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Regularization of a non-characteristic Cauchy problem for a parabolic equation

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Abstract. In this paper the non-characteristic Cauchy problem

$$\left.\begin{array}{ll} u_t - a(x)u_{xx} - b(x)u_x - c(x)u = 0 & x \in (0,l),\ t \in \mathbb{R} \\ u(0,t) = \varphi(t) & t \in \mathbb{R} \\ u_x(0,t) = 0 & t \in \mathbb{R} \end{array}\right\} \qquad (*)$$

is considered. The problem $(*)$ is well known to be severely ill-posed: a small perturbation in the Cauchy data may cause a dramatically large error in the solution. In this paper the following mollification method is suggested for this problem: if the Cauchy data are given inexactly then we mollify them by elements of well-posedness classes of the problem, namely by elements of an appropriate ∞-regular multiresolution approximation $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ which is generated by the father wavelet of Meyer. Within V_J the problem $(*)$ is well posed, and we can find a mollification parameter J depending on the noise level ε in the Cauchy data such that the error estimation between the exact solution and the mollified solution is of Hölder type. The method can be numerically implemented using fundamental results by Beylkin, Coifman and Rokhlin on representing (pseudo)differential operators in wavelet bases. A stable marching difference scheme based on this method is suggested. Several numerical examples are given.

1. Introduction

In this paper we consider the following non-characteristic Cauchy problem (see [10]):

$$\left.\begin{array}{ll} u_t - a(x)u_{xx} - b(x)u_x - c(x)u = 0 & x \in (0,l),\ t \in \mathbb{R} \\ u(0,t) = \varphi(t) & t \in \mathbb{R} \\ u_x(0,t) = 0 & t \in \mathbb{R}. \end{array}\right\} \qquad (1.1)$$

Here $l > 0$ and a, b, c are given functions such that for some $\lambda, A > 0$

$$\lambda \leqslant a(x) \leqslant A \qquad x \in [0,l] \qquad (1.2)$$

$$a \in W^{2,\infty}[0,l] \qquad b \in W^{1,\infty}[0,l] \qquad c \in L^\infty[0,l] \qquad c(x) \leqslant 0 \qquad x \in [0,l]. \qquad (1.3)$$

As we consider the problem in $L^2(\mathbb{R})$ with respect to time, we assume

$$\varphi \in L^2(\mathbb{R}). \qquad (1.4)$$

The problem (1.1) is well known to be severely ill-posed ([14]). In [10] a stability estimate of Hölder continuous fashion for the solution of this problem has been proved. In this paper we shall apply the mollification method suggested in [7] to the problem (1.1). In fact, if

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the Cauchy data are given inexactly then we mollify them by elements of well-posedness classes of the problem, namely by elements of an appropriate ∞-regular multiresolution approximation $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ which is generated by the father wavelet of Meyer. Within V_J our problem is well posed, and we can find a mollification parameter J depending on the noise level ε in the Cauchy data such that the error estimation between the exact solution and the mollified solution is of Hölder type. The method can be numerically implemented using fundamental results of Beylkin *et al* [1] on representing (pseudo)differential operators in wavelet bases. A stable marching difference scheme based on this method is suggested. Several various numerical experiments show that our method is effective.

Another mollification method has been suggested by Manselli and Miller [11], and Murio and further developed by Murio and his students (see [13] and references therein). However, as has been noted in [7], the method of these authors cannot be generalized to Banach spaces as in [7, 8], and they could not find reasonable mollification parameters which yield error estimates of the Hölder continuous fashion. For related papers, the reader is referred to [2–4, 15].

The paper is organized as follows. In section 2 we shall give some auxiliary results on wavelets and the Cauchy problem in the frequency domain and in section 3 we describe our regularization method and provide some error estimates. Section 4 is devoted to a stable marching difference scheme and, finally, in section 5 several numerical examples are presented.

Throughout the paper $C, C_1, C_2, \ldots, C', C'', \ldots$ are generic positive constants.

2. Auxiliary results

2.1. *The Cauchy problem in the frequency space*

Let $\mathcal{S}$ be the Schwartz space, and $\mathcal{S}'$ be its dual (the space of tempered distributions). For a function $\varphi \in \mathcal{S}$ its Fourier transform $\hat{\varphi}$ is defined by

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}x$$

while the Fourier transform of a tempered distribution $f \in \mathcal{S}'$ is defined by

$$(\hat{f}, \varphi) = (f, \hat{\varphi}) \qquad \forall \varphi \in \mathcal{S}.$$

For $s \in \mathbb{R}$ the Sobolev space $H^s(\mathbb{R})$ consist of all tempered distributions $f \in \mathcal{S}'$ for which $\hat{f}(\xi)(1+|\xi|^2)^{s/2}$ is a function in $L^2(\mathbb{R})$. The norm on this space is given by

$$\|f\|_{H^s} := \left(\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s \, \mathrm{d}\xi \right)^{1/2}.$$

Now we are ready to examine the Cauchy problem in frequency space. Set

$$A(x) := \int_0^x a(s)^{-1/2}\, \mathrm{d}s \qquad L := A(l)$$

and let $v(x, \xi)$ be a solution of the Cauchy problem

$$\left.\begin{array}{ll} a(x)v_{xx}(x,\xi) + b(x)v_x(x,\xi) + c(x)v(x,\xi) - \mathrm{i}\xi v(x,\xi) = 0 & x \in [0,l],\ \xi \in \mathbb{R} \\ v(0,\xi) = 1 \qquad \xi \in \mathbb{R} & \\ v_x(0,\xi) = 0 \qquad \xi \in \mathbb{R}. & \end{array}\right\} \qquad (2.1)$$

The following lemma can be found in [10].

