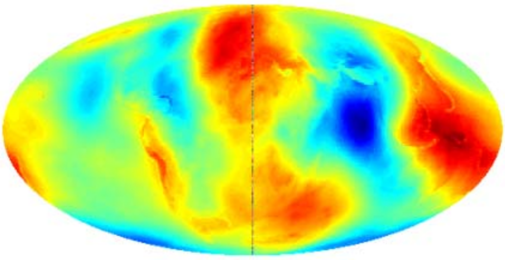
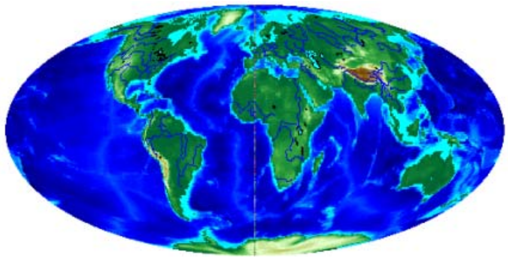

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A Study of Differential Operators for Complete Orthonormal Systems on a 3D Ball

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Abstract

In this article, we introduce a class of differential operators for two complete orthonormal systems in $\mathrm{L}^2(\mathcal{B})$, where $\mathcal{B}$ is a ball in $\mathbb{R}^3$, such that these orthonormal systems are eigenfunctions. We study further properties of these operators. It turns out, for instance, that the Sobolev norm, which is used for a spline interpolation method on $\mathcal{B}$, can be interpreted as the $\mathrm{L}^2(\mathcal{B})$-norm of the image of a (pseudo-)differential operator.

Key Words: Ball, complete orthonormal system, differential operators, eigenfunctions, self-adjoint operators, Sobolev space, spline.

AMS(2000) Classification: 33C45, 33C50, 33C55, 33E30, 46E35

1 Introduction

The approximation of functions on a 3d-ball has a series of important applications at present. In particular, such methods play an essential role in tomography problems of geophysics and medical imaging, see for example the survey article [24]. Since the structures to be determined usually consist of layers with (almost) spherical boundaries, the use of Euclidean approximation methods (restricted from $\mathbb{R}^3$ to the ball) are inappropriate. Therefore, functions based on a radial-angular decomposition are more useful. Classical approximation methods use a truncated singular value decomposition which requires an orthonormal basis for the relevant function space (e.g. $\mathrm{L}^2(\mathcal{B})$, where $\mathcal{B}$ is a ball with center at 0 and radius $R > 0$). Typically, orthogonal/orthonormal polynomials are chosen as such a basis. For intervals on the real line, such polynomials are well known and have already been investigated in detail for many decades (see for example, [28]). For instance, the Legendre polynomials represent a famous system of orthogonal polynomials on $[-1, 1]$. Moreover, for the unit sphere Ω in $\mathbb{R}^3$, the system of spherical harmonics

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$\{Y_{n,j}\}_{n=0,1,...;j=-n,...n}$ is also an established function system which is widely used and has many known properties (see, for instance, [13, 15, 27]). For the ball $\mathcal{B}$, the following two orthonormal basis systems are known:

$$G^{\mathrm{I}}_{m,n,j}(x) := \sqrt{\frac{4m+2n+3}{R^3}} P_m^{(0,n+1/2)} \left(2\frac{|x|^2}{R^2} - 1 \right) \left(\frac{|x|}{R} \right)^n Y_{n,j} \left(\frac{x}{|x|} \right), \qquad (1)$$

$x \in \mathcal{B}, m, n \in \mathbb{N}_0;\ j = -n, ..., n$ (see [6, 10, 14, 19]) as well as

$$G^{\mathrm{II}}_{m,n,j}(x) := \sqrt{\frac{2m+3}{R^3}} P_m^{(0,2)} \left(2\frac{|x|}{R} - 1 \right) Y_{n,j} \left(\frac{x}{|x|} \right), \qquad (2)$$

$x \in \mathcal{B} \setminus \{0\}, m, n \in \mathbb{N}_0;\ j = -n, ..., n$ (see [29]). Here, $P_m^{(a,b)}$ is a Jacobi polynomial of degree m corresponding to the parameters (a, b). Note that the system of type II is not an algebraic polynomial in x_1, x_2, x_3 and is discontinuous in $x = 0$ for $n > 0$. Both systems have their advantages and disadvantages as discussed in [24]. Based on the systems, advanced approximation methods such as spline and wavelet methods were developed in view of different applications, which are inverse gravimetry [11, 18, 19, 20, 21, 23, 25, 26], travel-time tomography [4, 5], normal mode tomography (in combination with inverse gravimetry) [7, 8, 9], MEG-EEG tomography [12] and seismic wave interpolation [17, 22], where in the latter two cases different orthonormal bases were used. Further, locally supported approximating structures on the unit ball were introduced in [1, 2, 3].

Unfortunately, the present knowledge about the orthonormal basis (1) and (2) is poor. Only a few properties such as a rough estimate of the maximum norm (see [14, 24]) are known. Since these systems represent the foundation for the named spline and wavelet methods, the derivation of further properties would improve the understanding of these tools and would open possibilities to prove further properties of the splines and wavelets. Moreover, the numerical implementation is still connected to some problems which have not been solved completely, yet. These problems are partially caused by the different behavior of type I in comparison to type II. For example, type II shows (expected) strong oscillations near the origin in combination with the discontinuity. A quantification of these phenomena would allow a better adaptation of the spline and wavelet kernels in the future.

