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METHOD OF LINES APPROXIMATIONS TO CAUCHY PROBLEMS FOR ELLIPTIC EQUATIONS IN TWO DIMENSIONS

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Abstract — In this paper, the method of lines approximation for a rather general elliptic equation containing a diffusion coefficient is considered. Our main results are the regularization of the ill-posed Cauchy problem and the proof of error estimates leading to convergence results for the method of lines. These results are based on the conditional stability of the continuous Cauchy problem and the approximation by appropriately chosen finite-dimensional spaces, onto which the possibly perturbed Cauchy data are projected. At the end of this paper, we present and discuss results of some of our numerical computations. There are multiple applications in material sciences, thermodynamics, medicine etc.; related problems are shape optimization problems which are important, e.g. for nondestructive testing, crack location, thermal tomography, and other applications.

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1. Introduction and Problem Setting

We consider the following Cauchy problem for an elliptic partial differential equation on a rectangle $\Omega := (0,1) \times (0,L)$,

$$a(x)\frac{\partial^2 u}{\partial y^2}(x,y) + \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}(x,y)\right) = f(x,y) \quad \text{in} \quad \Omega \tag{1.1}$$

with boundary conditions

$$u = f_i \quad \text{on} \quad \Sigma_i, \qquad i = 1,2,3, \tag{1.2}$$

$$\frac{\partial u}{\partial y} = \phi_1 \quad \text{on} \quad \Sigma_1, \tag{1.3}$$

where

$$\Sigma_1 = \big\{(x,0) \in \mathbb{R}^2 | 0 \leqslant x \leqslant 1\big\}, \quad \Sigma_2 = \big\{(0,y) \in \mathbb{R}^2 | 0 \leqslant y \leqslant L\big\},$$
$$\Sigma_3 = \big\{(1,y) \in \mathbb{R}^2 | 0 \leqslant y \leqslant L\big\}, \quad \Sigma_4 = \big\{(x,L) \in \mathbb{R}^2 | 0 \leqslant x \leqslant 1\big\},$$

Here one tries to identify u and $\partial u/\partial y$ on Σ_4. The functions f_1, ϕ_1 are the given *Cauchy data* (see Fig.1.1).

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This is a well-known improperly posed problem. In 1923 J. Hadamard [15] gave a classical example showing that the solution of the problem is not continuously dependent on the Cauchy data.

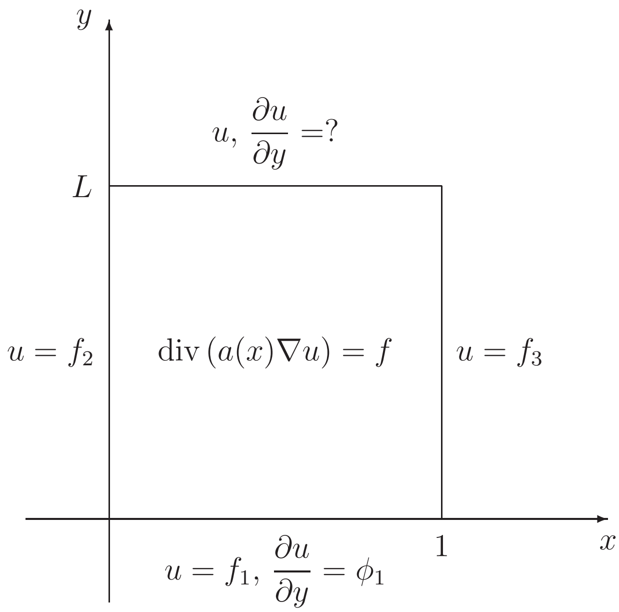

Fig. 1.1. Cauchy problem for an elliptic equation

It is impossible to solve this improperly posed problem by the classical theory of partial differential equations and, therefore, it has required the attention of many mathematicians in the last 50 years. M. M. Lavrent'ev [23] discussed bounded solutions of the Laplace equation with the Cauchy data in a special two-dimensional domain where the bounded solutions depend continuously on the Cauchy data. Fursikov [12] extended this approach later to domains in $\mathbb{R}^n$ proving an optimal stability estimate with respect to the H^0-norm. The latter is analogous to Hadamard's classical estimate for analytic functions which forms the content of the three-circles theorem. L. E. Payne [26,27] studied the solutions of more general second-order elliptic equations which are continuously dependent on the Cauchy data under some restrictions on the domains and on the solutions. In 1975, L. E. Payne outlined this problem in [28]. H. Han considered problem (1.1)–(1.3) in [16] in a somewhat more general setting and gave an $H^0(\Omega)$-stability estimate.

R. S. Falk [9] presented a three-lines theorem for the two-dimensional Laplace equation with the Cauchy data, for which a certain stability estimate is given. M. Kubo [21] obtained an H^0-stability estimate for the Cauchy problem for the Laplace equation on a doubly connected bounded domain. K. S. Fayazov and M. M. Lavrent'ev [11] studied the Cauchy problem for elliptic equations with operator coefficients in space. Using the method of logarithmic convexity, they proved the uniqueness and the H^0-stability estimate. In 1995, S. I. Kabanikhin and A. L. Karchevsky [19] presented an optimization method for solving the Cauchy problem for an elliptic equation numerically.

In the paper of H. Han and H.-J. Reinhardt [17], a series of stability estimates for problem (1.1) – (1.3) in Sobolev spaces are given, from which several regularization methods can be proposed for computing numerical approximations (see [30]).

Very recently, similar ideas as in our approach have been used by Zhi Qian and Chu-Li Fu [39] where two regularizations are studied, namely adding a fourth-order term multiplied by a small parameter to the Laplace equation as well as a truncated series of sine and cosh-terms. Dinh Nho Háo et al. [7] considered a similar problem as ours and proved stability estimates of the Hölder type for general L_p-norms. In [7], the Cauchy problem is studied in the frequency space and the mollification method is used to regularize the problem.

Using the method of lines approximations by discretizing the x-variable, the Cauchy problem is similar to the problem of determining a function by its Fourier coefficients. The latter problem is well-known to be ill-posed and can be regularized by various approaches as shown in the famous book of Tikhonov and Arsenin [37, Ch. V]. In the book of Samarskii and Vabishchevich [32, Ch. 7], the Cauchy problem for elliptic equations is studied under perturbations of the initial conditions as well as of the elliptic operator itself. For these, regularization algorithms are presented including difference schemes where both variables are discretized by the second-order difference quotients. Vabishchevich et al. [38] have considered the Cauchy problem for Laplace's equation on a circle using the idea of a regularized Fourier series as in [37] by appropriately defined stabilizing functionals. Contrary to the above-mentioned approaches our regularization is based on the stability estimate obtained by logarithmic convexity (see Sect. 3). Consequently, the optimal regularization parameter derived in Sect. 5 depends, among others, on the logarithm of reciprocal of the magnitude of data perturbations.

Especially, for the Laplace equation we have presented our results for the method of lines approximation in a short paper [5]. The basis for this and the present paper is the doctoral thesis [4].

This work is organized as follows. In Section 2, the method of lines approximation is introduced with lines perpendicular to the sides of the rectangle with the Cauchy data. The method of lines leads to a system of ordinary differential equations which can be decoupled when one solves an eigenvalue problem beforehand. The given Cauchy problem is shown to be conditionally well-posed by using the technique of logarithmic convexity (cf. Section 3). For the representation of the solution, the technique of separation of variables together with the well-known results for the Sturm — Liouville eigenvalue problem are utilized. The subject of Section 4 is the convergence of the eigenvalues and eigenvectors of discretized Sturm — Liouville eigenvalue problems. For the proofs of Theorem 4.1 and 4.3, we refer to the literature.

In Section 5, the main results concerning the convergence of the method of lines approximation are formulated and proved. In the first step, the data function ϕ_1 is projected into the space D_M of functions of truncated Fourier sine series of dimension M. We even allow perturbed data functions ϕ_1^ε such that $\|\phi_1 - \phi_1^\varepsilon\|_{L_2} \leqslant \varepsilon$. In this situation it is clear that, in general, $\phi_1^\varepsilon \notin D_M$ even if $\phi_1 \in D_M$. Then one has to estimate the projection error of the projected data and, additionally, the error between the true solution and the method of lines approximation with projected data in D_M. For the convergence, the magnitude of perturbations should depend on the discretization parameter by $h = O(\varepsilon^2)$ and the dimension of D_M has to be chosen in an optimal way.

In Section 6, we discuss several computational results for two examples. It can be shown that the choice of the regularization parameter M is essential for a good numerical approximation and the choice in Theorem 5.5 is indeed optimal. Finally, an appendix with 12 figures conclude this work.

The left-hand side of (1.1) defines a linear differential operator. It is well-known that one can set $f = f_1 = f_2 = f_3 = 0$ in (1.1)–(1.3). Indeed, the solution of the latter plus the solution of a (well-posed) direct problem gives the solution of the general inhomogeneous problem. Moreover, the Cauchy problem is ill-posed, which means that its solution is not continuously dependent on the Cauchy data.

Thus, in the following we seek a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of (1.1)–(1.3) with $f = 0$, $f_1 = f_2 = f_3 = 0$. We assume that $a \in C^1\big([0,1]\big)$ satisfies

$$a(x) \geqslant r_a > 0 \quad x \in [0,1]. \tag{1.4}$$

We denote the upper bounds of a and $|a'|$ as follows,

$$a(x) \leqslant R_a \text{ and } |a'(x)| \leqslant R'_a, \quad x \in [0,1]. \tag{1.5}$$

2. Method of lines approximation

In this section, the well-known technique of approximating an elliptic equation on a rectangle by the method of lines is outlined. One obtains a system of ordinary differential equations which can be decoupled when one solves an eigenvalue problem beforehand. The method of lines approach was successfully applied also to other types of ill-posed problems, e.g. inverse heat conduction problems in [29].

We use the well-known method of lines to approximate the elliptic boundary value problem. One has two choices, namely lines parallel to the x- or y-axis. Our approach requires that the lines should be chosen parallel to the y-axis or, in other words, perpendicular to the part Σ_1 of the boundary where the Cauchy data are given. With mesh points $x_i = ih$, $i = 0, \ldots, N, h = 1/N$, we approximate $\frac{\partial}{\partial x} a(x) \frac{\partial u}{\partial x}$ in the Laplace operator in (1.1) analogously to the central difference quotient of 2-nd order. Therefore, approximations $u_i(y)$ for the solution $u(x_i, y)$ of (1.1), (1.2) with $f_2 = 0, f_3 = 0, f = 0$ (see Fig. 2.1) can be obtained by the solution of the following system of ordinary differential equations, $u_0 = u_N = 0$,

$$\frac{1}{h^2}(a_{i+1}u_{i+1} - (a_i + a_{i+1})u_i + a_i u_{i-1}) + a_i u''_i = 0, \quad i = 1, \ldots, N-1, \tag{2.1}$$

with the initial conditions

$$u_i(0) = f_1(x_i), \quad u'_i(0) = \phi_1(x_i), \quad i = 1, \ldots, N-1. \tag{2.2}$$

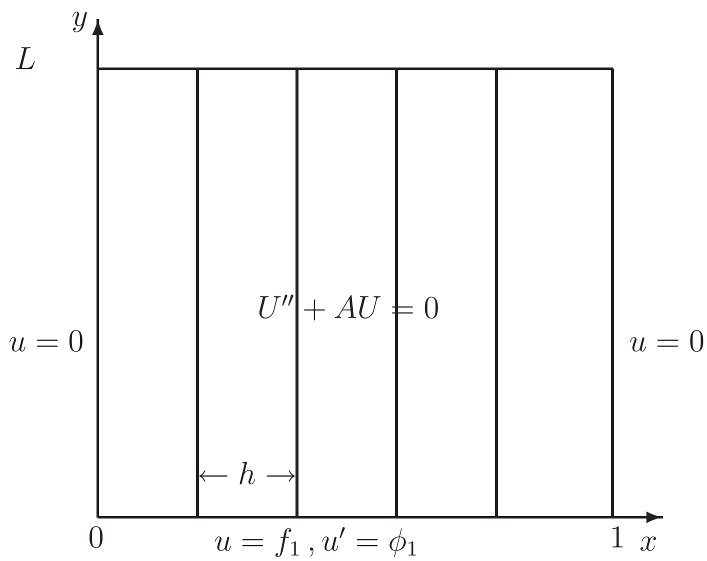

Fig. 2.1. Method of lines approximation

Here $a_i = a(x_i)$, $i = 0, \dots, N$, and $u_i = u_i(y)$ approximates $u(x_i, y)$. For convenience, we multiply system (2.1), (2.2) by $C^{-1} = \operatorname{diag}(1/a_i)_{i=1,\dots,N-1}$ and obtain (with $f_1 = 0$ as assumed in Section 1)

$$U'' + BU = 0, \tag{2.3}$$

$$U(0) = 0, \quad U'(0) = \Phi_1, \tag{2.4}$$

where $B = C^{-1}A$, $C = \operatorname{diag}(a_i)_{i=1,\dots,N-1}$ and $\Phi_1 = \big(\phi_1(x_1), \dots, \phi_1(x_{N-1})\big)^\top$.

