambda-\lambda'\right)\right]\right], \qquad (45)
\end{aligned}$$

where $\mathcal{P}_{n,j}$ and $\mathcal{Q}_{n,j}$ are, respectively, the associated Legendre functions of the first kind and of the second kind in terms of $\frac{u}{\varepsilon}$ as given in Equations (6) and (7) and $P_{n,j}$ are the associated Legendre functions of the first kind on $[-1,1]$ as given in Equation (3).
From Equation (12), we know that

$$\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)=i^{-n}\mathcal{P}_{n,|j|}\left(i\frac{u}{\varepsilon}\right), \tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u'}{\varepsilon}\right)=i^{n+1}\mathcal{Q}_{n,|j|}\left(i\frac{u'}{\varepsilon}\right),$$

therefore, with $i^{-n} \cdot i^{n+1} = i$, we consider the second part of Equation (45)

$$\begin{aligned}
& 2\sum_{j=1}^{n}(-1)^j\left(\frac{(n-j)!}{(n+j)!}\right)^2 \mathcal{Q}_{n,j}\left(i\frac{u'}{\varepsilon}\right)\mathcal{P}_{n,j}\left(i\frac{u}{\varepsilon}\right)P_{n,j}\left(\sin\varphi'\right)P_{n,j}\left(\sin\varphi\right) \\
& \times\cos\left[j\left(\lambda-\lambda'\right)\right] \\
= \; & \frac{2}{i}\sum_{j=1}^{n}(-1)^j\left(\frac{(n-j)!}{(n+j)!}\right)^2 \tilde{\mathcal{Q}}_{n,j}\left(\frac{u'}{\varepsilon}\right)\tilde{\mathcal{P}}_{n,j}\left(\frac{u}{\varepsilon}\right)P_{n,j}\left(\sin\varphi'\right)P_{n,j}\left(\sin\varphi\right) \\
& \times\cos\left[j\left(\lambda-\lambda'\right)\right] \\
= \; & \frac{2}{i}\sum_{j=1}^{n}(-1)^j\left(\frac{(n-j)!}{(n+j)!}\right)^2 \tilde{\mathcal{Q}}_{n,j}\left(\frac{u'}{\varepsilon}\right)\tilde{\mathcal{P}}_{n,j}\left(\frac{u}{\varepsilon}\right)P_{n,j}\left(\sin\varphi'\right)P_{n,j}\left(\sin\varphi\right) \\
& \times\cos j\lambda\cos j\lambda' \\
& +\frac{2}{i}\sum_{j=-n}^{-1}(-1)^{|j|}\left(\frac{(n-|j|)!}{(n+|j|)!}\right)^2 \tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u'}{\varepsilon}\right)\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)P_{n,|j|}\left(\sin\varphi'\right) \\
& \times P_{n,|j|}\left(\sin\varphi\right)\sin|j|\lambda\sin|j|\lambda'. \qquad (46)
\end{aligned}$$

Hence, using Equation (46), we can rewrite Equation (45) as follows:

$$\begin{aligned}
\frac{\varepsilon}{L} = & \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(2n+1)\left(2-\delta_{j0}\right)(-1)^{|j|}\left(\frac{(n-|j|)!}{(n+|j|)!}\right)^2 \tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u'}{\varepsilon}\right)\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right) \\
& \times P_{n,|j|}\left(\sin\varphi'\right)P_{n,|j|}\left(\sin\varphi\right)\begin{cases}\cos j\lambda\cos j\lambda', & \text{if } 0\le j\le n \\ \sin|j|\lambda\sin|j|\lambda', & \text{if } -n\le j\le -1.\end{cases} \qquad (47)
\end{aligned}$$

With Equations (16), (17) and (27), we know that

$$\begin{aligned}
H_{n,j}\left(x\left(u,\lambda,\varphi\right)\right) = & \frac{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)}{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)}P^*_{n,j}\left(\sin\varphi\right)\begin{cases}\cos j\lambda, & \text{if } 0\le j\le n \\ \sin|j|\lambda, & \text{if } -n\le j\le -1\end{cases} \\
= & \frac{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)}{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)}A\left(n,j\right)P_{n,|j|}\left(\sin\varphi\right)\begin{cases}\cos j\lambda, & \text{if } 0\le j\le n \\ \sin|j|\lambda, & \text{if } -n\le j\le -1\end{cases}
\end{aligned}$$

which gives that

$$\frac{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)}{A\left(n,j\right)}H_{n,j}\left(x\left(u,\lambda,\varphi\right)\right)=\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)P_{n,|j|}\left(\sin\varphi\right)\begin{cases}\cos j\lambda, & \text{if } 0\le j\le n \\ \sin|j|\lambda, & \text{if } -n\le j\le -1.\end{cases} \qquad (48)$$

Also, from Equations (16) and (18), we know that

$$\begin{aligned}
V_{n,j}\left(y\left(u',\lambda',\varphi'\right)\right) = & \frac{\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u'}{\varepsilon}\right)}{\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)}P^*_{n,j}\left(\sin\varphi'\right)\begin{cases}\cos j\lambda', & \text{if } 0\le j\le n \\ \sin|j|\lambda', & \text{if } -n\le j\le -1\end{cases} \\
= & \frac{\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u'}{\varepsilon}\right)}{\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)}A\left(n,j\right)P_{n,|j|}\left(\sin\varphi'\right)\begin{cases}\cos j\lambda', & \text{if } 0\le j\le n \\ \sin|j|\lambda', & \text{if } -n\le j\le -1\end{cases}
\end{aligned}$$

which gives that

$$\frac{\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)}{A\left(n,j\right)}V_{n,j}\left(y\left(u',\lambda',\varphi'\right)\right)=\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u'}{\varepsilon}\right)P_{n,|j|}\left(\sin\varphi'\right)\begin{cases}\cos j\lambda', & \text{if } 0\le j\le n \\ \sin|j|\lambda', & \text{if } -n\le j\le -1.\end{cases} \qquad (49)$$

Using the definition of $A(n,j)$ and substituting Equations (48) and (49) in Equation (47), we obtain the following form

$$\begin{aligned}
\frac{\varepsilon}{L} &= \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{|j|}\left(A(n,j)\right)^2\frac{(n-|j|)!}{(n+|j|)!}\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u'}{\varepsilon}\right)\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right)\\
&\quad\times P_{n,|j|}\left(\sin\varphi'\right)P_{n,|j|}\left(\sin\varphi\right)\begin{cases}\cos j\lambda\cos j\lambda', & \text{if } 0\le j\le n\\ \sin|j|\lambda\sin|j|\lambda', & \text{if } -n\le j\le -1\end{cases}\\
&= \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{|j|}\left(A(n,j)\right)^2\frac{(n-|j|)!}{(n+|j|)!}\\
&\quad\times\frac{\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)}{A(n,j)}\frac{\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)}{A(n,j)}H_{n,j}\left(x(u,\lambda,\varphi)\right)V_{n,j}\left(y(u',\lambda',\varphi')\right)\\
&= \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{|j|}\frac{(n-|j|)!}{(n+|j|)!}\\
&\quad\times\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)H_{n,j}\left(x(u,\lambda,\varphi)\right)V_{n,j}\left(y(u',\lambda',\varphi')\right)\\
&\overset{(44)}{=} \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{|j|}\frac{(n-|j|)!}{(n+|j|)!}\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\\
&\quad\times\left(c_{n,j}U_{n,j}\left(x(u,\lambda,\varphi)\right)+d_{n,j}U_{n-2,j}\left(x(u,\lambda,\varphi)\right)\right)V_{n,j}\left(y(u',\lambda',\varphi')\right) \qquad (50)\\
&= \sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(K_{n,j}U_{n,j}\left(x(u,\lambda,\varphi)\right)+L_{n,j}U_{n-2,j}\left(x(u,\lambda,\varphi)\right)\right)V_{n,j}\left(y(u',\lambda',\varphi')\right)
\end{aligned}$$

where we use the abbreviations

$$K_{n,j} := (-1)^{|j|}c_{n,j}\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\frac{(n-|j|)!}{(n+|j|)!}$$

and

$$L_{n,j} := (-1)^{|j|}d_{n,j}\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\frac{(n-|j|)!}{(n+|j|)!}.$$