As a contribution to the quest for a better understanding of the basis functions on $\mathcal{B}$, this paper addresses the problem of representing these functions as eigenfunctions of a differential operator. Such operators are known for 1D orthogonal polynomials (such as Jacobi polynomials) and for spherical orthogonal polynomials (the spherical harmonics are eigenfunctions of the Beltrami operator). It will be shown here that differential operators can be derived for which $G^{\mathrm{I}}_{m,n,j}$ and $G^{\mathrm{II}}_{m,n,j}$ are eigenfunctions. Further properties of these differential operators are proved. Another result of these investigations is a way to interpret particular Sobolev norms on the ball. This is important for the understanding of the smoothing properties of a spline method used in geomathematics.

2 Derivation of a Differential Operator for the Orthonormal Systems of Type I and Type II

Jacobi polynomials in general and Legendre polynomials in particular are known to be eigenfunctions of associated univariate differential operators (see, for example, [28]). Analogously, the spherical harmonics are eigenfunctions of the Beltrami operator, which is a spherical differential operator (see, for instance, [13]). This property already proved to be helpful to derive further properties of Jacobi polynomials and spherical harmonics, respectively. In this section, we will determine certain three-dimensional differential operators for which type I and type II, respectively, are eigenfunctions. The differential operators themselves shall be examined further.

First we find the differential operator for which the system of type I is an eigenfunction. The differential equation of the Jacobi polynomials $y = P_m^{(a,b)}$ is given by (see [28])

$$(1-x^2)\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_m^{(a,b)}(x) + (b-a-(a+b+2)x)\,\frac{\mathrm{d}}{\mathrm{d}x}P_m^{(a,b)}(x) \\ +m(m+a+b+1)P_m^{(a,b)}(x) = 0. \tag{3}$$

Putting $a = 0, b = n+\frac{1}{2}, x = u$, in the equation given above, we have

$$\left(1-u^2\right)\frac{\mathrm{d}^2}{\mathrm{d}u^2}P_m^{(0,n+\frac{1}{2})}(u) + \left(n+\frac{1}{2}-\left(n+\frac{5}{2}\right)u\right)\frac{\mathrm{d}}{\mathrm{d}u}P_m^{(0,n+\frac{1}{2})}(u) \\ +m\left(m+n+\frac{3}{2}\right)P_m^{(0,n+\frac{1}{2})}(u) = 0. \tag{4}$$

Substituting $u = \frac{2r^2}{R^2} - 1$, i.e. $r = R\sqrt{\frac{u+1}{2}}$ where $r \in [0, R]$, and using the chain rule for differentiation, equation (4) becomes

$$\left(R^2-r^2\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2}P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right) + 2\left(n\left(1-\frac{r^2}{R^2}\right)+1-\frac{2r^2}{R^2}\right)\frac{R^2}{r} \\ \times\frac{\mathrm{d}}{\mathrm{d}r}P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right) + 4m\left(m+n+\frac{3}{2}\right)P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right) = 0.$$

This is equivalent to

$$\left((R^2-r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2}+2\left(n\left(1-\frac{r^2}{R^2}\right)+1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)$$
$$=-4m\left(m+n+\frac{3}{2}\right)P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right). \quad (5)$$

Now, letting $Y(r):=P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n$ and $g(r):=P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)$, we get

$$Y=g\frac{r^n}{R^n}. \quad (6)$$

Differentiating with respect to r, we have

$$\begin{aligned} Y' &= g'\frac{r^n}{R^n}+n\frac{r^{n-1}}{R^n}g \\ &= g'\frac{r^n}{R^n}+\frac{n}{r}Y. \end{aligned} \quad (7)$$

Again differentiating equation (7) with respect to r, we have,

$$Y''=g''\frac{r^n}{R^n}+2n\frac{r^{n-1}}{R^n}g'+n(n-1)\frac{r^{n-2}}{R^n}g.$$

Using equations (6) and (7), we reformulate the equation given above as

$$\begin{aligned} Y'' &= g''\frac{r^n}{R^n}+\frac{2n}{r}\left(Y'-\frac{n}{r}Y\right)+\frac{n(n-1)}{r^2}Y \\ &= g''\frac{r^n}{R^n}+\frac{2n}{r}Y'-\frac{n(n+1)}{r^2}Y. \end{aligned}$$

Now multiplying the equation given above by R^2-r^2, we have

$$(R^2-r^2)Y''=(R^2-r^2)g''\frac{r^n}{R^n}+(R^2-r^2)\frac{2n}{r}Y'-(R^2-r^2)\frac{n(n+1)}{r^2}Y. \quad (8)$$

Putting the value of $(R^2-r^2)g''$ taken from (5) in (8) we get

$$\begin{aligned} (R^2-r^2)Y'' &= -2\left(n\left(1-\frac{r^2}{R^2}\right)+1-\frac{2r^2}{R^2}\right)g'\frac{R^2}{r}\frac{r^n}{R^n}-4m\left(m+n+\frac{3}{2}\right)g\frac{r^n}{R^n} \\ &\quad +(R^2-r^2)\frac{2n}{r}Y'-(R^2-r^2)\frac{n(n+1)}{r^2}Y. \end{aligned}$$

