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A note on the stability of the upwind scheme for ordinary differential equations

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Abstract — In this paper the stability of the upwind scheme for second order, linear ordinary differential equations with general boundary conditions of the third kind is proved. The proof is carried out via compactness arguments and is rather short. Moreover, no information on the behaviour of the solution of the differential equation itself is needed. This approach seems not to be present in the existing literature.

Keywords: upwind scheme, stability, compactness

1. Introduction

It is well-known that the upwind scheme for approximating second order, linear ordinary differential equations has superior stability properties compared to the central difference approximations. If a first order derivative is present in the differential equation, in the upwind scheme it is approximated by the forward or backward difference quotient in case the coefficient multiplying the first derivative is positive or negative, respectively. Especially, in cases where the coefficient is very large this strategy is preferable — otherwise in the central difference approximations the mesh widths must be chosen very small. Such cases appear in singularly perturbed ordinary differential equations where a small parameter multiplies the second derivative. Such differential equations are also called convection–diffusion equations where the convection term dominates the diffusion.

erty — assures a stability estimate for the solutions of the difference equations (cf., e.g., [14], 5.3). Such a stability estimates provides the uniform boundedness of the associated inverses of the difference operators. For the central difference approximations or the upwind scheme such a stability estimate must be proven separately. In case of the central difference approximations and general separated boundary conditions of the third kind a stability estimate via maximum principles is proved in [5], 3.8. For the special case of Dirichlet boundary conditions this is proved at many places (see, e.g., [13], 8.1). The stability of the upwind scheme is shown by means of maximum principle arguments in the excellent work of T. Linß [8]. There, in the mathematical analysis, detailed information on the behaviour of the solution of the singularly perturbed differential equation is needed. The essential point in this paper is the fact that the stability constant is independent of the small parameter multiplying the highest derivative. This is achieved by the fact that in the norm the discrete derivatives are weighted by a suitable power of the small parameter. It is also worth mentioning that in [8] it is shown that the stability constants for the differential operator and its discretization are equal w.r.t. the weighted norms.

Related results concerning the upwind scheme for approximating singularly perturbed convection–diffusion equations can be found in [1, 2, 7, 9, 10]. Standard references for numerical methods for convection–diffusion equations are [11, 12, 15]. Upwind schemes are also available for elliptic boundary value problems in higher spatial dimensions or parabolic initial-boundary value problems — also in higher dimensions (see [3, 4, 13]). For these, usually also maximum principles or positivity properties are exploited to assure stability.

Already by G. Vainikko in [16], 6.3 and 6.4, it is observed that compactness arguments can be used to provide an elegant proof of stability and convergence for difference approximations of boundary value problems for ordinary differential equations. Moreover, such properties are stronger than the one obtained by maximum principles because a stronger norm appears on the left-hand side without requiring stronger assumptions. In [13], 8.2, we have carried out the stability of the central difference approximation in the framework of discrete convergence using compactness arguments. It is not difficult to see that the same arguments can be also applied to the upwind scheme which, however, ordinary differential equations are presented. The boundary conditions are allowed to be of third kind including the function itself and its first derivative. The maximum principle and a related monotonicity property are formulated. Moreover the functional analytical framework with suitable norms and difference operators is introduced. In Section 3 of this paper we prove the stability of the upwind scheme. The proof is rather short and needs no information on the behaviour of the solution of the differential equation itself. The only assumption we use, is standard, namely that the homogeneous boundary value problem has only the trivial solution.

It should be mentioned that the proof — by contradiction using compactness arguments — seems not be able to guarantee that the constant in the stability estimate is independent of the small parameter in a singularly perturbed problem. As mentioned above, for a stability constant independent of the small parameter one must weight the discrete derivatives in the norm by powers of the small parameter. However, our proof by contradiction seems not to allow such a modified stability result where the stability constant is independent of a small parameter. Moreover, because our proof is not constructive it is obvious that we cannot make any statement which allows the comparison of the stability constant in a stability estimate for the differential operator and its discretization.

