EXPONENTIAL CONVERGENCE OF ADAPTIVE QUARKLET APPROXIMATION

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Abstract. This paper is concerned with approximation properties of polynomially enriched wavelet systems, so-called quarklet frames. We show that certain model singularities that arise in elliptic boundary value problems on polygonal domains can be approximated from the span of such quarklet systems at inverse-exponential rates. In order to realize these, we combine spatial refinement in the vicinity of the singularities with suitable growth of the polynomial degrees in regions where the solution is smooth, similar to adaptive hp-finite element approximation.

Key words: Adaptive approximation, h-p-refinement, reconstruction properties, quarkonial decompositions, wavelets.

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1. Introduction

The numerical treatment of elliptic partial differential equations is an intensively studied field. Finite-element-methods (FEM) are well-established as an important tool for this purpose. The classical h-FEM relies on a space refinement of the domain Ω under consideration, another possibility is to increase the polynomial degree of the ansatz functions, this is known as the p-method. Even a combination of both, so-called hp-FEM, is possible. For an overview of FEM we refer to [2, 7, 10, 13].

When it comes to practical applications, adaptive strategies are very often indispensable to increase efficiency. In the FEM setting, a lot of very efficient strategies have been derived, we refer to [13] for an overview. In the context of adaptivity, in particular the hp-method is appealing since in many cases exponential convergence has been observed. However, rigorous proofs of this fact are still quite rare.

A quite different approach is the use of wavelets. Wavelets have strong analytic properties and form stable bases of function spaces such as the classical Sobolev spaces. This fact can be used to design adaptive wavelet methods that are guaranteed to converge with optimal order, cf. [3, 12]. Essentially, these adaptive strategies are based on space refinement. Therefore, they can be interpreted as h-methods. Hence, the question arises whether it is possible to design hp-versions of adaptive wavelet schemes. This leads to the concept of *quarklets*, which have been studied in [5, 6].

This paper is concerned with the approximation power of highly nonlinear quarklet schemes which serve as the benchmark for the performance of adaptive numerical algorithms based on quarklets. It is well-known that for second order elliptic boundary value problems on polygonal domains with reentrant corners singular solutions of the form $S(r,\varphi) = r^\alpha \sigma(r)\eta(\varphi)$ occur. Here, (r,φ) denote polar coordinates with respect to the reentrant corner, $\eta(\varphi)$ is a smooth function, $\sigma(r)$ is a smooth cut-off function, and α depends on the interior angle. Having this relationship in mind, very often univariate functions of the form x^α serve as models for typical singularities, see, e.g. [1] for details. In this paper, we show that by means of a suitable refinement strategy the approximation error that can be achieved by a quarklet dictionary decreases exponentially with respect to the degrees of freedom. This result holds for both, the L_2 and the H^1 norm, and relies on the construction of a certain spline with varying polynomial degree and mesh size, which generalizes the concept of [1] to higher order splines. In order to quantify the number of quarklet functions needed to realize a certain approximation accuracy, we will transfer some concepts of wavelet theory to the quarklet setting. In particular we need to switch between single-scale and multi-scale functions. To this end we derive *reconstruction properties* for quarklets.

Our approach has one important advantage when compared, e.g., with the case of *hp* finite element dictionaries. After suitable rescaling, our univariate quarklets form frames in L_2 and H^1, and they are therefore stable under anisotropic tensor product approximation techniques, see [5] for details. In this way, we can generalize our findings to the multivariate case and obtain exponentially convergent quarklet approximations also for anisotropic edge singularities.

The paper is organized as follows. In Section 2 we recall the basic idea of quarklet frames as polynomially enriched wavelet bases. In Section 3 we derive the important reconstruction property of quarklets. To this end, in Subsection 3.1 we establish some general reconstruction properties of multi-wavelets which are applied to the quarklet case in Subsection 3.2. In Section 4, we study approximation to x^α-type singularity functions in terms of quarklets in the function spaces $L_2(I), H_1(I)$ and $H_1(I^2)$.

2. Quarklets

In this section we briefly recall the basic properties of quarkonial systems, as far as they are needed for our purposes. For fixed $m \in \mathbb{N}$ let $\varphi := N_m(\cdot + \lfloor \frac{m}{2} \rfloor)$ denote the symmetrized cardinal B-spline with $\operatorname{supp} \varphi = [-\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil]$. The *quark* φ_p is defined as

$$\varphi_p(x) := \left(\tfrac{x}{\lceil m/2 \rceil}\right)^p \varphi(x), \quad \text{for all } p \in \mathbb{N}_0, x \in \mathbb{R}. \tag{2.1}$$

Further we consider dilated and translated copies of φ_p:

$$\varphi_{p,j,k}(x) := 2^{j/2} \varphi_p(2^j x - k), \quad \text{for all } p, j \in \mathbb{N}_0, k \in \mathbb{Z}, x \in \mathbb{R}. \tag{2.2}$$

It is well known that the cardinal B-splines are refinable, i.e., for $x \in \mathbb{R}$ it holds that

$$\varphi(x) = \sum_{k=-\lfloor \frac{m}{2} \rfloor}^{\lceil \frac{m}{2} \rceil} a_k \varphi(2x - k), \quad a_k = 2^{1-m} \binom{m}{k + \lfloor \frac{m}{2} \rfloor}. \tag{2.3}$$

For later use, we cite a refinement property of the functions $\varphi_0, \ldots, \varphi_p$ from [6]: Although each individual φ_q is usually not a refinable function, the whole collection $(\varphi_0, \ldots \varphi_p)$ forms a refinable function vector.

Proposition 2.1 ([6], Prop. 5)**.** *For any $p \geq 0$, the vector $(\varphi_0, \dots, \varphi_p)$ is refinable with $(p+1) \times (p+1)$-refinement matrices A_k given by*

$$(A_k)_{q,l} := \frac{1}{2^q} a_k \binom{q}{l} k^{q-l}, \tag{2.4}$$

i.e.,

$$\begin{pmatrix} \varphi_0(x) \\ \vdots \\ \varphi_p(x) \end{pmatrix} = \sum_{k \in \mathbb{Z}} A_k \begin{pmatrix} \varphi_0(2x-k) \\ \vdots \\ \varphi_p(2x-k) \end{pmatrix}, \quad x \in \mathbb{R}. \tag{2.5}$$

Remark 2.2. *Property* (2.5) *sets the stage for the application of concepts from wavelet analysis. Usually, the construction of wavelets is based on a* multiresolution analysis*, which is a nested sequence $\{V_j\}_{j \in \mathbb{Z}}$ of closed subspaces whose union is dense while the intersection is zero. Defining*

$$V_{p,j} := \overline{\operatorname{span}\{\Phi_q(2^j \cdot -k) : 0 \leq q \leq p, j \in \mathbb{N}_0, k \in \mathbb{Z}\}}^{L_2}, \tag{2.6}$$

the relation (2.5) *immediately implies that $V_{p,j} \subset V_{p,j+1}$. We refer to Subsection 3.1 for further information.*

Let the CDF spline wavelet ψ with $\tilde{m}$ vanishing moments be defined by

$$\psi(x) := \sum_{k \in \mathbb{Z}} b_k \varphi(2x-k), \quad x \in \mathbb{R}, \tag{2.7}$$

where we refer to [4] for a detailed description. Then, we define the *quarklet* ψ_p by

$$\psi_p(x) := \sum_{k \in \mathbb{Z}} b_k \varphi_p(2x-k), \quad \text{for all } p \in \mathbb{N}_0, x \in \mathbb{R}. \tag{2.8}$$

Further we consider dilated and translated copies of ψ_p:

$$\psi_{p,j,k}(x) := 2^{j/2} \psi_p(2^j x - k), \quad \text{for all } p, j \in \mathbb{N}_0, k \in \mathbb{Z}, x \in \mathbb{R}. \tag{2.9}$$

It can be shown that the quarks and quarklets inherit crucial properties of the B-splines and B-spline wavelets, respectively. In particular, Jackson and Bernstein estimates are fulfilled and the quarklets possess the same amount of vanishing moments. For details we refer to [6].

Theorem 2.3 ([6], Thm. 3)**.** *Let $w_p \geq 0$ be chosen such that $w_0 = 1$ and $w_p(p+1)^{-1/2}$ is summable. Then, the system*

$$\Psi_{Q,w} := \{w_p \psi_{p,j,k} : p \in \mathbb{N}_0, j \in N_0 \cup \{-1\}, k \in \mathbb{Z}\}, \tag{2.10}$$

forms a frame for $L_2(\mathbb{R})$.

Theorem 2.4 ([6], Thm. 4)**.** *For a given $\gamma > 0$, let $\varphi = N_m(\cdot + \lfloor \frac{m}{2} \rfloor)$, $m > \gamma + 1/2$. Then, the system*

$$\Psi_{Q,w,s} = \{ w_{p,j,s} \psi_{p,j,k} : p \in \mathbb{N}_0, j \in N_0 \cup \{-1\}, k \in \mathbb{Z} \}, \tag{2.11}$$

with $w_{p,j,s} := 2^{-js}(p+1)^{-2s-\delta}$ for $j \in \mathbb{N}_0$ and $w_{p,-1,s} := w_{p,0,s}$ with $\delta > 1$ has the frame property in $H^s(\mathbb{R})$, $0 < s < \gamma$.

3. Reconstruction Properties

To establish the exponential convergence of quarklet expansions, it is necessary to express a fine quark $\varphi_{p,j,k}$ as a linear combination of coarse quarks $\varphi_{q,j-1,l}$ and quarklets $\psi_{q,j-1,n}$, i.e., we look for a decomposition relation

$$\varphi_{p,j,k}(x) = \sum_{q=0}^{p} \sum_{l \in \mathbb{Z}} c_{p,q,j-1,l} \varphi_{q,j-1,l}(x) + \sum_{q=0}^{p} \sum_{n \in \mathbb{Z}} d_{p,q,j-1,n} \psi_{q,j-1,n}(x), \quad x \in \mathbb{R}, \tag{3.1}$$

where $\boldsymbol{c}, \boldsymbol{d}$ are called *reconstruction sequences*. The existence of a relation (3.1) is nontrivial, and we will establish a sufficient criterion for such expansions to hold in the context of general multi-generators.