Following Equations (40) and (41), we note that for $|j|\le n-2$, $n\ge 2$,

$$\begin{aligned}
K_{n,j} &= (-1)^{|j|+1}\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\\
&\quad\times\frac{\left(H'_{n-2,j},H_{n,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}\left\|H'_{n,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}{\left\|H'_{n-2,j}\right\|^2_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}\frac{(n-|j|)!}{(n+|j|)!},\\
L_{n,j} &= (-1)^{|j|}\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\frac{\left(H'_{n-2,j},H_{n,j}\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}{\left\|H'_{n-2,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}}\frac{(n-|j|)!}{(n+|j|)!}
\end{aligned}$$

and for $|j|=n-1,n$,

$$K_{n,j} = (-1)^{|j|+1}\tilde{\mathcal{P}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{b}{\varepsilon}\right)\left\|H'_{n,j}\right\|_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}\frac{(n-|j|)!}{(n+|j|)!},\quad L_{n,j}=0.$$

Now, for $F\in\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)$, choose $y\,(=y(u',\lambda',\varphi'))\in E$ or $y\in E_{\mathrm{ext}}$ fixed and following Equation (50), we can write that

$$\frac{\varepsilon}{|\cdot-y|}=\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(K_{n,j}U_{n,j}(\cdot)+L_{n,j}U_{n-2,j}(\cdot)\right)V_{n,j}(y).$$

Now, the operator $T : \mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right) \longrightarrow T\left(\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)\right)$ given by

$$(TF)(y) := \int_{\overline{E_{\mathrm{int}}}} \frac{F(x)}{|x-y|}dx, \qquad y \in \overline{E_{\mathrm{ext}}}$$

takes the following form

$$\begin{aligned}
(TF)(y) = \left(\frac{1}{|\cdot - y|}, F\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} &= \frac{1}{\varepsilon}\left(\frac{\varepsilon}{|\cdot - y|}, F\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} \\
= \ \frac{1}{\varepsilon}&\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(K_{n,j}U_{n,j}(\cdot) + L_{n,j}U_{n-2,j}(\cdot)\right)V_{n,j}(y), F\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} \\
= \ \frac{1}{\varepsilon}&\int_{\overline{E_{\mathrm{int}}}}\left(\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(K_{n,j}U_{n,j}(x) + L_{n,j}U_{n-2,j}(x)\right)V_{n,j}(y)\right)F(x)\,dx. \qquad (51)
\end{aligned}$$

Since following, for example [17], [19] and [28], we know that for $x = x(u, \lambda, \varphi) \in E_{\mathrm{int}}$ with fixed $y = y(u', \lambda', \varphi') \in E_{\mathrm{ext}}$, an expansion of the form

$$\begin{aligned}
\frac{\varepsilon}{|x-y|} \ &= \ \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^{|j|}\left(A(n,j)\right)^2\frac{(n-|j|)!}{(n+|j|)!}\tilde{\mathcal{Q}}_{n,|j|}\left(\frac{u'}{\varepsilon}\right)\tilde{\mathcal{P}}_{n,|j|}\left(\frac{u}{\varepsilon}\right) \\
&\times P_{n,|j|}(\sin\varphi')\,P_{n,|j|}(\sin\varphi)\begin{cases}\cos j\lambda\cos j\lambda', & \text{if } 0 \le j \le n \\ \sin|j|\lambda\sin|j|\lambda', & \text{if } -n \le j \le -1\end{cases}
\end{aligned}$$

is uniformly convergent, following (50), we conclude that

$$\frac{1}{|x-y|} = \sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(K_{n,j}U_{n,j}(x) + L_{n,j}U_{n-2,j}(x)\right)V_{n,j}(y)$$

is uniformly convergent for $x \in E_{\mathrm{int}}$ with fixed $y \in E_{\mathrm{ext}}$. For this reason, the summation and integration in Equation (51) can be interchanged. We obtain

$$\begin{aligned}
(TF)(y) \ &= \ \frac{1}{\varepsilon}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\int_{\overline{E_{\mathrm{int}}}}\left(K_{n,j}U_{n,j}(x) + L_{n,j}U_{n-2,j}(x)\right)V_{n,j}(y)F(x)\,dx \\
&= \ \frac{1}{\varepsilon}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(K_{n,j}U_{n,j}(\cdot) + L_{n,j}U_{n-2,j}(\cdot), F\right)_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)}V_{n,j}(y).
\end{aligned}$$

If $F = U_{m,j}$ then

$$(TU_{m,j})(y) = K_{m,j}V_{m,j}(y) + L_{m+2,j}V_{m+2,j}(y). \qquad (52)$$

5 Spheroidal Harmonic Spline Approximation

This section deals with the construction of a regularization technique and a multiresolution analysis based on spheroidal harmonic splines. The appropriate space used for this purpose is a reproducing kernel Sobolev space which determines the (unique) interpolant with the smallest possible norm. The harmonic spline based multiresolution method satisfies the usual smoothing and best approximating properties. The theory and the proofs of the results presented here are motivated by, for example, [2], [3], [7], [8], [9], [12], [13], [14], [15], [24], where, however, a sphere or a ball is the treated domain.

Definition 5. *Let* $\{\mathcal{A}_n\}_{n\in\mathbb{N}_0}$ *be an arbitrary sequence of real numbers. We denote the space of all functions* $F \in \operatorname{Harm}\left(\overline{E_{\text{int}}}\right)$ *which satisfy the following conditions:*

1. $(F, U_{n,j})_{\mathrm{L}^2\left(\overline{E_{\text{int}}}\right)} = 0$ *for all* $n \in \mathbb{N}_0$ *with* $\mathcal{A}_n = 0$,

2. $\sum\limits_{\substack{n=0\\ \mathcal{A}_n\neq 0}}^{\infty} \mathcal{A}_n^{-2} (F, U_{n,j})^2_{\mathrm{L}^2\left(\overline{E_{\text{int}}}\right)} < +\infty$,

by $\chi := \chi\left(\{\mathcal{A}_n\}; \overline{E_{\text{int}}}\right)$.

For $F, G \in \chi$ we introduce the inner product

$$(F,G)_{\mathcal{H}\left(\{\mathcal{A}_n\};\overline{E_{\text{int}}}\right)} := \sum_{\substack{n=0\\ \mathcal{A}_n\neq 0}}^{\infty} \mathcal{A}_n^{-2} (F, U_{n,j})_{\mathrm{L}^2\left(\overline{E_{\text{int}}}\right)} (G, U_{n,j})_{\mathrm{L}^2\left(\overline{E_{\text{int}}}\right)}. \tag{53}$$

The associated norm is then given by $\|F\|_{\mathcal{H}\left(\{\mathcal{A}_n\};\overline{E_{\text{int}}}\right)} = \sqrt{(F,F)_{\mathcal{H}(\{\mathcal{A}_n\};\overline{E_{\text{int}}})}}$.
From these facts, we come to the following definition.

