Numerical Solution of Inverse Heat Conduction Problems in Two Spatial Dimensions

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Abstract

Inverse Heat Conduction Problems (IHCPs) have been extensively studied over the last 50 years. They have numerous applications in many branches of science and technology. The problem consists in determining the temperature and heat flux at inaccessible parts of the boundary of a 2- or 3-dimensional body from corresponding data – called 'Cauchy data' – on accessible parts of the boundary. It is well-known that IHCPs are severely illposed which means that small perturbations in the data may cause extremely large errors in the solution.

In this contribution we first present the problem and show examples of calculations for 2-dimensional IHCP's where the direct problems are solved with the Finite Element package DEAL. As solution procedure we use Tikhonov's regularization in combination with the conjugate gradient method.

1 Introduction

In [5] we have established the theoretical background for multidimensional inverse heat conduction problems. In this contribution, we present several 2-d. calculations.

The importance of inverse heat conduction problems and appropriate solution algorithms are established in numerous works and the books (see, e.g. [1], [3], [4], [8], [13], [18], [20], [21] and the references therein). The conjugate gradient method (CGM) as an iterative solution algorithm is itself a regularization method where the iteration index plays the role of the regularization parameter. For our calculations, we have used Tikhonov's regularization in combination with CGM. Our program is based on a C++ code of C. Fröbel [11] where, as a direct solver for the underlying parabolic problems, the Finite Element package DEAL [2], [21] is used. Our approach is very similar to the one used by Y. Jarny in a series of papers (see, e.g. [14], [16]).

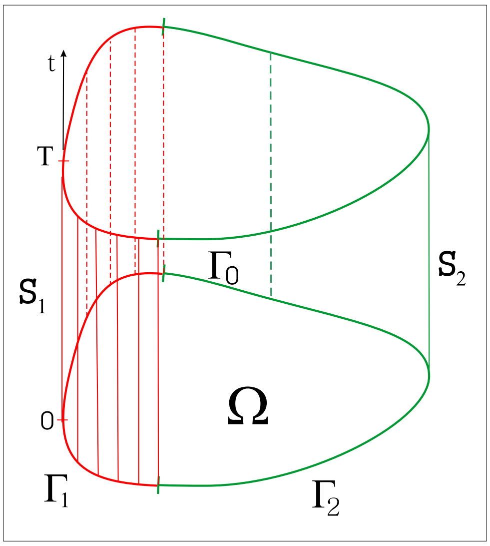

Figure 1: Problem setting

Let E_n be the n-dimensional Euclidian space, Ω be a bounded domain of E_n with a sufficiently smooth boundary $\partial\Omega$ (for example, $\partial\Omega \in C^2$, or Ω is a parallelepiped). For $t \in (0,T]$, set $Q_t := \Omega \times (0,t]$, $\Sigma_t := \partial\Omega \times (0,t)$, $\Sigma = \Sigma_T$. Suppose that $\partial\Omega$ consists of three parts Γ_0, Γ_1 and Γ_2, where Γ_1, Γ_2 are not empty, Γ_0 may be empty or not and $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$. We denote $\Gamma_i \times (0,T]$ by S_i for $i = 0,1,2$ (see Fig. 1). The multi-dimensional IHCPs can be formulated as a non-characteristic Cauchy problem for multi-dimensional parabolic equations as follows: Let $\partial u/\partial N$ be given only in S_1, and additionally the value of u at S_1 be also given. In addition, u or $\partial u/\partial N$ is given on Γ_0 – we restrict ourselves to a homogeneous flux condition in the following. It is desired to find $\partial u/\partial N|_{S_2}$ and $u|_{t=0}$ (and so the function u) from

$$\begin{aligned} \frac{\partial u}{\partial t} \;-\; &\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j} + a_i(x,t)u\right) \\ &+\sum_{i=1}^{n} b_i(x,t)u_{x_i} + a(x,t)u = f, \quad (x,t) \in Q_T, \end{aligned} \tag{1}$$

$$\frac{\partial u}{\partial N}|_{S_0} = 0, \quad (\xi,t) \in S_0, \tag{2}$$

$$u|_{S_1} = \varphi(\xi,t), \quad (\xi,t) \in S_1, \tag{3}$$

$$\frac{\partial u}{\partial N}|_{S_1} = g(\xi,t), \quad (\xi,t) \in S_1. \tag{4}$$

Here ν is the outer normal to Ω,

$$\frac{\partial u}{\partial N}|_{S_k} := \sum_{i,j=1}^{n} (a_{ij}(x,t)u_{x_j} + a_i(x,t)u)\cos(\nu,x_i)|_{S_k}, \quad k = 0,1,2,$$

$\lambda, \ \ \Lambda$ are positive constants, and $\mu_1 \ , \ \mu_2 \ \geq \ 0$. Further,

$$a_{ij}, \ a_i, \ b_i, \ a \ \in \ L_\infty(Q_T), \tag{5}$$

$$f \ \in \ L_2(Q_T), \ \varphi, g \in \ L_2(S_1) \tag{6}$$

$$a_{ij} = a_{ji}, \quad i, j \ \in \ \{1, 2, \dots, n\} \tag{7}$$

$$\lambda \|\xi\|^2_{E_n} \leq \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \leq \Lambda \|\xi\|^2_{E_n}, \ \ \forall \xi \ \in \ E_n, \tag{8}$$

$$|a(x,t)| \leq \mu_1, \quad \sqrt{\sum_1^n a_i^2}, \ \ \sqrt{\sum_1^n b_i^2} \ \leq \ \mu_2 \ \ a.e. \ \ in \ \ Q_T. \tag{9}$$

Later we assume that the b_i vanish.

The problem (1) - (4) is known to be severely ill-posed. In this paper we shall deal with this IHCP by a variational method suggested in [7]. The idea of our method is very simple: Since the initial condition and the heat flux $\partial u / \partial N|_{S_2}$ are not known, we consider them as a control v to minimize the defect $J_0(v) = \frac{1}{2}\|u|_{S_1} - \varphi\|^2_{L_2(S_1)}$.

2 The direct and inverse problem in weak formulation

In this section we first formulate the associated direct problem in a weak form and, then, deduce the above inverse problem (1) - (4) also in weak form. As a theoretical background we refer to our results on non-homogeneous second order boundary value problems for linear parabolic equations in [5] and [7].