Lemma 2.1. There exists an unique solution v of (2.1) such that

(i) $v(\cdot,\zeta) \in W^{2,\infty}[0,l]$ for every $\zeta \in \mathbb{C}$,

(ii) $v(x,\cdot)$ is an entire function for every $x \in [0,l]$,

(iii) $v(x,\zeta) \neq 0$ for every $\zeta \in \mathbb{C}$ with $\operatorname{Im}\zeta \leqslant 0$,

(iv) there exist constants C_i depending only on λ, A, a, b, c such that for $x \in [0,l]$, $\xi \in \mathbb{R}$

$$
\begin{aligned}
&|v(x,\xi)| \leqslant C_1 \exp\left(\sqrt{|\xi|/2}A(x)\right)\\
&|v(l,\xi)| \geqslant C_2 \exp\left(\sqrt{|\xi|/2}L\right)\\
&|v_x(x,\xi)| \leqslant C_3\sqrt{|\xi|}\exp\left(\sqrt{|\xi|/2}A(x)\right)\\
&|v_{xx}(x,\xi)| \leqslant C_4|\xi|\exp\left(\sqrt{|\xi|/2}A(x)\right).
\end{aligned}
$$

Furthermore it has been proved in [8] that if a solution of (1.1) exists in $L^2(\mathbb{R})$, then φ must be infinitely differentiable and

$$\|D^k\varphi\|_2 \leqslant C(2k)!s^{2k} \qquad \forall k \in \mathbb{N} \tag{2.2}$$

where $D = \mathrm{d}/\mathrm{d}t$ and C and s are some constants depending on the coefficients of the differential equation in (1.1) and $\|u(l,\cdot)\|_2$. Conversely, if (2.2) holds, then there exists a local solution u of (1.1) and $u(x,t)$ can be represented in the form

$$u(x,t) = v(x,D)\varphi(t) \qquad 0 \leqslant x \leqslant l$$

where $v(x,D)$ denotes the pseudodifferential operator associated with the analytic symbol $v(x,\xi)$. For simplicity, we will use the abbreviation

$$T_x := v(x,D) \qquad 0 \leqslant x \leqslant l \tag{2.3}$$

throughout this paper. It is clear that

$$\hat{u}(x,\xi) = v(x,\xi)\hat{\varphi}(\xi).$$

From this equation and lemma 2.1 it follows

$$\hat{u}(x,\xi) = \frac{v(x,\xi)}{v(l,\xi)}v(l,\xi)\hat{\varphi}(\xi) = \frac{v(x,\xi)}{v(l,\xi)}\hat{u}(l,\xi). \tag{2.4}$$

2.2. Properties of wavelets

In this section we recall some basic properties of wavelets which are derived from [9].

Throughout the paper let ϕ be an orthonormal scaling function with its corresponding wavelet ψ such that $\hat{\phi}$ has compact support. For simplicity we assume ϕ to be the Meyer scaling function defined by its Fourier transform

$$\hat{\phi}(\xi) = \begin{cases} (2\pi)^{-1/2} & |\xi| \leqslant \dfrac{2\pi}{3} \\ (2\pi)^{-1/2}\cos\left[\dfrac{\pi}{2}\nu\left(\dfrac{3}{4\pi}|\xi| - 1\right)\right] & \dfrac{2\pi}{3} \leqslant |\xi| \leqslant \dfrac{4\pi}{3} \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

where ν is a C^k function $(0 \leqslant k \leqslant \infty)$ with

$$\nu(x) = \begin{cases} 0 & x \leqslant 0 \\ 1 & x \geqslant 1 \end{cases}$$

and $\nu(x) + \nu(1-x) = 1$. Then $\hat{\phi}$ is a C^k function and the corresponding wavelet ψ is given by

$$\hat{\psi}(\xi) = \begin{cases} (2\pi)^{-1/2} e^{i\xi/2} \sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right] & \frac{2\pi}{3} \leqslant |\xi| \leqslant \frac{4\pi}{3} \\ (2\pi)^{-1/2} e^{i\xi/2} \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\xi| - 1\right)\right] & \frac{4\pi}{3} \leqslant |\xi| \leqslant \frac{8\pi}{3} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

We write down the supports of $\hat{\phi}$ and $\hat{\psi}$, since these will be essential later,

$$\operatorname{supp} \hat{\phi} = \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right] \tag{2.7}$$

$$\operatorname{supp} \hat{\psi} = \left[-\frac{8\pi}{3}, -\frac{2\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \frac{8\pi}{3}\right]. \tag{2.8}$$

From [5, 12] we see that the functions

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k) \qquad j, k \in \mathbb{Z}$$

constitute an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$. Furthermore the $\psi_{j,k}$ are entire functions, since their Fourier transforms have compact support. Consequently, the multiresolution analysis (MRA) $\{V_j\}_{j\in\mathbb{Z}}$ of Meyer generated by

$$V_j = \overline{\langle\{\phi_{j,k} : k \in \mathbb{Z}\}\rangle} \qquad \phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \qquad j, k \in \mathbb{Z}$$

is m-regular for all $m \in \mathbb{N}$ (see [9]). We will call such an MRA ∞-regular. The orthogonal projection on the space V_J is given by

$$P_J f := \sum_k (f, \phi_{J,k})\phi_{J,k}$$

while

$$Q_J f := \sum_k (f, \psi_{J,k})\psi_{J,k}$$

denotes the orthogonal projection on the wavelet space W_J with $V_{J+1} = V_J \oplus W_J$.

Remark on notation. Here and hereafter, $\sum_{k\in\mathbb{Z}}$ is denoted by $\sum_k$.

From [9] we have a Jackson-type inequality for the projection P_J on V_J.

Theorem 2.1. Let $\{V_j\}_{j\in\mathbb{Z}}$ be an m-regular MRA, and $r, s \in \mathbb{R}$ be such that $-m < r < s < m$. Then for any function $f \in H^s(\mathbb{R})$ and $J \in \mathbb{N}$ the following inequality holds:

$$\|f - P_J f\|_{H^r} \leqslant C2^{-J(s-r)}\|\|f\|_{H^s}.$$

Moreover, from [9] we see that some Bernstein-type inequalities hold for the differential operators D^k, $k \in \mathbb{N}$.

Theorem 2.2. Let $\{V_j\}_{j\in\mathbb{Z}}$ be Meyer's MRA and suppose $J \in \mathbb{N}$, $r \in \mathbb{R}$. Then for all $f \in V_J$ we have

$$\|D^k f\|_{H^r} \leqslant C2^{(J-1)k}\|f\|_{H^r} \qquad k \in \mathbb{N}.$$

Proof. From [9, 12], we have

$$\begin{aligned} &\|D^k f\|_{H^r} \leqslant C2^{jk}\|f\|_{H^r} && k\in\mathbb{N},\ f\in W_j,\ j\geqslant 0 \\ &\|D^k f\|_{H^r} \leqslant C\|f\|_{H^r} && k\in\mathbb{N},\ f\in V_0. \end{aligned} \tag{2.9}$$

First, for $f \in V_J,\ J \geqslant 0$ it follows

$$\begin{aligned} \|D^k f\|_{H^r} = \|D^k P_J f\|_{H^r} &= \left\| D^k P_0 f + \sum_{j=0}^{J-1} D^k Q_j f \right\|_{H^r} \\ &\leqslant C_0\|P_0 f\|_{H^r} + C_1 \sum_{j=0}^{J-1} 2^{jk}\|Q_j f\|_{H^r} \leqslant C2^{Jk}\|f\|_{H^r}. \end{aligned} \tag{2.10}$$

Since $P_J = P_{J-1} + Q_{J-1},\ J \in \mathbb{N}$, we see from (2.9) and (2.10), that

$$\|D^k f\|_{H^r} = \|D^k P_J f\|_{H^r} \leqslant \|D^k P_{J-1} f\|_{H^r} + \|D^k Q_{J-1} f\|_{H^r} \leqslant C2^{(J-1)k}\|f\|_{H^r}.$$

□

From this theorem a Bernstein-type inequality for the operator T_x (see (2.3)) follows immediately.