Definition 6. *The Sobolev space* $\mathcal{H} := \mathcal{H}\left(\{\mathcal{A}_n\}; \overline{E_{\text{int}}}\right)$ *is defined to be the completion of the space* $\chi = \chi\left(\{\mathcal{A}_n\}; \overline{E_{\text{int}}}\right)$ *with respect to the inner product* $(\cdot,\cdot)_{\mathcal{H}\left(\{\mathcal{A}_n\};\overline{E_{\text{int}}}\right)}$.

Clearly, $\mathcal{H}\left(\{\mathcal{A}_n\}; \overline{E_{\text{int}}}\right)$ equipped with the inner product (53) is a Hilbert space. From now on, if no confusion is likely to arise, we will simply write $\mathcal{H}$ instead of writing $\mathcal{H}\left(\{\mathcal{A}_n\}; \overline{E_{\text{int}}}\right)$.
In Lemma 8, we will prove that under certain conditions, the elements of the Sobolev space $\mathcal{H}$ can be interpreted as uniformly convergent Fourier series in terms of inner spheroidal harmonics $\{U_{n,j}\}_{n\in\mathbb{N}_0,\, j\in\{-n,\ldots,n\}}$. For this, we need to define the summable sequences.

Definition 7. *A sequence of real numbers* $\{\mathcal{A}_n\}_{n\in\mathbb{N}_0}$ *is called summable if*

$$\sum_{n=0}^{\infty}\sum_{j=-n}^{n} \mathcal{A}_n^2 \|U_{n,j}\|_\infty^2 < \infty \tag{54}$$

i.e. the series is convergent. The norm $\|\cdot\|_\infty$ *stands for the supremum norm.*

In the sequel, we will always assume that the sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}_0}$ is summable.
Using Definition 6, we formulate in the following the Sobolev Lemma.

Lemma 8 (Sobolev Lemma)**.** *Every* $F \in \mathcal{H}$ *corresponds to a continuous function on* $\overline{E_{\text{int}}}$. *Moreover,* F *is also harmonic on* $\overline{E_{\text{int}}}$ *and the Fourier series*

$$F(x) = \sum_{n=0}^{\infty}\sum_{j=-n}^{n} (F, U_{n,j})_{\mathrm{L}^2\left(\overline{E_{\text{int}}}\right)} U_{n,j}(x), \quad x \in \overline{E_{\text{int}}}$$

is uniformly convergent on $\overline{E_{\text{int}}}$.

Proof. For $F \in \mathcal{H}$, $N \in \mathbb{N}_0$, the application of the Cauchy-Schwarz inequality yields the estimate

$$\begin{aligned}
\left| \sum_{n=N}^{\infty} \sum_{j=-n}^{n} (F, U_{n,j})_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} U_{n,j}(x) \right| &= \left| \sum_{\substack{n=N \\ \mathcal{A}_n \neq 0}}^{\infty} \sum_{j=-n}^{n} (F, U_{n,j})_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} \mathcal{A}_n^{-1} \mathcal{A}_n U_{n,j}(x) \right| \\
&\leq \left(\sum_{\substack{n=N \\ \mathcal{A}_n \neq 0}}^{\infty} \sum_{j=-n}^{n} \mathcal{A}_n^{-2} (F, U_{n,j})^2_{\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)} \right)^{\frac{1}{2}} \left(\sum_{n=N}^{\infty} \sum_{j=-n}^{n} \mathcal{A}_n^2 \left(U_{n,j}(x)\right)^2 \right)^{\frac{1}{2}} \\
&\leq \|F\|_{\mathcal{H}\left(\{\mathcal{A}_n\};\overline{E_{\mathrm{int}}}\right)} \left(\sum_{n=N}^{\infty} \sum_{j=-n}^{n} \mathcal{A}_n^2 \|U_{n,j}\|_\infty^2 \right)^{\frac{1}{2}} \underset{N\to\infty}{\longrightarrow} 0, \qquad (55)
\end{aligned}$$

where the right hand side converges uniformly with respect to $x \in \overline{E_{\mathrm{int}}}$ as N approaches to infinity and this is because of the summability condition given in Equation (54). Since the functions $U_{n,j}$ are continuous on $\overline{E_{\mathrm{int}}}$, the uniform convergence of the series (55) implies that F itself is continuous on $\overline{E_{\mathrm{int}}}$. Moreover, since Harm $\left(\overline{E_{\mathrm{int}}}\right)$ is a closed subspace of $\mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)$ (see also [15, Theorem 6.4]), F is also harmonic. ☐

Corollary 9. *From the proof of Lemma* 8*, we can immediately deduce that*

$$\|F\|_\infty \leq \|F\|_{\mathcal{H}} \left(\sum_{n=0}^{\infty} \sum_{j=-n}^{n} \mathcal{A}_n^2 \|U_{n,j}\|_\infty^2 \right)^{\frac{1}{2}}$$

for all $F \in \mathcal{H}$.

The summability of the real sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}_0}$ guarantees the existence of the reproducing kernel which is essential for the construction of the harmonic splines discussed in this section. For a general theory on reproducing kernels, see, for example, [11].

Definition 10. *A function* $K_{\mathcal{H}} : \overline{E_{\mathrm{int}}} \times \overline{E_{\mathrm{int}}} \longrightarrow \mathbb{R}$ *is called a reproducing kernel of the space* $\mathcal{H}$ *if the following two conditions are satisfied*

1. $K_{\mathcal{H}}(x, \cdot) \in \mathcal{H}$ *for all* $x \in \overline{E_{\mathrm{int}}}$,
2. $\left(F(\cdot), K_{\mathcal{H}}(x, \cdot)\right)_{\mathcal{H}} = F(x)$ *for all* $F \in \mathcal{H}$ *and for all* $x \in \overline{E_{\mathrm{int}}}$.

Following functional analysis, we know that, if a Hilbert space has a reproducing kernel, it is unique. Moreover, due to Aronszajn's Theorem and Corollary 9, a reproducing kernel of $\mathcal{H}$ exists due to the summability condition. Furthermore, since $\{\mathcal{A}_n U_{n,j}\}_{n,j}$ is an orthonormal basis of $\mathcal{H}$ and $\sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left(\mathcal{A}_n U_{n,j}(x)\right)^2 < +\infty$ for all $x \in \overline{E_{\mathrm{int}}}$ due to the summability condition, Theorem 19 in [22] implies the following representation of the reproducing kernel.

Theorem 11. $\mathcal{H}$ *has a unique reproducing kernel* $K_{\mathcal{H}} : \overline{E_{\mathrm{int}}} \times \overline{E_{\mathrm{int}}} \longrightarrow \mathbb{R}$ *and, furthermore, its representation is given by*

$$K_{\mathcal{H}}(x, y) = \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \mathcal{A}_n^2 U_{n,j}(x) U_{n,j}(y), \quad x, y \in \overline{E_{\mathrm{int}}}. \qquad (56)$$

Corollary 12. *One can easily deduce from Equation* (56) *that* $K_{\mathcal{H}}$ *is symmetric, i.e.* $K_{\mathcal{H}}(x, y) = K_{\mathcal{H}}(y, x)$ *for all* $x, y \in \overline{E_{\mathrm{int}}}$.