System (2.1) can be decoupled using the eigenvalues and eigenvectors of the $(N-1) \times (N-1)$ – matrix

$$B = \frac{1}{h^2}\begin{pmatrix} \frac{-(a_1+a_2)}{a_1} & a_2 & 0 & \dots & 0 & 0 \\ a_2 & \frac{-(a_2+a_3)}{a_2} & a_3 & 0 & \dots & 0 \\ & & & \ddots & & \\ 0 & \dots & 0 & a_{N-2} & \frac{-(a_{N-2}+a_{N-1})}{a_{N-2}} & a_{N-1} \\ 0 & 0 & \dots & 0 & a_{N-1} & \frac{-(a_{N-1}+a_N)}{a_{N-1}} \end{pmatrix}$$

The matrices A and B are not necessarily symmetric. However (cf. [25]), tridiagonal matrices with positive side diagonals are known to be similar to symmetric ones. Here, B is similar to

$$B^{\text{sym}} = \frac{1}{h^2}\begin{pmatrix} -\frac{a_1+a_2}{a_1} & \sqrt{\frac{a_2}{a_1}} & 0 & \dots & 0 & 0 \\ \sqrt{\frac{a_2}{a_1}} & -\frac{a_2+a_3}{a_2} & \sqrt{\frac{a_3}{a_2}} & 0 & \dots & 0 \\ & & & \ddots & & \\ 0 & \dots & 0 & \sqrt{\frac{a_{N-2}}{a_{N-3}}} & -\frac{a_{N-2}+a_{N-1}}{a_{N-2}} & \sqrt{\frac{a_{N-1}}{a_{N-2}}} \\ 0 & 0 & \dots & 0 & \sqrt{\frac{a_{N-1}}{a_{N-2}}} & -\frac{a_{N-1}+a_N}{a_{N-1}} \end{pmatrix} \tag{2.5}$$

Indeed, we have

$$B^{\text{sym}} = S^{-1}BS$$

with

$$S = \operatorname{diag}(s_i)_{i=1,\dots,n}, \quad s_i = \left(\prod_{k=1}^{i-1} \frac{a_k}{a_{k+1}}\right)^{1/2} = \sqrt{\frac{a_1}{a_i}}, \quad i = 1, \dots, N-1.$$

The matrix B^{sym} has real eigenvalues $\tilde{\lambda}_{i,h}$, $i = 1, \dots, N-1$, and one can find a corresponding basis of orthonormal eigenvectors $\tilde{\tilde{w}}_h^{(i)}$ in $\mathbb{R}^{N-1}$ w.r.t. the Euklidean scalar product $\langle \cdot, \cdot \rangle$. B has the same eigenvalues as B^{sym} and the associated eigenvectors $\tilde{w}_h^{(i)}$ are obtained by $\tilde{w}_h^{(i)} = S\tilde{\tilde{w}}_h^{(i)}$.

With the matrix W^{sym} consisting of the column vectors $\tilde{\tilde{w}}_h^{(i)}$, $i = 1, \dots, N-1$, we define

$$W := \frac{1}{\sqrt{ha_1}}\, S\, W^{\text{sym}}, \quad W^{\text{sym}} = \left(\tilde{\tilde{w}}_h^{(1)} \middle| \dots \middle| \tilde{\tilde{w}}_h^{(N-1)}\right). \tag{2.6}$$

The columns of W form also a basis of eigenvectors, $w_h^{(i)} = \tilde{w}_h^{(i)}/\sqrt{ha_1}$, $i = 1, \dots, N-1$. With these definitions, we see that $W^{-1}BW = D$ where $D = \mathrm{diag}(\tilde{\lambda}_{i,h})_i$. We can assume that

$$\tilde{\lambda}_{1,h} \geqslant \dots \geqslant \tilde{\lambda}_{N-1,h}; \tag{2.7}$$

otherwise the rows and columns of B — and W — have to be sorted, respectively. Since all $\tilde{\lambda}_{i,h}$ are negative, B is negative definite and regular.

If we multiply (2.3) from the left by W^{-1}, we see that (2.3) is equivalent to the solution of the following system for $V = W^{-1}U$,

$$V'' + DV = 0\,. \tag{2.8}$$

This means that system (2.3) is decoupled,

$$v_i''(y) + \tilde{\lambda}_{i,h} v_i(y) = 0, \quad i = 1, \dots, N-1. \tag{2.9}$$

The eigenvalues $\tilde{\lambda}_{i,h}$ and the associated eigenvectors $w_h^{(i)}$ of $B, i = 1, \dots, N-1$ have to be determined analytically or numerically before one can solve (2.8), (2.9).

Explicit solutions of a system like (2.8), (2.9) are well-known and can be written as

$$v_i(y) = \xi_i \exp(\sqrt{-\tilde{\lambda}_{i,h}}y) + \eta_i \exp(-\sqrt{-\tilde{\lambda}_{i,h}}y), \quad i = 1, \dots, N-1.$$

The initial conditions at $y = 0$ determine the coefficients ξ_i, η_i using $f_1 = 0$,

$$\xi_i = \frac{(W^{-1}\Phi_1)_i}{2\sqrt{-\tilde{\lambda}_{i,h}}}, \quad \eta_i = -\frac{(W^{-1}\Phi_1)_i}{2\sqrt{-\tilde{\lambda}_{i,h}}}, \quad i = 1, \dots, N-1.$$

Here, $(W^{-1}\Phi_1)_i$ is the ith coordinate of the vector Φ_1 written as a linear combination of the basis of eigenvectors $\{w_h^{(i)}\}_{i=1,\dots,N-1}$. Denoting by $w_{i,k}$ the ith coordinate of $w_h^{(k)}$, for the solutions u_i of (2.3), (2.4) we obtain

$$u_i(y) = (WV)_i(y) = \sum_{k=1}^{N-1} w_{i,k} \left(\xi_k \exp\big(\sqrt{-\tilde{\lambda}_{k,h}}\, y\big) + \eta_k \exp\big(-\sqrt{-\tilde{\lambda}_{k,h}}\, y\big) \right) =$$

$$\sum_{k=1}^{N-1} w_{i,k} \frac{(W\Phi_1)_k}{\sqrt{-\tilde{\lambda}_{k,h}}} \sinh\left(\sqrt{-\tilde{\lambda}_{k,h}}\, y \right), \quad i = 1, \dots, N-1. \tag{2.10}$$

If we set $\lambda_{k,h} = -\tilde{\lambda}_{k,h}$, $k = 1, \dots, N-1$, then the $\lambda_{k,h}$ are all positive and (cf. (2.7)) $\lambda_{1,h} \leqslant \dots \leqslant \lambda_{N-1,h}$. Additionally, on the space of grid functions

$$V_{h,0} = \big\{ v_h : [0,1]_h \to \mathbb{R} \mid v_h(x_0) = v_h(x_N) = 0 \big\},$$

with $[0,1]_h = \{x_i \,|\, i = 0, \dots, N\}$, we define the scalar product

$$(v_h, w_h)_{0,a,h} = h \sum_{j=1}^{N-1} a_j v_h(x_j) w_h(x_j).$$

Then the functions $v_{k,h}(x_j) = (w_h^{(k)})_j = w_{j,k}$, $k, j = 1, \ldots, N-1$, form an orthonormal basis $\{v_{k,h}\}_{k=1,\ldots,N-1}$ w.r.t. to $(\cdot,\cdot)_{0,a,h}$. With the grid function $\phi_1^h = \phi_1 \,|\, [0,1]_h$, the solutions u_i of (2.3), (2.4) can be written in the form $\big($cf. (2.10)$\big)$

$$u_i(y) = \sum_{k=1}^{N-1} \frac{(\phi_1^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h}(x_i) \sinh(\sqrt{\lambda_{k,h}}\, y), \quad i = 1, \ldots, N-1. \tag{2.11}$$

If the function $a(\cdot)$ in (1.1) is constant, e.g., $a(x) = 1$, the eigenvalues and eigenvectors of the symmetric matrix

$$A_h = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ \vdots & & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

are explicitly known,

$$\tilde{\lambda}_{j,h} = -\frac{4}{h^2} \sin^2\left(jh\frac{\pi}{2}\right), \quad w_{j,k} = (w_h^{(j)})_k = \sqrt{2h}\,\sin(jkh\pi), \quad k, j = 1, \ldots, N-1.$$

The solutions of the method of lines approximation are then given by (cf.(2.11))

$$u_i(y) = h \sum_{k=1}^{N-1} \frac{\langle \Phi_1, w_h^{(k)} \rangle}{\sqrt{\lambda_{k,h}}} w_{i,k} \sinh\left(\sqrt{\lambda_{k,h}}\, y\right), \quad i = 1, \ldots, N-1,$$

with Φ_1 from the initial condition (2.4).

We now study an *example* similar to the classical Hadamard's example (cf. [15]) with Cauchy data

$$f_1 = 0, \quad \phi_1(x) = (m\pi)^{-1} \sin(m\pi x), \quad 0 \leqslant x \leqslant 1,$$

and with $m \in \mathbb{N}$, $m < N$. Again, we consider the case $a(x) = 1$. With respect to the maximum norm $\|\cdot\|_\infty$ in the x-interval, for the solution

$$(u =) u_m(x, y) = \frac{\sin(m\pi x)\sin(m\pi y)}{m^2\pi^2} \; 0 \leqslant x, y \leqslant 1\,,$$

of the original Cauchy problem (1.1)–(1.3) one can observe that

$$\|u_m(\cdot, y)\|_\infty = \frac{\sinh(m\pi y)}{(m\pi)^2} \to \infty \quad (m \to \infty)$$

for any $y > 0$, while, for the data function,

$$\|\phi_1\|_\infty = \frac{1}{m\pi} \to 0 \quad (m \to \infty).$$

For this example it is not difficult to show that the error for the method of lines approximation can be estimated as follows (cf. [4], 3.1):

$$|u(x_i, y) - u_i(y)| \leqslant |\sin(m\pi x_i)| \frac{m\pi y}{24} \exp(m\pi y) h^2, \quad i = 1, \ldots, N-1,$$

as long as $h \leqslant 4\sqrt{3}/(m\pi)$. Hence, for fixed m, the method of lines approximation converges for $h \to 0$.

Due to the ill-posedness of the original elliptic Cauchy problem, the method of lines approximation is still ill-posed. Indeed, for any $S > 0$ and every $y > 0$, for line $i = 1$ one can find a $h^* = h^*(y, S)$ such that

$$\big|u(x_1, y) - u_{1,\varepsilon}(y)\big| \geqslant S$$

where $u_{i,\varepsilon}$ denotes the solution given by (2.10) with the perturbed data function ($m = 2$)

$$\Phi_{1,\varepsilon} = \left(\frac{\sin(2\pi x_i) - \varepsilon}{2\pi}\right)_{i=1,\dots,N-1}.$$

For any $\varepsilon > 0$, similar results hold for any line $i = 1, \dots, N-1$.

We will study the convergence of the eigenvalues and eigenvectors for the general case of a not necessarily constant function $a = a(x)$ in the following Sections 4 and 5.

3. Conditional well-posedness of the Cauchy problem

As the main result of this section, the conditional stability of the Cauchy problem is shown. This result utilizes the logarithmic convexity of an appropriate norm of the solution. Moreover, the classical method of separation of variables for representing the solution is used. The method of logarithmic convexity for such problems w.r.t. various norms can be found, e.g., in [11, 17, 21].

Let us consider again the following semihomogeneous Cauchy problem in the rectangle $\Omega = (0,1) \times (0, L)$ (cf. (1.1)–(1.3))

$$\nabla\big(a(x)\nabla u\big)(x, y) = 0 \quad \text{in} \quad \Omega, \tag{3.1}$$

$$u = 0, \quad \frac{\partial u}{\partial y} = \phi_1 \quad \text{on} \quad \Sigma_1, \tag{3.2}$$

$$u = 0 \quad \text{on} \quad \Sigma_2 \cup \Sigma_3. \tag{3.3}$$

We want to determine $u|\Sigma_4$ from the Cauchy data on Σ_1. Solutions of (3.1)–(3.3) in the classical sense can be obtained via the method of separation of variables, $u(x, y) = v(x)s(y)$. Inserting this ansatz into the differential equation in (3.1) and using the relation

$$\nabla a(x)\nabla u = a\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)$$

leads to

$$\frac{(av')'}{av} = -\frac{\ddot{s}}{s} = -\lambda$$

with $\lambda > 0$, where $\ddot{s}$ and v' denote the differentiation w.r.t. y and x, respectively. For s, using $s(0) = 0$ we obtain

$$s(y) = C\sinh(\sqrt{\lambda}y) \tag{3.4}$$

for a nontrivial solution, while v and λ have to be determined from the following Sturm — Liouville eigenvalue problem for ordinary differential equations,

$$Lv + \lambda av = 0 \quad \text{in} \quad (0,1), \quad v(0) = v(1) = 0, \tag{3.5}$$

where $Lv := (av')'$. It is well-known that the boundary value problem (3.5) is equivalent to the Fredholm integral equation of the second kind. The reciprocals of the eigenvalues of the associated integral operator yield eigenvalues $0 < \lambda_1 < \lambda_2 < \ldots$ in (3.5) with an orthonormal system of eigenfunctions $v_i, i \in \mathbb{N}$; the underlying weighted scalar product is given by

$$(u, v)_{0,a} = \int_0^1 a(x)u(x)v(x)\,dx\,. \tag{3.6}$$

Moreover, all eigenvalues are simple, and are diverging $\lim_{i\to\infty} \lambda_i = \infty$. Denoting

$$V_0 := \big\{v \in C^2[0,1] \mid v(0) = v(1) = 0\big\}\,,$$

any function $v \in V_0$ can be represented as

$$v(x) = \sum_{n=1}^{\infty} (v, v_n)_{0,a} v_n(x) \tag{3.7}$$

with an absolutely and uniformly convergent series of eigenfunctions $\big\{v_n \mid n \in \mathbb{N}\big\}$. The eigenfunctions are orthonormal w.r.t. the norm $\|\cdot\|_{0,a}$ associated with (3.6)[1]. Moreover, one knows that all eigenvalues are pairwise distinct and, hence, the associated eigenspaces have dimension one. It should be noted that the representation by an absolute and uniformly convergent series like (3.7) is also valid for functions from

$$D = \big\{v \in C^1[0,1] \mid v(0) = v(1) = 0\big\}. \tag{3.8}$$

For L_2-functions one has a representation by a series converging w.r.t. the L_2-norm or the weighted L_2-norm. For L_2-functions the well-known *Parseval relation* holds

$$\|v\|_{0,a}^2 = \sum_{n=1}^{\infty} (v, v_n)_{0,a}^2$$

where, usually, the numbers $(v, v_n)_{0,a}$ are called *Fourier coefficients* of v w.r.t. to the a-orthonormal system $\{v_n\}$.