Theorem 2.3. Let $\{V_j\}_{j\in\mathbb{Z}}$ be Meyer's MRA and suppose $J \in \mathbb{N}$ and $r \in \mathbb{R}$, $0 \leqslant x \leqslant l$. Then for all $f \in V_J$ we have

$$\|T_x f\|_{H^r} \leqslant C\exp(2^{(J-1)/2}A(x))\|f\|_{H^r}. \tag{2.11}$$

Proof. According to lemma 2.1, the symbol $v(x,\cdot)$ of T_x satisfies

$$|v(x,\xi)| \leqslant C\exp\left(\sqrt{\frac{|\xi|}{2}}A(x)\right) \leqslant C\left|\cosh\left(\sqrt{\mathrm{i}\xi}A(x)\right)\right|. \tag{2.12}$$

Thus, theorem 2.2 shows that for $f \in V_J$

$$\begin{aligned} \|T_x f\|_{H^r} = \|T_x P_J f\|_{H^r} &= \left(\int_{-\infty}^{\infty} |v(x,\xi)\widehat{P_J f}(\xi)|^2(1+|\xi|^2)^r\,\mathrm{d}\xi\right)^{1/2} \\ &\leqslant C''\left(\int_{-\infty}^{\infty}\left|\cosh\left(\sqrt{\mathrm{i}\xi}A(x)\right)\widehat{P_J f}(\xi)\right|^2(1+|\xi|^2)^r\,\mathrm{d}\xi\right)^{1/2} \\ &= C''\left(\int_{-\infty}^{\infty}\left|\sum_{k=0}^{\infty}\frac{(A(x))^{2k}}{(2k)!}(\mathrm{i}\xi)^k\widehat{P_J f}(\xi)\right|^2(1+|\xi|^2)^r\,\mathrm{d}\xi\right)^{1/2} \\ &\leqslant C''\sum_{k=0}^{\infty}\frac{(A(x))^{2k}}{(2k)!}\left(\int_{-\infty}^{\infty}|(\mathrm{i}\xi)^k\widehat{P_J f}(\xi)|^2(1+|\xi|^2)^r\,\mathrm{d}\xi\right)^{1/2} \\ &= C''\sum_{k=0}^{\infty}\frac{(A(x))^{2k}}{(2k)!}\|D^k P_J f\|_{H^r} \\ &\leqslant C'\sum_{k=0}^{\infty}\frac{(A(x))^{2k}}{(2k)!}2^{(J-1)k}\|f\|_{H^r} \\ &= C'\cosh(2^{(J-1)/2}A(x))\|f\|_{H^r} \leqslant C\exp(2^{(J-1)/2}A(x))\|f\|_{H^r}. \end{aligned}$$

□

3. Regularization

Now we suppose that the function φ is given inexactly by φ_ε with $\|\varphi - \varphi_\varepsilon\|_{H^r} \leqslant \varepsilon$ for some $r \leqslant 0$. Since φ_ε belongs, in general, to $L^2(\mathbb{R})$, r should not be positive. We are interested in approximating the solution $u(t, x)$ of (1.1) from φ_ε in a stable way. For that it is necessary to require a condition on the solution of (1.1) at $x = l$. We assume that $f(t) := u(l, t)$ exists and belongs to $H^s(\mathbb{R})$ for some $s \in \mathbb{R}$, $s > r$, i.e.

$$\hat{f}(\xi) = v(l, \xi)\hat{\varphi}(\xi) \qquad \text{and} \qquad f \in H^s(\mathbb{R}).$$

From (2.4) it follows

$$\hat{u}(x, \xi) = v(x, \xi)\hat{\varphi}(\xi) = \frac{v(x, \xi)}{v(l, \xi)}\hat{f}(\xi).$$

Since the Cauchy data are given inexactly by φ_ε, we need a stable algorithm to approximate the solution of (1.1). Our method is as follows. We consider the operator

$$T_{x,J} := P_J T_x P_J$$

and show that it approximates T_x in a stable way for an appropriate choice of $J \in \mathbb{N}$ depending on ε. A discussion of the numerical implementation of this procedure will follow in section 5. At first, we see from the Bernstein-type inequality (2.11) that the problem of calculating $T_{x,J}\varphi_\varepsilon$ is well posed. Now we will prove some stability estimates for our method. We have

$$\|T_x\varphi - T_{x,J}\varphi_\varepsilon\|_{H^r} \leqslant \|T_x\varphi - T_{x,J}\varphi\|_{H^r} + \|T_{x,J}(\varphi - \varphi_\varepsilon)\|_{H^r}. \tag{3.1}$$

The second term of the right-hand side satisfies

$$\|T_{x,J}(\varphi - \varphi_\varepsilon)\|_{H^r} \leqslant \|T_x P_J(\varphi - \varphi_\varepsilon)\|_{H^r} \leqslant C\exp(2^{(J-1)/2}A(x))\varepsilon \tag{3.2}$$

as one can see from theorem 2.3. For the first term in (3.1), we have (see also [9])

$$\|T_x\varphi - T_{x,J}\varphi\|_{H^r} \leqslant \|(I - P_J)T_x\varphi\|_{H^r} + \|T_x(I - P_J)\varphi\|_{H^r}. \tag{3.3}$$

For the estimation of the terms on the right-hand side, it is essential that $\hat{\phi}$ and $\hat{\psi}$ have compact support. We introduce the operator M_J which is defined by the equation

$$\widehat{M_J g} := (1 - \chi_J)\hat{g} \qquad J \in \mathbb{N}$$

where χ_J denotes the characteristic function of the interval $I_J := [-\frac{2\pi}{3}2^J, \frac{2\pi}{3}2^J]$. Since

$$\hat{\psi}_{j,k}(\xi) = 2^{-j/2}\mathrm{e}^{-\mathrm{i}k2^{-j}\xi}\hat{\psi}(2^{-j}\xi) \tag{3.4}$$

we have $\hat{\psi}_{j,k}(\xi) = 0$, $\xi \in I_J$ for $j \geqslant J$ according to (2.8). This yields

$$(g, \psi_{j,k}) = (\hat{g}, \hat{\psi}_{j,k}) = ((1 - \chi_J)\hat{g}, \hat{\psi}_{j,k}) = (M_J g, \psi_{j,k}) \qquad j \geqslant J,\ k \in \mathbb{Z}$$

for arbitrary $g \in H^r(\mathbb{R})$, and we conclude

$$(I - P_J)g = \sum_{j \geqslant J}\sum_k (g, \psi_{j,k})\psi_{j,k} = \sum_{j \geqslant J}\sum_k (M_J g, \psi_{j,k})\psi_{j,k} = (I - P_J)M_J g.$$