Theorem 13. *Let $\mathcal{F}$ be an arbitrary bounded linear functional on $\mathcal{H}$. Then the function $y \mapsto \mathcal{F}_x K_{\mathcal{H}}(x,y)$ belongs to $\mathcal{H}$ and*

$$\mathcal{F}(F) = (F, \mathcal{F}_x K_{\mathcal{H}}(x,\cdot))_{\mathcal{H}} \text{ for all } F \in \mathcal{H}.$$

For a proof, see [11, Theorem 12.6.6].
We will see now that reproducing kernel representations may be used as a basis system in reproducing Sobolev spaces. In this context, we transform the inverse gravimetric problem into a harmonic spline interpolation problem to get a reasonably good approximation of the Earth's density distribution (under the constraint of harmonicity). To achieve this, we first define in the following harmonic splines as particular elements of the Sobolev space $\mathcal{H}$ and later, we will discuss some important theorems and properties corresponding to this spline. In this section, we will take $\mathcal{F}^N := \{\mathcal{F}_n\}_{n=1,\ldots,N}$, $N \in \mathbb{N}$ as a linearly independent system of bounded linear functionals from $\mathcal{H}$ into $\mathbb{R}$.

Definition 14 (Spheroidal Harmonic Spline)**.** *Let $\mathcal{F}^N := \{\mathcal{F}_n\}_{n=1,\ldots,N}$, $N \in \mathbb{N}$ be a system of linearly independent bounded linear functionals on $\mathcal{H}$. Then a function $S \in \mathcal{H}$ given by*

$$S(x) := \sum_{k=1}^{N} a_k \mathcal{F}_k K_{\mathcal{H}}(\cdot, x), \quad x \in \overline{E_{\text{int}}}, \tag{57}$$

with $a = (a_1, \ldots, a_N)^{\mathrm{T}} \in \mathbb{R}^N$ is called a (spheroidal harmonic) spline in $\mathcal{H}$ relative to $\mathcal{F}^N$. The corresponding linear space of all such spline functions is denoted by Spline $(\{\mathcal{A}_n\}; \mathcal{F}^N)$. *The scalars $a_1, \ldots, a_N$ are known as the coefficients of the spline S.*

Since this space is generated by $N \in \mathbb{N}$ linearly independent functionals, its dimension is N and Spline $(\{\mathcal{A}_n\}; \mathcal{F}^N) \subset \mathcal{H}$. Now keeping in mind the solution of the inverse gravimetric problem, we assume that $\mathcal{F}^N = \{\mathcal{F}_n\}_{n=1,\ldots,N}$, $N \in \mathbb{N}$ is a given system of linearly independent bounded linear functionals on $\mathcal{H}$ and $y = (y_1, \ldots, y_N)^{\mathrm{T}} \in \mathbb{R}^N$ is a given vector of values for which we have to find the Spline $S \in$ Spline $(\{\mathcal{A}_n\}; \mathcal{F}^N)$ such that

$$\mathcal{F}_i S = y_i \text{ for all } i = 1, \ldots, N. \tag{58}$$

These values y_i are, for example, 0th, 1st or 2nd order derivatives of the gravitational potential of the Earth.
Substituting the expression of S from Equation (57) into Equation (58), we get

$$\mathcal{F}_i \left(\sum_{k=1}^{N} a_k \mathcal{F}_k K_{\mathcal{H}}(\cdot,\cdot) \right) = y_i \text{ for all } i = 1, \ldots, N$$

or

$$\sum_{k=1}^{N} a_k \mathcal{F}_i \mathcal{F}_k K_{\mathcal{H}}(\cdot,\cdot) = y_i \text{ for all } i = 1, \ldots, N$$

which gives an equivalent formulation of the problem:
Determine $a = (a_1, \ldots, a_N)^{\mathrm{T}} \in \mathbb{R}^N$ such that the last equation holds, which gives rise to a system of linear equations with the matrix

$$M_N := (\mathcal{F}_i \mathcal{F}_k K_{\mathcal{H}}(\cdot,\cdot))_{i,k=1,\ldots,N}. \tag{59}$$

When we solve the system of linear equations given in (59), it is useful to know that M_N is positive definite.

Lemma 15 ($\mathcal{H}$-Spline Formula)**.** *Let* $S \in \operatorname{Spline}\left(\{\mathcal{A}_k\};\mathcal{F}^N\right)$*, i.e.*

$$S(x) = \sum_{k=1}^{N} a_k \mathcal{F}_k K_{\mathcal{H}}(\cdot, x), \quad x \in \overline{E_{\mathrm{int}}}.$$

Then, for arbitrary $F \in \mathcal{H}$*, the following holds true:*

$$(F, S)_{\mathcal{H}} = \sum_{k=1}^{N} a_k \mathcal{F}_k F.$$

Proof. This lemma is an immediate consequence of Theorem 13. □

Theorem 16. *Let* $\mathcal{F}^N = \{\mathcal{F}_n\}_{n=1,\ldots,N}$, $N \in \mathbb{N}$ *be a system of bounded linear functionals on* $\mathcal{H}$*. This system is linearly independent iff the matrix* M_N *in* (59) *is positive definite.*

Proof. Note that Theorem 13 yields

$$(\mathcal{F}_i)_x (\mathcal{F}_k)_y K_{\mathcal{H}}(x, y) = \Big((\mathcal{F}_k)_y K_{\mathcal{H}}(\cdot, y), (\mathcal{F}_i)_x K_{\mathcal{H}}(x, \cdot) \Big)_{\mathcal{H}}, \; i, k = 1, 2, ..., N.$$

Consequently, the matrix $M_N = (\mathcal{F}_i \mathcal{F}_k K_{\mathcal{H}}(\cdot, \cdot))_{i,k=1,\ldots,N}$ is a Gramian matrix and is positive definite if and only if the system $\{(\mathcal{F}_i)_x K_{\mathcal{H}}(x, \cdot)\}_{i=1,\ldots,N}$ is linearly independent. Furthermore, the system $\{(\mathcal{F}_i)_x K_{\mathcal{H}}(x, \cdot)\}_{i=1,\ldots,N}$ is linearly independent if and only if the identity

$$G(y) := \sum_{k=1}^{N} a_k (\mathcal{F}_k)_x K_{\mathcal{H}}(y, x) = 0 \text{ for all } y \in \overline{E_{\mathrm{int}}}$$

is equivalent to $a_k = 0$ for all $k = 1, ..., N$. Hence, (due to Lemma 15) the Gramian matrix is positive definite iff the following holds true:

$$(F, G)_{\mathcal{H}} = \sum_{k=1}^{N} a_k \mathcal{F}_k F = 0 \text{ for all } F \in \mathcal{H} \text{ iff } a_k = 0 \text{ for all } k = 1, ..., N,$$

which gives the linear independence of the system $\{\mathcal{F}_i\}_{i=1,\ldots,N}$. □

From Theorem 16, we can conclude that the spline interpolation problem is uniquely solvable.
The following theorem is advantageous if much time can be spent in the preprocessing and fast solutions are required as soon as the data are available.

Theorem 17 (Shannon Sampling Theorem)**.** *Any arbitrary spline function* $S \in \operatorname{Spline}\left(\{\mathcal{A}_k\};\mathcal{F}^N\right)$ *can be represented by its "samples"* $\mathcal{F}_i S$ *in the following way*

$$S(x) = \sum_{k=1}^{N} (\mathcal{F}_k S) L_k(x), \quad x \in \overline{E_{\mathrm{int}}} \tag{60}$$

with

$$L_k(x) = \sum_{j=1}^{N} a_j^{(k)} \mathcal{F}_j K_{\mathcal{H}}(x, \cdot), \quad x \in \overline{E_{\mathrm{int}}} \tag{61}$$

where $a_j^{(k)}$ *are the solutions of the following systems of linear equations*

$$\sum_{j=1}^{N} a_j^{(k)} \mathcal{F}_i \mathcal{F}_j K_{\mathcal{H}}(\cdot, \cdot) = \delta_{ik} \quad \textit{for all } i, k = 1, ..., N. \tag{62}$$

Proof. From Equations (61) and (62), we get $\mathcal{F}_i L_k = \delta_{ik}, i, k = 1, ..., N$. If we apply $\mathcal{F}_i$ on the right-hand side of Equation (60) then

$$\mathcal{F}_i \left(\sum_{k=1}^{N} \left(\mathcal{F}_k S \right) L_k \right) = \sum_{k=1}^{N} \mathcal{F}_k S \mathcal{F}_i L_k = \mathcal{F}_i S.$$

From the uniqueness of the interpolating spline, we conclude that

$$S\left(x\right) = \sum_{k=1}^{N} \left(\mathcal{F}_k S \right) L_k\left(x\right), \quad x \in \overline{E_{\mathrm{int}}}.$$

□

In practice, by a spline, we mean a function that minimizes a certain non-smoothness measure among all interpolating functions. In the following, we observe that the spheroidal harmonic spline minimizes the Sobolev norm among all interpolating functions which satisfy the interpolation conditions corresponding to the data set $y_1, ..., y_N$.