Let the conditions (5) - (9) be satisfied and $u_0 \in L_2(\Omega)$. Further, suppose that $\tilde{g}$ be a function defined on Σ such that $\tilde{g}|_{S_1} = g, \tilde{g}|_{S_0} = 0, \tilde{g}|_{S_2} \in L_2(S_2)$. As the *direct problem* consider the boundary value problem for the following parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t} \quad &- \quad \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial u}{\partial x_j} + a_i(x,t) u \right) \\ &+ \sum_{i=1}^n b_i(x,t) u_{x_i} + a(x,t) u = f, \quad (x,t) \ \in \ Q_T, \end{aligned} \tag{10}$$

$$u(x,0) \ = \ u_0(x), \quad x \ \in \ \Omega, \tag{11}$$

$$\frac{\partial u}{\partial N}|_\Sigma \ = \ \sum_{i,j=1}^n (a_{ij}(x,t) u_{x_j} + a_i(x,t) u) \cos(\nu, x_i)|_S = g(\xi, t), \quad (\xi, t) \in \Sigma. \tag{12}$$

Let $H^1(\Omega), H^{1,0}(Q_T), H^{1,1}(Q_T)$ be the standard Sobolev spaces ([17]) and $V^{1,0}(Q_T) := C([0,T]; L_2(\Omega)) \cap L_2((0,T); H^1(\Omega))$.

Definition

A weak solution $u(x,t)$ *of the problem (10)-(12) in* $V^{1,0}(Q_T)$ *is a function of* $V^{1,0}(Q_T)$ *satisfying the identity*

$$\int\limits_{Q_T}\left(-u\eta_t+\sum_{i,j=1}^{n}a_{ij}(x,t)u_{x_i}\eta_{x_j}+\sum_{i=1}^{n}a_iu\eta_{x_i}+\sum_{i=1}^{n}b_iu_{x_i}\eta+au\eta-f\eta\right)dxdt$$

$$=\int\limits_{\Omega}u_0\eta(x,0)dx+\int\limits_{S_1}g(\xi,t)\eta(\xi,t)d\xi dt+\int\limits_{S_2}\tilde{g}(\xi,t)\eta(\xi,t)d\xi dt, \tag{13}$$

for all $\eta \in H^{1,1}(Q_T)$, *such that* $\eta(\cdot,T) = 0.$

This is not the only way to formulate a weak solution (cf. [7], [5]). It can be shown that a unique weak solution of (13) exists and a stability estimate holds (see [7], [8]). If we are interested in higher regularity, one can define a weak solution in $H^{1,1}(Q_T)$.

Definition

A weak solution $u(x,t)$ *of the problem (10)-(12) in* $H^{1,1}(Q_T)$ *is a function of* $H^{1,1}(Q_T)$ *satisfying the identity*

$$\int\limits_{Q_t}\left(u_t\eta+\sum_{i,j=1}^{n}a_{ij}u_{x_i}\eta_{x_j}+\sum_{i=1}^{n}a_iu\eta_{x_i}+\sum_{i=1}^{n}b_iu_{x_i}\eta+au\eta-f\eta\right)dxd\tau$$

$$=\int\limits_{\Sigma_t}\tilde{g}(\xi,\tau)\eta(\xi,\tau)d\xi d\tau \tag{14}$$

$$\forall\ t\ \in\ (0,T],\ \ \forall\ \eta\ \in\ H^{1,0}(Q_T),\ \ \textit{and}\ u(x,0)=u_0(x). \tag{15}$$

Under the additional assumptions

$$\operatorname*{ess\,max}_{x\in\Omega}\left|\frac{\partial a_{ij}}{\partial t}\right|,\quad \operatorname*{ess\,max}_{x\in\Omega}\left|\frac{\partial a_i}{\partial t}\right|\ \le\ \mu(t), \tag{16}$$

$$\int\limits_0^T\mu(t)dt\ <\ \infty, \tag{17}$$

$$u_0\in H^1(\Omega),\quad g(\xi,t)\in H^{0,1}(\Sigma). \tag{18}$$

it can be proved (s. [7], [8]), that there exists a weak solution in $H^{1,1}(Q_T)$ which satisfies a stability estimate.

We have already formulated the associated *inverse problem* in the classical form in (1) - (4). In order to allow weak assumptions on the data we define the inverse problem in a weak form.

Definition

A pair of functions $v := \{v_0(x), v_1(\xi,t)\} := \{u(x,0), \partial u/\partial N|_{S_2}\} \in L_2(\Omega) \times L_2(S_2)$ *is said to be a* solution *of the inverse problem (1)-(3), if, for* φ *and* g*, there exists a function* $u(x,t) \in V^{1,0}(Q_T)$ *satisfying the identity*

$$\int_{Q_T} \left(-u\eta_t + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i}\eta_{x_j} + \sum_{i=1}^{n} a_i u \eta_{x_i} + \sum_{i=1}^{n} b_i u_{x_i}\eta + au\eta - f\eta \right) dxdt$$

$$= \int_{\Omega} v_0 \eta(x,0)dx + \int_{S_1} g(\xi,t)\eta(\xi,t)d\xi dt + \int_{S_2} v_1(\xi,t)\eta(\xi,t)d\xi dt,$$

for all $\forall \; \eta \in H^{1,1}(Q_T)$*, such that* $\eta(\cdot,T) = 0$*, and*

$$u|_{S_1} = \varphi(\xi,t), \quad (\xi,t) \in S_1. \tag{19}$$

Let us remark, that for the general case in n dimensions it is not clear when a solution of this inverse problem exists. Let us further mention that the inverse problem (1) - (4) is illposed which can be demonstrated in 1-d or 2-d by explicit examples (see [5] a. o.).

3 A variational approach

We want to solve the underlying inverse problem by discretization in combination with Tikhonov's regularization using a zeroth order penalty term as well as iterative regularization via an appropriate stopping rule. For this we have to reformulate the inverse problem as a minimization problem and determine the gradient of the minimizing functional.

As the underlying operator we choose the Neumann-to-Dirichlet mapping $A : L_2(\Omega) \times L_2(S_2) \longrightarrow L_2(S_1)$ which maps the (unknown) initial function v_0 and the heat flux $v_1 = \partial u/\partial N|S_2$ to $u|_{S_1}$ where u is the solution of the parabolic equation in a weak form.

The operator A is affine. It consists of a linear part A_L and a shift term w, $A = A_L + w$. Indeed, the solution u of the direct problem not only depends on v_0, v_1 but also on $f, g, u = u(x,t; v_0, v_1, f, g)$. With this notation,

$$A_L : L_2(\Omega) \times L_2(S_2) \ni (v_0, v_1) \mapsto u(\cdot,\cdot; v_0, v_1, 0, 0)\big|_{S_1} \in L^2(S_1)$$

$$w = u(\cdot,\cdot; 0, 0, f, \tilde{g})\big|_{S_1}$$

Later, we can consider the case when $f = 0$ and $g = 0$ because the shift term w can be obtained as the solution of a (well-posed) direct problem.