Hence, we have

$$(I - P_J) = (I - P_J)M_J \qquad J \in \mathbb{N} \tag{3.5}$$

and, by an analogous argument,

$$Q_J = Q_J M_J \qquad J \in \mathbb{N}. \tag{3.6}$$

Furthermore,

$$\widehat{P_J\varphi}(\xi) = \sum_k (\varphi, \phi_{J,k}) 2^{-J/2} \mathrm{e}^{-\mathrm{i}k2^{-J}\xi} \hat{\phi}(2^{-J}\xi)$$

and, according to (2.7),

$$\widehat{P_J\varphi}(\xi) = 0 \qquad |\xi| \geqslant \tfrac{4}{3}\pi 2^J.$$

Thus,

$$\|T_x(I - P_J)\varphi\|_{H^r} \leqslant \left(\int_{|\xi|\geqslant\frac{4\pi}{3}2^J} |v(x,\xi)\hat{\varphi}(\xi)|^2(1+|\xi|^2)^r \,\mathrm{d}\xi\right)^{1/2}$$
$$+\left(\int_{|\xi|<\frac{4\pi}{3}2^J} |v(x,\xi)((I-P_J)\varphi)^\wedge(\xi)|^2(1+|\xi|^2)^r \,\mathrm{d}\xi\right)^{1/2}. \tag{3.7}$$

The first integral on the right-hand side of (3.7) can be estimated by

$$\left(\int_{|\xi|\geqslant\frac{4\pi}{3}2^J} |v(x,\xi)\hat{\varphi}(\xi)|^2(1+|\xi|^2)^r \,\mathrm{d}\xi\right)^{1/2} = \left(\int_{|\xi|\geqslant\frac{4\pi}{3}2^J} \left|\frac{v(x,\xi)}{v(l,\xi)}\hat{f}(\xi)\right|^2 (1+|\xi|^2)^r \,\mathrm{d}\xi\right)^{1/2}$$
$$\leqslant \sup_{|\xi|\geqslant\frac{4\pi}{3}2^J} \left|\frac{v(x,\xi)}{v(l,\xi)}\frac{1}{(1+|\xi|^2)^{(s-r)/2}}\right| \left(\int_{|\xi|\geqslant\frac{4\pi}{3}2^J} |\hat{f}(\xi)|^2(1+|\xi|^2)^s \,\mathrm{d}\xi\right)^{1/2}$$
$$\leqslant C2^{-J(s-r)} \exp\left(\sqrt{\frac{2\pi}{3}}2^{J/2}(A(x)-L)\right)\|f\|_{H^s}$$
$$\leqslant C2^{-J(s-r)} \exp(2^{J/2}(A(x)-L))\|f\|_{H^s}.$$

From (2.8) and (3.4) it follows, that for $j > J$

$$\widehat{Q_j\varphi}(\xi) = 0 \qquad |\xi| < \tfrac{4}{3}\pi 2^J.$$

Since

$$(I - P_J)\varphi = \sum_{j\geqslant J} Q_j\varphi$$

we see that

$$((I-P_J)\varphi)^\wedge(\xi) = \widehat{Q_J\varphi}(\xi) \qquad |\xi| < \tfrac{4}{3}\pi 2^J.$$

Hence, the second integral on the right-hand side of (3.7) satisfies

$$\left(\int_{|\xi|<\frac{4\pi}{3}2^J} |v(x,\xi)((I-P_J)\varphi)^\wedge(\xi)|^2(1+|\xi|^2)^r \,\mathrm{d}\xi\right)^{1/2}$$
$$= \left(\int_{|\xi|<\frac{4\pi}{3}2^J} |v(x,\xi)\widehat{Q_J\varphi}(\xi)|^2(1+|\xi|^2)^r \,\mathrm{d}\xi\right)^{1/2}$$
$$\leqslant \|T_x Q_J\varphi\|_{H^r} \leqslant C\exp(2^{J/2}A(x))\|Q_J\varphi\|_{H^r} \tag{3.8}$$

since $Q_J\varphi \in V_{J+1}$. Furthermore,

$$\|Q_J\varphi\|_{H^r} = \|Q_J M_J\varphi\|_{H^r} \leqslant \|M_J\varphi\|_{H^r}$$
$$= \left(\int_{|\xi|\geqslant\frac{2\pi}{3}2^J} |\hat{\varphi}(\xi)|^2(1+|\xi|^2)^r \,\mathrm{d}\xi\right)^{1/2}$$
$$= \left(\int_{|\xi|\geqslant\frac{2\pi}{3}2^J} \left|\frac{\hat{f}(\xi)}{v(l,\xi)}\right|^2 (1+|\xi|^2)^r \,\mathrm{d}\xi\right)^{1/2}$$

$$\leqslant C2^{-J(s-r)} \exp\left(-\sqrt{\frac{\pi}{3}}2^{J/2}L\right) \|f\|_{H^s}$$
$$\leqslant C2^{-J(s-r)} \exp\left(-2^{J/2}L\right) \|f\|_{H^s}.$$

Together with (3.7) and (3.8) this leads to

$$\|T_x(I-P_J)\varphi\|_{H^r} \leqslant C2^{-J(s-r)} \exp(2^{J/2}(A(x)-L))\|f\|_{H^s}.$$