Theorem 18 (First Minimum Property)**.** *Let $y \in \mathbb{R}^N$ be given and $\mathcal{F}^N = \{\mathcal{F}_1, ..., \mathcal{F}_N\}$ be a linearly independent system of linear continuous functionals on $\mathcal{H}$. If $S^* = \sum_{i=1}^{N} a_i{}^* \left(\mathcal{F}_i\right)_x K_{\mathcal{H}}\left(\cdot, x\right)$ is the unique spline satisfying $\mathcal{F}_i S^* = y_i$ for all $i = 1, ..., N$, then S^* is the unique minimizer of*

$$\left\|S^*\right\|_{\mathcal{H}} = \min\left\{ \left\|F\right\|_{\mathcal{H}} \left| F \in \mathcal{H}, \mathcal{F}_i F = y_i, \, i = 1, ..., N \right. \right\}.$$

Theorem 19 (Second Minimum Property)**.** *Let $y \in \mathbb{R}^N$ be given and $\mathcal{F}^N = \{\mathcal{F}_1, ..., \mathcal{F}_N\}$ be a linearly independent system of linear continuous functionals on $\mathcal{H}$. If $S^* = \sum_{i=1}^{N} a_i^* \left(\mathcal{F}_i\right)_x K_{\mathcal{H}}\left(\cdot, x\right)$ with $\left(a_1^*, ..., a_N^*\right)^{\mathrm{T}} \in \mathbb{R}^N$ is the unique spline which satisfies $\mathcal{F}_i S^* = \mathcal{F}_i F$ where $F \in \mathcal{H}$ is given and $i = 1, ..., N$, then S^* is the unique minimizer of*

$$\left\|F - S^*\right\|_{\mathcal{H}} = \min\left\{ \left\|F - S\right\|_{\mathcal{H}} \left| S \in \mathrm{Spline}\left(\left\{\mathcal{A}_k\right\}; \mathcal{F}^N\right) \right. \right\}.$$

From this theorem, we conclude that S^* is the best approximating spline among all splines in $\mathrm{Spline}\left(\left\{\mathcal{A}_k\right\}; \mathcal{F}^N\right)$.
The proofs of Theorems 18 and 19 are analogues for the proofs of their counterparts on the sphere and ball.

6 Spline Scaling Functions and Multiresolution Analysis

In the following, we briefly describe the concept of the construction of a spline based multiresolution analysis. The principal ideas are the same as for the ball, (see [7], [8], [12]), see also [1] for the spheroid.

Definition 20. *Let $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, ...\}$ be a given countable system of bounded linear functionals on $\mathcal{H}$. Further, we denote linearly independent subsystems of these functionals by $\mathcal{F}^{N_J} := \{\mathcal{F}_1, ..., \mathcal{F}_{N_J}\}$ and $y^{(J)} \in \mathbb{R}^{N_J}$ stands for a given vector for every $J \in \mathbb{N}_0$ where $\left(N_J\right)_{J \in \mathbb{N}_0}$ is a monotonically increasing sequence of positive integers. We will continue to use these notations in the sequel.*

Definition 21. *If the family of sequences $\left\{\Phi_J^\wedge(n)\right\}_{n \in \mathbb{N}_0}$, $J \in \mathbb{N}_0$ satisfies the following conditions:*

1. $0 \leq \Phi_J^\wedge(n) \leq \Phi_{J+1}^\wedge(n) \leq 1$ *for all* $n, J \in \mathbb{N}_0$,

2. $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$ *is summable for all* $J \in \mathbb{N}_0$,

3. *for every fixed* $n \in \mathbb{N}_0$, *the sequence* $\{\Phi_J^\wedge(n)\}_{J\in\mathbb{N}_0}$ *is not identical to* 0, *i.e. there exists a positive integer* j_n *such that* $\Phi_J^\wedge(n) > 0$ *for all* $J \geq j_n$,

then the elements of the space $\mathcal{V}_J := \mathrm{Spline}\left(\{\Phi_J^\wedge(n)\}\,;\mathcal{F}^{N_J}\right)$ *are called spline-scaling functions.*

Sequences $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$, $J \in \mathbb{N}_0$ are divided into two categories.

1. band-limited sequences,

2. non-band-limited sequences.

Definition 22. *Let* $J \in \mathbb{N}_0$ *be fixed, then the sequence* $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$ *is called band-limited if only a finite number of its elements is different from zero, otherwise it is known as a non-band-limited sequence.*

Example 23. *Some examples of band-limited and non-band-limited sequences are given here (see also [14]).*
The Shannon sequences and the sequences generated by cubic polynomials fall in the band-limited category of the sequences and are given in the following:
Shannon Sequence

$$\Phi_J^\wedge(n) = \begin{cases} 1, & \text{for } 0 \leq n < 2^J, \\ 0, & \text{for } n \geq 2^J. \end{cases}$$

Sequence Generated by a Cubic Polynomial

$$\Phi_J^\wedge(n) = \begin{cases} \left(1 - 2^{-J}n\right)^2 \left(1 + 2^{-J+1}n\right), & \text{for } 0 \leq n < 2^J, \\ 0, & \text{for } n \geq 2^J. \end{cases}$$

Here, we also give some examples of non-band-limited sequences such as the Abel-Poisson sequence and the Gauß-Weierstraß sequence.
Abel-Poisson Sequence

$$\Phi_J^\wedge(n) = e^{-R2^{-J}n},\ R > 0.$$

Gauß-Weierstraß Sequence

$$\Phi_J^\wedge(n) = e^{-R2^{-J}n\left(2^{-J}n+1\right)},\ R > 0.$$

Note that the verification of the summability of the non-band-limited examples is still an open problem. In case of the spherical analogues, it is known to be satisfied.
The plots of these four sequences are given in Figure 1, *where each symbol is plotted from scale* $J = 1$ *to* $J = 7$ *(for continuous values* n*).*