We consider problem (1)-(3) as a variational problem. Since the initial condition $u|_{t=0}$ and the heat flux at S_2 are not known, we consider them as a control. We, therefore, reformulate (1)-(3) as the following optimal control problem:

Let V_0 and V_1 be subsets of $L_2(\Omega)$ and $L_2(S_2)$, respectively. Set $V := (V_0, V_1)$.

Minimize the functional

$$J_0(v) := \frac{1}{2}\|u|_{S_1} - \varphi\|^2_{L_2(S_1)}, v \in V, \tag{20}$$

subject to

$$u \in V^{1,0}(Q_T), \tag{21}$$

$$\int\limits_{Q_T} \left(-u\eta_t + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i}\eta_{x_j} + \sum_{i=1}^{n} a_i u \eta_{x_i} + \sum_{i=1}^{n} b_i u_{x_i}\eta + au\eta - f\eta \right) dxdt$$

$$= \int\limits_{\Omega} v_0 \eta(x,0)dx + \int\limits_{S_1} g(\xi,t)\eta(\xi,t)d\xi dt + \int\limits_{S_2} v_1(\xi,t)\eta(\xi,t)d\xi dt, \tag{22}$$

for all $\eta \in H^{1,1}(Q_T)$, such that $\eta(\cdot,T) = 0$.

The problem (20)-(22) is still unstable. A minimization problem is said to be unstable when there exists a minimizing sequence that does not converge to its minimizing element. In 2-d. such an example can be explicitly given (see [5]).

One can show (see [7], [8]), that for convex sets V_1, V_2 there exists a solution of problem (20)-(22) and every minimizing sequence converges weakly to the set of all minimum points of the problem.

Because of the possible unstable behaviour of the minimization problem, instead of (20)-(22) we add a penalty term to J_0 and minimize

$$J_\gamma(v) = \frac{1}{2}\left(\left\| u|_{S_1} - \varphi \right\|^2_{L_2(S_1)} + \gamma^2 \|v\|^2_{L_2(\Omega)\times L_2(S_2)} \right) \tag{23}$$

with a positive constant γ^2 which, in addition, has to be determined in an optimal way. This is the idea of Tikhonov's regularization.

To solve the control problem (20)-(22) or (21)-(23), we need the gradient of J_γ. For this, we assume in the following that

$$b_i(x,t) = 0, \; i = 1, \ldots, n. \tag{24}$$

Let $\psi(x,t) := \psi(x,t;v)$ be a weak solution from $V^{1,0}(Q_T)$ of the following *adjoint problem* to (21), (22),

$$\psi_t = -\left(\sum_{i,j=1}^{n} a_{i,j}\psi_{x_i} + \sum_{i=1}^{n} a_i \psi \right)_{x_j} + a(x,t)\psi, \quad x \in \Omega, \quad 0 \le t < T, \tag{25}$$

$$\psi(x,T) = 0, \quad x \in \Omega, \tag{26}$$

$$\frac{\partial \psi}{\partial N}|_{S_1} = u(v)|_{S_1} - \varphi, \quad 0 \le t < T, \tag{27}$$

$$\frac{\partial \psi}{\partial N}|_{S_2} = 0 \tag{28}$$

$$\frac{\partial \psi}{\partial N}|_{S_2} = 0, \quad 0 \le t < T, \tag{29}$$

where $u = u(x,t)$ be a solution of Problem (22). The function $\psi = \psi(x,t;v)$ satisfies the integral identity

$$\int\limits_{Q_T} \left(\psi\eta_t + \sum_{i,j=1}^{n} a_{i,j}\psi_{x_i}\eta_{x_j} + a\psi\eta \right) dxdt$$

$$= -\int\limits_{\Omega} \psi(x,0)\eta(x,0)dx + \int\limits_{S_1} [u(v)|_{S_1} - \varphi]\eta d\xi dt, \tag{30}$$

for all $\eta \in H^{1,1}(Q_T)$.

If we replace t in (25)-(29) by the new variable $\tau := T - t$, then we obtain a boundary value problem of the same type as (10)-(12). Since $u \in V^{1,0}(Q_T)$, and the available stability estimate for u, $u(v)|_{S_1} \in L_2(S_1)$. Moreover we see that there exists a unique solution in $V^{1,0}(Q_T)$ of Problem (25)-(29).

Based on the stability estimates for ψ we can prove that under the conditions (5)-(9) the functional J_0 is Fréchet differentiable in V and its gradient can be found as follows:

$$J_0'(v) = \left\{ \begin{array}{c} \psi(x,t;v)|_{t=0} \\ \psi(x,t;v)|_{(x,t)\in S_2} \end{array} \right\}. \tag{31}$$

Let us remark that the adjoint problem (25)-(29) is a parabolic equation backwards in time but wellposed due to the minus sign on the right-hand side of (25).

It is not difficult to see that the Fréchet derivative of J_γ is given by

$$J_\gamma'(v) = J_0'(v) + \gamma^2 v \tag{32}$$

If we notice that $u|_{S_1} = Av$ in (20) and (25), the Fréchet derivative of J_0 can be also written as

$$J_0'(v) = A_L^*(Av - \varphi) \tag{33}$$

where $A_L^* : L_2(S_1) \longrightarrow L_2(\Omega) \times L_2(S_2)$ is the adjoint operator to A_L which can be obtained via the weak solution of (25) - (29) and $A_L^* d$ as in the right-hand side of (31) with d on the right-hand side in the boundary condition (27) on S_1.

As an iterative algorithm to solve the minimization problem we use the CGM. We allow perturbations of the data, $\|\varphi - \varphi_\varepsilon\|_{L_2(S_1)} = O(\varepsilon)$ and the operator A is replaced by A_h which is a finite element or finite difference approximation – the same holds for A_L and $A_{L,h}$.

The CGM for minimizing J_γ has the form[1],

$$\begin{aligned} r_0 &= -d^{(0)} = A_{L,h}^*(A_h v^{(0)} - \varphi_\varepsilon) - \gamma^2 v^{(0)}, && (34)\\ \alpha_k &= \frac{\|r_k\|^2}{\|A_{L,h} d^{(k)}\|^2 + \gamma^2 \|d^{(k)}\|^2},\ v^{(k+1)} = v^{(k)} + \alpha_k d^{(k)}, && (35)\\ r_{k+1} &= A_{L,h}^*(A_h v^{(k+1)} - \varphi_\varepsilon) - \gamma^2 v^{(0)}, && (36)\\ \beta_k &= \frac{\|r_{k+1}\|^2}{\|r_k\|^2},\ d^{(k+1)} = -r_{k+1} + \beta_k d^{(k)}. && (37) \end{aligned}$$

The optimal regularization parameter γ is determined by a sequence $q^i\gamma_0$ with $0 < q < 1$. The optimal iteration index is determined as the first $k = k_{\max}$ such that the *stopping rule* (with $\gamma_1 = 1.1$)

$$\|A_h v^{(k)} - \varphi_\varepsilon\| \le \gamma_1 \big(h\|v^{(k)}\| + \varepsilon\big) \tag{38}$$

is fulfilled. In addition we stop if $\|r_k\| < \varepsilon$. The stopping rule (38) is a modification of that of Nemirovskii [19]. We like to emphasize the remarkable results of Nemirovskii [19] on the optimal order regularization properties of the conjugate gradient method for ill-posed problems.