3.1. **The General Setting.** Let $\Phi = (\Phi_0, \ldots, \Phi_p)^T$ be a vector of functions from $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. The function vector Φ is called *refinable* if there exists a sequence of $(p+1) \times (p+1)$-matrices A_k such that

$$\Phi(x) = \sum_{k \in \mathbb{Z}} A_k \Phi(2x - k), \quad x \in \mathbb{R}, \ \{A_k\}_{k \in \mathbb{Z}} \in \ell_2(\mathbb{Z})^{(p+1)\times(p+1)}. \tag{3.2}$$

To avoid technical difficulties, in the sequel we will always assume the stronger condition $\{A_k\}_{k \in \mathbb{Z}} \in \ell_1(\mathbb{Z})^{(p+1)\times(p+1)}$. Defining a second function vector $\Psi = (\Psi_0, \ldots, \Psi_p)^T$ by

$$\Psi(x) := \sum_{k \in \mathbb{Z}} B_k \Phi(2x - k), \quad x \in \mathbb{R}, \tag{3.3}$$

where B_k are $(p+1) \times (p+1)$-matrices, the question arises whether the functions defined in (3.3) span an algebraic complement $W_{p,j}$ such that $V_{p,j+1} = V_{p,j} \oplus W_{p,j}$. We will need the following proposition whose proof can be performed by standard arguments and will be presented in the appendix.

Proposition 3.1.

(i) *The Fourier transform* $\hat{\Phi} = (\hat{\Phi}_0, \ldots, \hat{\Phi}_p)^T$ *of a refinable function vector fulfills the matrix equation*

$$\hat{\Phi}(\xi) = \frac{1}{2}\mathscr{A}(z)\hat{\Phi}\left(\frac{\xi}{2}\right), \quad \xi \in \mathbb{R}, z = e^{i\frac{\xi}{2}} \in S_1, \tag{3.4}$$

where

$$\left(\mathscr{A}(z)\right)_{q,l} := \sum_{k\in\mathbb{Z}} \left(A_k\right)_{q,l} z^k \tag{3.5}$$

is called the symbol matrix *of* Φ.

(ii) *For a symbol matrix* $\mathscr{A}(z)$ *and* $\rho \in \{0,1\}$ *we define the* sub-symbol matrices $\mathscr{A}_\rho(z^2)$ *by*

$$\left(\mathscr{A}_\rho(z^2)\right)_{q,l} := \sum_{k\in\mathbb{Z}} \left(A_{2k+\rho}\right)_{q,l} z^{2k}. \tag{3.6}$$

Then, it holds that

$$\mathscr{A}_0(z^2) = \frac{1}{2}\left(\mathscr{A}(z) + \mathscr{A}(-z)\right), \quad \mathscr{A}_1(z^2) = \frac{1}{2z}\left(\mathscr{A}(z) - \mathscr{A}(-z)\right). \tag{3.7}$$

Applying the Fourier transform to (3.3) in an analogous way, it follows that

$$\hat{\Psi}(\xi) = \frac{1}{2}\mathscr{B}(z)\hat{\Phi}\left(\frac{\xi}{2}\right), \quad \left(\mathscr{B}(z)\right)_{q,l} := \sum_{k\in\mathbb{Z}} \left(B_k\right)_{q,l} z^k. \tag{3.8}$$

To achieve a decomposition relation (3.1), we need another preparatory result. By refinability, it suffices to consider $j = 1$. Because the spaces $V_{p,j}$ are shift-invariant and we have $\Phi_p(2x-k) = \Phi_p(2(x-\tilde{k})-\rho), \tilde{k} \in \mathbb{Z}, \rho \in \{0,1\}$, it is sufficient to derive a decomposition relation of $\Phi_p(2x-\rho)$.

Theorem 3.2. *Suppose that there exist* $(p+1)\times(p+1)$ *matrices* $C_{\rho+2k}, D_{\rho+2l}$, *such that*

$$\begin{pmatrix} \mathscr{C}_0(z^2) & \mathscr{D}_0(z^2) \\ \mathscr{C}_1(z^2) & \mathscr{D}_1(z^2) \end{pmatrix} \begin{pmatrix} \mathscr{A}_0(z^2) & \mathscr{A}_1(z^2) \\ \mathscr{B}_0(z^2) & \mathscr{B}_1(z^2) \end{pmatrix} = I, \tag{3.9}$$

where the sub-symbol matrices $\mathscr{C}_\rho(z^2)$, $\rho \in \{0,1\}$, *are defined by*

$$\left(\mathscr{C}_\rho(z^2)\right)_{q,l} = \sum_{k\in\mathbb{Z}} (C_{\rho+2k})_{q,l} z^{2k}, \quad \left(\mathscr{D}_\rho(z^2)\right)_{q,l} = \sum_{k\in\mathbb{Z}} (D_{\rho+2k})_{q,l} z^{2k}.$$

Then, each function vector $\Phi(2\cdot-\rho)$ *has a decomposition in terms of coarse generators* Φ *and wavelets* Ψ, *i.e., it holds*

$$\Phi(2x-\rho) = \sum_{k\in\mathbb{Z}} C_{\rho+2k}\Phi(x-k) + \sum_{n\in\mathbb{Z}} D_{\rho+2n}\Psi(x-n), \quad x \in \mathbb{R}. \tag{3.10}$$

Proof. Applying a component-wise Fourier-transform to (3.10) yields

$$\left(\frac{1}{2}e^{-i\frac{\xi}{2}\rho}I\right)\hat{\Phi}\left(\frac{\xi}{2}\right) = \mathscr{C}_\rho(z^2)\hat{\Phi}(\xi) + \mathscr{D}_\rho(z^2)\hat{\Psi}(\xi), \quad \xi \in \mathbb{R}, z \in S_1.$$

With (3.4) and (3.8) we get

$$\left(\frac{1}{2}e^{-i\frac{\xi}{2}\rho}I\right)\hat{\Phi}\left(\frac{\xi}{2}\right) = \frac{1}{2}\mathscr{C}_\rho(z^2)\mathscr{A}(z)\hat{\Phi}\left(\frac{\xi}{2}\right) + \frac{1}{2}\mathscr{D}_\rho(z^2)\mathscr{B}(z)\hat{\Phi}\left(\frac{\xi}{2}\right).$$

Hence, a sufficient condition for (3.10) is given by

$$z^\rho I = \mathscr{C}_\rho(z^2)\mathscr{A}(z) + \mathscr{D}_\rho(z^2)\mathscr{B}(z), \quad z \in S_1. \tag{3.11}$$

With (3.7) this is equivalent to

$$\begin{aligned} z^\rho I &= \mathscr{C}_\rho(z^2)\left(\sum_{\hat{\rho}=0,1} z^{\hat{\rho}}\mathscr{A}_{\hat{\rho}}(z^2)\right) + \mathscr{D}_\rho(z^2)\left(\sum_{\hat{\rho}=0,1} z^{\hat{\rho}}\mathscr{B}_{\hat{\rho}}(z^2)\right) \\ &= \sum_{\hat{\rho}=0,1} z^{\hat{\rho}}\left(\mathscr{C}_\rho(z^2)\mathscr{A}_{\hat{\rho}}(z^2) + \mathscr{D}_\rho(z^2)\mathscr{B}_{\hat{\rho}}(z^2)\right). \end{aligned}$$

Hence, by (3.9) the claim follows. □

Proposition 3.3. *Defining*

$$X(z) := \frac{1}{2}\begin{pmatrix} \mathscr{A}(z) & \mathscr{A}(-z) \\ \mathscr{B}(z) & \mathscr{B}(-z) \end{pmatrix}, \quad \mathscr{E}(z) := \begin{pmatrix} I & \frac{1}{z}I \\ I & -\frac{1}{z}I \end{pmatrix}, \tag{3.12}$$

it holds that

$$\begin{pmatrix} \mathscr{A}_0(z^2) & \mathscr{A}_1(z^2) \\ \mathscr{B}_0(z^2) & \mathscr{B}_1(z^2) \end{pmatrix} = X(z)\mathscr{E}(z). \tag{3.13}$$

Moreover, $\mathscr{E}(z)$ is invertible on S_1 with

$$\det \mathscr{E}(z) = 2^{p+1}(-z)^{-p-1}, \quad \mathscr{E}(z)^{-1} = \frac{1}{2}\begin{pmatrix} I & I \\ zI & -zI \end{pmatrix}. \tag{3.14}$$

Proof. One easily verifies (3.13) with the help of (3.7):

$$\begin{aligned} X(z)\mathscr{E}(z) &= \begin{pmatrix} \frac{1}{2}(\mathscr{A}(z) + \mathscr{A}(-z)) & \frac{1}{2z}(\mathscr{A}(z) - \mathscr{A}(-z)) \\ \frac{1}{2}(\mathscr{B}(z) + \mathscr{B}(-z)) & \frac{1}{2z}(\mathscr{B}(z) - \mathscr{B}(-z)) \end{pmatrix} \\ &= \begin{pmatrix} \mathscr{A}_0(z^2) & \mathscr{A}_1(z^2) \\ \mathscr{B}_0(z^2) & \mathscr{B}_1(z^2) \end{pmatrix}. \end{aligned}$$

Using block determinant formulas, cf. [11], we have

$$\begin{aligned}\det \mathscr{E}(z) &= \det(-\frac{1}{z}I)\det(I + zI\frac{1}{z}II)\\ &= \det(-\frac{1}{z}I)\det(2I)\\ &= (-z)^{-p-1}2^{p+1}.\end{aligned}$$

Additionally, one easily verifies $\mathscr{E}(z)\mathscr{E}(z)^{-1} = I$. □

Remark 3.4. *In case that the matrix*

$$X(z)\mathscr{E}(z) = \begin{pmatrix} \mathscr{A}_0(z^2) & \mathscr{A}_1(z^2) \\ \mathscr{B}_0(z^2) & \mathscr{B}_1(z^2) \end{pmatrix}$$

in (3.9) *is invertible on the torus S_1, it follows that all entries in*

$$\begin{pmatrix} \mathscr{C}_0(z^2) & \mathscr{D}_0(z^2) \\ \mathscr{C}_1(z^2) & \mathscr{D}_1(z^2) \end{pmatrix}$$

consist of symbols whose coefficients are contained in $\ell_1(\mathbb{Z})$. Indeed, by our assumption, every entry in $X(z)\mathscr{E}(z)$ has this property, therefore the same holds for the determinant. Consequently, if the determinant does not vanish on the torus, the result follows by an application of the Wiener lemma, see, e.g., [9], page 278, for details.