Returning to V_0, the Courant-Min-Max principle for characterizing the eigenvalues is available,

$$\lambda_k = \min_{\substack{W \subset V_0 \\ \dim W = k}} \; \max_{0 \neq v \in W} \rho(v) \tag{3.9}$$

where $\rho(v) = -(Lv, v)_{L_2} \,/\, \|v\|_{0,a}^2$ denotes the *Rayleigh coefficient*. For any $v \in V_0$, it can be represented by $\rho(v) = \|v'\|_{0,a}^2 \,/\, \|v\|_{0,a}^2$. In the constant case $a = 1$, one has $\lambda_k = k^2\pi^2$ (cf. the end of Sect. 2). In the general case, for $0 < r_a \leqslant a(x) \leqslant R_a$, $x \in [0,1]$, it is easy to see that the following estimates for the eigenvalues hold:

$$\frac{r_a}{R_a}\, k^2\pi^2 \leqslant \lambda_k \leqslant \frac{R_a}{r_a}\, k^2\pi^2, \quad k \in \mathbb{N}. \tag{3.10}$$

Having summarized the main properties of the Sturm — Liouville eigenvalue problem, we come back to the solution of the continuous Cauchy problem (3.1)–(3.3). Using the eigenfunctions $v_k, k \in \mathbb{N}$, of the Sturm — Liouville eigenvalue problem and setting

$$u_k(x, y) = v_k(x) \sin(\sqrt{\lambda_k} y),$$

[1] We also call $\{v_n\}$ *a-orthonormal*.

by superposition we obtain a solution in the form

$$u(x,y) = \sum_{k=1}^{\infty} C_k u_k(x,y) \tag{3.11}$$

which satisfies the boundary conditions

$$u(0,y) = u(1,y) = 0,\ 0 \leqslant y \leqslant L, \quad u(x,0) = 0, 0 \leqslant x \leqslant 1\,.$$

The coefficients C_k have to be determined such that the Neumann boundary condition at $y = 0$ is fulfilled,

$$\frac{\partial}{\partial y}\Big(\sum_{k=1}^{\infty} C_k u_k(x,y)\Big)\Big|_{y=0} = \phi_1(x), \quad 0 \leqslant x \leqslant 1.$$

Interchanging the differentiation and summation leads to

$$C_k = \frac{(\phi_1, v_k)_{0,a}}{\sqrt{\lambda_k}}, \quad k \in \mathbb{N},$$

and the solution u of (3.1)–(3.3) has the form

$$u(x,y) = \sum_{k=1}^{\infty} \frac{(\phi_1, v_k)_{0,a}}{\sqrt{\lambda_k}}\, v_k(x) \sinh(\sqrt{\lambda_k} y). \tag{3.12}$$

In the remaining part of this section, our aim is to prove a stability theorem for the Cauchy problem (3.1) - (3.3) utilizing the technique of logarithmic convexity. We will formulate the stability result w.r.t. the L_2-norm. Other results of this type are also shown w.r.t. the H^1- and H^2-norm in [17]. Since it is not immediately obvious how the L_2-stability can be obtained from Thm. 2.2 in [17], we will outline its proof here. The technique of logarithmic convexity for proving L_2-stability results of Cauchy problems for elliptic equations is well-known and have been used, e.g., by Kubo [21], Fayazov et al. [11], and others.

A function $f : [a,b] \to \mathbb{R}$ is called *logarithmic convex* if $ln(f)$ is convex. This can be characterized by

$$f\big(\lambda z + (1-\lambda)y\big) \leqslant f(z)^{\lambda} f(y)^{1-\lambda}, \quad \forall y, z \in [a,b], \quad \lambda \in [0,1].$$

Any positive, twice differentiable function is logarithmic convex if and only if

$$f''(y)f(y) - \big(f'(y)\big)^2 \geqslant 0 \quad \forall y \in [a,b]\,. \tag{3.13}$$

Theorem 3.1. *Let the classical solution of* (3.1)–(3.3) *fulfill the additional assumption*

$$\|u\|_{0,a,\sum_4} \leqslant E \tag{3.14}$$

with some nonnegative real constant E and let the series in representation (3.12) *of the solution converge pointwise in* $[0,1]\times[0,L]$ *and uniformly w.r.t. x in* $[0,1]$ *where the Fourier coefficients* $b_k := (\phi_1, v_k)_{0,a}$ *of* ϕ_1 *are so small that*

$$\sum_{k=1}^{\infty} b_k^2 \exp(2\sqrt{\lambda_k}\, L) \tag{3.15}$$

converges. Then for $R_1 := \max(L,1)$, $R_0 := \min(L,1)$ *the following estimates hold:*

$$\big\|u(\cdot,y)\big\|_{0,a} \leqslant \frac{y}{R_0}\|\phi_1\|_{0,a}^{1-y/L}\; E^{y/L} \leqslant R_1\|\phi_1\|_{0,a}^{1-y/L}\; E^{y/L}, \quad y \in [0,L]. \tag{3.16}$$

Proof. By means of our assumptions the functions

$$F_n(y) := ln\big(s_n(y)\big), \quad s_n(y) := \sum_{k=1}^{n} \frac{b_k^2}{\lambda_k} f_k(y), \quad n \in \mathbb{N},$$

$$f_k(y) := \frac{\sinh^2\left(\sqrt{\lambda_k}\, y\right)}{y^2}, \quad y \in [y_0, L], \quad y_0 \in (0, L),$$

have the following properties:

1) F_n is twice differentiable for every n;

2) $(F_n)_n$, $(F_n')_{n\in\mathbb{N}}$ and $(F_n'')_{n\in\mathbb{N}}$ converge (for $n \to \infty$) uniformly on $[y_0, L]$.

By the explicit representation of F_n'' one can show that $F_n''(y) > 0$, $y \in [y_0, L]$, so that $s_n(\cdot)$ is logarithmic convex. Because of the uniform convergence of F_n to $F = \ln(s)$ with $s(y) = \sum_{k=1}^{\infty} (b_k^2 \,/\, \lambda_k) f_k(y)$, together with its first and second derivatives, then also the limit function F is convex. The representation of solution (3.12) and the logarithmic convexity of

$$s(y) = \big\|u(\cdot, y)\big\|_{0,a}^2 \,/\, y^2$$

imply

$$\frac{N_a(y)}{y^2} \leqslant \left(\frac{N_a(y_0)}{y_0^2}\right)^{\lambda} \left(\frac{N_a(L)}{L^2}\right)^{1-\lambda}$$

for any y_0, y in $0 < y_0 \leqslant y$ with $\lambda = \lambda(y_0)$ and $N_a(y) := \big\|u(\cdot, y)\big\|_{0,a}^2$. For $y_0 \to 0$, we obtain

$$\frac{N_a(y)}{y^2} \leqslant \|\phi_1\|_{0,a}^{2(1-y/L)} \left(\frac{E^2}{L^2}\right)^{y/L}, \quad y \in [0, L),$$

which proves the desired estimate. □

4. Discrete Sturm — Liouville eigenvalue problems

In this section, we study the convergence of the eigenvalues and eigenvectors of discretized Sturm — Liouville eigenvalue problems. The corresponding results are classical and go back to the work of Bückner [1] of 1948. The paper of Keller [20] should also be mentioned as a fundamental one in this direction. Carasso [3] extended the approach of Keller to non-self-adjoint Sturm — Liouville operators. The results of Grigorieff [13, 14] and Stummel [33, 34] are based on the general perturbation theory of approximation methods for differential and integral equations.

We will state Theorem 4.1 to 4.3 as it is done in [4]. It is known that a better convergence rate, namely $O(h^2)$, can be achieved when a more appropriate difference approximation is used. However, this is not an essential point of this paper.

We approximate the Sturm — Liouville eigenvalue problem (3.5) by using an equidistant grid $G_h = [0, 1]_h = \{x_j = jh | j = 0, \dots, N\}$ where $Nh = 1$. With $a_i = a(x_i)$ a suitable approximation $L_h : G_{h,0} \to G_{h,0}$ of $Lv = (av')'$ is defined by (cf. (2.1))

$$(L_h v_h)(x_i) = \frac{1}{h^2}\big(a_{i+1} v_h(x_{i+1}) - (a_i + a_{i+1}) v_h(x_i) + a_i v_h(x_{i-1})\big), \quad i = 1, \dots, N-1,$$

$$(L_h v_h)(x_i) = 0, \quad i \leqslant 0, \quad i \geqslant N, \tag{4.1}$$

for functions from $\Gamma_h = \{v_h : \mathbb{R}^h \to \mathbb{R} | v_h(x_j) \neq 0$ for finitely many $j \in \mathbb{Z}\}$ where $\mathbb{R}^h = \{x_j = jh | j \in \mathbb{Z}\}$. Spaces of the grid functions are defined by

$$\mathcal{G}_h := \{v_h \in \Gamma_h | v_h(x_j) = 0,\ j \notin \{0, \dots, N\}\},$$

$$\mathcal{G}_{h,0} := \{v_h \in \mathcal{G}_h | v_h(x_0) = v_h(x_N) = 0\} (\subset \mathcal{G}_h \subset \Gamma_h).$$

There is a one-to-one correspondence between the grid functions in $\mathcal{G}_{h,0}$ and $\mathbb{R}^{n-1}$,

$$\mathcal{G}_{h,0} \ni v_h \longleftrightarrow \big(v_h(x_1), \dots, v_h(x_{N-1})\big)^\top \in \mathbb{R}^{N-1}.$$

In the following, we do not distinguish between vectors in $\mathbb{R}^{N-1}$ and grid functions in $\mathcal{G}_{h,0}$.

The forward and backward the first-order difference quotients are denoted by

$$(D_{1,h} v_h)(x_j) = \frac{v_h(x_{j+1}) - v_h(x_j)}{h},$$

and

$$(D_{-1,h} v_h)(x_j) = \frac{v_h(x_j) - v_h(x_{j-1})}{h}, \quad j \in \mathbb{Z}, \quad \text{resp.}$$

We associate the discrete scalar products on $\mathcal{G}_{h,0}$

$$(v_h, w_h)_{0,a,h} = h \sum_{j=1}^{N-1} a_j v_h(x_j) w_h(x_j), \quad (v_h, w_h)_{1,a,h} = (D_{1,h} v_h, D_{1,h} w_h)_{0,a,h},$$

$$(v_h, w_h)_{-1,a,h} = (D_{-1,h} v_h,\ D_{-1,h} w_h)_{0,a,h},$$

and denote the associated norms by $\|\cdot\|_{0,a,h}$, $\|\cdot\|_{1,a,h}$, $\|\cdot\|_{-1,a,h}$.

It is not difficult to see that the following relations hold:

$$L_h v_h(x_j) = D_{1,h}(a^h D_{-1,h} v_h)(x_j), \quad j \in \mathbb{Z},$$

$$(L_h v_h, w_h)_{0,h} = (v_h, L_h w_h)_{0,h} = -(v_h, w_h)_{-1,a,h}, \quad (L_h v_h, v_h)_{0,h} = -\|v_h\|^2_{-1,a,h}, \quad v_h, w_h \in \mathcal{G}_{h,0}.$$

Moreover, the *truncation error*

$$T_h(x_j) = \big((Lv)^h - L_h v^h\big)(x_j), \quad j = 1, \dots, N-1, \quad v \in V_0\,,$$

converges to zero, $\max\limits_{1 \leqslant j \leqslant N-1} |T_h(x_j)| \to 0$ $(h \to 0)$. Here, as always in the following, the superscript h denotes the restriction of a continuous function to the grid $[0,1]_h$. For smooth functions v, convergence rates of the truncation error can be achieved.

The continuous Sturm — Liouville eigenvalue problem was already introduced in Section 3, (3.5). The discrete Sturm — Liouville eigenvalue problem is given by

$$L_h v_h + \lambda_h a^h v_h = 0, \quad k = 1, \dots, N-1, \tag{4.2}$$

where grid functions v_h, i.e., the *eigenfunctions*, and the associated *eigenvalues* λ_h are sought. Compared to the notation in Section 1, L_h is represented by the matrix A, the eigenvectors in (4.2) are those of the matrix B denoted by $\tilde{w}_h^{(i)}$, when we multiply (4.2) by $C^{-1} = \operatorname{diag}(1/a_i)$, and the eigenvalues λ_h in (4.2) are denoted by $\tilde{\lambda}_{i,h}$ in Section 1 which are known to be simple and all negative. We assume that the eigenvalues are ordered by magnitude

$$0 > \tilde{\lambda}_{1,h} \geqslant \dots \geqslant \tilde{\lambda}_{N-1,h}\,.$$

The main results of this section are the convergence of the eigenvalues and eigenfunctions of the discrete Sturm — Liouville eigenvalue problems to the corresponding quantities of the continuous problem. The fundamental result for this is the following.

Theorem 4.1. *The discrete and continuous Rayleigh coefficients,*

$$\rho_h(v_h) = \frac{\|v_h\|^2_{-1,a,h}}{\|v_h\|^2_{0,a,h}}, \quad v_h \in V_{0,h},$$

and

$$\rho(v) = \frac{\|v'\|_{0,a}}{\|v\|_{0,a}}, \quad v \in V_0,$$

resp., satisfy the following relation:

$$\lim_{h\to 0}\Bigg(\min_{\substack{\dim M_h=j\\ M_h\subseteq \mathcal{G}_{h,0}}} \max_{0\neq v_h\in M_h} \rho_h(v_h)\Bigg) = \min_{\substack{\dim M=j\\ M\subset V_0}} \max_{0\neq v\in M} \rho(v)$$

and the estimate

$$\Bigg|\min_{\substack{\dim M_h=j\\ M_h\subset \mathcal{G}_{h,0}}} \max_{0\neq v_h\in M_h} \rho_h(v_h) - \min_{\substack{\dim M=j\\ M\subset V_0}} \max_{0\neq v\in M} \rho(v)\Bigg| \leqslant Ch \tag{4.3}$$

for any $j \in \mathbb{N}$, any h small enough, and some constant $C > 0$.

As a consequence we obtain the convergence of the eigenvalues and its reciprocals.