Now, we estimate $\|(I-P_J)T_x\varphi\|_{H^r}$. By (3.5) we have

$$\begin{aligned}
\|(I-P_J)T_x\varphi\|_{H^r} &= \|(I-P_J)M_JT_x\varphi\|_{H^r} \leqslant C'2^{-J(s-r)}\|M_JT_x\varphi\|_{H^s} \\
&= C'2^{-J(s-r)}\left(\int_{|\xi|\geqslant\frac{2\pi}{3}2^J} |v(x,\xi)\hat{\varphi}(\xi)|^2(1+|\xi|^2)^s\,\mathrm{d}\xi\right)^{1/2} \\
&= C'2^{-J(s-r)}\left(\int_{|\xi|\geqslant\frac{2\pi}{3}2^J} \left|\frac{v(x,\xi)}{v(l,\xi)}\hat{f}(\xi)\right|^2(1+|\xi|^2)^s\,\mathrm{d}\xi\right)^{1/2} \\
&\leqslant C'2^{-J(s-r)} \sup_{|\xi|\geqslant\frac{2\pi}{3}2^J}\left|\frac{v(x,\xi)}{v(l,\xi)}\right| \|f\|_{H^s} \\
&\leqslant C2^{-J(s-r)} \exp\left(\sqrt{\frac{\pi}{3}}2^{J/2}(A(x)-L)\right)\|f\|_{H^s} \\
&\leqslant C2^{-J(s-r)} \exp(2^{J/2}(A(x)-L))\|f\|_{H^s}.
\end{aligned}$$

Putting all together, we finally arrive at

$$\|T_x\varphi - T_{x,J}\varphi_\varepsilon\|_{H^r} \leqslant C_1\exp(2^{(J-1)/2}A(x))\varepsilon + C_2 2^{-J(s-r)}\exp(2^{J/2}(A(x)-L))\|f\|_{H^s}. \quad (3.9)$$

We want to obtain some stability estimates of Hölder-type for our method using (3.9). Choosing

$$J^* := \left[\log_2\left(2\left(\frac{1}{L}\ln\left(\frac{1}{\varepsilon}\left(\ln\frac{1}{\varepsilon}\right)^{-2(s-r)}\right)\right)^2\right)\right]$$

(where $[a]$ denotes the largest integer less than or equal to $a \in \mathbb{R}$), we get

$$\begin{aligned}
\exp(2^{(J^*-1)/2}A(x))\varepsilon &\leqslant \exp\left(\frac{1}{L}\ln\left(\frac{1}{\varepsilon}\left(\ln\frac{1}{\varepsilon}\right)^{-2(s-r)}\right)A(x)\right)\varepsilon \\
&= \varepsilon^{1-A(x)/L}\left(\ln\frac{1}{\varepsilon}\right)^{-2(s-r)A(x)/L}
\end{aligned}$$

and

$$\exp(2^{J^*/2}(A(x)-L)) \leqslant \varepsilon^{1-A(x)/L}\left(\ln\frac{1}{\varepsilon}\right)^{-2(s-r)(A(x)/L-1)}.$$

Furthermore,

$$\begin{aligned}
2^{-J^*(s-r)} &\leqslant \left(\left(\frac{1}{L}\ln\left(\frac{1}{\varepsilon}\left(\ln\frac{1}{\varepsilon}\right)^{-2(s-r)}\right)\right)^2\right)^{-(s-r)} \\
&= \left(\frac{L}{\ln\frac{1}{\varepsilon}+\ln((\ln\frac{1}{\varepsilon})^{-2(s-r)})}\right)^{2(s-r)}.
\end{aligned}$$

If C_1, C_2 and $\|f\|_{H^s}$ are known, with

$$J^{**} := \left[\log_2\left(2\left(\frac{1}{L}\ln\left(\frac{C_2\|f\|_{H^s}}{C_1\varepsilon}\left(\ln\frac{C_1\|f\|_{H^s}}{C_2\varepsilon}\right)^{-2(s-r)}\right)\right)^2\right)\right]$$

we get similar estimates. Now, we summarize our results.

Theorem 3.1. For every fixed $J \in \mathbb{N}$ the problem (1.1) with the Cauchy data φ in V_J is well posed. Suppose that the solution f of (1.1) exists at $x = l$ and belongs to $H^s(\mathbb{R})$ for some $s \in \mathbb{R}$, i.e. $f = T_l\varphi \in H^s(\mathbb{R})$. Suppose further that φ is given approximately by φ_ε with $\|\varphi - \varphi_\varepsilon\|_{H^r} \leqslant \varepsilon$, $r \leqslant \min\{0, s\}$, then the problem of calculating $T_{x,J}\varphi_\varepsilon$ is stable. Furthermore, with

$$J^* := \left[\log_2\left(2\left(\frac{1}{L}\ln\left(\frac{1}{\varepsilon}\left(\ln\frac{1}{\varepsilon}\right)^{-2(s-r)}\right)\right)^2\right)\right]$$

the following stability estimate holds

$$\|T_x\varphi - T_{x,J^*}\varphi_\varepsilon\|_{H^r} \leqslant \left(C_1 + C_2\|f\|_{H^s}\left(\frac{L\ln\frac{1}{\varepsilon}}{\ln\frac{1}{\varepsilon} + \ln((\ln\frac{1}{\varepsilon})^{-2(s-r)})}\right)^{2(s-r)}\right)$$
$$\times\varepsilon^{1-A(x)/L}\left(\ln\frac{1}{\varepsilon}\right)^{-2(s-r)A(x)/L}.$$

With

$$J^{**} := \left[\log_2\left(2\left(\frac{1}{L}\ln\left(\frac{C_2\|f\|_{H^s}}{C_1\varepsilon}\left(\ln\frac{C_1\|f\|_{H^s}}{C_2\varepsilon}\right)^{-2(s-r)}\right)\right)^2\right)\right]$$

we have

$$\|T_x\varphi - T_{x,J^{**}}\varphi_\varepsilon\|_{H^r} \leqslant \left(1 + \left(\frac{L\ln(C_2\|f\|_{H^s}/C_1\varepsilon)}{\ln(C_2\|f\|_{H^s}/C_1\varepsilon) + \ln((\ln(C_2\|f\|_{H^s}/C_1\varepsilon))^{-2(s-r)})}\right)^{2(s-r)}\right)$$
$$\times(C_2\|f\|_{H^s})^{A(x)/L}(C_1\varepsilon)^{1-A(x)/L}\left(\ln\frac{C_2\|f\|_{H^s}}{C_1\varepsilon}\right)^{-2(s-r)A(x)/L}.$$

4. A stable marching difference scheme

As in [8] we consider the system

$$\left.\begin{aligned}
&U_x = W \qquad x \in (0,l),\ t \in \mathbb{R}\\
&a(x)W_x + b(x)W + c(x)U = U_t \qquad x \in (0,l),\ t \in \mathbb{R}\\
&U(0,t) = \Psi(t) \qquad t \in \mathbb{R}\\
&W(0,t) = 0 \qquad t \in \mathbb{R}.
\end{aligned}\right\} \tag{4.1}$$