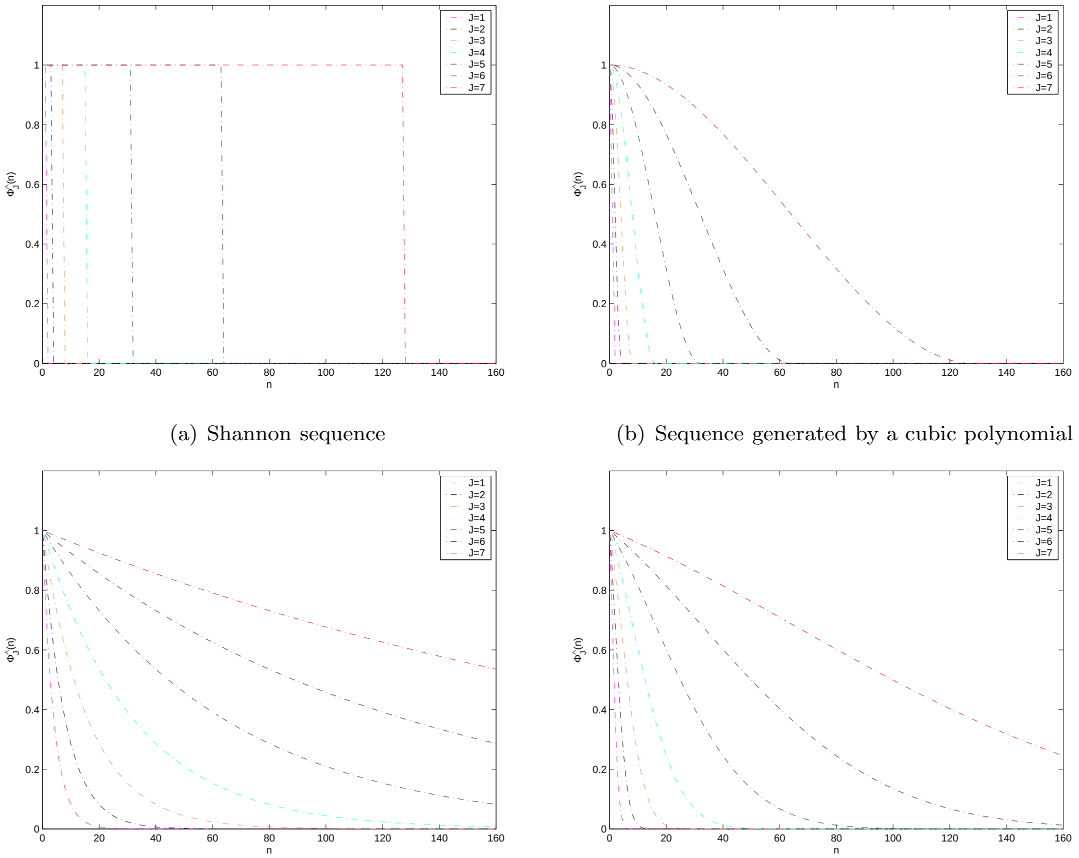
The Sobolev spaces $\mathcal{H}_J := \mathcal{H}\left(\{\Phi_J^\wedge(n)\}\,;\overline{E_{\mathrm{int}}}\right)$ represent a multiresolution analysis.

Theorem 24. *Let* $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$, $J \in \mathbb{N}_0$ *satisfy the conditions given in Definition 21, then the Sobolev spaces* $\mathcal{H}_J = \mathcal{H}\left(\{\Phi_J^\wedge(n)\}\,;\overline{E_{\mathrm{int}}}\right)$ *satisfy the following properties:*

1. $\mathcal{H}_J \subset \mathcal{H}_{J+1} \subset \mathcal{H}\left(\{\phi(n)\}\,;\overline{E_{\mathrm{int}}}\right) =: \mathcal{H}_\infty$ *for all* $J \in \mathbb{N}_0$,

2. $\mathcal{H}_\infty = \overline{\bigcup_{J\in\mathbb{N}_0} \mathcal{H}_J}^{\|\cdot\|_{\mathcal{H}\left(\{\phi(n)\};\overline{E_{\mathrm{int}}}\right)}}$,

where $\phi(n) = \lim\limits_{J\to\infty} \Phi_J^\wedge(n)$.



(a) Shannon sequence

(b) Sequence generated by a cubic polynomial

(c) Abel-Poisson sequence

(d) Gauß-Weierstraß sequence

Figure 1: Plot of the different sequences $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$, each calculated for the scales $J=1$ to $J=7$.

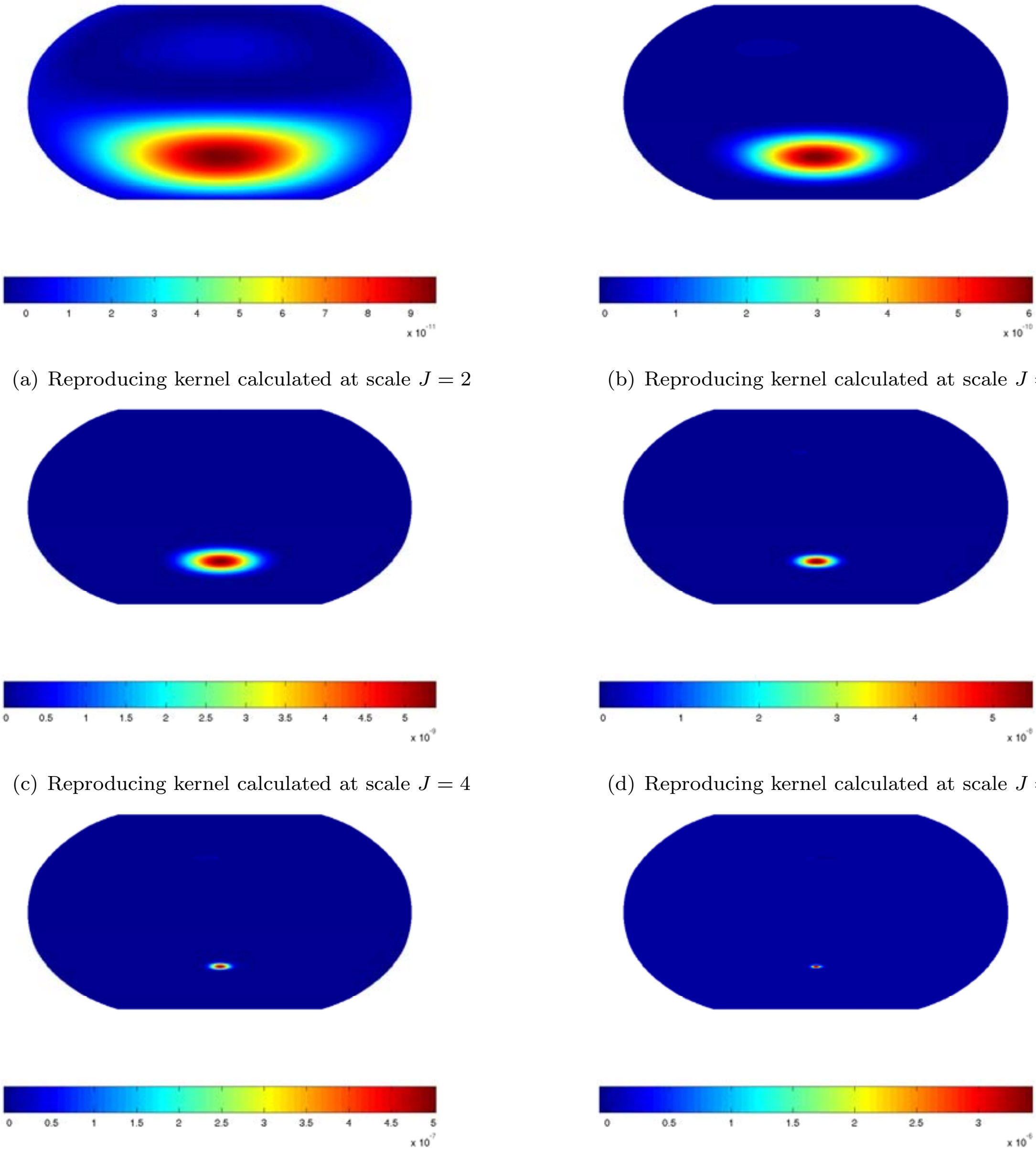
The statement of this theorem means that we can obtain a sequence of approxi-mating splines $S_J \in \mathcal{V}_J$ where each spline S_J is the smoothest function of the corresponding Sobolev space $\mathcal{H}_J$ and satisfies the equations $\mathcal{F}_n S_J = y_n^{(J)}$, $n = 1, ..., N_J$. Considering Figure 2, where the reproducing kernel is calculated from scale $J=2$ to $J=7$, one can observe that the kernel becomes more and more localized as the symbol $\Phi_J^\wedge(n)$ increases step by step with the increase of $J \in \mathbb{N}_0$. Hence, it is expected that the resolution will increase with the increase of J, and more data then have to be taken into account ($N_J < N_{J+1}$).
In our case, where