Theorem 4.2. *For any $j \in \{1, \ldots, N-1\}$, $N \in \mathbb{N}$, the jth eigenvalues $\tilde{\lambda}_{j,h}$ of L_h converge to the jth eigenvalue $\tilde{\lambda}_j$ of L of the Sturm — Liouville eigenvalue problem* (3.5) *with the convergence rate*

$$\big|\tilde{\lambda}_{i,h} - \tilde{\lambda}_j\big| = O(h) \quad (h \to 0). \tag{4.4}$$

The same holds for the reciprocals $\mu_{i,h} = 1/\tilde{\lambda}_{j,h}$, $\mu_j = 1/\tilde{\lambda}$

$$\big|\mu_{i,h} - \mu_j\big| = O(h) \quad (h \to 0), \quad j \in \{1, \ldots, N-1\}, \quad N \in \mathbb{N}. \tag{4.5}$$

Proof. The Courant-Min-Max principle (cf. (3.9)) in connection with Theorem 4.1 (cf. (4.3)) yields convergence (4.4). Since the eigenvalues $\tilde{\lambda}_j$ are bounded from below away from zero (cf. (3.10)), for sufficiently small h, (4.4) yields the corresponding estimate for $\mu_j, \mu_{j,h}$. □

As mentioned in the beginning of Section 3, the Sturm — Liouville eigenvalue problem (3.5) is equivalent to an eigenvalue problem for a Fredholm integral operator which is obtained via Green's function G associated with L. The eigenvalues of the integral operator are given by $\mu_j = 1/\tilde{\lambda}_j$. They are converging to zero, $\lim\limits_{j\to\infty} \mu_j = 0$. Accordingly, there exists a discrete Green's function F_h related to the difference operator L_h having the property that the solution $v_h \in \mathcal{G}_{h,0}$ of the discrete boundary value problem

$$L_h v_h = w_h, \quad w_h \in \mathcal{G}_{h,0}, \tag{4.6}$$

can be represented as follows:

$$v_h(x_j) = \begin{cases} h\displaystyle\sum_{\ell=1}^{N-1} F_h(x_j, t_\ell) w_h(t_\ell), & j = 0, \ldots, N, \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

For Green's functions G and F_h explicit formulae are available (cf., e.g., [4, 5.3.3, 6.2.1]). From these one can deduce that

$$\max_{1\leqslant j,\ell\leqslant N-1} |F_h(x_j,t_\ell) - G(x_j,t_\ell)| = O(h)(h\to 0)\,. \tag{4.8}$$

Using the equivalence of the Sturm — Liouville eigenvalue problems with the eigenvalue problems for the associated Fredholm integral operators in continuous and discrete form, one obtains the convergence of the eigenfunctions.

Theorem 4.3. *For any $j\in\{1,\dots,N-1\}$, $N\in\mathbb{N}$, the jth orthonormal eigenfunctions $v_{j,h}$ in the $(\cdot,\cdot)_{0,a,h}$ — orthonormal system $\{v_{1,h},\dots,v_{N-1,h}\}$ converge to the jth $(\cdot,\cdot)_{0,a}$ — orthonormal eigenfunction v_j with the following estimates:*

$$\|v_j^h - v_{j,h}\|_{0,\infty} = O(\sqrt{h}), \tag{4.9}$$

$$\|v_j^h - v_{j,h}\|_{0,a,h} = O(h) \quad (h\to 0)\,. \tag{4.10}$$

5. Convergence of the line approximation method

In the main section of this work, the convergence as well as the regularization of the line approximation method is proved. Here, the dimension M of the space of truncated Fourier series plays the role of a regularization parameter and, moreover, the mesh size of the line approximation method must go to zero when the magnitude of the data errors tends to zero. The optimal value of M is determined via the stability estimate of the logarithmic type.

The convergence for $h\to 0$ and $\varepsilon\to 0$ of the line approximation method is assured by the following steps. First, the data function ϕ_1 (note that we consider the case $f_1=0$) is projected into the space D_M of those functions which form a truncated series if one develops them w.r.t the a-orthonormal system $\{v_1,v_2,\dots\}$. We even allow perturbed data functions ϕ_1^ε such that $\|\phi_1-\phi_1^\varepsilon\|_{0,a}\leqslant\varepsilon$. In this situation it is clear that, in general, $\phi_1^\varepsilon\notin D_M$ even if $\phi_1\in D_M$. One has to estimate the projection error of projected data and then the error between the true solution and the line approximation method with projected data in D_M. We will show in this section that, for the convergence, the magnitude of perturbations should depend on the discretization parameter by $h=O(\varepsilon^2)$ and the dimension of D_M has to be chosen in an appropriate way.

As in Section 3, we denote the eigenvalues and eigenfunctions of the continuous Sturm — Liouville eigenvalue problem by λ_k and v_k, $k=1,2,\dots$; respectively (cf. Section 4), $\lambda_{k,h}$ and $v_{k,h}$, $k=1,\dots,N-1$ denote the eigenvalues and eigenfunctions of the discrete Sturm — Liouville eigenvalue problems for $h=1/N$ and any $N\in\mathbb{N}$. Their convergence for $h\to 0$ is ensured in the Theorems 4.2 and 4.3.

The eigenvalues are assumed to be ordered in magnitude,

$$0<\lambda_1<\lambda_2<\dots,\quad 0<\lambda_{1,h}<\dots<\lambda_{N-1,h.}$$

Note that in the general case of the function $a=a(x)$ we do not have $v_k^h=v_{k,h}$.

However, there is an isomorphism between the spaces of eigenfunctions. For $M,N,N>M,h=1/M$, and the space D given by (3.8) we define

$$D_M := \{\phi\in D \mid (\phi,v_k)_{0,a}=0,\quad k>M\}, \tag{5.1}$$

$$D_M^h := \{\phi_h \in \mathcal{G}_{h,0} \mid (\phi_h, v_{k,h})_{0,a,h} = 0, \quad N > k > M\}. \tag{5.2}$$

These spaces have orthonormal basis systems $v_1, \dots, v_M$ and $v_{1,h}, \dots, v_{M,h}$, and an isometric isomorphism $D_M \to D_M^h$ exists for any M and any h. The associated orthormal projections $P_M : D \to D_M$, $P_M^h : \mathcal{G}_{h,0} \to D_M^h$ are given by

$$P_M\phi = \sum_{k=1}^{M} (\phi, v_k)_{a,0} v_k, \quad \phi \in D, \quad P_M^h v_h = \sum_{l=1}^{M} (v_h, v_{l,h})_{0,a,h} v_{l,h}, \quad v_h \in \mathcal{G}_{h,0}. \tag{5.3}$$

They are linear and fulfill the minimum-norm property (here for P_M^h)

$$\min_{\psi_h \in D_M^h} \|\phi_h - \psi_h\|_{0,a}^2 = \|\phi_h - P_M^h \phi_h\|_{0,a}^2, \quad \phi_h \in \mathcal{G}_{h,0},$$

and the following estimate stated in Lemma 5.1. As in the previous section, we assume in the following that $0 < r_a \leqslant a(x) \leqslant R_a$, $x \in [0,1]$.

Lemma 5.1. *For any $\varepsilon > 0$ and any grid function $\delta_h \in \mathcal{G}_{h,0}$ with $\|\delta_h\|_{0,\infty} < \varepsilon$, for the orthormal projections the following estimate holds:*

$$\|P_M^h \delta_h - \delta_h\|_{0,\infty} \leqslant (C_0 M + 1)\varepsilon \tag{5.4}$$

with a constant $C_0 > 0$.

Proof. It is clear that

$$\|v_h\|_{0,a,h}^2 = h \sum_{j=1}^{N-1} v_h(x_j)^2 a(x_j) \leqslant R_a h(N-1) \|v_h\|_{0,\infty}^2 \leqslant R_a \|v_h\|_{0,\infty}^2$$

for any $v_h \in \mathcal{G}_{h,0}$. Furthermore, for the eigenfunctions we deduce by means of (4.9) that

$$\|v_{l,h}\|_{0,\infty} \leqslant \begin{cases} \|v_l^h\|_{0,\infty} + C^{(1)}\sqrt{h}, & h \leqslant h^{(1)}, \\ \max\limits_{h > h^{(1)}} \|v_{l,h}\|_{0,\infty}, & h > h^{(1)}, \end{cases}$$

$$\leqslant \underbrace{\max\left(\|v_l\|_{0,\infty} + C^{(1)}, \max_{h > h^{(1)}} \|v_{l,h}\|_{0,\infty} \right)}_{=:C^{(l)}} \leqslant \max_{l=1,\dots,M} C^{(l)} =: \tilde{C}_0$$

for some $h^{(1)}$ in $0 < h^{(1)} \leqslant 1$ and certain $C^{(1)}$. This implies

$$\|P_M \delta_h - \delta_h\|_{0,\infty} = \left\| \sum_{l=1}^{M} (\delta_h, v_{l,h})_{0,a} v_{l,h} - \delta_h \right\|_{0,\infty} \leqslant \sum_{l=1}^{M} (|(\delta_h, v_{l,h})_{0,a}| \cdot \|v_{l,h}\|_{0,\infty}) + \|\delta_h\|_{0,\infty} \leqslant$$

$$\underbrace{\|\delta_h\|_{0,a}}_{\leqslant \sqrt{R_a}\|\delta_h\|_{0,\infty}} \cdot \underbrace{\|v_{l,h}\|_{0,a}}_{=1} \cdot \underbrace{\|v_{l,h}\|_{0,\infty}}_{\leqslant \tilde{C}_0} + \|\delta_h\|_{0,\infty} \leqslant (\underbrace{\tilde{C}_0 \sqrt{R_a}}_{=:C_0} M + 1) \underbrace{\|\delta_h\|_{0,\infty}}_{\leqslant \varepsilon} \leqslant (C_0 M + 1)\varepsilon$$

which proves the desired estimate. □

Our next aim is to prove the convergence of the line approximations method in the case of exact data from D_M. From (3.12) we know that, for $\phi_1 \in D_M$, the solution of problem (3.1)–(3.3) is given by

$$u(x,y) = \sum_{k=1}^{M} \frac{(\phi_1, v_k)_{0,a}}{\sqrt{\lambda_k}} v_k(x) \sinh(\sqrt{\lambda_k} y). \tag{5.5}$$

For the grid function $\phi_{1,h} \in D_M^h$, the line method approximation on the ith line has the form

$$u_i^*(y) = \sum_{k=1}^{M} \frac{(\phi_{1,h}, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} \, v_{k,h}(x) \, \sinh\left(\sqrt{\lambda_k} y\right). \tag{5.6}$$

Formula (5.5) is also the solution of the Cauchy problem for general $\phi_1 \in D$ and data $\phi_1^* = P_M \phi_1$. Moreover, (5.6) yields the approximation on the ith line for data $P_M \phi_1^h$ again with general $\phi_1 \in D$ when we set $\phi_{1,h} = \phi_1^h$ in (5.6).

For the convergence proof we need the following lemma:

Lemma 5.2. *For fixed $M \in \mathbb{N}$ and any $k = 1, \ldots, M$ the eigenvalues and eigenvectors satisfy the estimates*

$$\left|\sqrt{\lambda_k} - \sqrt{\lambda_{k,h}}\right| = O(h), \quad \left|\frac{1}{\sqrt{\lambda_k}} - \frac{1}{\sqrt{\lambda_{k,h}}}\right| = O(h) \quad (h \to 0), \tag{5.7}$$

$$\left|\sinh\left(\sqrt{\lambda_k} y\right) - \sinh\left(\sqrt{\lambda_{k,h}} y\right)\right| = O(h) \quad (h \to 0), \quad \textit{uniformly in } \; y \in [0, L], \tag{5.8}$$

$$\left|\left(\phi_1^h, v_{k,h}\right)_{0,a,h} - (\phi_1, v_k)_{0,a}\right| = O(h) \quad (h \to \infty), \quad \phi_1 \in D_M. \tag{5.9}$$

Proof. The estimates in (5.7) are a consequence of the Courant Min-max-principle (3.10) and the convergence results in Theorem 4.1 and 4.2. Using the convergence $\lambda_{k,h} = \lambda_k + O(h)$ $(h \to 0)$, the definition of sinh and the second inequality in (3.10), the estimate (5.8) is not difficult to obtain. Here, one should distinguish between the cases $\lambda_{k,h} \leqslant \lambda_k$ and $\lambda_{k,h} \geqslant \lambda$. Finally, (5.9) follows from (4.10) in connection with the Cauchy — Schwarz inequality and the fact that $(\cdot,\cdot)_{0,a,h}$ is nothing but the *summed up* trapezoidal rule approximating the integral $(\cdot,\cdot)_{0,a}$ whose quadrature has magnitude $O(h)$. □

For the data in D_M we can now prove the convergence of u_i to $u(x_i, \cdot)$ with a certain power of h.

Theorem 5.1. *Let $\phi_1 \in D_M$. Then the approximation $u_i^*(y)$ in* (5.6) *with data $P_M \phi_1^h$ converges uniformly to $u(x_i, \cdot)$ given in* (5.5) *with the following asymptotic error estimate:*

$$|u(x_i, y) - u_i^*(y)| = O(\sqrt{h}) \quad (h \to 0) \tag{5.10}$$

for $i \in \{1, \ldots, N\}$, $N \in \mathbb{N}$, and $y > 0$. Furthermore,

$$\|u(\cdot, y)^h - u_h^*(y)\|_{0,a,h} = O(h) \tag{5.11}$$

where $u_h^*(y) = (u_1^*(y), \ldots, u_{N-1}^*(y))$.

Proof. With the representations of u and u_i^* in (5.5) and (5.6), resp., we can estimate

$$\left|\frac{(\phi_1, v_k)_{0,a}}{\sqrt{\lambda_k}} v_k(x_i) \sinh(\sqrt{\lambda_k}\, y) - \frac{(\phi_1^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h}(x_i) \sinh(\sqrt{\lambda_{k,h}}\, y)\right| \leqslant C\sqrt{h}$$

using the results of Lemma 5.2. This implies

$$|u(x_i, y) - u_i^*(y)| =$$

$$\left|\sum_{k=1}^{M} \left(\frac{(\phi_1, v_k)_{0,a}}{\sqrt{\lambda_k}} v_k(x_i) \sinh\left(\sqrt{\lambda_k} y\right) - \frac{(\phi_1^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h}(x_i) \sinh\left(\sqrt{\lambda_{k,h}} y\right)\right)\right| \leqslant MC\sqrt{h}$$

uniformly for $y \geqslant 0$.