Now using (6) and (7) in the equation above, we have,

$$
\begin{aligned}
(R^2-r^2)Y'' &= -2\left(n\left(1-\frac{r^2}{R^2}\right)+1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\left(Y'-\frac{n}{r}Y\right) \\
&\quad -4m\left(m+n+\frac{3}{2}\right)Y+(R^2-r^2)\frac{2n}{r}Y'-(R^2-r^2)\frac{n(n+1)}{r^2}Y \\
&= -2\left(n\left(1-\frac{r^2}{R^2}\right)+1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}Y'+2\left(n\left(1-\frac{r^2}{R^2}\right)+1-\frac{2r^2}{R^2}\right) \\
&\quad \times\frac{R^2}{r}\frac{n}{r}Y-4m\left(m+n+\frac{3}{2}\right)Y+(R^2-r^2)\frac{2n}{r}Y'-(R^2-r^2)\frac{n(n+1)}{r^2}Y \\
&= \left(-\frac{2R^2}{r}+4r\right)Y'+\left(\frac{n^2R^2}{r^2}+\frac{nR^2}{r^2}-n^2-3n-4m\left(m+n+\frac{3}{2}\right)\right)Y \\
&= -2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}Y'+\left(n(n+1)\frac{R^2}{r^2}-n(n+3)-4m\left(m+n+\frac{3}{2}\right)\right)Y.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&(R^2-r^2)Y''+2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}Y'-n(n+1)\frac{R^2}{r^2}Y \\
&= -\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right)Y.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\left((R^2-r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2}+2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-n(n+1)\frac{R^2}{r^2}\right)Y \\
&\qquad = -\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right)Y. \qquad (9)
\end{aligned}
$$

This shows that

$$
D_r^{\mathrm{I}} := (R^2-r^2)\frac{\mathrm{d}^2}{\mathrm{d}r^2}+2\left(1-\frac{2r^2}{R^2}\right)\frac{R^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-n(n+1)\frac{R^2}{r^2}
$$

is a differential operator with eigenfunction $Y(r)=P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n, r=|x|, r\in[0,R]$ and the corresponding eigenvalue is $-\left(n(n+3)+4m(m+n+\frac{3}{2})\right)$. Let us define another operator ${}^*\Delta^{\mathrm{I}}$ by

$$
{}^*\Delta_x^{\mathrm{I}} := D_{|x|}^{\mathrm{I}}\circ\Delta^*_{\frac{x}{|x|}},
$$

where Δ^* is the Beltrami operator, for which the spherical harmonics Y_n of degree n are the eigenfunctions corresponding to the eigenvalue $-n(n+1)$ (for further details see [13]), i.e.

$$
\Delta^* Y_{n,j} = -n(n+1)Y_{n,j}.
$$

Now, it is easy to show that ${}^*\Delta^{\mathrm{I}}$ is a differential operator for which the basis functions $G^{\mathrm{I}}_{m,n,j}$ are eigenfunctions and the corresponding eigenvalues are

$$n(n+1)\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right).$$

Theorem 2.1 *The basis function $G^{\mathrm{I}}_{m,n,j}$ is an eigenfunction of the differential operator ${}^*\Delta^{\mathrm{I}}$ defined above corresponding to the eigenvalue*

$$n(n+1)\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right),$$

i.e.,

$$\begin{aligned}
{}^*\Delta^{\mathrm{I}}_x\left(G^{\mathrm{I}}_{m,n,j}(x)\right) &= \left(D^{\mathrm{I}}_{|x|}\circ\Delta^*_{\frac{x}{|x|}}\right)\left(G^{\mathrm{I}}_{m,n,j}(x)\right)\\
&= D^{\mathrm{I}}_{|x|}\circ\Delta^*_{\frac{x}{|x|}}\left(\sqrt{\frac{4m+2n+3}{R^3}}P_m^{(0,n+\frac{1}{2})}\left(\frac{2|x|^2}{R^2}-1\right)\left(\frac{|x|}{R}\right)^n Y_{n,j}\left(\frac{x}{|x|}\right)\right)\\
&= n(n+1)\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right)\left(G^{\mathrm{I}}_{m,n,j}(x)\right), \quad x\in\mathcal{B}.
\end{aligned}$$

Note that the differential operator depends on the degree of the used spherical harmonic, such that a reference to n would actually be necessary in the notation. We omit this, however, for the sake of a better readability. We, therefore, do not have one fixed differential operator for which all basis functions are eigenfunctions. Now, we derive the differential operator for the system of type II. If we put $a=0$, $b=2$ and $x=u$ in equation (3), we get a differential equation for the Jacobi polynomials $P_m^{(0,2)}$ as follows

$$\left((1-u^2)\frac{\mathrm{d}^2}{\mathrm{d}u^2}+(2-4u)\frac{\mathrm{d}}{\mathrm{d}u}\right)P_m^{(0,2)}(u)+m(m+3)P_m^{(0,2)}(u)=0. \tag{10}$$

Now substituting $u=\frac{2r}{R}-1$, i.e., $r=\frac{R(u+1)}{2}$ in (10) and using the chain rule we get

$$\left(rR\left(1-\frac{r}{R}\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2}+(3R-4r)\frac{\mathrm{d}}{\mathrm{d}r}\right)P_m^{(0,2)}\left(\frac{2r}{R}-1\right)=-m(m+3)P_m^{(0,2)}\left(\frac{2r}{R}-1\right).$$

This implies that $rR\left(1-\frac{r}{R}\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2}+(3R-4r)\frac{\mathrm{d}}{\mathrm{d}r}$ is a differential operator for which $P_m^{(0,2)}\left(\frac{2r}{R}-1\right)$ is an eigenfunction with the corresponding eigenvalue $-m(m+3)$, where $r\in[0,R]$.