Remark 3.5. *In practice, one usually works with compactly supported generators that possess finitely supported masks $\{A_k\}_{k\in\mathbb{Z}}$. Then, it is of course desirable to find wavelets such that the entries of the matrix*

$$\begin{pmatrix} \mathscr{A}(z) & \mathscr{A}(-z) \\ \mathscr{B}(z) & \mathscr{B}(-z) \end{pmatrix}^{-1}$$

consist of Laurent polynomials, for then the reconstruction sequences in (3.10) *are also finitely supported. Fortunately, in the quarklet case, this is indeed the case, see Subsection 3.2.*

3.2. **Application to Quarks.** We apply the general theory of reconstruction to the quarklet case. The assumptions (3.2) and (3.3) simplify as follows: The matrices A_k defined in (2.4) and hence the symbol matrix $\mathscr{A}(z)$ are lower triangular matrices. The definition of the quarklets (2.8) can be interpreted as a matrix equation with

$$B_k = \operatorname{diag}(b_k, \dots, b_k).$$

Accordingly (3.8) becomes

$$\hat{\Psi}(\xi) = \frac{1}{2} b(z) I \hat{\Phi}\left(\frac{\xi}{2}\right), \tag{3.15}$$

where $b(z)$ is the symbol of the wavelet ψ.

Remark 3.6. *For later use let us recall some basic facts concerning the construction of biorthogonal wavelets. Let $a(z), \tilde{a}(z)$ be the symbols of the primal (dual) generator. The biorthogonality of the generators φ und $\tilde{\varphi}$ implies that*

$$a(z)\overline{\tilde{a}(z)} + a(-z)\overline{\tilde{a}(z)} = 4. \tag{3.16}$$

The wavelet symbols are chosen as

$$b(z) = -z\overline{\tilde{a}(-z)}, \quad \tilde{b}(z) = -z\overline{a(-z)}, \tag{3.17}$$

Hence, (3.16) *and* (3.17) *imply the fundamental identity*

$$a(z)b(-z) - b(z)a(-z) = 4z. \tag{3.18}$$

Theorem 3.7. *Let $z \in S_1$. With the definition*

$$X(z) := \frac{1}{2}\begin{pmatrix} \mathscr{A}(z) & \mathscr{A}(-z) \\ b(z)I & b(-z)I \end{pmatrix}, \tag{3.19}$$

it holds that

$$\det X(z) = 2^{-p(p+1)/2} z^{p+1}. \tag{3.20}$$

If $b(-z) \neq 0$, then $b(-z)I$ is invertible on S_1 and

$$X(z)^{-1} = 2\begin{pmatrix} T(z)^{-1} & -\frac{1}{b(-z)}T(z)^{-1}\mathscr{A}(-z) \\ -\frac{b(z)}{b(-z)}T(z)^{-1} & \frac{1}{b(-z)}T(z)^{-1}\mathscr{A}(z) \end{pmatrix}, \tag{3.21}$$

where

$$T(z) = \frac{1}{2}\left(\mathscr{A}(z) - \frac{b(z)}{b(-z)}\mathscr{A}(-z)\right). \tag{3.22}$$

Otherwise, if $b(-z) = 0$*, it holds*

$$X(z)^{-1} = 2\begin{pmatrix} 0 & \frac{1}{b(z)}I \\ \mathscr{A}(-z)^{-1} & -\frac{1}{b(z)}\mathscr{A}(-z)^{-1}\mathscr{A}(z) \end{pmatrix}. \tag{3.23}$$

Proof. Since $b(z)I$ and $b(-z)I$ commute, the determinant can by computed by, cf. [11]

$$\det X(z) = \left(\frac{1}{2}\right)^{2p+2} \det(\mathscr{A}(z)b(-z)I - b(z)I\mathscr{A}(-z)).$$

The matrix $\mathscr{A}(z)b(-z) - b(z)\mathscr{A}(-z)$ is of lower triangular shape with diagonal entries

$$\begin{aligned} a_{qq}(z)b(-z) - b(z)a_{qq}(-z) &= 2^{-q}(a(z)b(-z) - b(z)a(-z)) \\ &= 2^{-q}4z, \end{aligned}$$

where we used (3.5), (2.4), (3.18). Hence we conclude

$$\begin{aligned} \det X(z) &= 2^{-2p-2}\prod_{q=0}^{p} 2^{-q}4z \\ &= 2^{-2p-2}(4z)^{p+1}2^{-\sum_{q=0}^{p} q} \\ &= z^{p+1}2^{-p(p+1)/2}. \end{aligned}$$

First let us consider the case $b(-z) \neq 0$. Again, $T(z)$ is a lower triangular matrix, hence its determinant is given by

$$\begin{aligned} \det T(z) &= \left(\frac{1}{2}\right)^{p+1} \prod_{q=0}^{p}\left(a_{qq}(z) - \frac{b(z)}{b(-z)}a_{qq}(-z)\right) \\ &= 2^{-p-1}\prod_{q=0}^{p} 2^{-q}\frac{a(z)b(-z) - b(z)a(-z)}{b(-z)} \\ &= 2^{-p-1}2^{-p(p+1)/2}\left(\frac{4z}{b(-z)}\right)^{p+1}. \end{aligned}$$

We then calculate

$$
\begin{aligned}
&X(z)^{-1}X(z)\\
&= \begin{pmatrix} T(z)^{-1}\left(\mathscr{A}(z) - \frac{b(z)}{b(-z)}\mathscr{A}(-z)\right) & T(z)^{-1}\mathscr{A}(-z) - T(z)^{-1}\mathscr{A}(-z) \\ T(z)^{-1}\left(-\frac{b(z)}{b(-z)}\mathscr{A}(z) + \frac{b(z)}{b(-z)}\mathscr{A}(z)\right) & T(z)^{-1}\left(-\frac{b(z)}{b(-z)}\mathscr{A}(-z) + \mathscr{A}(z)\right) \end{pmatrix}\\
&= \begin{pmatrix} T(z)^{-1}T(z) & 0 \\ 0 & T(z)^{-1}T(z) \end{pmatrix}.
\end{aligned}
$$

Now let $b(-z) = 0$. Then, with (3.18) it holds that $b(z) \neq 0, a(-z) \neq 0$ and hence $b(z)I$ and $\mathscr{A}(-z)$ are invertible. From (3.18) we also conclude $a(-z) = -\frac{4z}{b(z)}$. We compute

$$
\det \mathscr{A}(-z) = \prod_{q=0}^{p} 2^{-q} a(-z) = 2^{-p(p+1)/2}\left(-\frac{4z}{b(z)}\right)^{p+1}. \tag{3.24}
$$

Again one easily verifies (3.23). □

Theorem 3.8. *Let $\rho \in \{0,1\}$. The matrix*

$$
\begin{pmatrix} \mathscr{C}_0(z^2) & \mathscr{D}_0(z^2) \\ \mathscr{C}_1(z^2) & \mathscr{D}_1(z^2) \end{pmatrix}
$$

in (3.9) *consists of Laurent polynomials only. Furthermore, the length of the reconstruction sequences $(C_{\rho+2k})_{k\in\mathbb{Z}}$, $(D_{\rho+2k})_{k\in\mathbb{Z}}$ scales linearly with p.*

Proof. By (3.13), invertibility of $X(z)$ implies invertibility of the matrix

$$
S(z^2) := \begin{pmatrix} \mathscr{A}_0(z^2) & \mathscr{A}_1(z^2) \\ \mathscr{B}_0(z^2) & \mathscr{B}_1(z^2) \end{pmatrix}
$$

from (3.9). Furthermore, (3.20) implies that $\det S(z^2)$ is a monomial. Hence, since any subdeterminant of $S(z^2)$ is also a Laurent polynomial, each entry of $S(z^2)^{-1}$ is a Laurent polynomial as well. The claim follows with another application of (3.9). The dimension of the matrix $S(z^2)$ is of order p, hence the degree of its subdeterminants is linearly increasing with respect to p and the second claim follows. □

Theorem 3.9. *Each multi-quark $\Phi(2^j \cdot -\rho)$ on level j has a decomposition in terms of multi-quarks Φ and multi-quarklets Ψ, i.e., it holds*

$$\Phi(2^j \cdot -\rho) = \sum_{k=jpk_-}^{jpk_+} C_{j,\rho+2k}\Phi(\cdot - k) + \sum_{i=0}^{j-1} \sum_{n=ipk_-+pn_-}^{ipk_++pn_+} D_{i,\rho+2n}\Psi(2^{j-1-i} \cdot -n). \tag{3.25}$$

Furthermore, the length of the sequences C_j (D_i) is of order jp (ip). The overall length of the reconstruction sequences is of order j^2p.

Proof. To prove (3.25), we iteratively apply the decomposition relation (3.10). In particular we have to determine the number of multi-quarks and multi-quarklets, respectively. With the length of the reconstruction sequences being of order p, see Theorem 3.8, we conclude that the translations corresponding to the nontrivial coefficients are contained in the interval $[pk_-, pk_+]$ and $[pn_-, pn_+]$, respectively. Without loss of generality we assume $k_-, n_- < 0$. We have

$$\Phi(2^j \cdot -\rho) = \sum_{k=pk_-}^{pk_+} C_{0,\rho+2k}\Phi(2^{j-1} \cdot -k) + \sum_{n=pn_-}^{pn_+} D_{0,\rho+2n}\Psi(2^{j-1} \cdot -n).$$

Again decomposing the multi-quarks leads to

$$\begin{aligned}\sum_{k=pk_-}^{pk_+} C_{0,\rho+2k}\Phi(2^{j-1} \cdot -k) &= \sum_{k=2pk_-}^{2pk_+} C_{1,\rho+2k}\Phi(2^{j-2} \cdot -k) \\ &\quad + \sum_{n=pk_-+pn_-}^{pk_++pn_+} D_{1,\rho+2n}\Psi(2^{j-2} \cdot -n).\end{aligned}$$

Inductively we get

$$\Phi(2^j \cdot -\rho) = \sum_{k=jpk_-}^{jpk_+} C_{j,\rho+2k}\Phi(\cdot - k) + \sum_{i=0}^{j-1} \sum_{n=ipk_-+pn_-}^{ipk_++pn_+} D_{i,\rho+2n}\Psi(2^{j-1-i} \cdot -n).$$

Counting the nontrivial entries of C_j, D_i leads to $|\operatorname{supp} C_j| = jp(k_+ - k_-)$, $|\operatorname{supp} D_i| = ip(k_+ - k_-) + p(n_+ - n_-)$. Since k_-, k_+, n_-, n_+ depend on the reconstruction properties of the underlying wavelet basis, only, we get $|\operatorname{supp} C_j| \sim jp$, $|\operatorname{supp} D_i| \sim ip$. Summation over i yields a total number of function vectors of order j^2p. □

TABLE 1. Refinement coefficients of Φ in z-notation.