The norm $\|\cdot\|_{0,a,h}$ can be estimated analogously:

$$\left\| \frac{(\phi_1, v_k)_{0,a}}{\sqrt{\lambda_k}} v_k^h \sinh\left(\sqrt{\lambda_k} y\right) - \frac{(\phi_1^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h} \sinh\left(\sqrt{\lambda_{k,h}} y\right) \right\|_{0,a,h} \leqslant Ch$$

which yields

$$\|(u(.,y))^h - u_h^*(y)\|_{0,a,h} =$$

$$\left\| \sum_{k=1}^{M} \left(\frac{(\phi_1, v_k)_{0,a}}{\sqrt{\lambda_k}} v_k \sinh\left(\sqrt{\lambda_k} y\right) - \frac{(\phi_1^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h} \sinh\left(\sqrt{\lambda_{k,h}} y\right) \right) \right\|_{0,a,h} \leqslant MCh.$$

□

Now we study perturbed data, i.e., $\phi_1 \in M$ and $\phi_{1,\varepsilon} \in D$ with

$$\|\phi_1^h - \phi_{1,\varepsilon}^h\|_{0,\infty} \leqslant \varepsilon \tag{5.12}$$

for some $\varepsilon > 0$. After projecting the data into D_M^h, we compare the solution u_i^* in (5.6) with the corresponding solution having data $P_M \phi_{1,\varepsilon}^h$, i.e.,

$$u_{i,\varepsilon}^*(y) = \sum_{k=1}^{M} \frac{(\phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} \, v_{k,h}(x_i) \sinh(\sqrt{\lambda_{k,h}} y)\,. \tag{5.13}$$

We can prove the following result.

Theorem 5.2. *Let $M < N$, $\phi_1 \in D_M$, and let $\phi_{1,\varepsilon}$ satisfy* (5.12). *Then, for sufficiently small h, the following estimates hold:*

$$|u_i^*(y) - u_{i,\varepsilon}^*(y)| \leqslant C\varepsilon, \quad i = 1, \ldots, N-1, \tag{5.14}$$

$$\|u_h^*(y) - u_{\varepsilon,h}^*(y)\|_{0,a,h} \leqslant C\varepsilon \tag{5.15}$$

uniformly for $0 < y \leqslant L$.

Proof. In view of (5.6) and (5.13) we have

$$|u_i^*(y) - u_{i,\varepsilon}^*(y)| = \left| \sum_{k=1}^{M} \frac{(\phi_1^h - \phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h}(x_i) \sinh\left(\sqrt{\lambda_{k,h}} y\right) \right| \leqslant$$

$$\sum_{k=1}^{M} \left| \frac{(\phi_1^h - \phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h}(x_i) \sinh\left(\sqrt{\lambda_{k,h}} y\right) \right|,$$

where

$$\left|(\phi_1^h - \phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}\right| \leqslant \|\phi_1^h - \phi_{1,\varepsilon}^h\|_{0,a,h} \cdot \underbrace{\|v_{k,h}\|_{0,a,h}}_{=1} \leqslant \sqrt{R_a} \|\phi_1^h - \phi_{1,\varepsilon}^h\|_\infty \leqslant \sqrt{R_a}\varepsilon\,.$$

Using this, we obtain

$$\left| \frac{(\phi_1^h - \phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h}(x_i) \sinh\left(\sqrt{\lambda_{k,h}} y\right) \right| \leqslant$$

$$\sinh\left(\sqrt{\lambda_{k,h}} y\right) |v_{k,h}(x_i)| \frac{1}{\sqrt{\lambda_{k,h}}} \left|(\phi_1^h - \phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}\right| \leqslant \sinh\left(\sqrt{\lambda_{k,h}} y\right) |v_{k,h}(x_i)| \frac{1}{\sqrt{\lambda_{k,h}}} \sqrt{R_a}\varepsilon.$$

The first term on the right-hand side can be estimated by means of (5.8), the second one by (4.9), and the third one by (5.7). Bounding y by $y \leqslant L$ and taking h sufficiently small, (5.14) is proved. Since

$$\|v_h\|_{0,a,h} \leqslant \sqrt{R_a}\|v_h\|_{0,\infty}, \quad v_h \in \mathcal{G}_{h,0},$$

(5.15) follows from (5.14). □

Together with Theorem 5.1 the last theorem shows that

$$|u(x_i, y) - u^*_{i,\varepsilon}(y)| = O(\sqrt{h} + \varepsilon) \tag{5.16}$$

and

$$\|u^h(\cdot, y) - u^*_{\varepsilon,h}(y)\|_{0,a,h} = O(h + \varepsilon) \tag{5.17}$$

for $(h, \varepsilon) \to 0$.

It will be recalled that these estimates hold for the case $\phi_1 \in D_M$, $\phi_{1,\varepsilon} \in D$. For more general ϕ_1, analogous estimates will be provided in the following.

For this purpose the total error will be split into three parts:

$$u - u^*_{\varepsilon,h} = u - u^* + u^* - u^*_\varepsilon + u^*_\varepsilon - u^*_{\varepsilon,h} \tag{5.18}$$

Here, we use the following notations:

u — solution of (3.1)–(3.3) for data $\phi_1 \in D$;

u^* — solution of (3.1)–(3.3) for data $\phi_1^* = P_M\phi_1$;

u_ε — solution of (3.1)–(3.3) for data $\phi_{1,\varepsilon} \in D$;

u^*_ε — solution of (3.1)–(3.3) for data $\phi^*_{1,\varepsilon} = P_M\phi_{1,\varepsilon}$ with $\|\phi_1 - \phi_{1,\varepsilon}\|_{0,\infty} \leqslant \varepsilon$.

Additionally, we use

u^*_i — solution by the line method on the ith line with data $\phi^*_1 = P_M\phi_1$ (see (5.6) with $\phi_{1,h} = \phi^h_1$);

$u^*_{i,\varepsilon}$ — solution by the line method on the ith line with data $\phi^*_{1,\varepsilon} = P_M\phi_{1,\varepsilon}$ (see (5.13));

u^*_h and $u^*_{\varepsilon,h}$ denote grid functions corresponding to u^*_i and $u^*_{i,\varepsilon}$, $i = 1, \ldots, N-1$, resp.;

u^h, ϕ^h_1, etc., denote restrictions of functions in (x, y) to the x-grid (with continuous y).

The error contribution $u - u^*_\varepsilon$ has already been estimated in Theorem 5.2 for data $\phi_1 \in D_M$ and perturbations $\phi_{1,\varepsilon}$. In the same situation, Theorem 5.1 provides an estimate for $u^h - u^*_h$. Note that, according to our notation, $u = u^*$ for data $\phi_1 \in D_M$. For the general case, namely $\phi_1 \in D$ but not necessarily $\phi_1 \in D_M$, we need an estimate for the projection error w.r.t. $\|\cdot\|_{0,a}$.

Lemma 5.3. *Let $M < N$ and, for Cauchy data $\phi_1 \in D$, let the boundedness condition (3.14), $\|u\|_{a,\Sigma_4} \leqslant E$, be fulfilled for the solution u of the Cauchy problem. Then for*

$$\phi^*_1 = P_M\phi_1 = \sum_{k=1}^{M} (\phi_1, v_k)_{0,a} v_k$$

the following projection error estimate holds:

$$\|\phi_1 - \phi^*_1\|_{0,a} \leqslant C_1 \frac{EM}{\exp(\sqrt{r_a/R_a}M\pi L)} \tag{5.19}$$

where the constant C_1 only depends on the length L of the y-interval and the ratio $\gamma := r_a/R_a$.

Proof. Since ϕ_1 has the representation

$$\phi_1(x) = \sum_{k=1}^{\infty} (\phi_1, v_k)_{0,a} v_k(x)\,,$$

which is uniformly and absolutely convergent, for the projection error we obtain

$$\|\phi_1 - \phi_1^*\|_{0,a}^2 = \sum_{k=M+1}^{\infty} (\phi_1, v_k)_{0,a}^2$$

using Parseval's relation. By the boundedness assumption (3.14),

$$\|u(\cdot, L)\|_{0,a}^2 = \sum_{k=1}^{\infty} \frac{(\phi_1, v_k)_{0,a}^2}{\lambda_k} \sinh^2(\lambda_k L) \leqslant E^2.$$

In particular any summand in this series can be estimated by E^2 which yields

$$(\phi_1, v_k)_{0,a}^2 \leqslant \frac{E^2 \lambda_k}{\sinh^2(\sqrt{\lambda_k} L)}.$$

We now use the fact that

$$\sinh(k\alpha) \geqslant \beta_0 \exp(k\alpha)$$

for any $\alpha > 0$ with $\beta_0 = \big(1 - \exp(-4\alpha)\big)/2$, and that

$$\sum_{k=1}^{\infty} \frac{(k+M)^2}{\exp(2\gamma k \pi L)} \leqslant C_0^2 M^2$$

with a constant $C_0 = C_0(L) \leqslant R_a/(r_a \pi L^2)$ (see [4, Lemma 4.2.1]). Inserting this into the above projection error representation and using (3.10), we obtain

$$\|\phi_1 - \phi_1^*\|_{0,a}^2 \leqslant E^2 \sum_{k=M+1}^{\infty} \frac{\lambda_k}{\sinh^2(\sqrt{\lambda_k} L)} \leqslant$$

$$\frac{R_a}{r_a} E^2 \pi^2 \sum_{k=M+1}^{\infty} \frac{k^2}{\sinh(\sqrt{r_a/R_a} k \pi L)} \leqslant \frac{1}{\gamma} \left(\frac{E\pi}{\beta_0}\right)^2 \sum_{k=M+1}^{\infty} \frac{k^2}{\exp(2\gamma k \pi L)} =$$

$$\frac{1}{\gamma} \left(\frac{E\pi}{\beta_0 \exp(\gamma M \pi L)}\right)^2 \sum_{k=1}^{\infty} \frac{(k+M)^2}{\exp(2\gamma k \pi L)} \leqslant \frac{1}{\gamma} \left(\frac{E \pi C_0 M}{\beta_0 \exp(\gamma M \pi L)}\right)^2 \leqslant \left(\frac{R_a}{r_a}\right)^3 \left(\frac{EM}{\beta_0 L^2 \exp(\gamma M \pi L)}\right)^2,$$

where $\gamma := r_a/R_a$. With

$$C_1 := \left(\frac{R_a}{r_a}\right)^{3/2} \frac{2E}{\big(1 - \exp(-4\sqrt{r_a/R_a} \pi L\big) L^2}$$

the desired estimate (5.18) is proved. □

Using the stability estimate of Theorem 3.1 and the projection error estimate in the last theorem, we can estimate the two contributions in the error representation

$$u - u_\varepsilon^* = u - u^* + u^* - u_\varepsilon^*$$

w.r.t the norm $\|\cdot\|_{0,a}$.

Theorem 5.3. *Let the assumption of Lemma 5.3 be fulfilled. Then the following estimates hold:*

$$\|(u-u^*)(\cdot,y)\|_{0,a} \leqslant C_2 E\left(\frac{M}{\exp(\sqrt{\gamma}M\pi L)}\right)^{1-y/L}, \tag{5.20}$$

$$\|(u^*-u^*_\varepsilon)(\cdot,y)\|_{0,a} \leqslant \frac{R_a C_3 \sinh(\sqrt{\gamma}M\pi y)}{\sqrt{\gamma}M\pi}\varepsilon, \tag{5.21}$$

where $\gamma := r_a/R_a$ *and the constants* C_2, C_3 *depend on* γ *and* L *while* C_3 *additionally depends on* y.

Proof. We first see that

$$\|(u-u^*)(\cdot,L)\|^2_{0,a} = \sum_{k=M+1}^{\infty} \frac{(\phi_1,v_k)^2_{0,a}}{\lambda_k}\sinh^2(\sqrt{\lambda_k}L) \leqslant \sum_{k=1}^{\infty} \frac{(\phi_1,v_k)^2_{0,a}}{\lambda_k}\sinh^2(\sqrt{\lambda_k}L) =$$
$$\|u(\cdot,L)\|^2_{0,a} \leqslant E^2.$$

The function $u-u^*$ belongs to the Cauchy data $\phi_1-\phi_1^*$ and can be inserted into the stability estimate (3.16). Together with the projection error estimate (5.19) we obtain

$$\|(u-u^*)(\cdot,y)\|^2_{0,a} \leqslant R_1 E^{y/L}\|\phi_1-\phi_1^*\|^{1-y/L}_{0,a} \leqslant$$
$$R_1 E^{y/L}\left(C_1\frac{EM}{\exp(\sqrt{\gamma}M\pi L)}\right)^{1-y/L} = R_1 E C_1^{1-y/L}\left(\frac{M}{\exp(\sqrt{\gamma}M\pi L)}\right)^{1-y/L}.$$

This proves (5.20) with $C_2 = R_1\max\{1,C_1\}$.

To prove (5.21), we first observe that the difference of the projections $\phi_1^*-\phi_{1,\varepsilon}^*$ has the representation

$$\phi_1^*-\phi_{1,\varepsilon}^* = \sum_{k=1}^{\infty}(z\varepsilon,v_k)_{0,a}v_k,$$

where $\phi_{1,\varepsilon} = \phi_1+\varepsilon z$, with $\phi_1, z\in D, \|z\|_{0,\infty}\leqslant 1$. The coefficients in the sum can be estimated as follows:

$$\left|(z\varepsilon,v_k)_{0,a}\right| \leqslant \|z\varepsilon\|_{0,a}\|v_k\|_{0,a} \leqslant R_a\varepsilon\,.$$

Using this estimate, estimate (3.10), and the representation (3.12) of the solution of the Cauchy problem, we obtain

$$\|(u^*-u^*_\varepsilon)(\cdot,y)\|^2_{0,a} = \sum_{k=1}^{M}\frac{(z\varepsilon,v_k)^2_{0,a}}{\lambda_k}\sinh^2(\sqrt{\lambda_k}y) \leqslant R_a^2\varepsilon^2\sum_{k=1}^{M}\frac{\sinh^2(\sqrt{\lambda_k}y)}{\lambda_k} \leqslant$$
$$R_a^2\varepsilon^2\sum_{k=1}^{M}\frac{\sinh^2(\sqrt{R_a/r_a}k\pi y)}{(r_a/R_a)k^2\pi^2} = \frac{R_a^3}{r_a}\varepsilon^2\sum_{k=1}^{M}\frac{\sinh^2(\sqrt{R_a/r_a}k\pi y)}{(k\pi)^2}.$$

A careful analysis (cf. [4, p. 153, (4.61)]) ensures that with a constant $C_3 = C_3(y)$ as a factor the above sum can be estimated by its last term

$$\sum_{k=1}^{M}\frac{\sinh^2(\sqrt{1/\gamma}k\pi y)}{(k\pi)^2} \leqslant C_3\frac{\sinh^2(\sqrt{1/\gamma}M\pi y)}{(M\pi)^2}\,.$$

This proves (5.21). □

In order to estimate the total error $u-u^*_{\varepsilon,h}$, it remains to study $u^*_\varepsilon - u^*_{\varepsilon,h}$. The following theorem provides an estimate for this error contribution w.r.t. $\|\cdot\|_{0,a,h}$.