Our stability result can be generalized to the case of nonuniform grids. When one considers a sequence of nonuniform grids for which the maximal mesh widths tend to zero, our Lemma 3.2 is still valid and, therefore, the stability result in Theorem 3.1 also holds with a stability constant independent of the mesh widths. In principle, the application of compactness arguments should be also possible for higher dimensional convection–diffusion equations.

2. Problem setting

Let us consider boundary value problems for ordinary differential equations of the form

$$u'' + pu' + qu = f \quad \text{in } I = [a,b] \tag{2.1}$$

with general separated boundary conditions of the third kind,

$$\alpha_0 u(a) + \alpha_1 u'(a) = \eta_0$$

We denote the linear differential operator and the linear functionals on the left-hand sides of (2.1) and (2.2), respectively, by

$$\begin{aligned}(Lv)(x) &:= v''(x) + p(x)v'(x) + q(x)v(x),\, x \in I\\ \ell_0(v) &:= \alpha_0 v(a) + \alpha_1 v'(a)\\ \ell_1(v) &:= \beta_0 v(b) + \beta_1 v'(b).\end{aligned}$$

For the approximation by difference methods we consider a null sequence Λ of mesh widths $h > 0$ and equidistant grids,

$$\begin{aligned}I_h &:= \left\{x \in [a,b] \,\middle|\, x = x_j := a + jh,\ j = 0,\dots,N_h\right\}\\ I_h' &:= \left\{x \in I \,\middle|\, x = x_j,\ j = 1,\dots,N_h - 1\right\}\\ I_h^0 &:= \left\{x \in I \,\middle|\, x = x_j,\ j = 0,\dots,N_h - 1\right\}\\ I_h^1 &:= \left\{x \in I \,\middle|\, x = x_j,\ j = 1,\dots,N_h\right\}\end{aligned}$$

where $N_h \in \mathbb{N}$, $hN_h = b - a$. The differential quotients are approximated by difference quotients,

$$\begin{aligned}(D_h v_h)(x) &= \frac{1}{2h}\big(v_h(x+h) - v_h(x-h)\big), \quad x \in I_h', \text{ (central)}\\ (D_h^+ v_h)(x) &= \frac{1}{h}\big(v_h(x+h) - v_h(x)\big), \quad x \in I_h^0, \text{ (forward)}\\ (D_h^- v_h)(x) &= \frac{1}{h}\big(v_h(x) - v_h(x-h)\big), \quad x \in I_h^1, \text{ (backward)}\\ (D_h^2 v_h)(x) &= \frac{1}{h^2}\big(v_h(x+h) - 2v_h(x) + v_h(x-h)\big), \quad x \in I_h' \text{ (central, 2nd order)}\end{aligned}$$

where $v_h \in C(I_h)$ denotes a grid function.

The *upwind scheme* approximates the first order differential quotient by the forward or backward difference quotient whenever the value of the function $p(x)$ is positive or negative; hence pu' is approximated by

$$p(x)D_h^+u_h(x), \quad \text{if } p(x) > 0$$

where

$$p_h^+(x) := \max\big(0, p(x)\big), \quad p_h^-(x) := \min\big(0, p(x)\big), \quad x \in I_h'$$
$$q_h = q\big|I_h', \quad f_h = f\big|I_h'.$$

One can also write the difference operator on the left-hand side of (2.4) in the following form,

$$(L_h v_h)(x) = a_{-1,h}(x)v_h(x-h) + a_{0,h}(x)v_h(x) + a_{1,h}(x)v_h(x+h), \quad x \in I_h' \tag{2.5}$$

with

$$\begin{aligned} a_{-1,h}(x) &= h^{-2}\big(1 - hp_h^-(x)\big), \quad a_{-1,h}(x) = h^{-2}\big(1 + hp_h^+(x)\big) \\ a_{0,h}(x) &= -h^{-2}\big(2 - h^2 q_h(x) + h\big|p_h(x)\big|\big), \quad x \in I_h'. \end{aligned} \tag{2.6}$$