m	p	$\mathscr{A}(z)$
1	0	$\begin{pmatrix} z+1 \end{pmatrix}$
	1	$\begin{pmatrix} z+1 & 0 \\ \frac{1}{2}z & \frac{1}{2}z+\frac{1}{2} \end{pmatrix}$
	2	$\begin{pmatrix} z+1 & 0 & 0 \\ \frac{1}{2}z & \frac{1}{2}z+\frac{1}{2} & 0 \\ \frac{1}{4}z & \frac{1}{2}z & \frac{1}{4}z+\frac{1}{4} \end{pmatrix}$
2	0	$\begin{pmatrix} \frac{\frac{1}{2}z^2+z+\frac{1}{2}}{z} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{\frac{1}{2}z^2+z+\frac{1}{2}}{z} & 0 \\ \frac{\frac{1}{4}z^2-\frac{1}{4}}{z} & \frac{\frac{1}{4}z^2+\frac{1}{2}z+\frac{1}{4}}{z} \end{pmatrix}$
	2	$\begin{pmatrix} \frac{\frac{1}{2}z^2+z+\frac{1}{2}}{z} & 0 & 0 \\ \frac{\frac{1}{4}z^2-\frac{1}{4}}{z} & \frac{\frac{1}{4}z^2+\frac{1}{2}z+\frac{1}{4}}{z} & 0 \\ \frac{\frac{1}{8}z^2+\frac{1}{8}}{z} & \frac{\frac{1}{4}z^2-\frac{1}{4}}{z} & \frac{\frac{1}{8}z^2+\frac{1}{4}z+\frac{1}{8}}{z} \end{pmatrix}$
3	0	$\begin{pmatrix} \frac{\frac{1}{4}z^3+\frac{3}{4}z^2+\frac{3}{4}z+\frac{1}{4}}{z} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{\frac{1}{4}z^3+\frac{3}{4}z^2+\frac{3}{4}z+\frac{1}{4}}{z} & 0 \\ \frac{\frac{1}{4}z^3+\frac{3}{8}z^2-\frac{1}{8}}{z} & \frac{\frac{1}{8}z^3+\frac{3}{8}z^2+\frac{3}{8}z+\frac{1}{8}}{z} \end{pmatrix}$

TABLE 2. Reconstruction coefficients of $\Phi(2\cdot)$ in z-notation.

m	p	$\mathscr{C}_0(z^2)$
1	0	$\begin{pmatrix} \frac{1}{2} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{4} & 1 \end{pmatrix}$
	2	$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$
2	0	$\begin{pmatrix} \frac{-\frac{1}{8}z^4+\frac{3}{4}z^2-\frac{1}{8}}{z^2} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{-\frac{1}{8}z^4+\frac{3}{4}z^2-\frac{1}{8}}{z^2} & 0 \\ \frac{\frac{1}{64}z^8-\frac{3}{32}z^6+\frac{3}{32}z^2-\frac{1}{64}}{z^4} & \frac{-\frac{1}{4}z^4+\frac{3}{2}z^2-\frac{1}{4}}{z^2} \end{pmatrix}$
	2	$\begin{pmatrix} \frac{-\frac{1}{8}z^4+\frac{3}{4}z^2-\frac{1}{8}}{z^2} & 0 & 0 \\ \frac{\frac{1}{64}z^8-\frac{3}{32}z^6+\frac{3}{32}z^2-\frac{1}{64}}{z^4} & \frac{-\frac{1}{4}z^4+\frac{3}{2}z^2-\frac{1}{4}}{z^2} & 0 \\ \frac{-\frac{1}{256}z^{12}+\frac{5}{128}z^{10}-\frac{15}{256}z^8-\frac{13}{64}z^6-\frac{15}{256}z^4+\frac{5}{128}z^2-\frac{1}{256}}{z^6} & \frac{\frac{1}{16}z^8-\frac{3}{8}z^6+\frac{3}{8}z^2-\frac{1}{16}}{z^4} & \frac{-\frac{1}{2}z^4+3z^2-\frac{1}{2}}{z^2} \end{pmatrix}$
3	0	$\begin{pmatrix} \frac{-\frac{9}{64}z^6+\frac{45}{64}z^4-\frac{7}{64}z^2+\frac{3}{64}}{z^4} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{-\frac{9}{64}z^6+\frac{45}{64}z^4-\frac{7}{64}z^2+\frac{3}{64}}{z^4} & 0 \\ \frac{-\frac{81}{16384}z^{14}+\frac{999}{16384}z^{12}-\frac{1881}{16384}z^{10}-\frac{5865}{16384}z^8+\frac{3749}{16384}z^6-\frac{1251}{16384}z^4+\frac{261}{16384}z^2-\frac{27}{16384}}{z^8} & \frac{-\frac{9}{32}z^6+\frac{45}{32}z^4-\frac{7}{32}z^2+\frac{3}{32}}{z^4} \end{pmatrix}$

TABLE 3. Reconstruction coefficients of $\Phi(2\cdot)$ in z-notation.

m	p	$\mathscr{D}_0(z^2)$
1	0	$\begin{pmatrix} \frac{1}{2} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}$
	2	$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
2	0	$\begin{pmatrix} \frac{\frac{1}{4}z^2+\frac{1}{4}}{z^2} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{\frac{1}{4}z^2+\frac{1}{4}}{z^2} & 0 \\ \frac{-\frac{1}{32}z^6+\frac{7}{32}z^4-\frac{7}{32}z^2+\frac{1}{32}}{z^4} & \frac{\frac{1}{4}z^2+\frac{1}{4}}{z^2} \end{pmatrix}$
	2	$\begin{pmatrix} \frac{\frac{1}{4}z^2+\frac{1}{4}}{z^2} & 0 & 0 \\ \frac{-\frac{1}{32}z^6+\frac{7}{32}z^4-\frac{7}{32}z^2+\frac{1}{32}}{z^4} & \frac{\frac{1}{4}z^2+\frac{1}{4}}{z^2} & 0 \\ \frac{\frac{1}{128}z^{10}-\frac{11}{128}z^8+\frac{13}{64}z^6+\frac{13}{64}z^4-\frac{11}{128}z^2+\frac{1}{128}}{z^6} & \frac{-\frac{1}{16}z^6+\frac{7}{16}z^4-\frac{7}{16}z^2+\frac{1}{16}}{z^4} & \frac{\frac{1}{4}z^2+\frac{1}{4}}{z^2} \end{pmatrix}$
3	0	$\begin{pmatrix} \frac{\frac{3}{8}z^2+\frac{1}{8}}{z^2} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{\frac{3}{8}z^2+\frac{1}{8}}{z^2} & 0 \\ \frac{\frac{27}{2048}z^{10}-\frac{189}{2048}z^8+\frac{159}{1024}z^6-\frac{93}{1024}z^4+\frac{39}{2048}z^2-\frac{9}{2048}}{z^6} & \frac{\frac{3}{8}z^2+\frac{1}{8}}{z^2} \end{pmatrix}$

TABLE 4. Reconstruction coefficients of $\Phi(2\cdot -1)$ in z-notation.

m	p	$\mathscr{C}_1(z^2)$
1	0	$\begin{pmatrix} \frac{1}{2} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{4} & 1 \end{pmatrix}$
	2	$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$
2	0	$\begin{pmatrix} \frac{1}{4}z^2 + \frac{1}{4} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{1}{4}z^2 + \frac{1}{4} & 0 \\ \frac{-\frac{1}{32}z^6 - \frac{1}{32}z^4 + \frac{1}{32}z^2 + \frac{1}{32}}{z^2} & \frac{1}{2}z^2 + \frac{1}{2} \end{pmatrix}$
	2	$\begin{pmatrix} \frac{1}{4}z^2 + \frac{1}{4} & 0 & 0 \\ \frac{-\frac{1}{32}z^6 - \frac{1}{32}z^4 + \frac{1}{32}z^2 + \frac{1}{32}}{z^2} & \frac{1}{2}z^2 + \frac{1}{2} & 0 \\ \frac{\frac{1}{128}z^{10} - \frac{3}{128}z^8 - \frac{7}{64}z^6 - \frac{7}{64}z^4 - \frac{3}{128}z^2 + \frac{1}{128}}{z^4} & \frac{-\frac{1}{8}z^6 - \frac{1}{8}z^4 + \frac{1}{8}z^2 + \frac{1}{8}}{z^2} & z^2 + 1 \end{pmatrix}$
3	0	$\begin{pmatrix} \frac{\frac{3}{64}z^6 - \frac{7}{64}z^4 + \frac{45}{64}z^2 - \frac{9}{64}}{z^2} \end{pmatrix}$
	1	$\begin{pmatrix} \frac{\frac{3}{64}z^6 - \frac{7}{64}z^4 + \frac{45}{64}z^2 - \frac{9}{64}}{z^2} & 0 \\ \frac{\frac{27}{16384}z^{14} - \frac{261}{16384}z^{12} + \frac{483}{16384}z^{10} - \frac{1957}{16384}z^8 - \frac{5655}{16384}z^6 + \frac{4185}{16384}z^4 - \frac{999}{16384}z^2 + \frac{81}{16384}}{z^6} & \frac{\frac{3}{32}z^6 - \frac{7}{32}z^4 + \frac{45}{32}z^2 - \frac{9}{32}}{z^2} \end{pmatrix}$

TABLE 5. Reconstruction coefficients of $\Phi(2\cdot -1)$ in z-notation.

m	p	$\mathscr{D}_1(z^2)$
1	0	$\begin{pmatrix} -\frac{1}{2} \end{pmatrix}$
	1	$\begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$
	2	$\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$
2	0	$\begin{pmatrix} -\frac{1}{2} \end{pmatrix}$
	1	$\begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{\frac{1}{16}z^4-\frac{1}{16}}{z^2} & -\frac{1}{2} \end{pmatrix}$
	2	$\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{\frac{1}{16}z^4-\frac{1}{16}}{z^2} & -\frac{1}{2} & 0 \\ \frac{-\frac{1}{64}z^8+\frac{1}{16}z^6+\frac{5}{32}z^4+\frac{1}{16}z^2-\frac{1}{64}}{z^4} & \frac{\frac{1}{8}z^4-\frac{1}{8}}{z^2} & -\frac{1}{2} \end{pmatrix}$
3	0	$\begin{pmatrix} -\frac{1}{8}z^2-\frac{3}{8} \end{pmatrix}$
	1	$\begin{pmatrix} -\frac{1}{8}z^2-\frac{3}{8} & 0 \\ \frac{-\frac{9}{2048}z^{10}+\frac{39}{2048}z^8-\frac{93}{1024}z^6+\frac{159}{1024}z^4-\frac{189}{2048}z^2+\frac{27}{2048}}{z^4} & -\frac{1}{8}z^2-\frac{3}{8} \end{pmatrix}$

4. Quarklet Approximation

Usually, the function

$$\begin{aligned} u_\alpha : I &\to \mathbb{R}, \\ x &\mapsto x^\alpha, \end{aligned} \tag{4.1}$$

with $\alpha > \frac{1}{2}$ serves as a typical model example of singular solutions to elliptic PDEs. In this section, we show that the function (4.1) can be approximated in L_2 and in H^1, respectively, with exponential order by the elements of our quarklet frame. We proceed in the following way. First of all, we choose a highly nonuniform partition of $[0,1]$ and approximate u_α by means of a Hermite spline with respect to the partition. Then we show that this spline can be written as a linear combination of quarks on different refinement levels. Finally, by using the decomposition relation derived in Section 3.2, we rewrite the spline in terms of quarklets and count the necessary degrees of freedom.