Theorem 5.4. *Under the assumption of Lemma 5.3 we have*

$$\|\big((u_\varepsilon^*)^h - u_{\varepsilon,h}^*\big)(\cdot,y)\|_{0,a,h} \leqslant MCh\,\frac{\sinh(\sqrt{1/\gamma}M\pi y)}{\sqrt{\gamma}\pi} \tag{5.22}$$

for sufficiently small h and a constant C independent of y and M.

Proof. Let ε be fixed in $0 < \varepsilon \leqslant 1$. By means of the representations

$$u_\varepsilon^*(x_j,y) = \sum_{k=1}^M \frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}}\, v_k(x_j)\sinh(\sqrt{\lambda_k}y),$$

$$u_{\varepsilon,h}^*(x_j,y) = u_{j,\varepsilon}^*(y) = \sum_{k=1}^M \frac{(\phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}}\, v_{k,h}(x_j)\sinh(\sqrt{\lambda_{k,h}}y)$$

one can estimate

$$\|u_\varepsilon^*(\cdot,y) - u_{\varepsilon,h}^*(\cdot,y)\|_{0,a,h} \leqslant$$

$$\sum_{k=1}^M \left\| \frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}}\, v_k^h \sinh(\sqrt{\lambda_k}y) - \frac{(\phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}}\, v_{k,h}\sinh(\sqrt{\lambda_{k,h}}y)\right\|_{0,a,h}$$

Each summand can be estimated as follows:

$$\left\| \frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}} v_k^h \sinh\left(\sqrt{\lambda_k}y\right) - \frac{(\phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h}\sinh\left(\sqrt{\lambda_{k,h}}y\right)\right\|_{0,a,h} \leqslant$$

$$\left\| \frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}} v_k^h \sinh\left(\sqrt{\lambda_k}y\right) - \frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}} v_{k,h}\sinh\left(\sqrt{\lambda_k}y\right)\right\|_{0,a,h} +$$

$$\left\| \frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}} v_{k,h} \sinh\left(\sqrt{\lambda_k}y\right) - \frac{(\phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} v_{k,h}\sinh\left(\sqrt{\lambda_{k,h}}y\right)\right\|_{0,a,h} =$$

$$\left| \frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}} \sinh\left(\sqrt{\lambda_k}y\right)\right| \|v_k^h - v_{k,h}\|_{0,a,h} +$$

$$\left| \frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}} \sinh\left(\sqrt{\lambda_k}y\right) - \frac{(\phi_{1,\varepsilon}^h, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}} \sinh\left(\sqrt{\lambda_{k,h}}y\right)\right| \|v_{k,h}\|_{0,a,h}.$$

Herein, according to (4.10), we have

$$\|v_k^h - v_{k,h}\|_{0,a,h} = O(h) \quad (h \to 0)\,.$$

Furthermore, $\{v_{k,h}\}$ is an orthonormal system, $\|v_{k,h}\|_{0,a,h} = 1$, $k = 1,\ldots,M$. With $\gamma = r_a/R_a$, we have

$$\left| \frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}} \sinh\left(\sqrt{\lambda_k}y\right)\right| \leqslant \big(\|\phi_1\|_{0,a} + \varepsilon R_a\big)\,\frac{\sinh(\sqrt{1/\gamma}M\pi y)}{\sqrt{\gamma}\pi}$$

which results from the facts that

$$|(\phi_{1,\varepsilon}, v_k)_{0,a}| \leqslant \|\phi_{1,\varepsilon}\|_{0,a} \leqslant \|\phi_1\|_{0,a} + \varepsilon R_a\,,$$

and, since the eigenvalues are all positive and ordered in magnitude, and using (3.10)

$$\left|\frac{\sinh(\sqrt{\lambda_k}y)}{\sqrt{\lambda_k}}\right| \leqslant \frac{\sinh(\sqrt{\lambda_M}y)}{\sqrt{\lambda_1}} \leqslant \frac{\sinh(\sqrt{R_a/r_a}M\pi)}{\sqrt{r_a/R_a}\pi}.$$

Now, Lemma 5.2 ensures that

$$\left|\frac{(\phi_{1,\varepsilon}, v_k)_{0,a}}{\sqrt{\lambda_k}}\sinh\left(\sqrt{\lambda_k}y\right) - \frac{(\phi^h_{1,\varepsilon}, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}}\sinh\left(\sqrt{\lambda_{k,h}}y\right)\right| \|v_{k,h}\|_{0,a,h} \leqslant \tilde{C}h$$

for sufficiently small h. Altogether, we come to the result that

$$\left\|u^*_\varepsilon(\cdot,y) - u^*_{\varepsilon,h}(\cdot,y)\right\|_{0,a,h} \leqslant M\left\{\left(\|\phi_1\|_{0,a} + \varepsilon R_a\right)\frac{\sinh(\sqrt{1/\gamma}M\pi y)}{\sqrt{\gamma}\pi} + \tilde{C}\right\}h$$

which proves (5.22) with

$$C = \max\left(\|\phi_1\|_{0,a} + R_a, \tilde{C}, 1\right).$$

□

We cannot compare u with $u^*_{\varepsilon,h}$ because the first is a continuous function w.r.t. x and the latter is a grid function. We can either estimate $\|u^h - u^*_{\varepsilon,h}\|_{0,a,h}$ or we have to extend $u^*_{\varepsilon,h}$ to a continuous function $\overline{u^*_{\varepsilon,h}}$ in x and measure $\|u - \overline{u^*_{\varepsilon,h}}\|_{0,a,}$. To estimate the grid functions, we need a discrete analog of the stability estimate proved in Theorem 3.1. This is very likely available by means of the same techniques as in [30] but is beyond the scope of this study.

We therefore use the second approach and the discrete line method approximation $u^*_{\varepsilon,h}$ (cf. (5.13)) will be extended to a function for all $x \in [0,1]$ by extending the discrete eigenfunctions $v_{k,h}$ to functions $\overline{v_{k,h}} \in C^2[0,1]$. Without proof we state that this can be done by a polynomial such that

$$\|v_k - \overline{v_{k,h}}\|_{0,a} = O\left(\sqrt{h}\right), \quad k = 1,\dots,M. \tag{5.23}$$

Note that compared to (4.10) we loose the power $h^{1/2}$. With $\overline{v_{k,h}}$ the extension $\overline{u^*_{\varepsilon,h}}$ is given by

$$\overline{u^*_{\varepsilon,h}}(x,y) = \sum_{k=1}^{M} \frac{(\phi^h_{1,\varepsilon}, v_{k,h})_{0,a,h}}{\sqrt{\lambda_{k,h}}}\overline{v_{k,h}}(x)\sinh\left(\sqrt{\lambda_{k,h}}y\right). \tag{5.24}$$

Now we are able to estimate $u - \overline{u^*_{\varepsilon,h}}$ w.r.t. $\|\cdot\|_{0,a}$. When we use (5.21), (5.22) and the same techniques as in the proof of Theorem 5.4 with (5.23) instead of (4.10), we obtain

$$\|u - \overline{u^*_{\varepsilon,h}}\|_{0,a} \leqslant C\left\{E\left(\frac{M}{\exp(\sqrt{\gamma}M\pi L)}\right)^{1-\frac{y}{L}} + \frac{R_a}{\sqrt{\gamma}\pi}\frac{\sinh(\sqrt{1/\gamma}M\pi y)}{M}\varepsilon + M\sqrt{h}\frac{\sinh(\sqrt{1/\gamma}M\pi y)}{\sqrt{\gamma}\pi}\right\} \tag{5.25}$$

where the constant C can be chosen uniformly for all $y \in (0, L]$.

Our final aim is to choose M and h depending on ε such that the total error converges to zero when $\varepsilon \to 0$.

Theorem 5.5. *Let the assumptions of Lemma* 5.3 *be fulfilled and let the extensions* $\overline{v_{k,h}}$ *satisfy* (5.23). *Then for*

$$M = \left\lceil \frac{\ln(1/\varepsilon)}{\pi L\sqrt{R_a/r_a}} \right\rceil \quad \textit{and} \quad h \leqslant \varepsilon^2, \tag{5.26}$$

with $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geqslant x\}$, $x \in \mathbb{R}$ *the following error estimate holds* (*with* $\gamma = r_a/R_a$)*:*

$$\left\|(u - \overline{u^*_{\varepsilon,h}})(\cdot, y)\right\|_{0,a} \leqslant C\left\{E\left(\frac{\varepsilon^\gamma \ln(1/\varepsilon)}{\pi L\sqrt{1/\gamma}} + \varepsilon^\gamma\right)^{1-\frac{y}{L}} + \frac{R_a^2 L}{r_a}\exp(\sqrt{1/\gamma}\pi y)\frac{\varepsilon^{1-\frac{y}{L}}}{\ln(1/\varepsilon)} + \varepsilon^{1-\frac{y}{L}}\ln(1/\varepsilon)\right\}, \tag{5.27}$$

where $y \in (0, L]$.

Proof. With the choice of M in (5.26) we obtain (with $\gamma = r_a/R_a$)

$$\frac{\ln(1/\varepsilon)}{\pi L\sqrt{1/\gamma}} < M \leqslant \frac{\ln(1/\varepsilon)}{\pi L\sqrt{1/\gamma}} + 1, \quad \sqrt{\gamma}M\pi L > \sqrt{\gamma}\,\frac{\ln(1/\varepsilon)}{\pi L\sqrt{1/\gamma}} = \gamma \ln\frac{1}{\varepsilon},$$

and

$$\left(\frac{M}{\exp(\sqrt{\gamma}M\pi L)}\right)^{1-y/L} \leqslant \left(\left(\frac{\ln(1/\varepsilon)}{\pi L\sqrt{1/\gamma}} + 1\right)\frac{1}{\exp(\gamma\ln(1/\varepsilon))}\right)^{1-y/L} = \left(\frac{\varepsilon^\gamma \ln(1/\varepsilon)}{\pi L\sqrt{1/\gamma}} + \varepsilon^\gamma\right)^{1-y/L}.$$

For the second term in (5.25), we use

$$\sinh\left(\sqrt{1/\gamma}M\pi y\right) \leqslant \exp\left(\sqrt{1/\gamma}M\pi y\right) \leqslant \exp\left(\sqrt{1/\gamma}\left(\frac{\ln(1/\varepsilon)}{\pi L\sqrt{1/\gamma}} + 1\right)\pi y\right) =$$

$$\exp\left(\ln\left(\frac{1}{\varepsilon}\right)\frac{y}{L}\right)\exp\left(\sqrt{1/\gamma}\pi y\right) = \varepsilon^{-y/L}\exp\left(\sqrt{1/\gamma}\,\pi y\right) \tag{5.28}$$

and obtain

$$\frac{\varepsilon\sinh(\sqrt{1/\gamma}M\pi y)}{\sqrt{\gamma}M\pi} \leqslant \frac{\varepsilon^{1-y/L}\exp(\sqrt{1/\gamma}\pi y)}{\ln(1/\varepsilon)\gamma\frac{1}{L}}.$$

For the third term in (5.25), we first assume $\varepsilon \leqslant 1/e$ and obtain $1 \leqslant \ln(1/\varepsilon)$. Further, from $0 < \varepsilon \leqslant 1$ we get $\varepsilon \leqslant \varepsilon^{1-y/L}$ and, using the second condition in (5.26)

$$M\sqrt{h}\gamma^{-1}\pi^{-1}\sinh\left(\sqrt{1/\gamma}M\pi y\right) \overset{(5.28)}{\leqslant} M\varepsilon^{1-y/L}\exp\left(\sqrt{1/\gamma}\pi y\right)(\gamma\pi)^{-1} \leqslant$$

$$\frac{1}{\gamma\pi}\exp\left(\sqrt{1/\gamma}\pi y\right)\varepsilon^{1-y/L}\left(1 + \frac{\ln(1/\varepsilon)}{\pi L\sqrt{1/\gamma}}\right) \leqslant C\varepsilon^{1-y/L}\ln(1/\varepsilon).$$

Altogether, estimate (5.27) is proved. □

6. Numerical examples

In this final section, the computational results for two model examples of Cauchy problems are presented. The first one is Hadamard's example and the second one includes a variable coefficient functional $a(\cdot)$. In our tests, the optimal M is rather small (2 or 4). The optimal M depends, as shown theoretically, on the magnitude of the data errors and also on the solution itself. Also the influence of the grid size on the regularization parameter has been studied for the second example. The rounding error analysis of the underlying algorithms is not a subject of this paper; rounding errors lead to highly nonlinear relations which are difficult to analyze even for simple problems (see, e.g., [36]).

In [5] we have presented a numerical solution of the Hadamard example (cf. the end of section 2) in the constant coefficient case for $m = 4$ with noise $\varepsilon = 10^{-2}$ and various discretization parameters h. In the present paper, we give some more computational results for the Hadamard example to verify experimentally that the choice of M in (5.26) is appropriate indeed. As a second example, we consider problem (1.1) with

$$a(x) = 1 + x, \quad x \in [0,1], \tag{6.1}$$

and the solution $u(x,y) = x^2 + y^2$; in this case, $f = 6x + 4$. As mentioned in Section 1, we split this problem into a direct one and an ill-posed Cauchy problem:

$$L_a v = 0 \text{ in } \Omega = [0,1]\times[0,1], \quad v = 0 \text{ on } \Sigma_1 \cup \Sigma_2 \cup \Sigma_3, \quad \frac{\partial v}{\partial y}(x,0) = -(1+x) \text{ on } \Sigma_1, \tag{6.2}$$

where L_a denotes the differential operator on the right-hand side of (1.1).