Setting $J := J^*$ (J^{**}) and $\Psi(t) := P_J\varphi_\varepsilon(t)$, this system is obviously the mollified version of problem (1.1) with noisy Cauchy data $\varphi_\varepsilon(t)$. Hence, the solution of (4.1) is given by

$$U(x,t) = T_x P_J \varphi_\varepsilon(t)$$
$$W(x,t) = U_x(x,t).$$

From the proof of theorem 3.1 we easily obtain that $\|T_x\varphi - T_xP_J\varphi_\varepsilon\|_{H^r}$ satisfies the same stability estimate as $\|T_x\varphi - T_{x,J}\varphi_\varepsilon\|_{H^r}$. This shows that it is enough, to solve the mollified problem (4.1), to get an approximate solution of (1.1). We are now interested in an unconditionally stable finite difference scheme, that approximates the solution of (4.1). Consider a uniform grid on the $[0,l]\times\mathbb{R}$ plane:

$$\{x_n = nh,\ t_m = m\tau : n = 0, 1, \dots, N,\ h = l/N,\ m \in \mathbb{Z}\}.$$

For a function $\mathcal{W}$ defined on $[0,l]\times\mathbb{R}$ we set

$$\mathcal{W}_m^n := \mathcal{W}(x_n, t_m).$$

Following Carasso [2] and Murio [13], (4.1) is discretized by

$$\left.\begin{aligned}
&\frac{U_m^{n+1}-U_m^n}{h} = W_m^{n+1} \qquad n = 0,1,\dots,N,\ m\in\mathbb{Z}\\
&a^n\frac{W_m^{n+1}-W_m^n}{h} + b^n W_m^n + c^n U_m^n = \frac{U_{m+1}^n - U_{m-1}^n}{2\tau} \qquad n = 0,1,\dots,N,\ m\in\mathbb{Z}\\
&U_m^0 = \Psi_m \qquad m\in\mathbb{Z}\\
&W_m^0 = 0 \qquad m\in\mathbb{Z}.
\end{aligned}\right\} \tag{4.2}$$

It is clear that this difference scheme has a local truncation error which behaves like $\mathrm{O}(h+\tau^2)$. Taking the discrete Fourier transform (see [8, 13]) in (4.2), we get

$$\left.\begin{aligned}
&\frac{\widetilde{U}^{n+1}(\xi)-\widetilde{U}^n(\xi)}{h} = \widetilde{W}^{n+1}(\xi) \qquad n = 0,1,\dots,N\\
&a^n\frac{\widetilde{W}^{n+1}(\xi)-\widetilde{W}^n(\xi)}{h} + b^n\widetilde{W}^n(\xi) + c^n\widetilde{U}^n(\xi) = \mathrm{i}\frac{\sin(\tau\xi)}{\tau}\widetilde{U}^n(\xi) \qquad n = 0,1,\dots,N\\
&\widetilde{U}^0(\xi) = \widetilde{\Psi}(\xi)\\
&\widetilde{W}^0(\xi) = 0.
\end{aligned}\right\} \tag{4.3}$$

Thus, for $n = 0, 1, \dots, N$

$$\begin{aligned}
\widetilde{W}^{n+1}(\xi) &= \left(-\frac{b^n}{a^n}h + 1\right)\widetilde{W}^n(\xi) + \frac{h}{a^n}\left(-c^n + \mathrm{i}\frac{\sin(\tau\xi)}{\tau}\right)\widetilde{U}^n(\xi)\\
\widetilde{U}^{n+1}(\xi) &= h\widetilde{W}^{n+1}(\xi) + \widetilde{U}^n(\xi)\\
&= \left(1 + \frac{h^2}{a^n}\left(-c^n + \mathrm{i}\frac{\sin(\tau\xi)}{\tau}\right)\right)\widetilde{U}^n(\xi) + h\left(-\frac{b^n}{a^n}h + 1\right)\widetilde{W}^n(\xi).
\end{aligned}$$

Let $|b(x)| \leqslant B$, $|c(x)| \leqslant C$ and $0 < \lambda \leqslant |a(x)|$. For $0 \leqslant |\xi| \leqslant \pi/\tau$, we have

$$\begin{aligned}
|\widetilde{W}^{n+1}(\xi)| &\leqslant \left(1 + \frac{B}{\lambda}h + \frac{C}{\lambda}h + \frac{h}{\lambda}\frac{\sin(\tau|\xi|)}{\tau}\right)\max\{|\widetilde{U}^n(\xi)|, |\widetilde{W}^n(\xi)|\}\\
|\widetilde{U}^{n+1}(\xi)| &\leqslant \left(1 + \frac{B}{\lambda}h^2 + \frac{C}{\lambda}h^2 + \frac{h^2}{\lambda}\frac{\sin(\tau|\xi|)}{\tau} + h\right)\max\{|\widetilde{U}^n(\xi)|, |\widetilde{W}^n(\xi)|\}.
\end{aligned}$$

Taking h sufficiently small and using $\sin(\tau|\xi|)/\tau \leqslant |\xi|$, we obtain

$$\begin{aligned}
\max\{|\widetilde{U}^n(\xi)|, |\widetilde{W}^n(\xi)|\} &\leqslant \left(1 + \frac{B+C+|\xi|}{\lambda}h + h\right)^n |\widetilde{\Psi}(\xi)|\\
&\leqslant \mathrm{e}^{nh(B+C+|\xi|+\lambda)/\lambda}|\widetilde{\Psi}(\xi)|\\
&\leqslant \mathrm{e}^{l(\lambda+B+C+|\xi|)/\lambda}|\widetilde{\Psi}(\xi)|.
\end{aligned}$$

Since the Fourier transform of $P_J\varphi$ has compact support in the interval $[-\nu, \nu] = [-\frac{4\pi}{3}2^J, \frac{4\pi}{3}2^J]$, taking $\tau \leqslant \pi/\nu$ (that is $\tau \leqslant \frac{3}{4}2^{-J}$), we have $\widetilde{\Psi} = \widehat{\Psi}$ by the Poisson summation formula. It follows that

$$\begin{aligned}\max\{\|U^n\|_{l^2}, \|W^n\|_{l^2}\} &= \max\{\|\widetilde{U}^n\|_{L^2(-\pi/\tau,\pi/\tau)}, \|\widetilde{W}^n\|_{L^2(-\pi/\tau,\pi/\tau)}\} \\ &\leqslant \|e^{l(\lambda+B+C+|\cdot|)/\lambda}|\widetilde{\Psi}|\|_{L^2(-\pi/\tau,\pi/\tau)} \\ &\leqslant e^{l(\lambda+B+C+\nu)/\lambda}\|\Psi\|_2. \end{aligned} \tag{4.4}$$

Hence, we conclude

Theorem 4.1. The difference scheme (4.2) approximates the problem (4.1) with a truncation error behaving like $O(h + \tau^2)$. Furthermore, if h is sufficiently small and $\tau \leqslant \frac{3}{4}2^{-J}$, then it is unconditionally stable and the estimate (4.4) is valid.