As a 'zeroth approximation' we usually take $v^{(0)} = (v_0^{(0)}, v_1^{(0)}) = 0$. Then, according to the vanishing 'terminal value' at $t = T$ of the adjoint problem (see (26)), the iterative

[1] This is also called CGLS (=conjugate gradient least square) in [12].

approximations $v^{(k)} = (v_0^{(k)}, v_1^{(k)})$ in (35) leave the solutions $u^{(k)}$ of the associated direct problems at $t = T$ nearly unchanged as long as γ^2 is close to zero,

$$\begin{aligned} u_t^{(k)} - Lu^{(k)} &= 0 \text{ in } Q_T, \\ \frac{\partial u^{(k)}}{\partial N}|_{S_i} &= 0 \text{ on } S_i,\ i = 0, 1, \\ \frac{\partial u^{(k)}}{\partial N}|_{S_2} &= v_1^{(k)} \text{ on } S_2, \\ u^{(k)}|_{t=0} &= v_0^{(k)} \text{ in } \Omega. \end{aligned} \tag{39}$$

Note that f and g in (1) and (4), resp., are set to zero. Here, L denotes the elliptic operator on the left-hand side of (10) written as $u_t - Lu = f$.

We are now able to compute the solutions of IHCPs with an iterative method which is a regularization method in the sense of Tikhonov when the stopping rule (38) is applied (see Nemirovskii [19]). Additionally, the regularization parameter γ makes the computations more stable and the above stopping rule helps to determine it in an optimal way. In our computations we leave the discretization parameter h fixed.

4 Numerical Calculations

The computations are performed by using the Finite Element Package DEAL [2]. For the inverse problem calculations we use the *Crank-Nicolson method* for the time integration and the conjugate gradient method plus Tikhonov's regularization to solve the minimization problem described above. The direct solution of the parabolic problems with DEAL uses bilinear ansatzfunctions.

In the numerical experiments presented in this paper, we study the identifiability of the boundary condition only; the initial condition is assumed to be given. From the theory we know that the initial condition can also be identified. This will be verified by numerical experiments in a forthcoming study.

As a **first example** we study the half-ring problem from [6]. The geometry is displayed in Fig. 2.

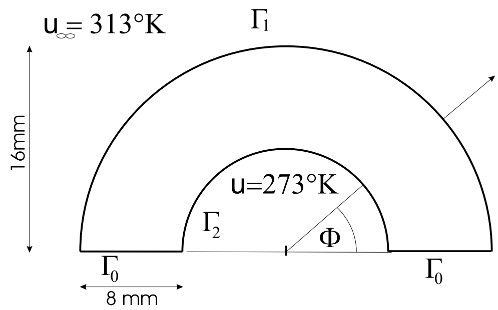

Figure 2: Geometrical setting

In addition to Γ_1 (Cauchy data on outer circle), Γ_2 (inner circle, inaccessible) we have a symmetry condition on Γ_0. The governing equations are

$$\frac{\partial u}{\partial t} - \alpha \Delta u = 0 \quad \text{in } Q_T, \tag{40}$$

$$\lambda \frac{\partial u}{\partial N} = 0 \quad \text{on } \Gamma_0, \tag{41}$$

$$\lambda \frac{\partial u}{\partial N} + \theta(u - u_\infty) = 0 \quad \text{on } \Gamma_1, \tag{42}$$

where the constants are given by

$$\alpha = 1.43 * 10^{-7} m^2 s^{-1}, \quad \lambda = 0.3 W m^{-1} K^{-1},$$

The heat transfer coefficient θ on Γ_1 is spatial dependent,

$$\theta(\phi) = \begin{cases} 10, & 0 \leq \phi \leq 40- \\ 4.8\ \phi - 182, & 40- \leq \phi \leq 90- \\ 682 - 4.8\ \phi, & 90- \leq \phi \leq 140- \\ 10, & 140- \leq \phi \leq 180- \end{cases}$$

In this problem, the initial condition is given and is assumed to be constant in space, $u|_{t=0} = 273^0 K$. The temperature u_∞ outside of the domain under consideration is given and assumed to be constant, $u_\infty = 313 - K$. Since $u|_{\Gamma_1}$ and θ are known, the form (42) of the boundary condition on Γ_1, resp. S_1, can be rewritten in the form (4).

In the inverse problem, the boundary Γ_2 is considered to be inaccessible while on Γ_1 Cauchy data are given. We want to recover the constant temperature $u|_{\Gamma_2} = 273^0 K$ on Γ_2 from temperature data together with the heat flux caused by the heat transfer coefficient $\theta(\phi)$ on Γ_1. The data on Γ_1 are obtained by a direct computation using the heat flux on Γ_1 and the exact temperature on Γ_2.

Fig. 3 shows the spatial discretization utilizing 320 rectangles and 369 grid points. The spatial distribution of the temperature data on Γ_1, obtained by the above mentioned direct calculation in the beginning, are displayed in Fig. 4 at three different time values.