We note that $p_h^+ - p_h^- = |p_h|$, and that under the assumption

$$q(x) \leqslant 0, \quad x \in I \tag{2.7}$$

the difference operator L_h in (2.5) is of *positive type*. This means

$$a_{-1,h} > 0, \quad a_{1,h} > 0, \quad a_{-1,h} + a_{0,h} + a_{1,h} \leqslant 0 \quad \text{in } I_h'. \tag{2.8}$$

It is important to note that this property holds without any restriction on the mesh widths. An important consequence from the positivity (2.8) is the following *monotonicity property* (see, e.g., [13], 8.1):

$$\begin{aligned} &\textit{If} \quad L_h v_h(x) \leqslant L_h u_h(x), \quad x \in I_h', \quad v_h(a) \geqslant u_h(a), \quad v_h(b) \geqslant u_h(b) \\ &\textit{then} \quad v_h(x) \geqslant u_h(x) \quad \forall x \in I_h. \end{aligned} \tag{2.9}$$

Maximum principles can also be formulated and proved for general boundary conditions of the form (2.10) (see [5]).

If one approximates the differential equation (2.1) by central difference quotients also for the first order derivative, the positive type property is present if the mesh width is small enough, namely, $h < 2/\max|p(x)|$. Such a difference approximation has the advantage that the truncation error is of order $O(h^2)$ while, for the upwind scheme, the truncation error is only $O(h)$. The restriction on the mesh widths, however, makes the central difference approximation where $\eta_{i,h}$ are approximations of $\eta_i, i = 0,1$, in (2.2).

Together with the difference operators L_h defined in (2.5), (2.6) we define linear operators $\widehat{L}_h : E_h \to F_h$, $E_h = F_h = C(I_h)$, by

$$\big(\widehat{L}_h v_h\big)(x) = \begin{cases} L_h v_h(x), & x \in I_h' \\ \ell_{0,h}(v_h), & x = a \\ \ell_{1,h}(v_h), & x = b, \quad h \in \Lambda. \end{cases} \tag{2.11}$$

Moreover, we supply E_h and F_h with the following norms,

$$\|v_h\|_{E_h} := \|v_h\|_{2,\infty,h} := \max_{x\in I_h} |v_h(x)| + \max_{x\in I_h^0} |D_h^+ v_h(x)| + \max_{x\in I_h'} |D_h^2 v_h(x)|, \quad v_h \in E_h$$

$$\|w_h\|_{F_h} := \|w_h\|_{0,\infty,h} := |w_h(a)| + |w_h(b)| + \max_{x\in I_h'} |w_h(x)|, \quad w_h \in F_h.$$

Moreover, we set $\|\cdot\|_{0,\infty}$ for the maximum norm in $C(I)$. It is clear that

$$\max_{x\in I_h'} |L_h v_h(x)| \leqslant \max\big(1, \|p\|_{0,\infty}, \|q\|_{0,\infty}\big) \|v_h\|_{2,\infty,h}, \quad v_h \in E_h. \tag{2.12}$$

Indeed, in (2.4) one has

$$\max_{x\in I_h^1} |D_h^- v_h(x)| = \max_{x\in I_h^0} |D_h^+ v_h(x)|$$

and $|p_h^+(x)| \leqslant |p(x)|$, $|p_h^-(x)| \leqslant |p(x)|$, $x \in I_h'$.

Moreover, the $\ell_{i,h}$, $i = 0,1$, are uniformly bounded linear functionals because the following estimates are fulfilled,

$$|\ell_{0,h}(v_h)| \leqslant \max\big(|\alpha_0|, |\alpha_1|\big)\big(|v_h(a)| + |D_h^+ v_h(a)|\big)$$

$$\leqslant \max\big(|\alpha_0|, |\alpha_1|\big)\Big(\max_{x\in I_h} |v_h(x)| + \max_{x\in I_h^0} |D_h^+ v_h(x)|\Big)$$

$$\leqslant \max\big(|\alpha_0|, |\alpha_1|\big)\|v_h\|$$

Combining the last estimates with (2.12), one obtains

$$\begin{aligned} \|\widehat{L}_h v_h\|_{F_h} &= \left|\ell_{0,h}(v_h)\right| + \left|\ell_{1,h}(v_h)\right| + \max_{x\in I_h'} |L_h v_h(x)| \\ &\leqslant c_1 \|v_h\|_{2,\infty,h} \end{aligned} \tag{2.13}$$

with a constant c_1 independent on the mesh width h.