4.1. Approximation in L_2.

4.1.1. *Construction of the Spline.* We study approximations to (4.1) in terms of quarks and quarklets. First we are going to construct a piecewise polynomial approximation by Hermite interpolation. This generalizes the spline from [1]. Thus, for $i = 1, \ldots, J$ we define

- a finite geometric sequence of points: $x_0 := 0$, $x_i := 2^{i-J}$;
- intervals $I_i := [x_{i-1}, x_i]$;
- a local maximal refinement level $j_i := J - i + 1 + \lceil \log_2(m) \rceil$;
- a local maximal polynomial degree $p_i := i + m - 3$.

Theorem 4.1. *Let g be the piecewise polynomial on $[0,1]$ which on each subinterval I_i is defined as the Hermite interpolant g_i with respect to*

$$\underbrace{x_{i-1}, \ldots, x_{i-1}}_{m-1 \text{ times}}, \underbrace{y_i, \ldots, y_i}_{i-1 \text{ times}}, \underbrace{x_i, \ldots, x_i}_{m-1 \text{ times}}, \tag{4.2}$$

where $y_i := \frac{1}{2}(x_i + x_{i-1})$. Let

$$E_i := \|u_\alpha - g_i\|^2_{L_2(I_i)} \tag{4.3}$$

denote the squared L_2 approximation error. Then, it holds that

$$E_i \lesssim 2^{-2i} 2^{(i-J)(2\alpha+1)}, \quad i = 2, \dots, J, \tag{4.4}$$

$$E_1 \lesssim 2^{(1-J)(2\alpha+1)}. \tag{4.5}$$

Furthermore the squared global L_2 error is bounded by

$$\sum_{i=1}^{J} E_i \lesssim 2^{-2J}. \tag{4.6}$$

Proof. Let $i \geq 2$. We construct g_i as the Hermite interpolant with respect to the $i + 2m - 3$ knots from (4.2). For $x \in I_I$ the pointwise error can be estimated by

$$\begin{aligned}
|u_\alpha(x) - g_i(x)| &\leq \frac{|u_\alpha^{(i+2m-3)}(\xi)|}{(i+2m-3)!} |x - x_{i-1}|^{m-1} |x - y_i|^{i-1} |x - x_i|^{m-1} \\
&\leq \frac{|u_\alpha^{(i+2m-3)}(\xi)|}{(i+2m-3)!} |I_i|^{i+2m-3} 2^{-i+1} \\
&= \frac{|u_\alpha^{(i+2m-3)}(\xi)|}{(i+2m-3)!} 2^{(i-J-1)(i+2m-3)} 2^{-i+1}.
\end{aligned}$$

Now we estimate the derivative for $\xi \in I_i$. With the absolute convergence of the binomial series it can be bounded by

$$\frac{|u_\alpha^{(i+2m-3)}(\xi)|}{(i+2m-3)!} = \frac{\alpha |\alpha - 1| \cdots |\alpha - i - 2m + 4|}{(i+2m-3)!} \xi^{\alpha - i - 2m + 3} \leq 2^\alpha \xi^{\alpha - i - 2m + 3}.$$

Combining these estimates with the monotonicity of u_α leads to

$$\begin{aligned}
|u_\alpha(x) - g_i(x)| &\leq 2^\alpha 2^{(i-J-1)(i+2m-3)} 2^{-i+1} \begin{cases} x_{i-1}^{\alpha - i - 2m + 3}, & \alpha < i + 2m - 3 \\ x_i^{\alpha - i - 2m + 3}, & \alpha > i + 2m - 3 \end{cases} \\
&\leq 2^\alpha \begin{cases} 2^{(i-J-1)\alpha} 2^{-i+1}, & \alpha < i + 2m - 3 \\ 2^{(i-J)\alpha} 2^{-2i-2m+3}, & \alpha > i + 2m - 3 \end{cases} \\
&\lesssim 2^{(i-J)\alpha} 2^{-i},
\end{aligned}$$

with a constant depending on m and α. For $E_i, i = 2, \dots, J$ we conclude

$$E_i = \int_{I_i} |u_\alpha(x) - g_i(x)|^2 \mathrm{d}x \lesssim 2^{i-J-1} 2^{2(i-J)\alpha} 2^{-2i} \lesssim 2^{(i-J)(2\alpha+1)} 2^{-2i}.$$

It remains to treat the case $i = 1$. We directly compute

$$E_1 \leq \|u_\alpha\|^2_{L_2(I_1)} = \int_0^{2^{1-J}} |x^\alpha|^2 dx = \left[\frac{1}{2\alpha+1} x^{2\alpha+1}\right]_0^{2^{1-J}} \leq 2^{(2\alpha+1)(1-J)}.$$

Finally we consider the global error.

$$\begin{aligned}
\sum_{i=2}^{J} E_i &\lesssim \sum_{i=2}^{J} 2^{-2i} 2^{(2\alpha+1)(i-J)} = 2^{-(2\alpha+1)J} \sum_{i=2}^{J} \left(2^{-2} 2^{2\alpha+1}\right)^i \\
&= 2^{-(2\alpha+1)J} \left(2^{-2} 2^{2\alpha+1}\right)^2 \sum_{i=0}^{J-2} \left(2^{-2} 2^{2\alpha+1}\right)^i \\
&\lesssim 2^{-(2\alpha+1)J} \left(2^{-2} 2^{2\alpha+1}\right)^2 \left(2^{-2} 2^{2\alpha+1}\right)^{J-1} \\
&\lesssim 2^{-2J}.
\end{aligned}$$

With the asymptotic behaviour of E_1 the claim follows. □

4.1.2. *Quarkonial Decomposition.* The next step is to show that the approximation $\sum_{i=1}^{J} g_i \chi_{I_i}$ can be expanded in terms of quarklet frame elements. Firstly, we consider a decomposition into fine quarks which are not elements of the frame. Secondly, we use the reconstruction properties derived in Section 3 to get a decomposition into frame elements.

Proposition 4.2. *The functions $\varphi_0(\cdot - k), \ldots, \varphi_p(\cdot - k)$, $|k| < m$, span a spline space that contains the polynomial space $\Pi_{p+m-1}(-\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil)$.*

Proof. Let $q \in \mathbb{N}_0$. If $q \leq m-1$, x^q has a representation in terms of B-splines, i.e., for $x \in (-\lfloor \frac{m}{2} \rfloor, \lceil \frac{m}{2} \rceil)$ it holds

$$x^q = \sum_{-m<k<m} c_{q,k} \varphi_0(x-k).$$

Otherwise we split $q = p + m - 1$ such that

$$\begin{aligned}
x^q = x^p x^{m-1} &= x^p \sum_{-m<k<m} c_{m-1,k} \varphi_0(x-k) \\
&= \sum_{-m<k<m} \sum_{l=0}^{p} \binom{p}{l} k^{p-l} (x-k)^l c_{m-1,k} \varphi_0(x-k) \qquad (4.7) \\
&= \sum_{-m<k<m} \sum_{l=0}^{p} \binom{p}{l} \lceil m/2 \rceil^l k^{p-l} c_{m-1,k} \varphi_l(x-k).
\end{aligned}$$

□

By a change of variables, we can expand any polynomial on the intervals I_i in terms of quark generators:

Proposition 4.3. *The quarks* $\varphi_{l,j_i,k}, l = 0,\dots,p_i$, $2^{\lceil\log_2(m)\rceil} - \lceil m/2\rceil < k < 2^{\lceil\log_2(m)\rceil+1} + \lfloor m/2\rfloor$ *span a spline space including the polynomial space* $\Pi_{i+2m-4}(I_i)$.

Proof. We have $\varphi_{p,j_i,k} = 2^{j_i/2}\varphi_p(2^{j_i}\cdot - k)$ and hence

$$\operatorname{supp}\varphi_{p,j_i,k} = [2^{-j_i}(k - \lfloor m/2\rfloor), 2^{-j_i}(k + \lceil m/2\rceil)]. \tag{4.8}$$

Comparing the interval bounds with $I_i = [2^{i-J-1}, 2^{i-J}]$, we conclude that only the suppports of those quarks intersect with I_i which satisfy

$$2^{\lceil\log_2(m)\rceil} - \lceil m/2\rceil < k < 2^{\lceil\log_2(m)\rceil+1} + \lfloor m/2\rfloor. \tag{4.9}$$

For quarks lying completely inside I_i the condition on the translation parameter reads as follows:

$$2^{\lceil\log_2(m)\rceil} + \lfloor m/2\rfloor \le k \le 2^{\lceil\log_2(m)\rceil+1} - \lceil m/2\rceil. \tag{4.10}$$

Hence, the B-splines fulfilling (4.9) generate all polynomials with degree $m-1$ on I_i. The remainder of the proof is analogous to the previous one. □

So far, we have shown that each g_i on I_i can be decomposed in terms of quarks on level j_i. But, for $m \ge 2$ the supports of certain quarks intersect with the intervals I_{i-1}, I_{i+1}. As a consequence we have to thin out the amount of quarks in the neighbourhood of the knots x_i to construct a globally $m-1$-times differentiable approximation to u_α. We do this in the following way. We allow the supports of the quarks $\varphi_{p,j_i,k}$ to intersect with the interval I_{i+1}, but not with I_{i-1}. At the left boundary of I_i, for each p we insert $m-1$ quarks on level $j_i - 1$. By proceeding this way and using the refinability of the B-splines the gap in the spline space of degree $m-1$ is filled. In the following we use the abbreviations

$$K_0 := 2^{\lceil\log_2(m)\rceil}, \quad K_1 := 2^{\lceil\log_2(m)\rceil+1}. \tag{4.11}$$

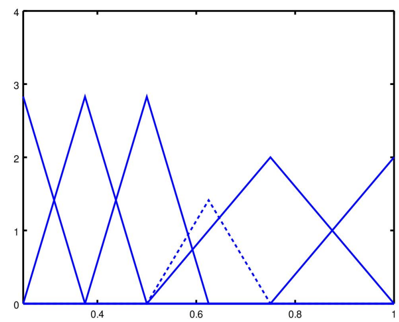

FIGURE 1. Closing the 'gap' in the spline space at the right hand side of $x_i = \frac{1}{2}$ with an additional fine hat.