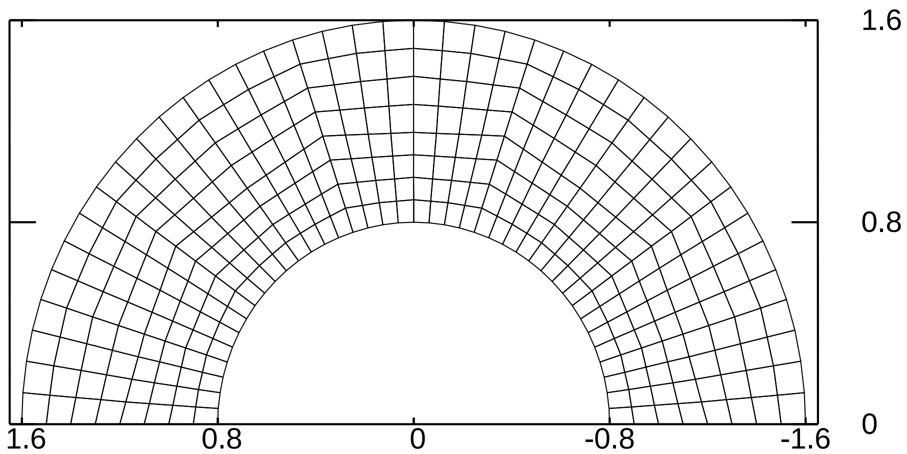

Figure 3: 369 grid points, 320 rectangles, 41 grid points on Γ_1 and Γ_2

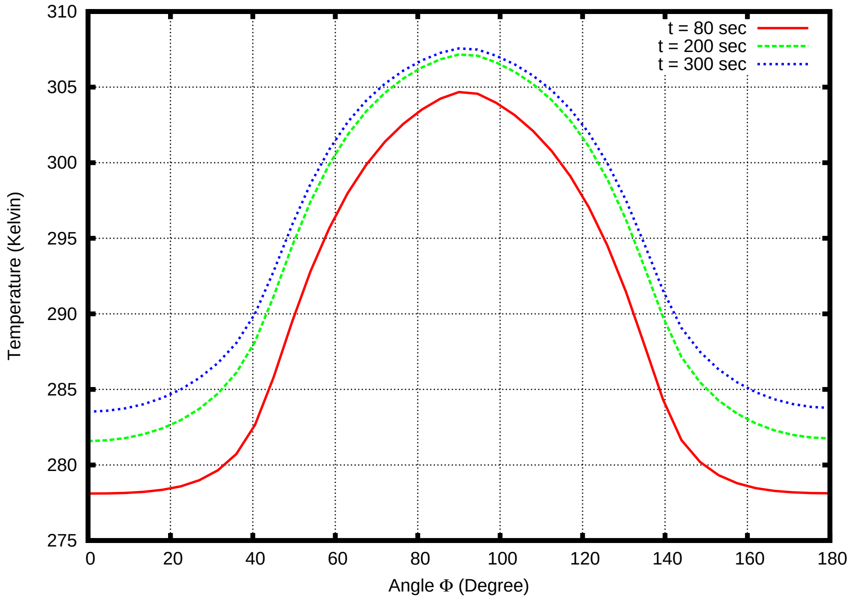

Figure 4: Spatial temperature distribution on Γ_1 (Cauchy data) at different times

In Fig. 5 we show the decrease of the defect functional $J_\gamma(v^{(k)})$, the absolute value of the gradient $|J'_\gamma(v^{(k)})|$ as well as the L_2-norms $\left\|(u^{(k+1)} - u^{(k)})|_{\Gamma_1}\right\|^2$ during the iteration. The latter can be also utilized for a stopping rule. Here, the optimal regularization parameter $\gamma = 5 \cdot 10^{-4}$ is used; a maximum of 50 iterations is allowed.

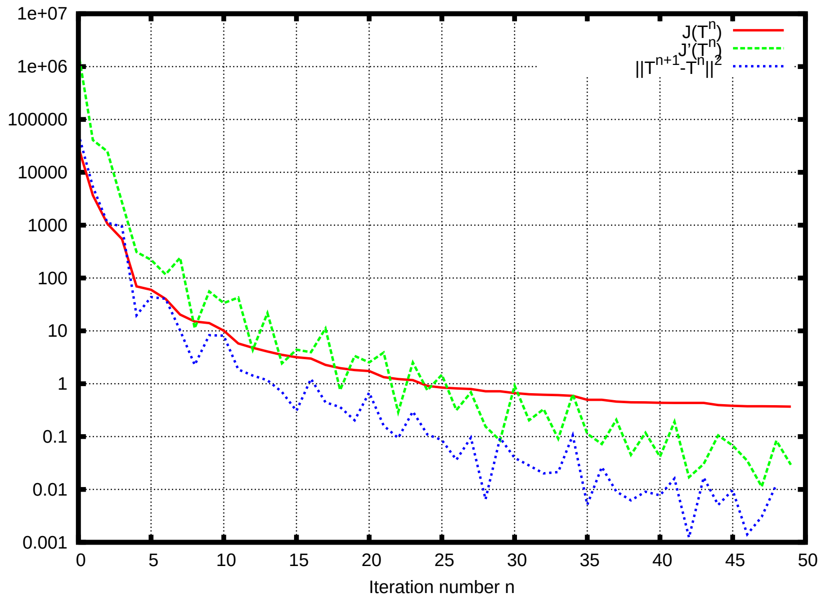

Figure 5: Tikhonov-Regularization with $\gamma^2 = 5 * 10^{-4}$ for $J(u^n)$, $J'(u^n)$, $\|(u^{n+1} - u^n)|_{\Gamma_0}\|^2$

Fig. 6 displays the time plot for the resulting temperature (after 50 iterations) at different spatial positions on the inner circle Γ_2. It is clearly observed that the results are bad for times greater than 480 sec. The reason lies in the starting function of the CGM which is set to zero. Therefore, at the final time $T = 600$ sec. the initial guess of the temperature remains nearly unchanged due to the adjoint problem during the iteration. As a remedy, we suggest to use more data in time or to take the resulting temperature only at 80 % of the time interval where data are available. (Compare also [9], [10] for a discussion of the same phenomenon).

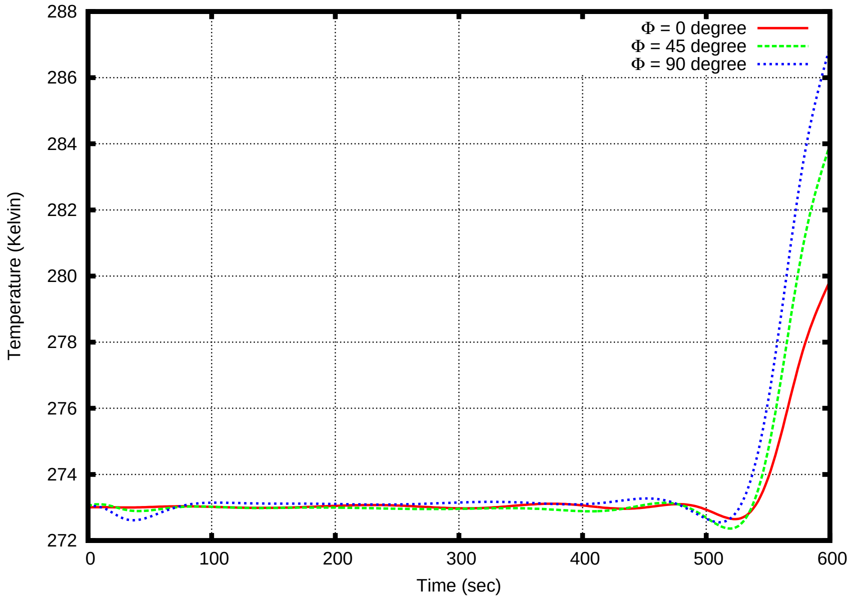

Figure 6: Result of Tikhonov regularization with $\gamma^2 = 5 * 10^{-4}$ on Γ_1 at different positions

Fig. 7 and Fig. 8, show the spatial distribution of the temperature and of the error in the temperature on the complete half ring after $t = 300$ sec. It is remarkable, that in our calculations we have a maximal error of $0.2 - K$ which is $1/8$ of the error obtained by the calculations in [6] with MODULEF on 400 triangles and 231 modal points. As expected the maximal error appears at $\phi = 90-$ on Γ_2 opposite to the highest temperature on the outer circle Γ_1.