Example 6.1 (Hadamard's example). For $m \in \mathbb{N}$ the solution is

$$(u =) u_m(x,y) = \frac{\sin(mx)\sin(m\pi y)}{m^2\pi^2}, \quad 0 \leqslant x, y \leqslant 1,$$

where the Cauchy data are given by

$$(u\big|_{y=0}) f_1 = 0, \quad \left(\frac{\partial u}{\partial y}\Big|_{y=0} = \right) \phi_1 = \frac{1}{m\pi}\sin(m\pi x).$$

Figures 6.1 and 6.2 show the exact data as well as perturbed data with $\varepsilon = 0.1$ together with the data projected onto the space D_M with $M = 2$ and $M = 4$, respectively, The perturbations are obtained by adding at each grid point a randomly distributed function with values in $[-1,1]$ multiplied by $\varepsilon = 0.1$. For Hadamard's example with $M = 2$, obviously $M = 2$ is the best choice, which is clearly shown in Fig. 6.1 and 6.2. Figures 6.3–6.6 demonstrate that the choice of M is essential for the quality of the approximation. According to (5.26), the optimal M should be chosen as $M \approx \ln(1/\varepsilon)$. Thus $M = 2$ is the best choice for perturbations of the magnitude $\varepsilon = 10^{-2}$ (see Fig. 6.3 and 6.4). $M = 4$ is certainly a wrong choice for such perturbations (see Fig. 6.5), while it is a good choice when the perturbations are of magnitude $\varepsilon = 10^{-4}$ (see Fig. 6.6).

Example 6.2 (Polynomial example (6.1), (6.2)). Figures 6.7 and 6.8 demonstrate very clearly how the magnitude of the perturbation ε and the grid width h in connection with the dimension M of the space D_M of the approximate solutions influence the quality of the approximations.

In Fig. 6.7, the data error caused by the perturbation ε obviously dominates over the other contributions to the total error. Since ε is of the same magnitude as the data themselves, it is not surprising that the total error is relatively large even with small $h's$. To get better results, we have to provide more precise data, as is shown in Fig. 6.8 for $\varepsilon = 10^{-3}$. In this context, it makes sense to use smaller $h's$ as well, which also clearly follows from Fig. 6.8.

Since this problem has to be split into a direct one and an ill-posed Cauchy problem with $f = 0$ and $f_i = 0, i = 1,2,3$, the direct problem has to be solved numerically by the same line approximation method as the inverse one. To this end, we use a step width $\bar{h} > 0$ in the x-interval which has to be chosen small enough in order that the discretization error of the direct problem remains small compared to the error of the inverse problem. The figures for this example give also the size of $\bar{h}$.

Figures 6.9 and 6.10 show the same example for $\varepsilon = 10^{-3}$, $M = 2$. Figure 6.9 demonstrated that for $\bar{h} = 1/50$ the discretization of the direct problem dominates over the total error which can be improved by choosing $\bar{h} = 1/200$ (see Fig. 6.10).

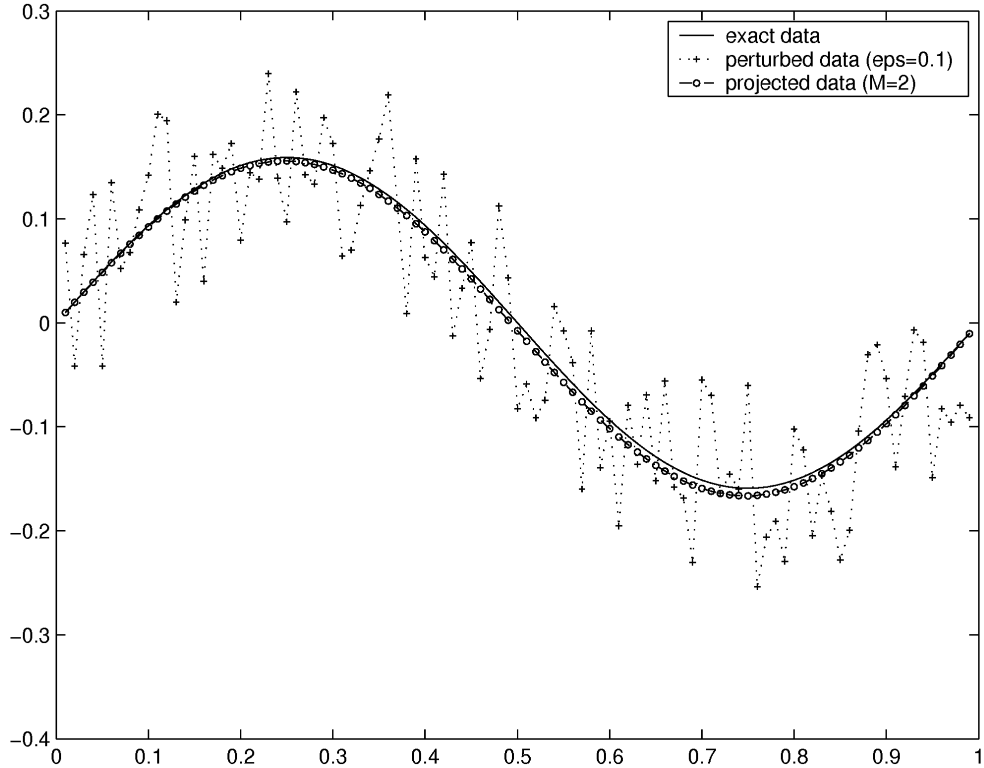

Fig. 6.1. Data ϕ_1 (from Hadamard Example) at $y = 0$ projected onto D_M with $\varepsilon = 0.1$, $h = 1/100$, $m = 2$, $M = 2$

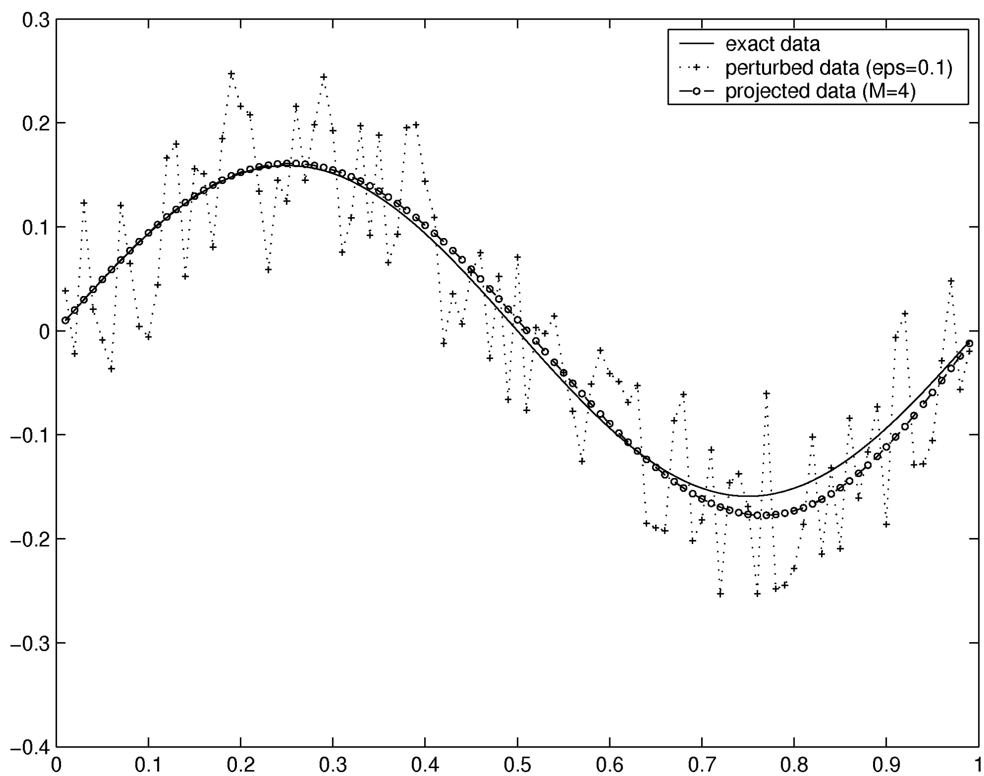

Fig. 6.2. Same as Fig. 6.1 with $M = 4$

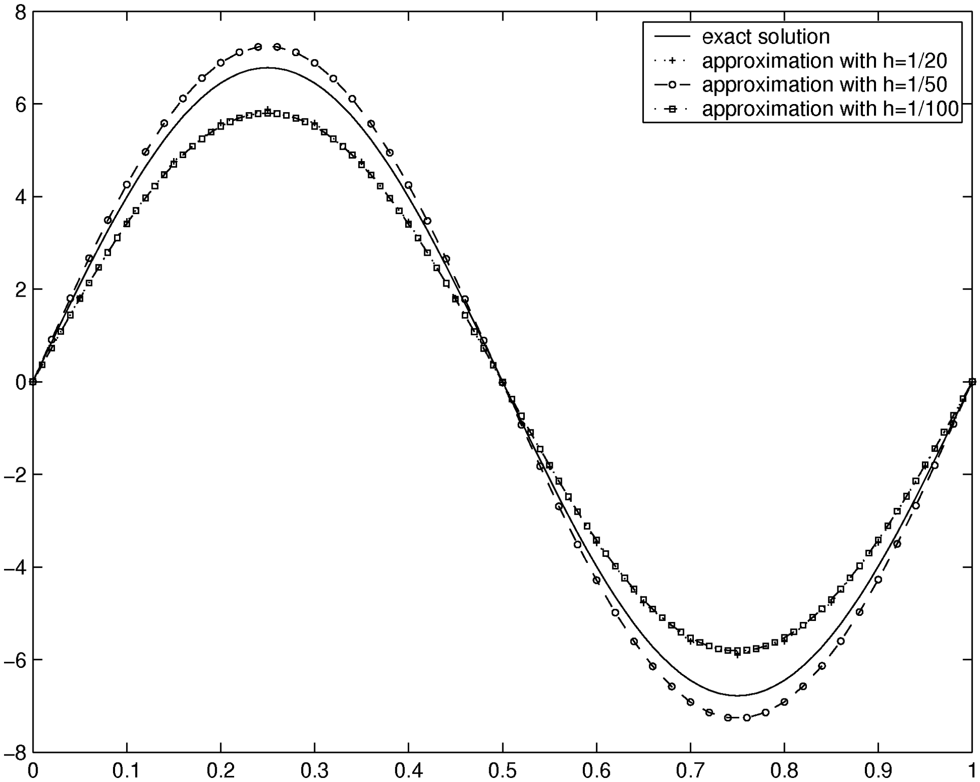

Fig. 6.3. Solution and line method approximations of Hadamard Example at $y = 1$ for $\varepsilon = 0.1$, $m = 2$, $M = 2$ and different $h's$

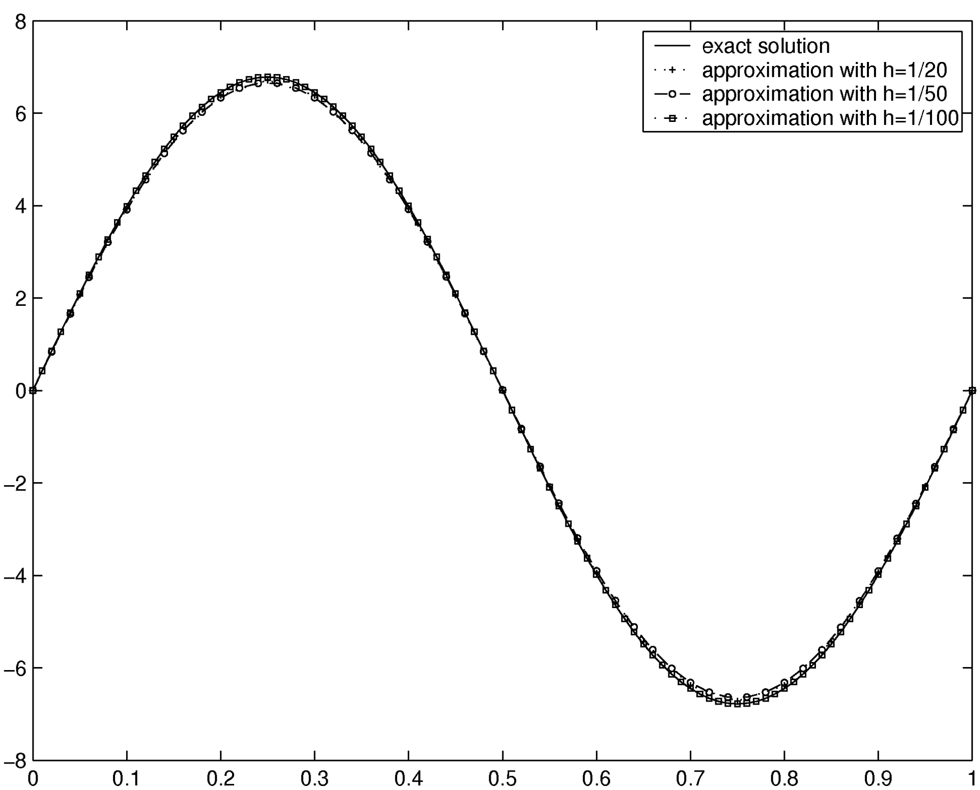

Fig. 6.4. Same as Fig. 6.3 with $\varepsilon = 10^{-2}$

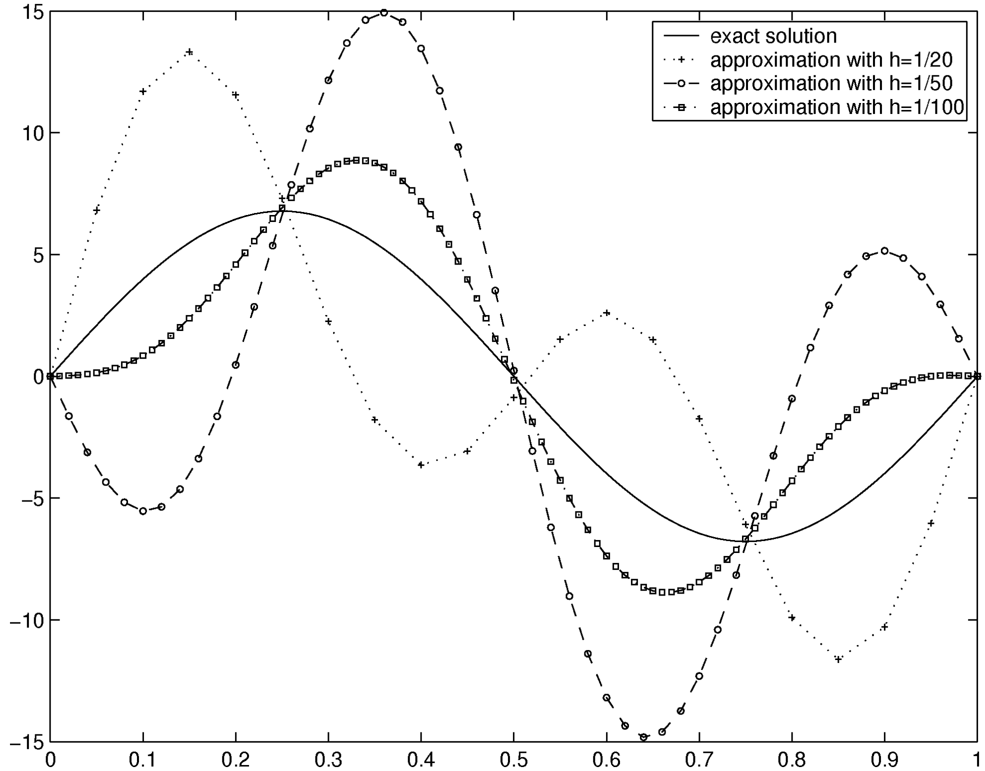

Fig. 6.5. Solution and line method approximations of Hadamard Example at $y = 1$ for $\varepsilon = 10^{-2}$, $m = 2$, $M = 4$ and different $h's$