Proposition 4.4. *Let $2 \leq i < J$, $j_i = J - i + 1 + \lceil \log_2(m) \rceil$ and $p_i = i + m - 3$. Then, the coarse quarks $\varphi_{p,j_i,k}, p = 0, \ldots, p_i$, $K_0 + \lfloor m/2 \rfloor \leq k < K_1 + \lfloor m/2 \rfloor$ and the fine quarks $\varphi_{p,j_i+1,k}, p = 0, \ldots, p_i - 1, K_1 - \lceil m/2 \rceil < k \leq K_1 + \lfloor m/2 \rfloor + m - 2$ span a spline space including the polynomial space $\Pi_{i+2m-5}(I_i)$.*

Proof. As seen in the proof of Proposition 4.2, it suffices to show that each polynomial of degree $m - 1$ can be decomposed into a sum of B-splines on fine and coarse scales. Let $P \in \Pi_{m-1}(I_i)$. Of course, P has a decomposition in terms of coarse B-splines

$$\begin{aligned} P(x) &= \sum_{K_0 - \lceil m/2 \rceil < k < K_1 + \lfloor m/2 \rfloor} c_k \varphi_{0,j_i,k}(x) \\ &= \sum_{k=K_0-\lceil m/2 \rceil+1}^{K_0+\lfloor m/2 \rfloor-1} c_k \varphi_{0,j_i,k}(x) + \sum_{k=K_0+\lfloor m/2 \rfloor}^{K_1+\lfloor m/2 \rfloor-1} c_k \varphi_{0,j_i,k}(x), \end{aligned}$$

where the latter sum consists only of B-splines not intersecting with I_{i-1}. We have a look at the first sum. Inserting the refinement equation for B-splines (2.3) yields

$$\sum_{k=K_0-\lceil m/2 \rceil+1}^{K_0+\lfloor m/2 \rfloor-1} c_k \varphi_0(2^{j_i}x - k) = \sum_{k=K_0-\lceil m/2 \rceil+1}^{K_0+\lfloor m/2 \rfloor-1} c_k \left(\sum_{l=-\lfloor m/2 \rfloor}^{\lceil m/2 \rceil} a_l \varphi_0(2 \cdot 2^{j_i}x - 2k - l) \right).$$

With an index shift we obtain

$$\sum_{k=K_0-\lceil m/2 \rceil+1}^{K_0+\lfloor m/2 \rfloor-1} c_k \varphi_0(2^{j_i}x - k) = \sum_{\tilde{l}=K_1-m-\lceil m/2 \rceil+2}^{K_1+m+\lfloor m/2 \rfloor-2} \left(\sum_{2k+l=\tilde{l}} c_k a_l \right) \varphi_0(2^{j_i+1}x - \tilde{l}).$$

With (4.10) we can omit the fine B-splines lying completely inside I_{i-1} and get

$$\sum_{k=K_0-\lceil m/2\rceil+1}^{K_0+\lfloor m/2\rfloor-1} c_k\varphi_0(2^{j_i}x-k) = \sum_{\tilde{l}=K_1-\lceil m/2\rceil+1}^{K_1+m+\lfloor m/2\rfloor-2} \left(\sum_{2k+l=\tilde{l}} c_k a_l\right) \varphi_0(2^{j_i+1}x-\tilde{l}).$$

Hence we have for $x \in I_i$:

$$\begin{aligned} P(x) = &\sum_{\tilde{l}=K_1-\lceil m/2\rceil+1}^{K_1+\lfloor m/2\rfloor+m-2} \left(\sum_{2k+l=\tilde{l}} c_k a_l\right) \varphi_{0,j_i+1,\tilde{l}}(x) \\ &+ \sum_{k=K_0+\lfloor m/2\rfloor}^{K_1+\lfloor m/2\rfloor-1} c_k\varphi_{0,j_i,k}(x) \end{aligned}$$

By polynomial enrichment as in (4.7), the claim follows . $\square$

Theorem 4.5. *Let the spline constructed in Theorem 4.1 be given by* $g = \sum_{i=1}^{J} g_i\chi_{I_i}$ *and let* j_i *be defined as at the beginning of this section. We define the collection of quark indices*

$$\begin{aligned} \Lambda_{j_i} &:= \{(p, j_i, k) : p \le p_i, K_0 + \lfloor m/2\rfloor \le k \le K_1 + \lfloor m/2\rfloor + m - 2\}, \\ \Lambda_{j_1} &:= \{(p, j_1, k) : p \le p_1, \lfloor m/2\rfloor \le k \le K_1 + \lfloor m/2\rfloor + m - 2\}, \\ \Lambda_{j_J} &:= \{(p, j_J, k) : p \le p_J, K_0 + \lfloor m/2\rfloor \le k < K_1 + \lfloor m/2\rfloor\}, \\ \Lambda &:= \bigcup_{i=1}^{J} \Lambda_{j_i}. \end{aligned} \tag{4.12}$$

Then, there exist $c_\lambda \in \mathbb{R}$, *such that*

$$g(x) = \sum_{\lambda\in\Lambda} c_\lambda\varphi_\lambda(x), \quad x \in I. \tag{4.13}$$

Proof. Let $2 \le i \le J$. We proceed in the following way: Suppose, that g_{i-1} is given on I_{i-1} as a polynomial of degree $i + 2m - 5$. Then, we decompose g_i into

$$g_i(x) = P_i(x) + Q_i(x),$$

where $P_i \in \Pi_{i+2m-5}$, $Q_i \in \Pi_{i+2m-4}$ and Q_i has a root with multiplicity $m-1$ in x_{i-1}. P_i is constructed as the extension of $g_{i-1}|_{I_i}$ to a polynomial on I_i. With (4.9) and (4.10) we can rewrite Λ as $\Lambda = \cup\Lambda_{I_i}$,

$$\begin{aligned}\Lambda_{I_i} :=&\{(p, j_{i-1}, k) : p \leq p_{i-1}, K_1 - \lceil m/2\rceil < k < K_1 + \lfloor m/2\rfloor\}\\ &\cup\{(p, j_{i-1}, k) : p \leq p_{i-1}, K_1 + \lfloor m/2\rfloor \leq k \leq K_1 + \lfloor m/2\rfloor + m - 2\}\\ &\cup\{(p, j_i, k) : p \leq p_i, K_0 + \lfloor m/2\rfloor \leq k < K_1 + \lfloor m/2\rfloor\}.\end{aligned}$$

Since g_{i-1} is a polynomial on I_{i-1}, with Proposition 4.3 we have for $x \in I_{i-1}$:

$$g_{i-1}(x) = \sum_{\lambda\in\Lambda_{I_{i-1}}} c_\lambda\varphi_\lambda(x).$$

Considering g_{i-1} on I_i, we can omit the nonoverlapping quarks and get for $x \in I_i$

$$g_{i-1}|_{I_i}(x) = \sum_{k=K_1-\lceil m/2\rceil+1}^{K_1+\lfloor m/2\rfloor-1} \sum_{p=0}^{p_i-1} c_{k,p}\varphi_{p,j_i+1,k}(x).$$

With Proposition 4.4, $g_{i-1}|_{I_i}$ can be extended to a polynomial $P_i \in \Pi_{i+2m-5}(I_i)$:

$$P_i(x) = \sum_{k=K_1-\lceil m/2\rceil+1}^{K_1+\lfloor m/2\rfloor+m-2} \sum_{p=0}^{p_i-1} c_{k,p}\varphi_{p,j_i+1,k}(x) + \sum_{k=K_0+\lfloor m/2\rfloor}^{K_1+\lfloor m/2\rfloor-1} \sum_{p=0}^{p_i-1} c_{k,p}\varphi_{p,j_i,k}(x).$$

By construction, the polynomial Q_i interpolates $u_\alpha - P_i$ in x_i and y_i and has a decomposition

$$Q_i(x) = (x - x_{i-1})^{m-1} \sum_{p=0}^{p_i} a_p x^p.$$

Obviously, the first part consists only of B-splines lying completely inside I_i. Hence, Q_i has a decomposition in coarse quarks

$$Q_i(x) = \sum_{k=K_0+\lfloor m/2\rfloor}^{K_1+\lfloor m/2\rfloor-1} \sum_{p=0}^{p_i} b_{k,p}\varphi_{p,j_i,k}(x)$$

The case $i = 1$ is already covered by Proposition 4.3, since no overlapping quarks have to be considered. □

After all these preparations, we are now able to state and to prove the main result of this section. By expanding the Hermite interpolation spline with respect to the elements of the quarklet frame, we show that the model function u_α can indeed be approximated with exponential order.

Theorem 4.6. *Let $\Delta := \{(p, j, k) : p \in \mathbb{N}_0, j \in \mathbb{N}_0 \cup \{-1\}, k \in \mathbb{Z}\}$ be the index set of the full quarklet system and let g be the spline constructed in Theorem 4.1 . For $N \sim J^5$, $J \in \mathbb{N}$ there exist $c_\lambda \in \mathbb{R}$ such that*

$$g(x) = \sum_{\lambda \in \Delta' \subset \Delta : |\Delta'| \leq N} c_\lambda \psi_\lambda(x), \quad x \in I, \tag{4.14}$$

$$\|u_\alpha - g\|^2_{L_2(I)} \lesssim \left(2^2\right)^{-N^{1/5}} = e^{-2\ln(2)N^{1/5}}. \tag{4.15}$$

Proof. First we have a look at (4.14). Since we have a decomposition of g in terms of quarks, see (4.13), and finite reconstruction sequences derived in Section 3, (4.14) follows. It remains to estimate the asymptotic number of frame elements. With Proposition 4.4, each polynomial p_i on I_i has a decomposition in terms of $C_0(m)p_i$ fine quarks on level j_i. With Theorem 3.9, each fine multi-quark consists of $j_i^2 p_i$ frame element vectors. Hence, summation over i gives a total number of J^5 frame elements. Inserting this into the estimate (4.6) gives (4.15). □