$$\mathcal{F}_i F = \int_{\overline{E_{\mathrm{int}}}} \frac{F(y)}{|x_i - y|} dy, \quad F \in \mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right),$$

for a fixed $x_i \in E_{\mathrm{ext}}$, following Equation (52), the spline basis functions are given by

$$\begin{aligned} \mathcal{F}_i K_{\mathcal{H}_J}(y,\cdot) &= \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left(\Phi_J^\wedge(n)\right)^2 U_{n,j}(y)\, \mathcal{F}_i U_{n,j}(\cdot) \\ &= \sum_{n=0}^{\infty} \sum_{j=-n}^{n} \left(\Phi_J^\wedge(n)\right)^2 U_{n,j}(y) \left(K_{n,j} V_{n,j}(x_i) + L_{n+2,j} V_{n+2,j}(x_i)\right), \end{aligned}$$

where $\{U_{n,j}\}_{n\in\mathbb{N}_0,\, j\in\{-n,\dots,n\}}$ and $\{V_{n,j}\}_{n\in\mathbb{N}_0,\, j\in\{-n,\dots,n\}}$ represent the orthonormalized inner har-



(a) Reproducing kernel calculated at scale $J = 2$

(b) Reproducing kernel calculated at scale $J = 3$

(c) Reproducing kernel calculated at scale $J = 4$

(d) Reproducing kernel calculated at scale $J = 5$

(e) Reproducing kernel calculated at scale $J = 6$

(f) Reproducing kernel calculated at scale $J = 7$

Figure 2: Plots of the reproducing kernel $K_{\mathcal{H}_J}$, calculated by the sequence generated by a cubic polynomial for the scales $J = 2$ to $J = 7$, where one argument is kept fixed.

monics (see Section 3) and the outer harmonics (see Equation (18)), respectively. It is also clear that on imposing the condition of $\lim_{J\to\infty} \Phi_J^\wedge(n) = 1$ on the multiresolution analysis (see Theorem 24), we get the "limit space" $\mathcal{H}\left(\{\varphi(n)\}\,;\overline{E_{\text{int}}}\right) = \mathcal{H}\left(\{1\}\,;\overline{E_{\text{int}}}\right) = \text{Harm}\left(\overline{E_{\text{int}}}\right)$, i.e. with this condition we get the set of all harmonic functions on $\overline{E_{\text{int}}}$.

Now at scale $J \in \mathbb{N}_0$, we can calculate the approximating harmonic spline by solving the system of linear equations given by

$$\sum_{k=1}^{N_J} a_k^{(J)} \mathcal{F}_l \mathcal{F}_k K_{\mathcal{H}_J}(\cdot,\cdot) = y_l^{(J)}; \quad l = 1, ..., N_J$$

and the corresponding spline has the following expression:

$$S_J(x) = \sum_{k=1}^{N_J} a_k^{(J)} \mathcal{F}_k K_{\mathcal{H}_J}(\cdot, x).$$

Note that the difference $S_{J+1} - S_J$ can be interpreted as a spline-wavelet and the described method represents a regularization of the ill-posed problem due to the finite dimension of the spline spaces.

Theorem 25. *Let $S_J \in \mathcal{H}_J$ be the unique spline satisfying the conditions $\mathcal{F}_k S_J = y_k^{(J)}; k = 1, ..., N_J$. Then the spline depends continuously on the given data.*

7 Convergence Theorem

In this section, we formulate a convergence theorem which tells us that by adding more and more data and decreasing the hat-width of the basis functions at the same time, we can obtain a sequence of approximating splines. This sequence converges to the unknown function of our problem. The proof is again analogous to the case on the ball, see also [1] for the spheroid.
Let $\mathcal{H}_J^*$ stand for the space of bounded linear functionals $\mathcal{T} : \mathcal{H}_J \longrightarrow \mathbb{R}$. The space $\mathcal{H}_J^*$ is known as the dual space.

Theorem 26 (Convergence Theorem)**.** *Let $F \in \cup_{J\in\mathbb{N}_0} \mathcal{H}_J$ be a given function and $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ be a linearly independent system of linear and continuous functionals in $\mathcal{H}_\infty^*$ such that* span $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ *is dense in $\mathcal{H}_\infty^*$. Let there exist subsystems $\mathcal{F}^{N_J} = \{\mathcal{F}_1, ..., \mathcal{F}_{N_J}\} \subset \mathcal{F}$, $J \in \mathbb{N}_0$, with $N_J \leq N_{J+1}$ for all $J \in \mathbb{N}_0$ and $\lim_{J\to\infty} N_J = \infty$. Moreover, let the sequence of spline-scaling functions $(S_J)_{J\in\mathbb{N}_0}$ be given by*

$$\begin{aligned} S_J \in \mathcal{V}_J = \mathrm{Spline}\left(\left\{\Phi_J^\wedge(n)\right\}; \mathcal{F}^{N_J}\right), \\ \mathcal{F}_i S_J = \mathcal{F}_i F \text{ for all } i = 1, ..., N_J. \end{aligned}$$

Then $\lim_{J\to\infty} \|S_J - F\|_{\mathcal{H}_\infty} = 0$.

8 Conclusions and Outlook

The main aim of this work was to develop the theory of a reproducing kernel based spline method for the regularization of the spheroidal inverse gravimetric problem. This problem is mathematically given by a Fredholm integral equation of the first kind (see Equation (1)). Analogous methods proved to be useful for such tomographic problems in geophysics in case of a spherical Earth model. To develop this method we first constructed an orthonormal basis system in $\left(\mathrm{Harm}\left(\overline{E_{\mathrm{int}}}\right), \mathrm{L}^2\left(\overline{E_{\mathrm{int}}}\right)\right)$, which is one of the main results of this paper.

The constructed spline interpolation method represents a regularization, i.e. every spline continuously depends on the given gravitational data set. Each spline function satisfies the first and second minimum properties which means that each spline is the smoothest among all the elements of the (reproducing kernel) Sobolev space which fit to the given data, i.e. which generate the same gravitational potential at given data points, and the interpolating spline is the best approximation in the spline space.

Moreover, Sobolev spaces defined by particular symbols which are monotonically increasing with respect to the scale $J \in \mathbb{N}_0$ represent a multiresolution analysis, i.e. we get a sequence of approximating splines which get closer and closer to the unknown density function as the scale J increases (see Theorem 24). Finally, a convergence theorem is proved which actually assures that the approximating splines converge to the underlying unknown harmonic density function.

In future research, a numerical implementation of the developed method will be necessary. In this context, also practical problems such as the choice of an appropriate gravitational field model have to be solved. Moreover, the search for efficient numerical algorithms for the calculation of the spline basis functions and the matrix of the corresponding equation system still represents a challenge (also for the spherical case).

Moreover, we know that the solution of the Fredholm integral equation given in (1) consists of a harmonic and an anharmonic part. In this paper, we have developed a spline interpolation theory to reconstruct the harmonic part only of this solution. Although we find rare publications (such as [6], [20] and [21]) which discuss the anharmonic part, it is an interesting topic for further research to study the combined determination of the harmonic part and the anharmonic part of the solution by means of constructive approximation to get a better approximation to the density function (see also the approaches in [7], [8] and [9], where seismic data are used to constrain the anharmonic part). Moreover, anharmonic basis functions for the spheroidal case still have to be constructed. An important initial step in this respect was done in [6].