Let us denote the differential operator of $P_m^{(0,2)}\left(\frac{2r}{R}-1\right)$ by D^{II}_r, i.e.,

$$D^{\mathrm{II}}_r := rR\left(1-\frac{r}{R}\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2}+(3R-4r)\frac{\mathrm{d}}{\mathrm{d}r}.$$

Now, we define another operator by ${}^*\Delta_x^{\mathrm{II}} := D_{|x|}^{\mathrm{II}} \circ \Delta^*_{\frac{x}{|x|}}$. Also here it is easy to show that ${}^*\Delta^{\mathrm{II}}$ is a differential operator for which $G_{m,n,j}^{\mathrm{II}}$ is an eigenfunction and the corresponding eigenvalue is $n(n+1)m(m+3)$.

Theorem 2.2 *The basis functions $G_{m,n,j}^{\mathrm{II}}$ are eigenfunctions of the differential operator ${}^*\Delta^{\mathrm{II}}$, where $G_{m,n,j}^{\mathrm{II}}$ corresponds to the eigenvalue $m(m+3)n(n+1)$, i.e.*

$$ {}^*\Delta_x^{\mathrm{II}} \left(G_{m,n,j}^{\mathrm{II}}(x)\right) = m(m+3)n(n+1)G_{m,n,j}^{\mathrm{II}}(x),\ x \in \mathcal{B}. $$

3 Some properties of the differential operators ${}^*\Delta_x^{\mathrm{I}}$ and ${}^*\Delta_x^{\mathrm{II}}$

We denote the eigenvalues of $P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n$ corresponding to D_r^{I} by

$$ D^{\mathrm{I}\wedge}(m,n) := -\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right). $$

We can observe that $D^{\mathrm{I}\wedge}(0,0) = 0$. This shows that D^{I} is not invertible. As a consequence, ${}^*\Delta_x^{\mathrm{I}} = D_{|x|}^{\mathrm{I}} \circ \Delta^*_{\frac{x}{|x|}}$ with the eigenvalues

$$ \left({}^*\Delta^{\mathrm{I}}\right)^\wedge(m,n) := n(n+1)\left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right) $$

is also not invertible. Similarly, we set $\left({}^*\Delta^{\mathrm{II}}\right)^\wedge(m,n) := mn(n+1)(m+3)$, which also shows that ${}^*\Delta^{\mathrm{II}}$ is not invertible, because

$$ \left({}^*\Delta^{\mathrm{II}}\right)^\wedge(m,0) = \left({}^*\Delta^{\mathrm{II}}\right)^\wedge(0,n) = 0. $$

Next, with the help of the differential operators given above, we find two new operators which are invertible. Note that there exist similar results for spherical (pseudo-)differential operators (see [13]).

Theorem 3.1 *The differential operators ${}^{**}\Delta_x^{\mathrm{I}} := (-D_{|x|}^{\mathrm{I}} + \frac{9}{4}) \circ (\Delta^*_{\frac{x}{|x|}} + \frac{1}{4})$ and ${}^{**}\Delta_x^{\mathrm{II}} := (-D_{|x|}^{\mathrm{II}} + \frac{9}{4}) \circ (\Delta^*_{\frac{x}{|x|}} + \frac{1}{4})$ as well as the iterated operators $({}^{**}\Delta^{\mathrm{I}})^l$ and $({}^{**}\Delta^{\mathrm{II}})^l$, $l \in \mathbb{N}$, are invertible. Moreover, for any $l \in \mathbb{N}$, their eigenvalues corresponding to $G_{m,n,j}^{\mathrm{I}}$ and $G_{m,n,j}^{\mathrm{II}}$, respectively, satisfy*

$$ \begin{aligned} \left(\left({}^{**}\Delta^{\mathrm{I}}\right)^l\right)^\wedge(m,n) &= \left(\left(n+\frac{1}{2}\right)\left(n+2m+\frac{3}{2}\right)\right)^{2l}, \\ \left(\left({}^{**}\Delta^{\mathrm{II}}\right)^l\right)^\wedge(m,n) &= \left(\left(n+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)\right)^{2l}. \end{aligned} $$

Proof. The differential operator $(-D^{\mathrm{I}}+\frac{9}{4})$ has the symbol (i.e. the eigenvalues)

$$\left(-D^{\mathrm{I}}+\frac{9}{4}\right)^{\wedge}(m,n)=\left(n+2m+\frac{3}{2}\right)^{2},$$

since equation (9) yields

$$\begin{aligned}
&\left(-D^{\mathrm{I}}_r+\frac{9}{4}\right)\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right)\\
&= -D^{\mathrm{I}}_r\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right)+\frac{9}{4}\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right)\\
&= \left(n(n+3)+4m\left(m+n+\frac{3}{2}\right)\right)\cdot\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right)\\
&\quad+\frac{9}{4}\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right)\\
&= \left(n+2m+\frac{3}{2}\right)^2\left(P_m^{(0,n+\frac{1}{2})}\left(\frac{2r^2}{R^2}-1\right)\left(\frac{r}{R}\right)^n\right).
\end{aligned}$$

Hence, we conclude that $(-D^{\mathrm{I}}+\frac{9}{4})^\wedge(m,n)=\left(n+2m+\frac{3}{2}\right)^2\neq 0$ for all $m,n\in\mathbb{N}_0$. This shows that $(-D^{\mathrm{I}}+\frac{9}{4})$ is invertible. By applying induction, we have

$$\left(\left(-D^{\mathrm{I}}+\frac{9}{4}\right)^l\right)^{\wedge}(m,n)=\left(n+2m+\frac{3}{2}\right)^{2l}.$$