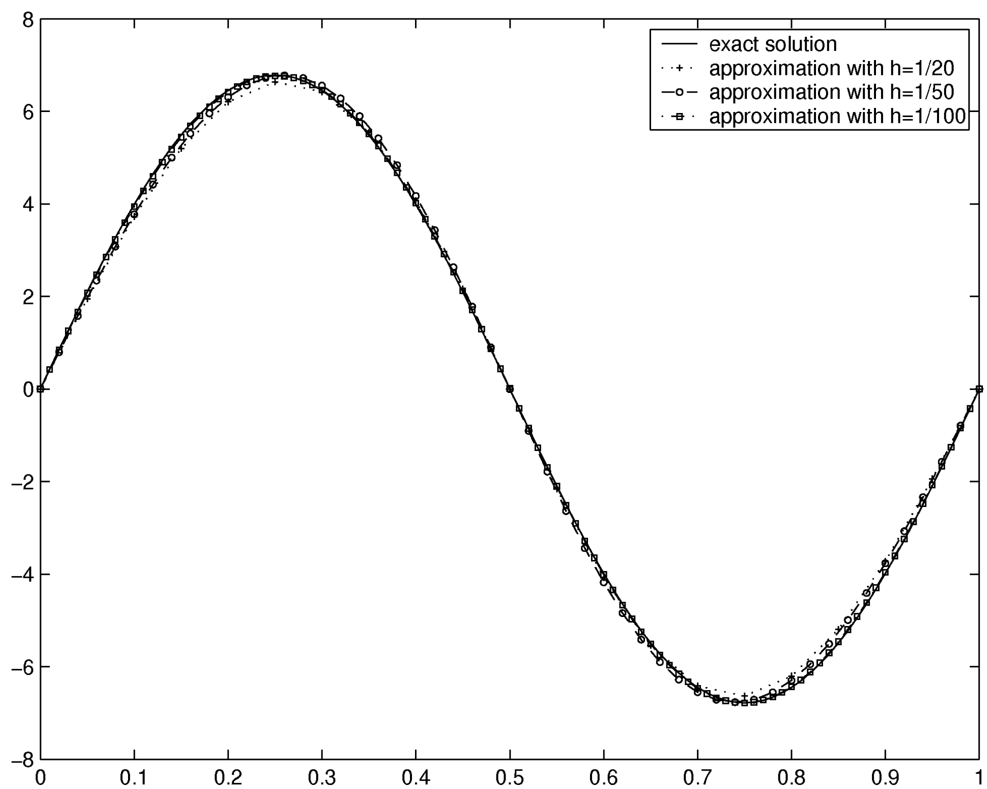

Fig. 6.6. Same as Fig. 6.5 with $\varepsilon = 10^{-4}$ (Note: $\varepsilon \approx 1/\exp M$)

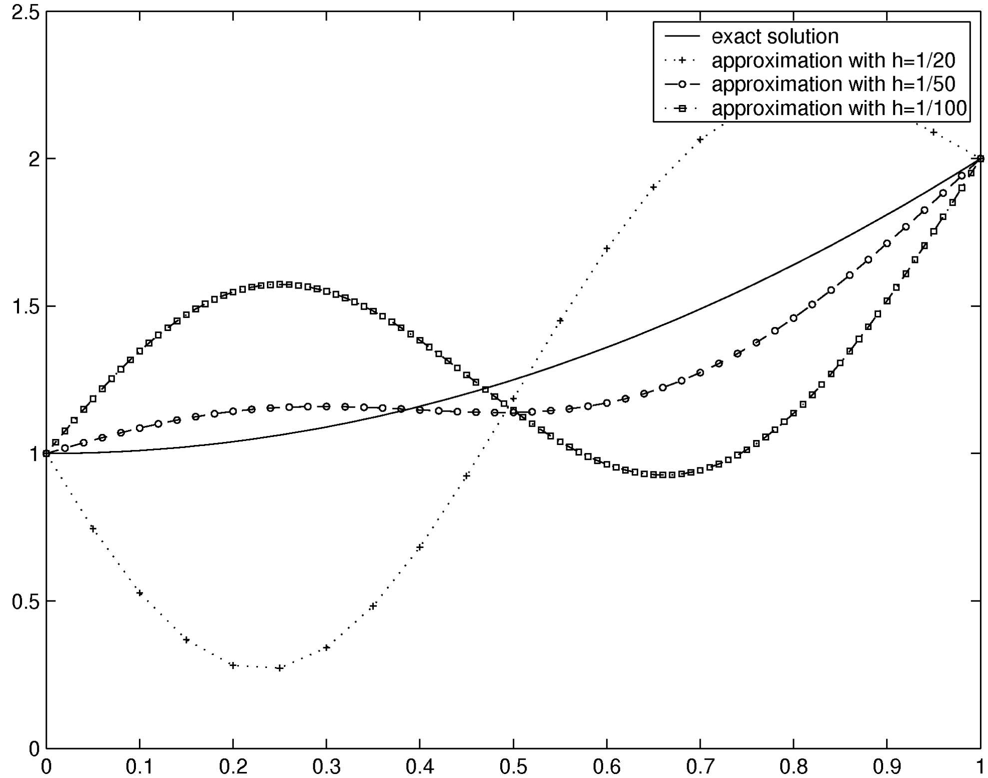

Fig. 6.7. Solution $u = x^2 + y^2$ and line method approximations in the case of $a(x) = x + 1$ at $y = 1$ for $\varepsilon = 10^{-1}$, $M = 2$, different $h's$ (and $\bar{h} = 1/200$)

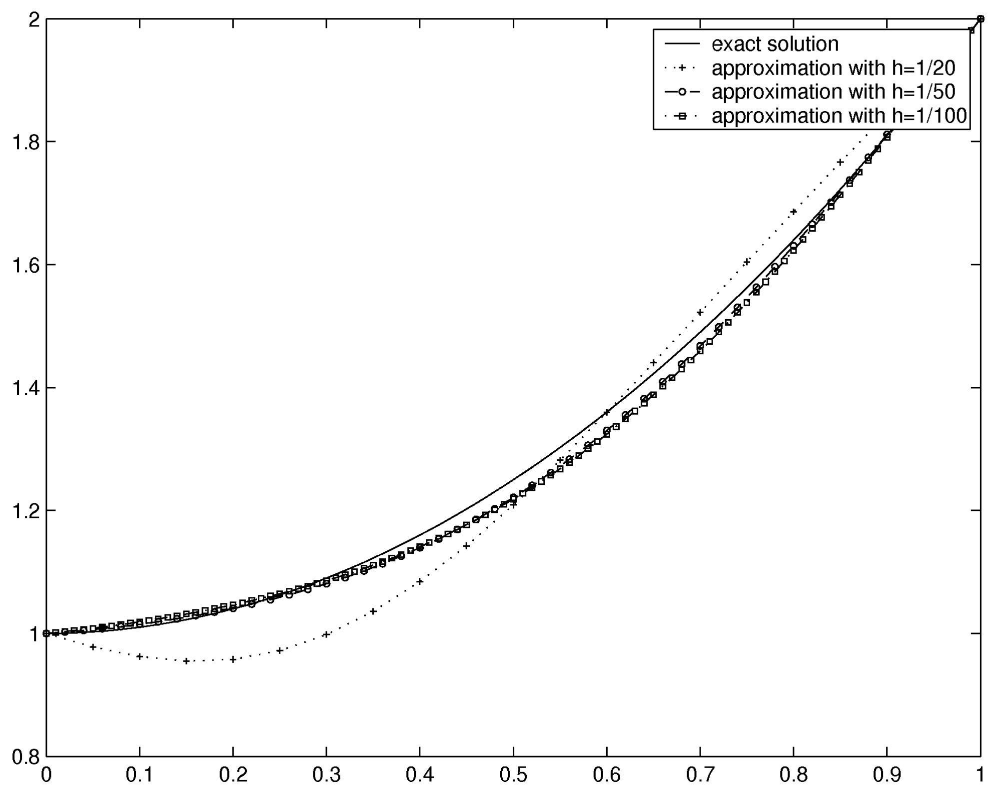

Fig. 6.8. Same as Fig. 6.7 with $\varepsilon = 10^{-3}$

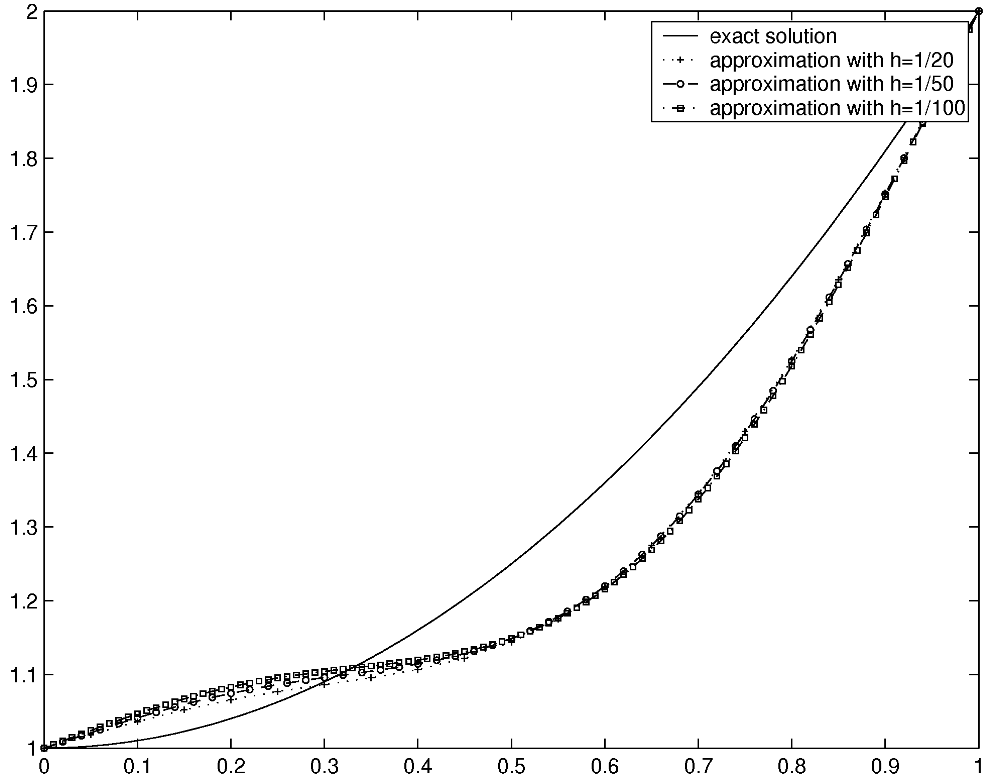

Fig. 6.9. Exact and approximate solutions at $y = 1$ for $\bar{h} = 1/50$, $M = 2$, $\varepsilon = 10^{-3}$ and different $h's$

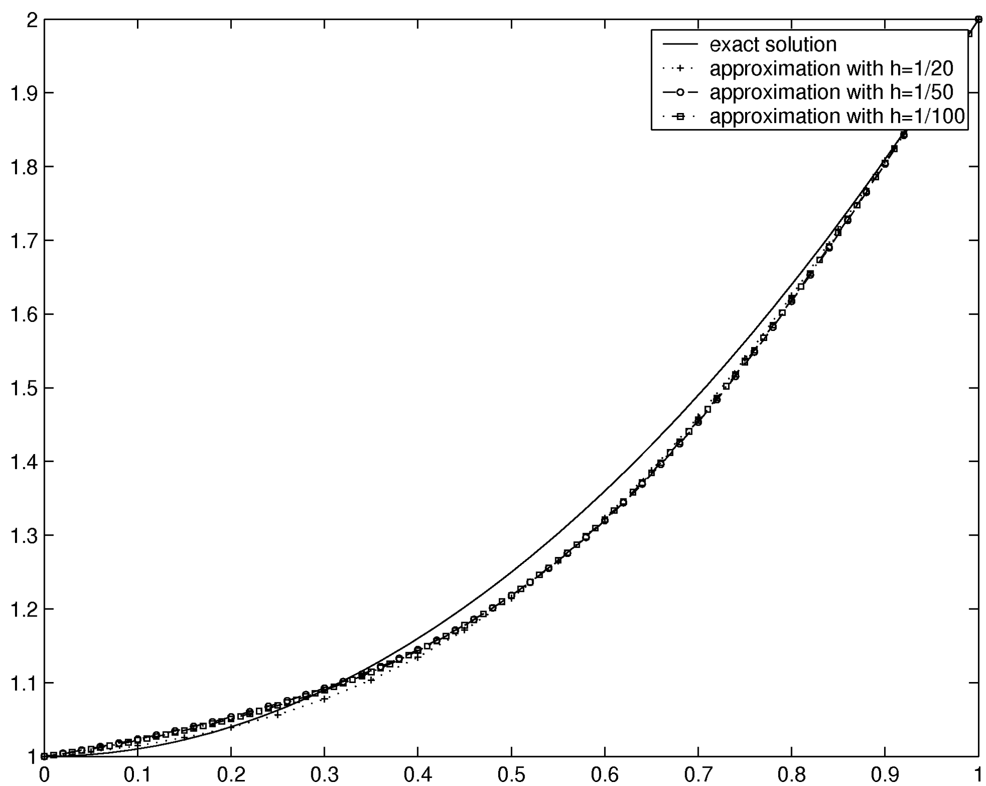

Fig. 6.10. Same as Fig. 6.9 with $\bar{h} = 1/200$

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