The estimate (2.13) bounds the operators $\widehat{L}_h$, $h \in \Lambda$, from above which is important but not difficult to show. In the following section we shall prove a uniform estimate from below for $\widehat{L}_h$ which, in other words, yields the 'stability of the upwind scheme'.

3. Stability via compactness arguments

An important tool for the following considerations is the theorem of Arzela–Ascoli. We use a formulation as it is stated in [13], 5.4. In the following we do not distinguish between a sequence of grid functions $u_h, v_h \in C(I_h)$, $h \in \Lambda$, and its linear interpolants which are functions in $C(I)$. If a grid function is defined only on inner grid points, e.g., on I_h', we extend the function piecewise constant in $[a, a+h]$ or $[b-h, b]$. At some places we denote the restriction of a continuous function v to I_h (or I_h^0, I_h') by v^h. A sequence of functions $v_h \in C(I)$, $h \in \Lambda$, is called *equicontinuous*, if, for every $\varepsilon > 0$ there exists a positive number $\delta > 0$ such that for every $h \in \Lambda$ and every $x, y \in I$ with $|x-y| < \delta$ the estimate

$$|v_h(x) - v_h(y)| < \varepsilon$$

holds. One may also call this property 'uniform equicontinuous' because $\delta > 0$ is also independent of the positions $x, y \in I$. One can formulate the equicontinuity pointwise and knows that, due to the compactness of $I = [a, b]$, the equicontinuity also holds uniformly.

We need a lemma which is well-known and can be found in many textbooks (see, e.g., [6]).

Lemma 3.1. *Let $v_h, h \in \Lambda$, an equicontinuous sequence in $C(I)$. Then the following statements hold:*

(i) If $(v_h(x))$ converges for every $x \in J$, J dense in I, then there exists a

The following lemma is essential for the stability theorem and uses the Arzela–Ascoli theorem as main tool. The proof is very similar to the one of Lemma 5.10 in [14] but instead of central differences we consider here the difference approximation (2.5), (2.6) using the upwind scheme.

Lemma 3.2. *Let the sequence* $(L_h v_h)_{h\in\Lambda}$ *be compact and* $\big(\|v_h\|_{2,\infty,h}\big)_{h\in\Lambda}$ *be bounded. Then the following statements hold:*

(i) $(v_h)_{h\in\Lambda}$, $(D_h^{\pm} v_h)_{h\in\Lambda}$ *and* $(D_h^2 v_h)_{h\in\Lambda}$ *are compact in* $C(I)$.

(ii) For all subsequences $\Lambda' \subset \Lambda$ *there exists a subsequence* $\Lambda'' \subset \Lambda'$ *and a function* $v \in C^2(I)$ *such that*

$$\|v_h - v^h\|_{0,\infty,h} + \|D_h^+ v_h - (v')^h\|_{0,\infty,h} + \|D_h^2 v_h - (v'')^h\|_{0,\infty,h} \to 0\ (h \to 0, h \in \Lambda'').$$

Proof. (i) Let $\|v_h\|_{2,\infty,h} \leqslant 1, h \in \Lambda$, without restriction of generality. Since v_h is considered to be piecewise differentiable with

$$|v_h'(x)| \leqslant \max_{x\in I_h^0} |D_h^+ v_h(x)| \leqslant 1, \quad h \in \Lambda$$

one obtains

$$|v_h(x) - v_h(y)| \leqslant |x-y| \max_{s\in I_h^0} |D_h^+ v_h(s)| \leqslant |x-y|, \quad x,y \in I, \quad h \in \Lambda.$$