Furthermore, the operator $-\Delta^*+\frac{1}{4}$ has the eigenvalues

$$\left(-\Delta^*+\frac{1}{4}\right)^{\wedge}(n)=\left(n+\frac{1}{2}\right)^2$$

where $n=0,1,\ldots,$ and hence has an inverse $(-\Delta^*+\frac{1}{4})^{-1}$ which is a rational pseudo-differential operator of order -2 (for further details see [13]). More generally, $(-\Delta^*+\frac{1}{4})^l$ is a rational pseudo-differential operator of order $2l$ and has the spherical symbol (i.e. the eigenvalues)

$$\left(\left(-\Delta^*+\frac{1}{4}\right)^l\right)^{\wedge}(n)=\left(n+\frac{1}{2}\right)^{2l},n=0,1,\ldots. \tag{11}$$

Therefore, from the discussion given above, we can conclude that

$$^{**}\Delta^{\mathrm{I}}_x=\left(-D^{\mathrm{I}}_{|x|}+\frac{9}{4}\right)\circ\left(-\Delta^*_{\frac{x}{|x|}}+\frac{1}{4}\right)$$

is invertible and the eigenvalues corresponding to $G^{\mathrm{I}}_{m,n,j}$ are

$$\begin{aligned}\left(\left({}^{**}\Delta^{\mathrm{I}}\right)^l\right)^\wedge(m,n) &= \left(\left(-D^{\mathrm{I}}+\frac{9}{4}\right)^l\right)^\wedge(m,n)\left(\left(-\Delta^*+\frac{1}{4}\right)^l\right)^\wedge(n)\\ &= \left(\left(n+\frac{1}{2}\right)\left(n+2m+\frac{3}{2}\right)\right)^{2l}\end{aligned}$$

for all $m,n\in\mathbb{N}_0$ and all $j=-n,...,n$.
Similarly, $-D^{\mathrm{II}}+\frac{9}{4}$ has the eigenvalues

$$\left(-D^{\mathrm{II}}+\frac{9}{4}\right)^\wedge(m)=\left(m+\frac{3}{2}\right)^2,\ m=0,1,\ldots,$$

since

$$\begin{aligned}\left(-D^{\mathrm{II}}+\frac{9}{4}\right)P_m^{(0,2)}\left(\frac{2r}{R}-1\right) &= -D^{\mathrm{II}}\left(P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\right)+\frac{9}{4}P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\\ &= m(m+3)\left(P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\right)+\frac{9}{4}P_m^{(0,2)}\left(\frac{2r}{R}-1\right)\\ &= \left(m+\frac{3}{2}\right)^2P_m^{(0,2)}\left(\frac{2r}{R}-1\right).\end{aligned}$$

By induction, we obtain

$$\left(\left(-D^{\mathrm{II}}+\frac{9}{4}\right)^l\right)^\wedge(m)=\left(m+\frac{3}{2}\right)^{2l},\quad m=0,1\ldots. \tag{12}$$

From (12), one can conclude that $(-D^{\mathrm{II}}+\frac{9}{4})$ is invertible.
As $-\Delta^*+\frac{1}{4}$ and $-D^{\mathrm{II}}+\frac{9}{4}$ both are invertible, the combined operator

$${}^{**}\Delta^{\mathrm{II}}=\left(-D^{\mathrm{II}}+\frac{9}{4}\right)\circ\left(\Delta^*+\frac{1}{4}\right)$$

is also invertible. Therefore, using equations (11) and (12), we finally have the eigenvalues of $\left({}^{**}\Delta^{\mathrm{II}}\right)^l$ corresponding to $G^{\mathrm{II}}_{m,n,j}$,

$$\begin{aligned}\left(\left({}^{**}\Delta^{\mathrm{II}}\right)^l\right)^\wedge(m,n) &= \left(\left(\left(-D^{\mathrm{II}}+\frac{9}{4}\right)\circ\left(-\Delta^*+\frac{1}{4}\right)\right)^l\right)^\wedge(m,n)\\ &= \left(\left(n+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)\right)^{2l}.\end{aligned}$$

■

Definition 3.2 *For any $s \in \mathbb{R}$, we define the operators $({}^{**}\Delta^{\mathrm{I}})^s$ and $({}^{**}\Delta^{\mathrm{II}})^s$ by their eigenvalues*

$$\begin{aligned}\left(\left({}^{**}\Delta^{\mathrm{I}}\right)^s\right)^{\wedge}(m,n) &:= \left(\left(n+\frac{1}{2}\right)\left(n+2m+\frac{3}{2}\right)\right)^{2s}\\ \left(\left({}^{**}\Delta^{\mathrm{II}}\right)^s\right)^{\wedge}(m,n) &:= \left(\left(n+\frac{1}{2}\right)\left(m+\frac{3}{2}\right)\right)^{2s}\end{aligned}$$

corresponding to $G^{\mathrm{I}}_{m,n,j}$ and $G^{\mathrm{II}}_{m,n,j}$, respectively.

Note that all eigenvalues are independent of the order j of the chosen spherical harmonic $Y_{n,j}$, i.e. the operators are isotropic.

4 Further Properties of the Operators

We first summarize the introduction of Sobolev spaces on $\mathcal{B}$ from [24].