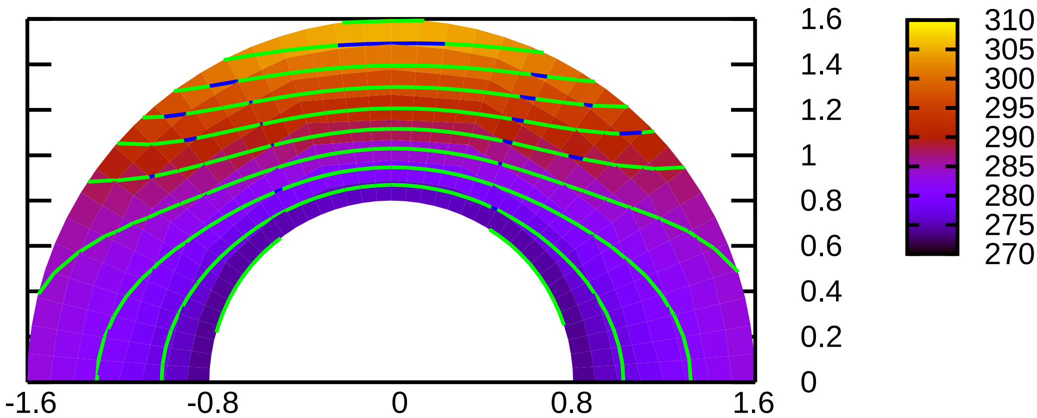

Figure 7: Spatial temperature distribution at $t = 300$ sec.

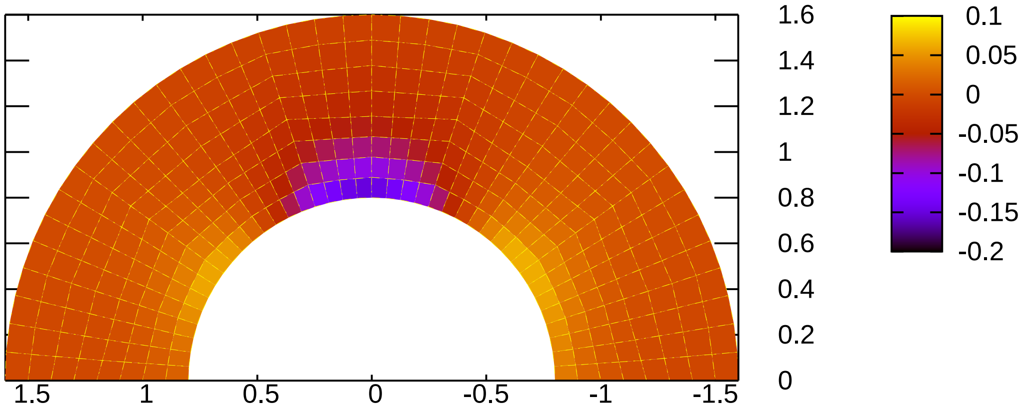

Figure 8: Spatial distribution of the error (in the temperature) at $t = 300$ sec.

In Fig. 9 the results of a stability analysis are displayed. The temperature data on Γ_1 are pointwise perturbed by ε multiplied by a random function having values between -1 and 1. The time plot compares the temperature on Γ_2 at position $\phi = 90-$ for $\varepsilon = 10^{-4}, 10^{-3}, 10^{-2}$ with the temperature calculated with $\varepsilon = 0$. The observed errors are pointwise within the same magnitude as the perturbations which demonstrates the stability of our algorithm.

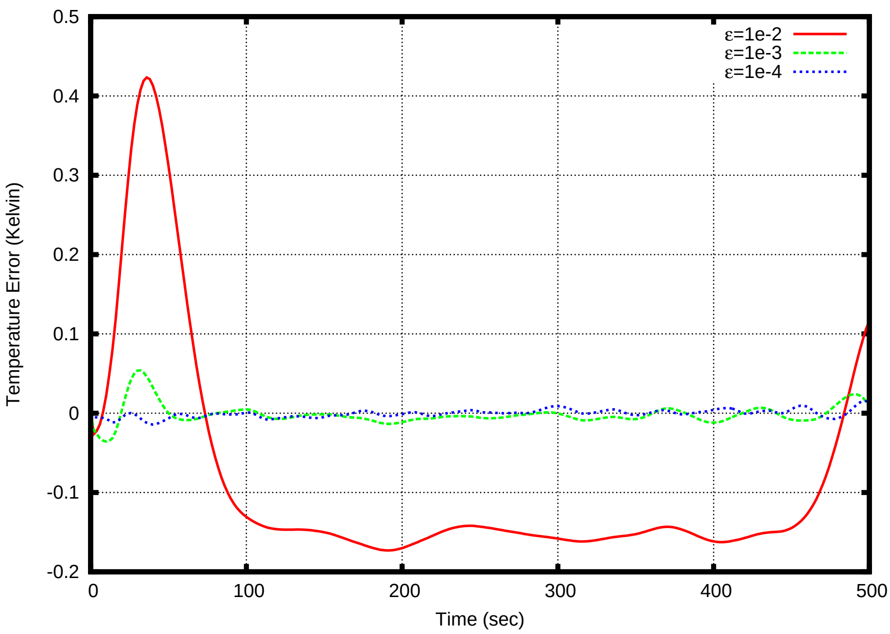

Figure 9: Reconstruction errors on Γ_2 at $\phi = 90-$ with perturbed data

For the **second example** we show analogous figures for calculations on a rectangular domain (s. Fig. 10).