Acknowledgements

NA gratefully acknowledges the support by the Higher Education Commission Pakistan. Moreover, VM gratefully acknowledges the support by the German Research Foundation (projects MI 655/2-1 and MI 655/2-2).

References

[1] N. Akhtar: A Multiscale Harmonic Spline Interpolation Method for the Inverse Spheroidal Gravimetric Problem. PhD Thesis, Geomathematics Group, Department of Mathematics, University of Siegen, Shaker Verlag, Aachen, 2010.

[2] A. Amirbekyan: The Application of Reproducing Kernel Based Spline Approximation to Seismic Surface and Body Wave Tomography: Theoretical Aspects and Numerical Results. PhD Thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, Germany, 2007.
(http://kluedo.ub.uni-kl.de/volltexte/2007/2103/pdf/ThesisAbel.pdf).

[3] A. Amirbekyan, V. Michel: Splines on the Three-Dimensional Ball and their Application to Seismic Body Wave Tomography. Inverse Problems, 24 (2008), doi:10.1088/0266-5611/24/1/015022.

[4] A. A. Ardalan, E. W. Grafarend: Reference Ellipsoidal Gravity Potential Field and Gravity Intensity Field of Degree/Order 360/360, Department of Geodesy and GeoInformatics, University of Stuttgart, 2000.

[5] A. A. Ardalan, E. W. Grafarend: Somigliana - Pizzetti Gravity: the International Gravity Formula Accurate to the Sub-nanoGal Level. Journal of Geodesy, 75 (2001), 424-437.

[6] L. Ballani, J. Engels, E. W. Grafarend: Global Base Functions for the Mass Density in the Interior of a Massive Body (Earth). Manuscripta Geodaetica, 18 (1993), 99-114.

[7] P. Berkel: Multiscale Methods for the Combined Inversion of Normal Mode and Gravity Variations. PhD Thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, Shaker Verlag, Aachen, 2009.

[8] P. Berkel, D. Fischer, V. Michel: Spline Multiresolution and Numerical Results for Joint Gravitation and Normal Mode Inversion with an Outlook on Sparse Regularisation. International Journal on Geomathematics, accepted for publication, 2010.

[9] P. Berkel, V. Michel: On Mathematical Aspects of a Combined Inversion of Gravity and Normal Mode Variations by a Spline Method. Mathematical Geosciences, 42 (2010), 795-816.

[10] S. Chandrasekhar: Ellipsoidal Figures of Equilibrium. Yale University Press, 1969, Dover Publications, New York, 1987.

[11] P. J. Davis: Interpolation and Approximation. Dover Publications, Inc. New York, 1975.

[12] M. J. Fengler, D. Michel, V. Michel: Harmonic Spline-Wavelets on the 3-dimensional Ball and their Application to the Reconstruction of the Earth's Density Distribution from Gravitational Data at Arbitrarily Shaped Satellite Orbits. Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), 86 (2006), 856-873.

[13] W. Freeden: On Approximation by Harmonic Splines. Manuscripta Geodaetica, 6 (1981), 193-244.

[14] W. Freeden, T. Gervens, M. Schreiner: Constructive Approximation on the Sphere - With Applications to Geomathematics. Oxford Science Publications, Oxford University Press, Oxford, 1998.

[15] W. Freeden, V. Michel: Multiscale Potential Theory - With Applications to Geoscience. Birkhäuser, Boston, 2004.

[16] D. Gilbarg, N. S. Trudinger: Elliptic Partial Differential Equations of Second Order. Springer-Verlag, Berlin, 1977.

[17] E. W. Grafarend, A. A. Ardalan: World Geodetic Datum 2000. Journal of Geodesy, 73 (1999), 611-623.

[18] W. A. Heiskanen, H. Moritz: Physical Geodesy. Reprint, Institute of Physical Geodesy, Technical University Graz, Austria, 1981.

[19] E. W. Hobson: The Theory of Spherical and Ellipsoidal Harmonics. Chelsea Publishing Company, New York, 1965.

[20] V. Michel: A Multiscale Method for the Gravimetry Problem: Theoretical and Numerical Aspects of Harmonic and Anharmonic Modelling. PhD Thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, Shaker Verlag, Aachen, 1999.

[21] V. Michel: A Multiscale Approximation for Operator Equations in Separable Hilbert Spaces - Case Study: Reconstruction and Description of the Earth's Interior. Habilitation Thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, Shaker Verlag, Aachen, 2002.

[22] V. Michel: Tomography - Problems and Multiscale Solutions, Contribution to the Monograph "Handbook of Geomathematics" (W. Freeden, M.Z. Nashed, T. Sonar, eds.), 2010, 949-972.

[23] V. Michel, A. S. Fokas: A Unified Approach to Various Techniques for the Non-uniqueness of the Inverse Gravimetric Problem and Wavelet-Based Methods. Inverse Problems, 24 (2008), 045019 (25pp), doi: 10.1088/0266-5611/24/4/045019.

[24] V. Michel, K. Wolf: Numerical Aspects of a Spline-Based Multiresolution Recovery of the Harmonic Mass Density out of Gravity Functionals. Geophysical Journal International, 173 (2008), 1-16.

[25] P. H. Moon, D. E. Spencer: Recent Investigations of the Separation of Laplace's Equation. Proc. Amer. Math. Soc. 4 (1953), 302-307.

[26] P. H. Moon, D. E. Spencer: Field Theory for Engineers. D. Van Nostrand Comp., Princeton, 1961.

[27] H. Moritz: Computation of Ellipsoidal Mass Distributions. Report No. 206, Department of Geodetic Science and Surveying, The Ohio State University, Columbus, USA, 1973.

[28] J. Neumann: Entwicklung der in elliptischen Koordinaten ausgedrückten reziproken Entfernung zweier Punkte in Reihen. Journal für die reine und angewandte Mathematik, 37 (1848), 21-50.

[29] G. Sona: Numerical Problems in the Computation of Ellipsoidal Harmonics. Journal of Geodesy, 70 (1995), 117-126.

[30] H. Sünkel, C. C. Tscherning: A Method for the Construction of Spheroidal Mass Distributions Consistent With the Harmonic Part of the Earth's Gravity Potential. Manuscripta Geodaetica, 6 (1981), No. 2, 131-156.

[31] N. C. Thong, E. W. Grafarend: A Spheroidal Harmonic Model of the Terrestrial Gravitational Field. Manuscripta Geodaetica, 14 (1989), 285-304.

Siegen Preprints on Geomathematics

The preprint series “Siegen Preprints on Geomathematics” was established in 2010. See www.geomathematics-siegen.de for details and a contact address. At present, the following preprints are available:

1. P. Berkel, D. Fischer, V. Michel: *Spline multiresolution and numerical results for joint gravitation and normal mode inversion with an outlook on sparse regularisation*, 2010.
2. M. Akram, V. Michel: *Regularisation of the Helmholtz decomposition and its application to geomagnetic field modelling*, 2010.
3. V. Michel: *Optimally Localized Approximate Identities on the 2-Sphere*, 2010.
4. N. Akhtar, V. Michel: *Reproducing Kernel Based Splines for the Regularization of the Inverse Spheroidal Gravimetric Problem*, 2011.

Geomathematics Group Siegen
Prof. Dr. Volker Michel

Contact at:
Geomathematics Group
Department of Mathematics
University of Siegen
Walter-Flex-Str. 3
57068 Siegen
www.geomathematics-siegen.de