4.2. **Approximation in H^1.**

Theorem 4.7. *Let g be the piecewise polynomial on $[0, 1]$ which on each subinterval I_i is defined as the Hermite interpolant g_i with respect to u'_α and the knots*

$$\underbrace{x_{i-1}, \ldots, x_{i-1}}_{m-1-times}, \underbrace{y_i, \ldots, y_i}_{i-2-times}, \underbrace{x_i, \ldots, x_i}_{m-1-times}, \tag{4.16}$$

where $y_i := \frac{1}{2}(x_{i-1} + x_i)$. Let

$$E_i := |u_\alpha - g|^2_{H^1(I_i)} \tag{4.17}$$

denote the squared H^1 approximation error. Then, it holds that

$$E_i \lesssim 2^{-2i} 2^{(i-J)(2\alpha-1)}, \quad i = 2, \ldots, J, \tag{4.18}$$

$$E_1 \lesssim 2^{(1-J)(2\alpha-1)}. \tag{4.19}$$

Furthermore the global squared H^1 error is bounded by

$$\sum_{i=1}^{J} E_i \lesssim \min\left(2^2, 2^{2\alpha-1}\right)^{-J}. \tag{4.20}$$

Proof. We use

$$|u|_{H^1(I_i)} = \|u'\|_{L_2(I_i)}.$$

Now the remainder of the proof is analogous to the previously treated L_2-case. Let $i \geq 2$ and consider the $i + 2m - 4$ knots in (4.16). With $u'_\alpha(x) = \alpha x^{\alpha-1}$ and $g_i \in \Pi_{i+2m-5}$ we conclude for $x \in I_i$

$$\begin{aligned}
|u'_\alpha(x) - g_i(x)| &\leq \frac{|u_\alpha^{\prime(i+2m-4)}(\xi)|}{(i+2m-4)!}|x - x_{i-1}|^{m-1}|x - y_i|^{i-2}|x - x_i|^{m-1} \\
&\leq \frac{|u_\alpha^{(i+2m-3)}(\xi)|}{(i+2m-4)!}|I_i|^{i+2m-4}2^{-i+2} \\
&= \frac{|u_\alpha^{(i+2m-3)}(\xi)|}{(i+2m-4)!}2^{(i-J-1)(i+2m-4)}2^{-i+2}.
\end{aligned}$$

Now we have a look at the derivative for $\xi \in I_i$. With the absolute convergence of the binomial series it can be bounded by

$$\begin{aligned}
\frac{|u_\alpha^{(i+2m-3)}(\xi)|}{(i+2m-4)!} &= \frac{\alpha|\alpha-1|\cdots|\alpha-i-2m+4|}{(i+2m-4)!}\xi^{\alpha-i-2m+3} \\
&\leq \alpha 2^{\alpha-1}\xi^{\alpha-i-2m+3}.
\end{aligned}$$

Combining these estimates with the monotonicity of u_α leads to

$$\begin{aligned}
|u'_\alpha(x) - g_i(x)| &\leq \alpha 2^{\alpha-1}2^{(i-J-1)(i+2m-4)}2^{-i+2}\begin{cases} x_{i-1}^{\alpha-i-2m+3}, & \alpha < i+2m-3 \\ x_i^{\alpha-i-2m+3}, & \alpha > i+2m-3 \end{cases} \\
&\leq \alpha 2^{\alpha-1}\begin{cases} 2^{(i-J-1)(\alpha-1)}2^{-i+2}, & \alpha < i+2m-3 \\ 2^{(i-J)(\alpha-1)}2^{-2i-2m+6}, & \alpha > i+2m-3 \end{cases} \\
&\lesssim 2^{(i-J)(\alpha-1)}2^{-i},
\end{aligned}$$

with a constant depending on m and α. For $E_i, i = 2, \ldots, J$ we conclude

$$E_i = \int_{I_i}|u'_\alpha(x) - g_i(x)|^2 \mathrm{d}x \lesssim 2^{i-J-1}2^{2(i-J)(\alpha-1)}2^{-2i} \lesssim 2^{(i-J)(2\alpha-1)}2^{-2i}.$$

Now let $i = 1$. We directly compute

$$E_1 \leq \|u'_\alpha\|^2_{L_2(I_1)} = \alpha^2\int_0^{2^{1-J}}|x^{\alpha-1}|^2 dx = \left[\frac{\alpha^2 x^{2\alpha-1}}{2\alpha-1}\right]_0^{2^{1-J}} \lesssim 2^{(2\alpha-1)(1-J)}.$$

Next we consider the global error.

$$\begin{aligned}
\sum_{i=2}^{J} E_i &\lesssim \sum_{i=2}^{J} 2^{-2i} 2^{(2\alpha-1)(i-J)} = 2^{-(2\alpha-1)J} \sum_{i=2}^{J} \left(2^{-2} 2^{2\alpha-1}\right)^i \\
&= 2^{-(2\alpha-1)J} \left(2^{-2} 2^{2\alpha-1}\right)^2 \sum_{i=0}^{J-2} \left(2^{-2} 2^{2\alpha-1}\right)^i \\
&\lesssim 2^{-(2\alpha-1)J} \left(2^{-2} 2^{2\alpha-1}\right)^2 \left(2^{-2} 2^{2\alpha-1}\right)^{J-1} \\
&\lesssim 2^{-2J}.
\end{aligned}$$

With the asymptotic behaviour of E_1 the claim follows. □

Theorem 4.8. *Let $\Delta = \{(p, j, k) : p \in \mathbb{N}_0, j \in \mathbb{N}_0 \cup \{-1\}, k \in \mathbb{Z}\}$ be the index set of the full quarklet system and let g be the spline constructed in Theorem 4.7 . For $N \sim J^5$, $J \in \mathbb{N}$ there exist $c_\lambda \in \mathbb{R}$ such that*

$$g(x) = \sum_{\lambda \in \Delta' \subset \Delta : |\Delta'| \le N} c_\lambda \psi_\lambda(x), \quad x \in I, \tag{4.21}$$

$$|u_\alpha - g|^2_{H^1(I)} \lesssim \min\left(2^2, 2^{2\alpha-1}\right)^{-N^{1/5}} = e^{-\min(2,2\alpha-1)\ln(2) N^{1/5}}. \tag{4.22}$$

Proof. To derive a quarkonial decomposition of the polynomial approximation, we differentiate

$$x^{q-1} = \left(\frac{1}{q} x^q\right)' = \sum_\lambda \tilde{c}_\lambda \varphi'_\lambda(x), \quad q = 1, \ldots, i + 2m - 4.$$

That means that every polynomial on I_i can be decomposed with respect to derivatives of quarks. Using the decomposition of quarks on a fine level in terms of frame elements as in Theorem 4.6, we get the asymptotic behaviour. □

4.3. **Tensor product quarklet approximation.** Let us now consider the case of the unit cube $I^2 = [0, 1]^2$. As a model for edge singularities that might occur in higher dimensions we consider the function

$$\begin{aligned} u_\alpha : I^2 &\to \mathbb{R}, \\ x &\mapsto x_1^\alpha, \end{aligned} \tag{4.23}$$

with $\alpha > \frac{1}{2}$. We expect that anisotropic singularities of the form (4.23) can be very efficiently approximated by anisotropic tensor product quarklets. This is indeed the case, as we shall see below.

Theorem 4.9. *Let $\tilde{g}$ be the univariate spline constructed in Theorem 4.7. Then we define the function $g(x_1, x_2) := \tilde{g}(x_1)\chi_{[0,1]}(x_2)$. Let*

$$E_i := |u_\alpha - g|^2_{H^1(I_i \times I)}, \tag{4.24}$$

denote the squared H^1 approximation error. Then, it holds that

$$E_i \lesssim 2^{-2i} 2^{(i-J)(2\alpha-1)}, \quad i = 2, \dots, J, \tag{4.25}$$

$$E_1 \lesssim 2^{(1-J)(2\alpha-1)}. \tag{4.26}$$

Furthermore the global squared H^1 error is bounded by

$$\sum_{i=1}^{J} E_i \lesssim \min\left(2^2, 2^{2\alpha-1}\right)^{-J}. \tag{4.27}$$

Proof. The proof is analogous to the univariate case. We use

$$|u|^2_{H^1(I_i \times I)} = \sum_{\beta \in \mathbb{N}^2, |\beta|=1} \|D^\beta u\|_{L_2(I_i \times I)}, \tag{4.28}$$

where $D^{(0,1)} u_\alpha = 0$. Hence it suffices to consider derivatives with respect to x_1, i.e. $\frac{\partial}{\partial x_1} u$. We construct the Hermite interpolation polynomial g_i as $g_i(x) := \tilde{g}_i(x_1)$. Following the lines of the proof of Theorem 4.7, we obtain

$$|\frac{\partial}{\partial x_1} u_\alpha(x) - g_i(x)| \lesssim 2^{(i-J)(\alpha-1)} 2^{-i}. \tag{4.29}$$

Now we conclude for E_i:

$$\begin{aligned} E_i &= \int_I \int_{I_i} |\frac{\partial}{\partial x_1} u_\alpha(x) - g_i(x)|^2 \mathrm{d}x \lesssim \int_I 2^{i-J-1} 2^{2(i-J)(\alpha-1)} 2^{-2i} \mathrm{d}x_2 \\ &\lesssim 2^{(i-J)(2\alpha-1)} 2^{-2i}. \end{aligned}$$

Similar computations for E_1 and summation over i complete the proof. ☐

Theorem 4.10. *Let $\Delta = \{(p, j, k) : p \in \mathbb{N}_0, j \in \mathbb{N}_0 \cup \{-1\}, k \in \mathbb{Z}\}$ be the index set of the full univariate quarklet system and let g be the spline defined in Theorem 4.9 . For $N \sim J^5$, $J \in \mathbb{N}$ there exist $c_\lambda \in \mathbb{R}$ such that*

$$g(x) = \sum_{\boldsymbol{\lambda} \in \Delta' \subset \Delta^2 : |\Delta'| \le N} c_{\boldsymbol{\lambda}} \psi_{\boldsymbol{\lambda}}(x), \quad x \in I, \tag{4.30}$$

$$|u_\alpha - g|^2_{H^1(I^2)} \lesssim \min\left(2^2, 2^{2\alpha-1}\right)^{-N^{1/5}} = e^{-\min(2,2\alpha-1)\ln(2) N^{1/5}}. \tag{4.31}$$

Proof. Again, we derive a decomposition of the polynomials g_i with respect to elements of the tensor quarklet frame. From the tensor product structure

$$\psi_{\boldsymbol{\lambda}}(x) = \psi_{p_1,j_1,k_1}(x_1)\psi_{p_2,j_2,k_2}(x_2),$$

the partition of unity

$$1 = \sum_{-m<k<m} \varphi_0(x_2 - k), \quad x_2 \in I,$$

and $g_i(x) = \tilde{g}_i(x_1)\chi_{[0,1]}(x_2)$ we conclude that only those frame elements of the form

$$\psi_{\boldsymbol{\lambda}}(x) = \psi_{p_1,j_1,k_1}(x_1)\varphi_0(x_2 - k_2),$$

are needed for the decomposition of g_i. In particular we have the same decomposition as in the univariate case, i.e., g_i consists of i fine quarks and hence of $(J-i)^2 i^2$ frame elements. Summation over i gives an asymptotic number of degrees of freedom of J^5. □

5. Discussion

In this paper, we have shown that typical singularity functions that may arise in the context of elliptic boundary value problems on nonsmooth domains can be approximated with exponential order by means of quarklet expansions. This result is similar to the univariate approximation result based on *hp*-dictionaries as outlined in [1]. These facts indicate the potential of quarklet frames for the numerical treatment of elliptic operator equations. Moreover, our approach has important advantages compared to *hp* finite element systems. Since the quarklet frames give rise to stable expansions in scales of Sobolev spaces including L_2, there is a very natural way to generalize our results to higher-dimensional problems by means of tensor products. This results in highly anisotropic dictionaries which again give rise to exponentially convergent approximation schemes for specific anisotropic functions that serve as models for typical edge singularities. We are convinced that these exponential approximation properties can also be realized algorithmically by adaptive quarklet schemes, and we will attack this task in the near future.