5. Numerical examples

5.1. *Implementation*

We want to discuss some numerical aspects in this section. If the function $v(x, \xi)$ can be found explicitly, one can directly use the method of section 3. For a function g, we have

$$T_{x,J}g = \sum_l \left(\sum_k (T_x\phi_{J,k}, \phi_{J,l})(g, \phi_{J,k}) \right) \phi_{J,l}$$

and

$$(T_x\phi_{J,k}, \phi_{J,l}) = \int_{|\xi|\leqslant\frac{4\pi}{3}} v(x, 2^J\xi)|\hat{\phi}(\xi)|^2 e^{i\xi(l-k)}\, d\xi.$$

Hence we have only to compute the numbers

$$r_{J,l} := \int_{|\xi|\leqslant\frac{4\pi}{3}} v(x, 2^J\xi)|\hat{\phi}(\xi)|^2 e^{i\xi l}\, d\xi$$

and convolve this sequence with the wavelet coefficients $(g, \phi_{J,k})$ of g, to get the wavelet coefficients of the function $T_{x,J}g$. The authors of [1] have proposed this method in the case of exactly given g. To approximate the solution of (1.1), we use it with $g = \varphi_\varepsilon$ and $J = J^*$.

If $v(x, \xi)$ is difficult to find, we only compute $P_{J^*}\varphi_\varepsilon$. Then the marching difference scheme (4.2) from section 4 is applied to solve (1.1).

5.2. *Examples*

We have placed some figures at the end of the paper which show that our method works well. In all examples the inverse heat conduction problem

$$u_{xx} = u_t \qquad u(t, 0) = \varphi(t) \qquad u_x(t, 0) = 0 \qquad t \in \mathbb{R} \qquad 0 \leqslant x \leqslant 1 \tag{5.1}$$

is considered. In this case the symbol v is given by

$$v(x, \xi) = \cosh\left(x\sqrt{i\xi}\right).$$

Since the Meyer wavelet decreases rapidly at infinity, in general, if we are interested in the solution over a finite interval of t, there is no need to care so much whether $\varphi \in L^2(\mathbb{R})$.

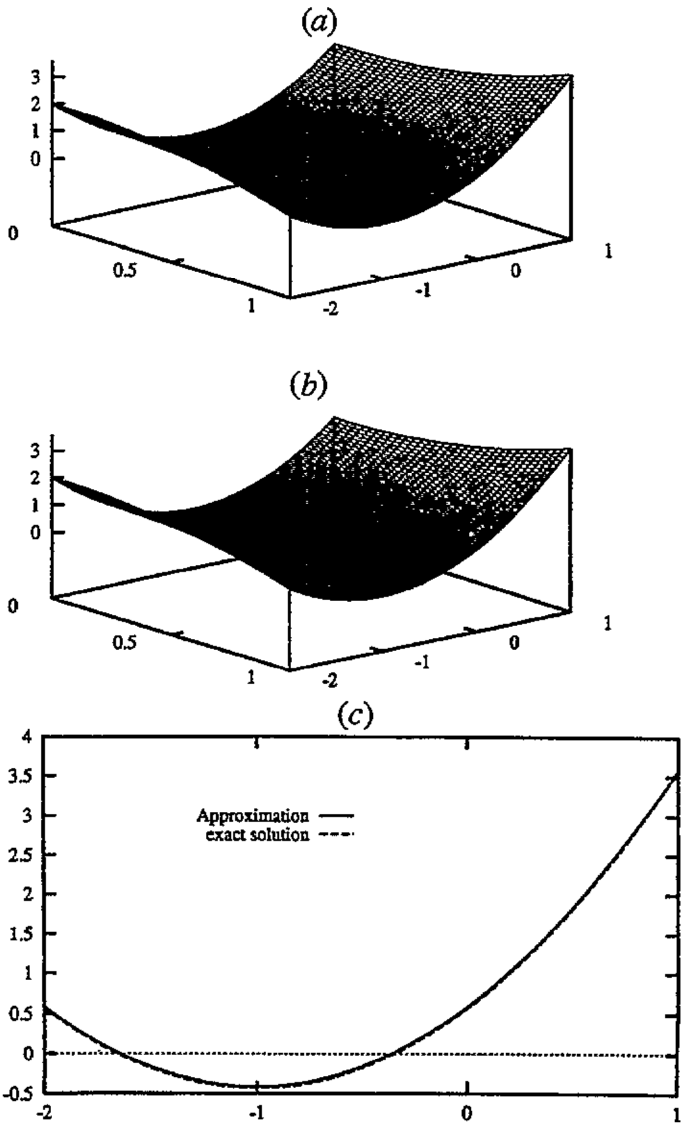

Figure 1. Example 1: $\varphi(t) = t^2 + t$: (a) exact solution; (b) approximation with $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant \varepsilon = 0.01$; and (c) exact solution and approximation at $x = 1$ ($\varepsilon = 0.01$).

Example 1. Let $\varphi(t) := t^2 + t$ (not in $L^2(\mathbb{R})$). Then

$$u(t, x) = t^2 + t + (t + \tfrac{1}{2})x^2 + x^4/12.$$

We are interested in approximations of $u(t, x)$ for $t \in [-2, 1]$ only. We allow a random noise in φ, in fact, $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant \varepsilon = 10^{-2}$. In this example the approximation is perfectly good (figure 1). One cannot see any difference between the exact solution and our approximation. We have also applied our marching scheme to this example and had the same results as for the method above but we do not present them here.