Therefore, $(v_h)_{h\in\Lambda}$ is equicontinuous and uniformly bounded and, according to the Arzela–Ascoli theorem, it is compact. Similarly, the same argument for $z_h := D_h^+ v_h$ and $D_h^+ z_h(x) = D_h^2 v_h(x+h)$ yields

$$|z_h(x) - z_h(y)| \leqslant |x-y| \max_{s\in I_h'} |D_h^+ z_h(s)| \leqslant |x-y|, \quad x,y \in I, \quad h \in \Lambda.$$

Hence, $(z_h)_{h\in\Lambda}$ is also compact. Analogously, one sees that $(D_h^- v_h)_{h\in\Lambda}$ is com-

(ii) Summarizing (i), for all $\Lambda' \subset \Lambda$ there exist a $\Lambda'' \subset \Lambda'$ and functions $v, w, z \in C(I)$ such that

$$\|v_h - v\|_{0,\infty} \to 0, \quad \|D_h^+ v_h - w\|_{0,\infty} \to 0$$

and

$$\|D_h^2 v_h - z\|_{0,\infty} \to 0 \; (h \to 0, h \in \Lambda'').$$

Because one has

$$v_h(x) = v_h(a) + \int_a^x v_h'(s)\,\mathrm{d}s = v_h(a) + h \sum_{\substack{a \leqslant y < x \\ y \in I_h^0}} D_h^+ v_h(y)$$

in the limit $h \to 0$ one obtains (see, e.g., [4], 104.4)

$$v(x) = v(a) + \int_a^x w(s)\,\mathrm{d}s.$$

This shows that $v'(x) = w(x)$. Analogously, one sees that $w' = z$ which completes the proof of this lemma. □

Let us recall that a sequence $(g_h)_{h \in \Lambda}$ of functions in $C(I)$ is *compact* if, for every subsequence $\Lambda' \subset \Lambda$ there exist a subsequence $\Lambda'' \subset \Lambda'$ and a function $g \in C(I)$ such that $g_h \to g$ ($h \to 0$, $h \in \Lambda''$) uniformly.

We are now able to prove that stability of the upwind scheme for general boundary conditions of the form (2.10).

Theorem 3.1. *Let us assume that the homogeneous boundary value problem* (2.1), (2.2) *has only the trivial solution. Then the difference approximations* $\widehat{L}_h$*,* $h \in \Lambda$*, defined in* (2.11) *fulfill the estimate*

$$\|v_h\|_{2,\infty,h} \leqslant \gamma\big(|\ell_{0,h}(v_h)| + |\ell_{1,h}(v_h)| + \|L_h v_h\|_{0,\infty,h}\big), \quad v_h \in E_h, \quad h \in \Lambda_0 \quad (3.1)$$

where $\Lambda_0 = \{h \in \Lambda \,|\, 0 < h \leqslant h_0\}$ *and* $h_0 > 0$ *and* $\gamma > 0$ *are independent of* h

We choose $\gamma = n$, $n = 1, 2, \ldots$, and set $h = h_1, h_2, \ldots$. Then for every $n \in \mathbb{N}$, $n \geqslant 2$, one can find $h^{(n)} \in \Lambda$, $h^{(n)} < h_{n-1}$, and $v_h^{(n)} \in E_h$, $h = h^{(n)}$, such that

$$\|v_h^{(n)}\|_{2,\infty,h} > n\big(|\ell_{0,h}(v_h^{(n)})| + |\ell_{1,h}(v_h^{(n)})| + \|L_h v_h^{(n)}\|_{0,\infty,h}\big), \quad h = h^{(n)}.$$

We obtain a subsequence $\Lambda' = \{h^{(1)}, h^{(2)}, \ldots\}$ of Λ and elements

$$z_h = v_h^{(n)} / \|v_h^{(n)}\|_{2,\infty,h}, \quad h = h^{(n)} \in \Lambda'$$

with the properties

$$\|z_h\|_{2,\infty,h} = 1,\ |\ell_{0,h}(z_h)| + |\ell_{1,h}(z_h)| + \|L_h z_h\|_{0,\infty,h} < \frac{1}{n}, \quad h = h^{(n)}, \quad n \in \mathbb{N}.$$