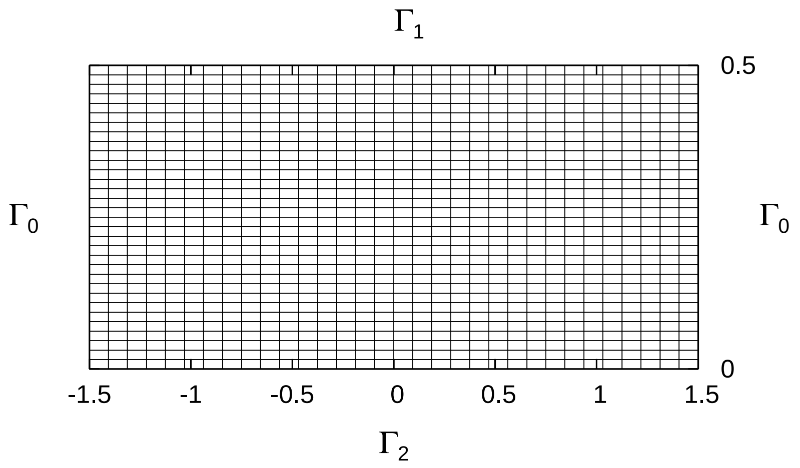

Figure 10: 1089 grid points, 1024 rectangles, 33 grid points on Γ_1 and Γ_2

The underlying equations are the same as in (40) where, here, $\alpha = \lambda = \theta = 1$ and $u_\infty = 20-$C. The Cauchy data are given on the upper horizontal side Γ_1 while the lower horizontal side Γ_2 is considered to be inaccessible. Additionally, on the vertical sides Γ_0 a zero flux condition is imposed. The 'exact' temperature values on Γ_2 are obtained by an inverse calculation with certain fictitious temperature data on Γ_1. The Cauchy data for u on Γ_1 are then obtained by a direct calculation with the prescribed temperature flux on Γ_2 and the Neumann boundary condition on Γ_1. Fig. 11 displays the temperature data on S_1 as a function of space and time.

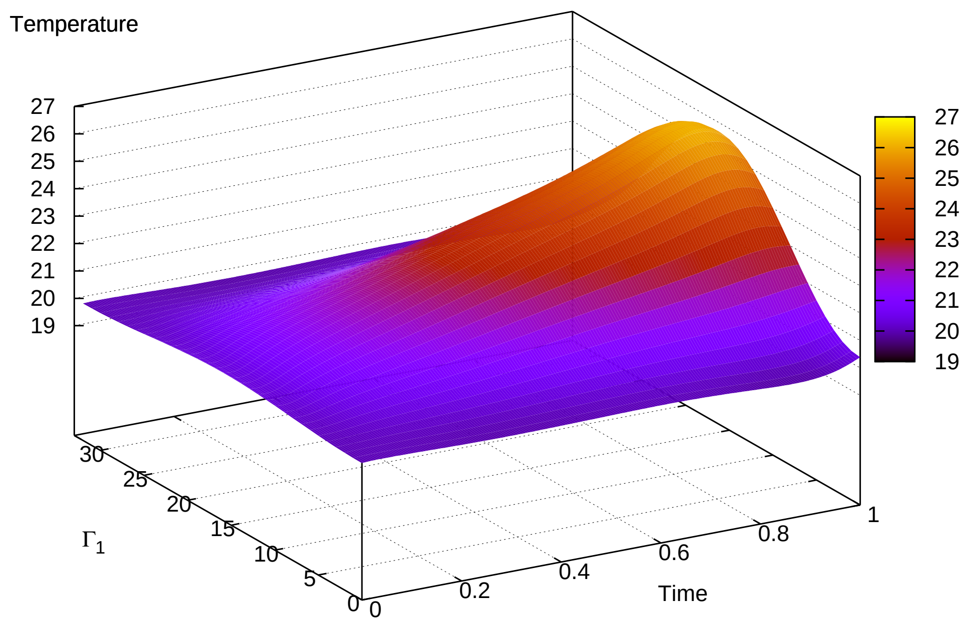

Figure 11: Temperature distribution on Γ_1 (Cauchy data) in time and space

Our result of the Tikhonov regularisation with $\gamma^2 = 5 \cdot 10^{-5}$ after 10 iterations are displayed in Fig. 12 and Fig. 13, resp., where the former shows the temperature on two spatial positions on Γ_1 and Γ_2 while the latter shows the error at the same positions on Γ_2.

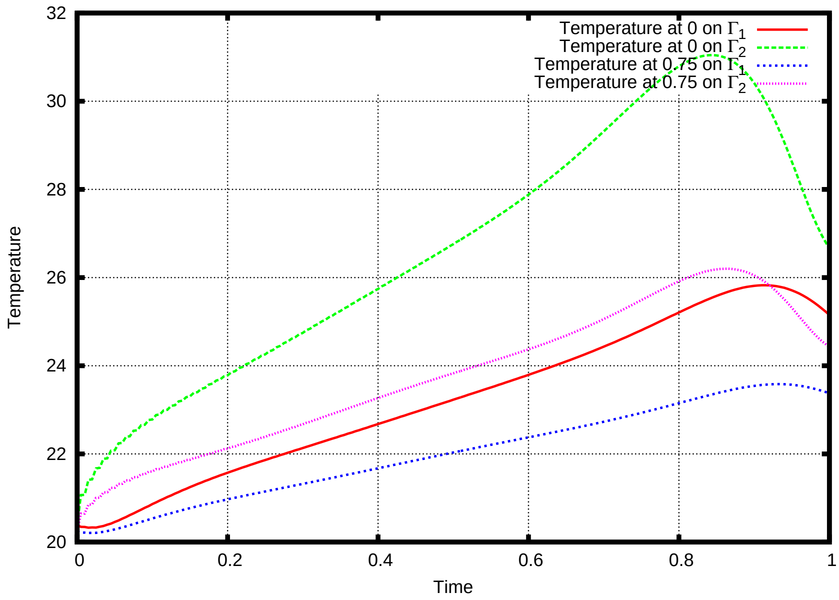

Figure 12: Result of Tikhonov regularization (with $\gamma^2 = 5 \cdot 10^{-5}$) at two different positions after 10 iterations

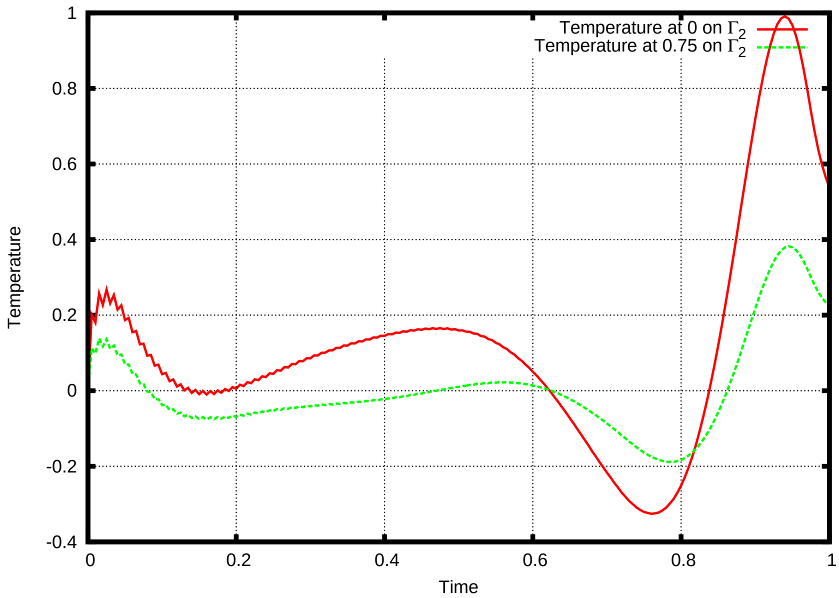

Figure 13: Error in temperature at position 0 and 0.75 on Γ_2

Here, again, the largest errors occur near the final time $T = 1$. Until $t = 0.8$ the errors are less than 1-. Similarly to Fig. 9, the error of a stability analysis at position 0 for four different magnitude of perturbations ε of the Cauchy data are shown in Fig. 14.