Definition 4.1 *A sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ satisfies the summability condition of type* I *if*

$$\sum_{m,n=0}^{\infty} A_{m,n}^2 n(2m+n)\frac{\left(n+m+\frac{1}{2}\right)^{2m}}{(m!)^2} < +\infty,$$

where as the summability condition of type II *is given by*

$$\sum_{m,n=0}^{\infty} A_{m,n}^2 n m^5 < +\infty.$$

If a sequence satisfies the summability condition of type I *or* II*, respectively, we say that the sequence is* I- *or* II- *summable, respectively.*

Definition 4.2 *Let the sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ be bounded and $\mathrm{X} \in \{\mathrm{I},\mathrm{II}\}$ be given. Then the space $\mathcal{H} := \mathcal{H}\left((A_{m,n}),\mathrm{X},\mathcal{B}\right)$ contains all $F \in \mathrm{L}^2(\mathcal{B})$ such that $\langle F, G^{\mathrm{X}}_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})} = 0$ for all (m,n,j) with $A_{m,n} = 0$ and*

$$\sum_{\substack{m,n=0\\ A_{m,n}\neq 0}}^{\infty} A_{m,n}^{-2} \sum_{j=1}^{2n+1} \left\langle F, G^{\mathrm{X}}_{m,n,j}\right\rangle^2_{\mathrm{L}^2(\mathcal{B})} < +\infty.$$

Moreover, $\mathcal{H}$ is equipped with the inner product

$$\langle F_1, F_2\rangle_{\mathcal{H}} := \sum_{\substack{m,n=0\\ A_{m,n}\neq 0}}^{\infty} A_{m,n}^{-2} \sum_{j=1}^{2n+1} \left\langle F_1, G^{\mathrm{X}}_{m,n,j}\right\rangle^2_{\mathrm{L}^2(\mathcal{B})} \left\langle F_2, G^{\mathrm{X}}_{m,n,j}\right\rangle^2_{\mathrm{L}^2(\mathcal{B})}.$$

Theorem 4.3 *The spaces $\mathcal{H} := \mathcal{H}\left((A_{m,n}), \mathrm{X}, \mathcal{B}\right)$, which we defined in Definition 4.2, are reproducing kernel Hilbert spaces with the unique reproducing kernel*

$$K_{\mathcal{H}}(x,y) = \sum_{m,n=0}^{\infty} \sum_{j=1}^{\infty} A_{m,n}^2 G_{m,n,j}^{\mathrm{X}}(x) G_{m,n,j}^{\mathrm{X}}(y); \quad x, y \in \mathcal{B},$$

if the sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ is X*-summable.*

Based on these Sobolev spaces, we can now clarify the domain of the operators $^{**}\Delta^{\mathrm{I}}$ and $^{**}\Delta^{\mathrm{II}}$.

Definition 4.4 *For any $s \in \mathbb{R}_0^+$, we define the spaces*

$$\mathcal{H}_s^{\mathrm{I}}(\mathcal{B}) := \mathcal{H}\left(\left(\left(n + 2m + \frac{3}{2}\right)^{-s} \left(n + \frac{1}{2}\right)^{-s}\right), \mathrm{I}, \mathcal{B}\right)$$

and

$$\mathcal{H}_s^{\mathrm{II}}(\mathcal{B}) := \mathcal{H}\left(\left(\left(m + \frac{3}{2}\right)^{-s} \left(n + \frac{1}{2}\right)^{-s}\right), \mathrm{II}, \mathcal{B}\right).$$

Obviously, $\mathcal{H}_{s_1}^{\mathrm{X}}(\mathcal{B}) \subset \mathcal{H}_{s_2}^{\mathrm{X}}(\mathcal{B})$ for $s_1 \geq s_2$ and $\mathrm{X} \in \{\mathrm{I}, \mathrm{II}\}$. Furthermore, $\mathcal{H}_0^{\mathrm{I}}(\mathcal{B}) = \mathcal{H}_0^{\mathrm{II}}(\mathcal{B}) = \mathrm{L}^2(\mathcal{B})$.
We can now reformulate Definition 3.2 more precisely.

Definition 4.5 *Let $s, t \in \mathbb{R}_0^+$ with $s \geq 2t$. Then we formally define the operators*

$$\left(^{**}\Delta^{\mathrm{I}}\right)^t : \mathcal{H}_s^{\mathrm{I}}(\mathcal{B}) \to \mathcal{H}_{s-2t}^{\mathrm{I}}(\mathcal{B})$$

and

$$\left(^{**}\Delta^{\mathrm{II}}\right)^t : \mathcal{H}_s^{\mathrm{II}}(\mathcal{B}) \to \mathcal{H}_{s-2t}^{\mathrm{II}}(\mathcal{B})$$

by

$$\left(^{**}\Delta^{\mathrm{I}}\right)^t F_1 = \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\left(n + \frac{1}{2}\right)\left(n + 2m + \frac{3}{2}\right)\right)^{2t} \left\langle F_1, G_{m,n,j}^{\mathrm{I}} \right\rangle_{\mathrm{L}^2(\mathcal{B})} G_{m,n,j}^{\mathrm{I}},$$

$$\left(^{**}\Delta^{\mathrm{II}}\right)^t F_2 = \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\left(n + \frac{1}{2}\right)\left(m + \frac{3}{2}\right)\right)^{2t} \left\langle F_2, G_{m,n,j}^{\mathrm{II}} \right\rangle_{\mathrm{L}^2(\mathcal{B})} G_{m,n,j}^{\mathrm{II}}$$

for $F_1 \in \mathcal{H}_s^{\mathrm{I}}(\mathcal{B})$ and $F_2 \in \mathcal{H}_s^{\mathrm{II}}(\mathcal{B})$.