Thus

$$L_h z_h \to 0 \text{ and } \ell_{i,h}(z_h) \to 0\ (h \to 0, h \in \Lambda'), \quad i = 0, 1.$$

In particular, $(L_h z_h)_{h \in \Lambda'}$ is compact and, because $\|z_h\|_{2,\infty,h}$, $h \in \Lambda'$, is bounded, Lemma 3.2 can be applied. We obtain a subsequence $\Lambda'' \subset \Lambda'$ and a $z \in C^2(I)$ such that

$$\|z_h - z^h\|_{0,\infty,h} + \|D_h^+ z_h - (z')^h\|_{0,\infty,h} + \|D_h^2 z_h - (z'')^h\|_{0,\infty,h} \to 0\ (h \to 0, h \in \Lambda'').$$

Since $z \in C^2(I)$, we know that $\|L_h z^h - (Lz)^h\|_{0,\infty,h} \to 0$ $(h \to 0,\ h \in \Lambda'')$, and the estimate (2.13) yields

$$\|L_h z_h - (Lz)^h\|_{0,\infty,h} \to 0\ (h \to 0,\ h \in \Lambda'').$$

Indeed, we have

$$\begin{aligned} \|L_h z_h - (Lz)^h\|_{0,\infty,h} &\leqslant \|L_h z_h - L_h z^h\|_{0,\infty,h} + \|L_h z^h - (Lz)^h\|_{0,\infty,h} \\ &\underset{(2.13)}{\leqslant} c_1 \|z_h - z^h\|_{2,\infty,h} + \|L_h z^h - (Lz)^h\|_{0,\infty,h} \end{aligned}$$

and

$$\|z_h - z^h\|_{2,\infty,h} \leqslant \|z_h - z^h\|_{0,\infty,h} + \|D_h^+(z_h - z^h)\|_{0,\infty,h} + \|D_h^2(z_h - z^h)\|_{0,\infty,h}$$

$$\leqslant \|z_h - z^h\|_{0,\infty,h} + \|D_h^+ z_h - (z')^h\|_{0,\infty,h} + \|(z')^h - D_h^+ z^h\|$$

for $i = 0, 1$ and because $|\ell_i(z) - \ell_{i,h}(z)| \to 0$ $(h \to 0)$ for $z \in C^2(I)$. By assumption, $z = 0$ must be the trivial solution of the homogeneous boundary value problem (2.1), (2.2) which contradicts $\|z_h\|_{2,\infty,h} = 1$ and

$$\big|\|z_h\|_{2,\infty,h} - \|z^h\|_{2,\infty,h}\big| \to 0 \quad (h \to 0,\ \ h \in \Lambda'').$$

This proves the stability theorem. □

As a consequence of the stability theorem the difference equations (2.4), (2.10) are uniquely solvable for every right-hand side $f_h, \eta_{0,h}, \eta_{1,h}$ and every $h \in \Lambda_0$. Moreover the uniform boundedness of the inverses $\widehat{L}_h^{-1} : F_h \to E_h$, $h \in \Lambda_0$, is assured. The estimate (3.1) is stronger than corresponding estimates obtained by the maximum principles because the stronger norm $\|\cdot\|_{2,\infty,h}$ — instead of $\|\cdot\|_{0,\infty,h}$ — appears on the left-hand side. However, the constant γ is not explicitly known.

As a further consequence of the stability theorem one obtains convergence in the usual way whenever $f_h \to f$, $\eta_{i,h} \to \eta_i$, $i = 0, 1$ $(h \to 0)$. Indeed, with u the solution of (2.1), (2.2), one inserts $u_h - u^h$ in the stability estimate (3.1) and on the right-hand side there essentially appear the truncation errors in the differential equation and the boundary conditions. These are known to converge to zero with an order $O(h)$.

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