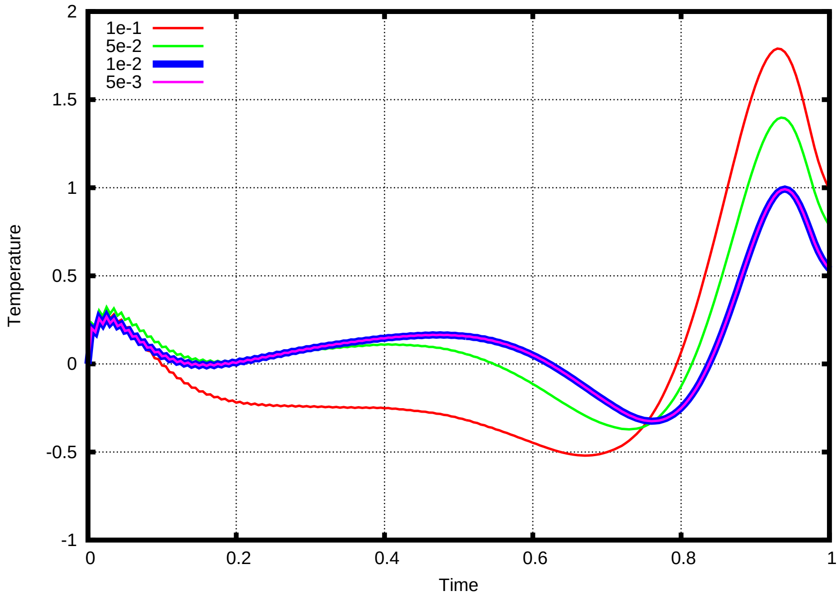

Figure 14: Absolute errors in temperature at position 0 on Γ_2

As a **third example** we consider the half ring situation as in Example 1. However, Cauchy data on the outer circle Γ_1 are only available for $\phi \in [0-, 142-]$, while the rest Γ_3 of the outer circle is considered to be inaccessible – as the inner circle Γ_2 – and should now belong to that part of the boundary where the temperature and flux should be determined. Fig. 15 shows the spatial temperature distribution on Γ_1 at three different times.

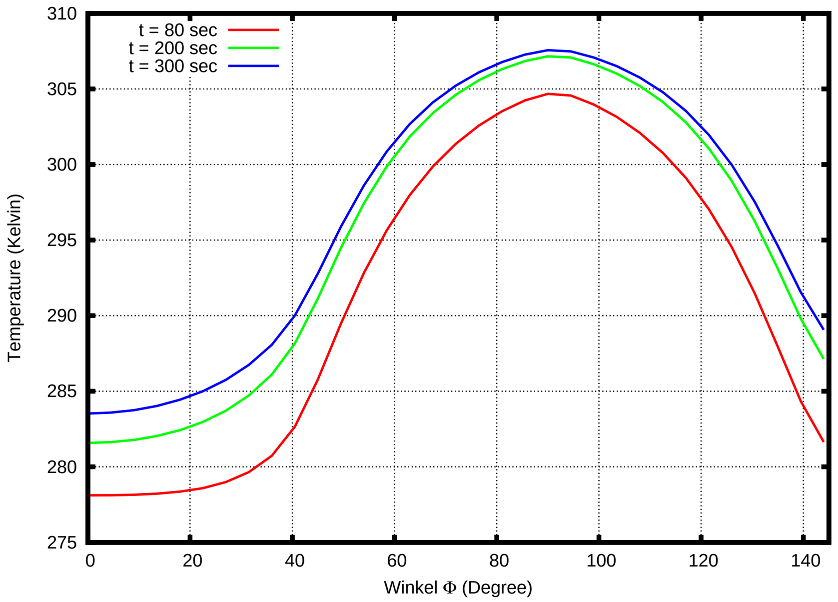

Figure 15: Temperature on Γ_1

The next and final figure shows the result of our inverse calculations after 300 sec. on the complete half-ring.

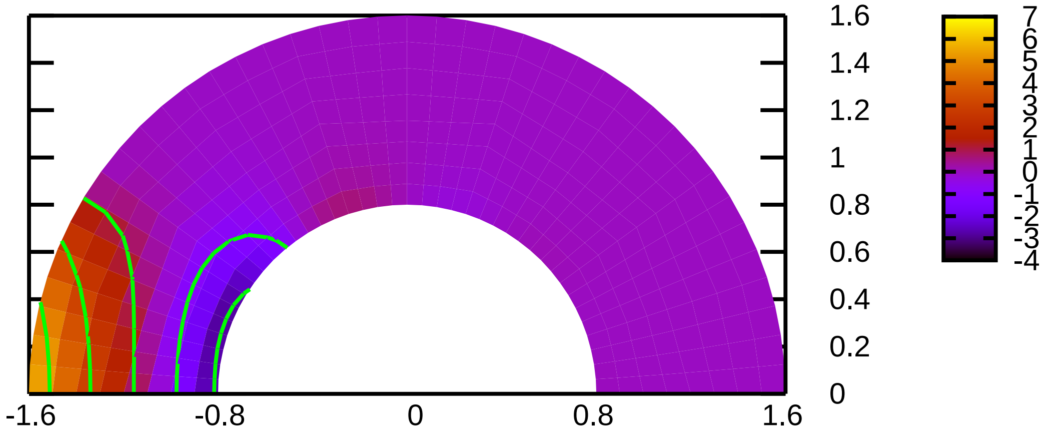

Figure 16: Result of inverse calculation at $t = 300sec$.

We observe astonishing good results on the inner circle Γ_2 while unacceptable errors occur on the inaccessible part of the outer circle. In these calculations no perturbations of data are present. Possible improvements may be obtained by the approach suggested by Johansson [15].

5 Conclusions

Our calculations show that CGM together with Tikhonov's regularization provides a fast and efficient method for solving Inverse Heat Conduction problems in two spatial dimensions. The stability analysis ensures that the method is stable w.r.t. perturbations in the data. The third example with a reduced amount of data leave some theoretical questions open; a hint for improvements is given.

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