To verify that the operators are well-defined, we show that

$$\begin{aligned}\left\| \left({}^{**}\Delta^{\mathrm{I}}\right)^t F_1 \right\|^2_{\mathcal{H}^{\mathrm{I}}_{s-2t}(\mathcal{B})} &= \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} \left(n+\frac{1}{2}\right)^{2s-4t} \left(n+2m+\frac{3}{2}\right)^{2s-4t} \\ &\qquad \times \left\langle \left({}^{**}\Delta^{\mathrm{I}}\right)^t F_1, G^{\mathrm{I}}_{m,n,j} \right\rangle^2_{\mathrm{L}^2(\mathcal{B})} \\ &= \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} \left(n+\frac{1}{2}\right)^{2s} \left(n+2m+\frac{3}{2}\right)^{2s} \left\langle F_1, G^{\mathrm{I}}_{m,n,j} \right\rangle^2_{\mathrm{L}^2(\mathcal{B})} \\ &= \|F_1\|^2_{\mathcal{H}^{\mathrm{I}}_s(\mathcal{B})} < +\infty.\end{aligned}$$

Analogous considerations can be made for type II. This also yields the following result.

Theorem 4.6 *Let $s,t \in \mathbb{R}^+_0$ with $s \geq 2t$. If $F_1 \in \mathcal{H}^{\mathrm{I}}_s(\mathcal{B})$ and $F_2 \in \mathcal{H}^{\mathrm{II}}_s(\mathcal{B})$, then*

$$\left\| \left({}^{**}\Delta^{\mathrm{I}}\right)^t F_1 \right\|_{\mathcal{H}^{\mathrm{I}}_{s-2t}(\mathcal{B})} = \|F_1\|_{\mathcal{H}^{\mathrm{I}}_s(\mathcal{B})},$$

$$\left\| \left({}^{**}\Delta^{\mathrm{II}}\right)^t F_2 \right\|_{\mathcal{H}^{\mathrm{II}}_{s-2t}(\mathcal{B})} = \|F_2\|_{\mathcal{H}^{\mathrm{II}}_s(\mathcal{B})}.$$

In particular,

$$\left\| \left({}^{**}\Delta^{\mathrm{I}}\right)^{s/2} F_1 \right\|_{\mathrm{L}^2(\mathcal{B})} = \|F_1\|_{\mathcal{H}^{\mathrm{I}}_s(\mathcal{B})},$$

$$\left\| \left({}^{**}\Delta^{\mathrm{II}}\right)^{s/2} F_2 \right\|_{\mathrm{L}^2(\mathcal{B})} = \|F_2\|_{\mathcal{H}^{\mathrm{II}}_s(\mathcal{B})}.$$

Therefore, the Sobolev norms $\|\cdot\|_{\mathcal{H}^{\mathrm{I}}_s(\mathcal{B})}$ and $\|\cdot\|_{\mathcal{H}^{\mathrm{II}}_s(\mathcal{B})}$ can be interpreted as the $\mathrm{L}^2(\mathcal{B})$-norms of certain generalized derivatives (i.e. in the sense of pseudo-differential operators). This is an important new result, since norms of this kind are used to measure the non-smoothness of interpolating functions on $\mathcal{B}$, where it is known that the interpolating spline minimizes the Sobolev norm among all interpolants (see [4, 5, 7, 8, 9, 24] for further details). Hence, this minimum property can, indeed, be considered as an analogue of corresponding results on the real line (see [16, Theorem 2.3.3]) and the sphere (see [13, Lemma 6.1.4]).
Moreover, since we defined that

$$\left\langle \left({}^{**}\Delta^{\mathrm{X}}\right)^t F, G^{\mathrm{X}}_{m,n,j} \right\rangle_{\mathrm{L}^2(\mathcal{B})} = \left(\left({}^{**}\Delta^{\mathrm{X}}\right)^\wedge (m,n)\right)^t \left\langle F, G^{\mathrm{X}}_{m,n,j} \right\rangle_{\mathrm{L}^2(\mathcal{B})}$$

for $\mathrm{X} \in \{\mathrm{I},\mathrm{II}\}$, $F \in \mathcal{H}^{\mathrm{X}}_s(\mathcal{B})$, $s \geq 2t$, $m,n \in \mathbb{N}_0, j = 1,...,2n+1$, we immediately get the following result.

Theorem 4.7 *The operators* $({}^{**}\Delta^{\mathrm{I}})^t$ *and* $({}^{**}\Delta^{\mathrm{II}})^t$ *are self-adjoint in the sense that*

$$\left\langle \left({}^{**}\Delta^{\mathrm{X}}\right)^t F_1, F_2 \right\rangle_{\mathrm{L}^2(\mathcal{B})} = \left\langle F_1, \left({}^{**}\Delta^{\mathrm{X}}\right)^t F_2 \right\rangle_{\mathrm{L}^2(\mathcal{B})}$$

for all $F_1, F_2 \in \mathcal{H}_s^{\mathrm{X}}(\mathcal{B})$ *with* $\mathrm{X} \in \{\mathrm{I}, \mathrm{II}\}$*.*

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References

[1] M. Akram: Constructive Approximation on the 3-dimensional Ball With Focus on Locally Supported Kernels and the Helmholtz Decomposition, Ph.D. Thesis, Department of Mathematics of the University of Kaiserslautern, 2008. Shaker Verlag, Aachen, 2009.

[2] M. Akram, V. Michel: Locally Supported Approximate Identities on the Unit Ball, Rev. Mat. Complut., 23 (2010), 233 - 249.