Appendix A

The following properties are well-known, we refer to [8] for details.

Proposition A.1.

(i) *The Fourier transform* $\hat{\Phi} = (\hat{\Phi}_0, \dots, \hat{\Phi}_p)^T$ *of a refinable function vector fulfils the matrix equation*

$$\hat{\Phi}(\xi) = \frac{1}{2}\mathscr{A}(z)\hat{\Phi}\left(\frac{\xi}{2}\right), \quad \xi \in \mathbb{R}, z = e^{i\frac{\xi}{2}} \in S_1, \tag{A.1}$$

where

$$\left(\mathscr{A}(z)\right)_{q,l} := \sum_{k\in\mathbb{Z}} \left(A_k\right)_{q,l} z^k, \tag{A.2}$$

is called the symbol matrix *of* Φ.

(ii) *For a symbol matrix* $\mathscr{A}(z)$ *and* $\rho \in \{0,1\}$ *we define the* sub-symbol matrices $\mathscr{A}_\rho(z^2)$ *by*

$$\left(\mathscr{A}_\rho(z^2)\right)_{q,l} := \sum_{k\in\mathbb{Z}} \left(A_{2k+\rho}\right)_{q,l} z^{2k}. \tag{A.3}$$

Then, it holds that

$$\mathscr{A}_0(z^2) = \frac{1}{2}\left(\mathscr{A}(z) + \mathscr{A}(-z)\right), \quad \mathscr{A}_1(z^2) \;= \frac{1}{2z}\left(\mathscr{A}(z) - \mathscr{A}(-z)\right). \tag{A.4}$$

Proof.

(i) Applying the Fourier transform to the q-th component of Φ, we obtain

$$\begin{aligned}\hat{\Phi}_q(\xi) &= \sum_{k\in\mathbb{Z}} \sum_{l=0}^{q} \left(A_k\right)_{q,l} \widehat{\Phi_l(2\cdot -k)}(\xi) \\ &= \sum_{k\in\mathbb{Z}} \sum_{l=0}^{q} \left(A_k\right)_{q,l} \frac{1}{2}\hat{\Phi}_l\left(\frac{\xi}{2}\right) e^{-i\frac{\xi}{2}k} \\ &= \frac{1}{2}\sum_{l=0}^{q} \underbrace{\left(\sum_{k\in\mathbb{Z}} \left(A_k\right)_{q,l} z^k\right)}_{:=\left(\mathscr{A}(z)\right)_{q,l}} \hat{\Phi}_l\left(\frac{\xi}{2}\right)\end{aligned}$$

(ii) We consider an arbitrary entry of the matrices. Let $0 \leq q, l \leq p$. We calculate

$$\begin{aligned}\left(\frac{1}{2}\left(\mathscr{A}(z)+\mathscr{A}(-z)\right)\right)_{q,l} &= \frac{1}{2}\left(\sum_{k\in\mathbb{Z}}(A_k)_{q,l}z^k + \sum_{k\in\mathbb{Z}}(A_k)_{q,l}(-z)^k\right)\\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}}2(A_{2k})_{q,l}z^{2k} = \left(\mathscr{A}_0(z^2)\right)_{q,l},\\ \left(\frac{1}{2z}\left(\mathscr{A}(z)-\mathscr{A}(-z)\right)\right)_{q,l} &= \frac{1}{2z}\left(\sum_{k\in\mathbb{Z}}(A_k)_{q,l}z^k - \sum_{k\in\mathbb{Z}}(A_k)_{q,l}(-z)^k\right)\\ &= \frac{1}{2z}\sum_{k\in\mathbb{Z}}2(A_{2k+1})_{q,l}z^{2k+1} = \left(\mathscr{A}_1(z^2)\right)_{q,l}.\end{aligned}$$

□

In Section 3.2 we have shown that the reconstruction sequences are finitely supported. In addition, we show how to iteratively compute the symbol matrices $\mathscr{C}_\rho(z^2), \mathscr{D}_\rho(z^2)$ and hence the reconstruction coefficients.

Theorem A.2. *Let $b(-z) \neq 0$. Then, $X(z)^{-1}$ consists of Laurent polynomials only. In particular, the entries of $T(z)^{-1}$ are products of a Laurent polynomial $L(z)$ and $b(-z)$.*

Proof. We proceed by induction. Let $p = 1$. We have $\det T(z) = 2(\frac{z}{b(-z)})^2$ and

$$\begin{aligned} T(z) &= \frac{1}{2}\begin{pmatrix} a_{00}(z) - \frac{b(z)}{b(-z)}a_{00}(-z) & 0 \\ a_{10}(z) - \frac{b(z)}{b(-z)}a_{10}(-z) & a_{11}(z) - \frac{b(z)}{b(-z)}a_{11}(-z)\end{pmatrix},\\ T(z)^{-1} &= \frac{b(-z)^2}{4z^2}\begin{pmatrix} a_{11}(z) - \frac{b(z)}{b(-z)}a_{11}(-z) & 0 \\ \frac{b(z)}{b(-z)}a_{10}(-z) - a_{10}(z) & a_{00}(z) - \frac{b(z)}{b(-z)}a_{00}(-z)\end{pmatrix}\\ &= \frac{b(-z)}{4z^2}\begin{pmatrix} a_{11}(z)b(-z) - a_{11}(-z)b(z) & 0 \\ a_{10}(-z)b(z) - a_{10}(z)b(-z) & a_{00}(z)b(-z) - a_{00}(-z)b(z)\end{pmatrix}\end{aligned}$$

Now let T_1 of dimension $p \times p$ and consist of Laurent polynomials times $b(-z)$ only. T and T^{-1} are given by

$$T = \begin{pmatrix} T_1 & 0 \\ x & \alpha_{p+1,p+1}\end{pmatrix}, \quad T^{-1} = \begin{pmatrix} T_1^{-1} & 0 \\ y & \alpha^{-1}_{p+1,p+1}\end{pmatrix}.$$

We can compute $\alpha^{-1}_{p+1,p+1}$ explicitly by

$$\alpha_{p+1,p+1} = a_{p+1,p+1}(z) - \frac{b(z)}{b(-z)} a_{p+1,p+1}(-z) = 2^{-p-1}\frac{4z}{b(-z)}.$$

For the i-th column of T^{-1} it is

$$0 = \langle \begin{pmatrix} x \\ \alpha_{p+1,p+1} \end{pmatrix}, \begin{pmatrix} (T_1^{-1})_i \\ y_i \end{pmatrix} \rangle,$$

from which we conclude

$$y_i = -\frac{\langle x, (T_1^{-1})_i \rangle}{\alpha_{p+1,p+1}} = -\frac{\langle x, (T_1^{-1})_i\rangle b(-z)}{2^{-p-1}4z}.$$

Since $\langle x, (T_1^{-1})_i\rangle$ consists of Laurent polynomials only, the claim follows. $\square$

Theorem A.3. *Let $b(-z) = 0$. Then, $X(z)^{-1}$ consists of Laurent polynomials only. In particular, the entries of $\mathscr{A}(-z)^{-1}$ are products of a Laurent polynomial $L(z)$ and $b(z)$.*

Proof. The proof is analogous to the previous one. From (3.18) we conclude $\frac{1}{b(z)} = -\frac{a(-z)}{4z}$. Now let $p = 1$. We have $\det \mathscr{A}(z) = 8\left(-\frac{z}{b(z)}\right)^2$ and

$$\mathscr{A}(z) = \begin{pmatrix} a_{00}(-z) & 0 \\ a_{10}(-z) & a_{11}(-z) \end{pmatrix},$$

$$\mathscr{A}(z)^{-1} = -\frac{b(z)^2}{8z^2}\begin{pmatrix} a_{11}(-z) & 0 \\ -a_{10}(-z) & a_{00}(-z) \end{pmatrix}.$$

Now let $\mathscr{A}_1$ of dimension $p \times p$ and consist of Laurent polynomials times $b(z)$ only. $\mathscr{A}$ and $\mathscr{A}^{-1}$ are given by

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}_1 & 0 \\ x & \alpha_{p+1,p+1} \end{pmatrix}, \quad \mathscr{A}^{-1} = \begin{pmatrix} \mathscr{A}_1^{-1} & 0 \\ y & \alpha^{-1}_{p+1,p+1} \end{pmatrix}.$$

We can compute $\alpha^{-1}_{p+1,p+1}$ explicitly by

$$\alpha_{p+1,p+1} = a_{p+1,p+1}(-z) = -2^{-p-1}\frac{4z}{b(z)}.$$

For the i-th column of $\mathscr{A}^{-1}$ it is

$$0 = \langle \begin{pmatrix} x \\ \alpha_{p+1,p+1} \end{pmatrix}, \begin{pmatrix} (\mathscr{A}_1^{-1})_i \\ y_i \end{pmatrix} \rangle,$$

from which we conclude

$$y_i = -\frac{\langle x, (\mathscr{A}_1^{-1})_i \rangle}{\alpha_{p+1,p+1}} = \frac{\langle x, (\mathscr{A}_1^{-1})_i \rangle b(z)}{2^{-p-1} 4z}.$$

Since $\langle x, (\mathscr{A}_1^{-1})_i \rangle$ consists of Laurent polynomials only, the claim follows. $\square$

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