[3] M. Akram, V. Michel: Regularisation of the Helmholtz Decomposition and its Application to Geomagnetic Field Modelling, Int. J. Geomath., 1 (2010), 101-120.

[4] A. Amirbekyan: The Application of Reproducing Kernel Based Spline Approximation to Seismic Surface and Body Wave Tomography: Theoretical Aspects and Numerical Results, PhD Thesis, Department of Mathematics of the University of Kaiserslautern, 2007, http://kluedo.ub.uni-kl.de/volltexte/2007/2103.

[5] A. Amirbekyan, V. Michel: Splines on the Three-dimensional Ball and Their Application to Seismic Body Wave Tomography, Inverse Problems, 24 (2008), 015022 (25pp).

[6] L. Ballani, J. Engels, E.W. Grafarend: Global Base Functions for the Mass Density in the Interior of a Massive Body (Earth), Manuscr. Geod., 18 (1993), 99-114.

[7] P. Berkel: Multiscale Methods for the Combined Inversion of Normal Mode and Gravity Variations, PhD thesis, Department of Mathematics of the University of Kaiserslautern, Shaker Verlag, Aachen, 2009.

[8] P. Berkel, D. Fischer, V. Michel: Spline Multiresolution and Numerical Results for Joint Gravitation and Normal Mode Inversion with an Outlook on Sparse Regularisation, Int. J. Geomath., 1 (2011), 167-204.

[9] P. Berkel, V. Michel: On Mathematical Aspects of a Combined Inversion of Gravity and Normal Mode Variations by a Spline Method, Mathematical Geosciences, 42 (2010), 795-816.

[10] H. M. Dufour: Fonctions orthogonales dans la sphère: résolution théorique du problème du potentiel terrestre, Bull. Geod., 51 (1977), 227-237.

[11] M. J. Fengler, D. Michel, V. Michel: Harmonic Spline-wavelets on the 3-dimensional Ball and Their Application to the Reconstruction of the Earth's Density Distribution from Gravitational Data at Arbitrarily Shaped Satellite Orbits, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), 86 (2006), 856-873.

[12] A. S. Fokas, O. Hauk, V. Michel: Electro-magneto-encephalography for the Three-Shell Model: Numerical Implementation for Distributed Current in Spherical Geometry, Siegen Preprints on Geomathematics, 6 (2011).

[13] W. Freeden, T. Gervens, M. Schreiner: Constructive Approximation on the Sphere — With Applications to Geomathematics, Oxford University Press, Oxford, 1998.

[14] W. Freeden, V. Michel: Multiscale Potential Theory with Applications to Geoscience, Birkhäuser, Boston, 2004.

[15] W. Freeden, M. Schreiner: Spherical Functions of Mathematical Geosciences, a Scalar, Vectorial and Tensorial Setup, Springer, Berlin, 2009.

[16] W. Gautschi: Numerical Analysis — an Introduction, Birkhäuser, Boston, 1997.

[17] P. Kammann, V. Michel: Time-dependent Cauchy-Navier Splines and Their Application to Seismic Wave Front Propagation, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), 88 (2008), 155-178.

[18] V. Michel: A Wavelet Based Method for the Gravimetry Problem, in Progress in geodetic science, proceedings of the Geodetic Week 1998, edited by W. Freeden, 283-298, Shaker Verlag, Aachen, 1998.

[19] V. Michel: A Multiscale Method for the Gravimetry Problem - Theoretical and Numerical Aspects of Harmonic and Anharmonic Modelling, PhD thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, Shaker, Aachen, 1999.

[20] V. Michel: Scale Continuous, Scale Discretized and Scale Discrete Harmonic Wavelets for the Outer and the Inner Space of a Sphere and Their Application to an Inverse Problem in Geomathematics, Appl. Comput. Harmon. Anal. (ACHA), 12 (2002), 77-99.

[21] V. Michel: A Multiscale Approximation for Operator Equations in Separable Hilbert Spaces — Case Study: Reconstruction and Description of the Earth's Interior, Habilitation thesis, Shaker Verlag, Aachen, 2002.

[22] V. Michel: Theoretical Aspects of a Multiscale Analysis of the Eigenoscillations of the Earth, Rev. Mat. Complut., 16 (2003), 519-554.

[23] V. Michel: Regularized Wavelet-based Multiresolution Recovery of the Harmonic Mass Density Distribution from Data of the Earth's Gravitational Field at Satellite Height, Inverse Problems, 21 (2005), 997-1025.

[24] V. Michel: Tomography: Problems and Multiscale Solutions, in "Handbook of Geomathematics" (By W. Freeden, M.Z. Nashed, T. Sonar, eds), 2010, pp. 949-972.

[25] V. Michel, A. S. Fokas: A Unified Approach to Various Techniques for the Non-uniqueness of the Inverse Gravimetric Problem and Wavelet-based methods, Inverse Problems, 24 (2008), 045019 (25pp).

[26] V. Michel, K. Wolf: Numerical Aspects of a Spline-based Multiresolution Recovery of the Harmonic Mass Density out of Gravity Functionals, Geophysical Journal International, 173 (2008), 1-16.

[27] C. Müller: Spherical Harmonics, Springer, Berlin, 1996.

[28] G. Szegö: Orthogonal Polynomials, AMS Colloquium Publications, Volume XXIII, Providence, Rhode Island, 1939.

[29] C.C. Tscherning: Isotropic Reproducing Kernels for the Inner of a Sphere or Spherical Shell and Their Use as Density Covariance Functions, Math. Geol., 28 (1996), 161-168.

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