It is interesting to observe what happens if we use the measured Cauchy data only in the interval in which our approximation is found. Thus, in this case we suppose that

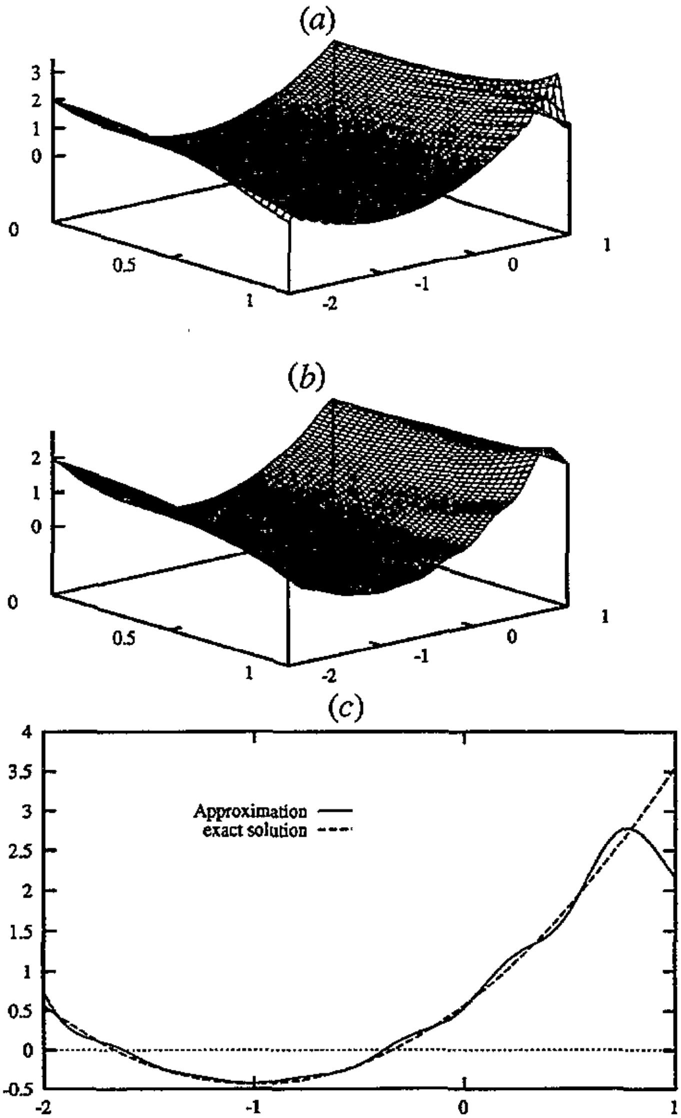

Figure 2. Example 1: $\varphi(t) = t^2 + t$, $-2 \leqslant t \leqslant 1$: ($a$) approximation with $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant \varepsilon = 10^{-6}$; ($b$) approximation with $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant \varepsilon = 0.01$; and ($c$) exact solution and approximation at $x = 1$ ($\varepsilon = 0.01$).

$\varphi_\varepsilon(t) = \varphi_\varepsilon(-2)$, $t < -2$ and $\varphi_\varepsilon(t) = \varphi_\varepsilon(1)$, $t > 1$. Numerical results of this example are shown in figure 2, where in the first plot $\varepsilon = 10^{-6}$, and in the second one $\varepsilon = 10^{-2}$. The last plot compares the exact solution with its approximation at $x = 1$ with $\varepsilon = 10^{-2}$.

The last two examples are constructed as follows. Choose a function f as the solution of (5.1) at $x = 1$. Then solve the well-posed problem

$$u_{xx} = u_t \qquad u_x(t, 0) = 0 \qquad u(t, 1) = f(t) \qquad t \in \mathbb{R} \qquad 0 \leqslant x \leqslant 1. \tag{5.2}$$

We use the FFT to compute the solution $u(t, x)$ of (5.2). After that we take $\varphi(t) := u(t, 0)$ as the Cauchy data in (5.1), where a random noise in φ is allowed. Then we try to reconstruct

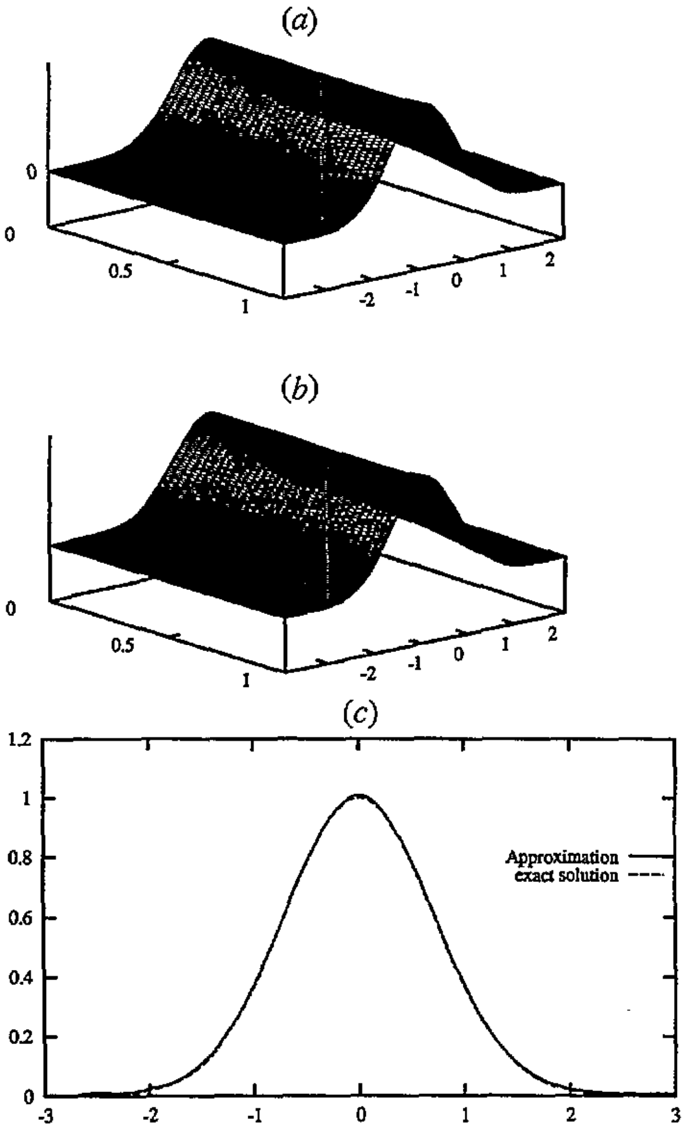

Figure 3. Example 2: $f(t) = \exp(-t^2)$: (a) exact solution; (b) approximation with $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant \varepsilon = 0.01$; and (c) exact solution and approximation at $x = 1$ ($\varepsilon = 0.01$).

f from the inexact Cauchy data.

Example 2. $f(t) = \exp(-t^2)$, $\varepsilon = 10^{-2}$. The solution is to be found in $[-3, 3]$. Note that in this example $\varphi(t) := u(t, 0)$ (where $u(t, x)$ denotes the solution of (5.2) with $f(t) = \exp(-t^2)$) is a function in $L^2(\mathbb{R})$. The numerical results are outlined in figure 3, where the first plot indicates the exact solution, the second one shows the approximation with a random noise on φ by our method, and the last one compares f with $T_{1,J}\varphi_\varepsilon$. We also applied the marching difference scheme (4.2) and obtained the same results. So we do not give them here.

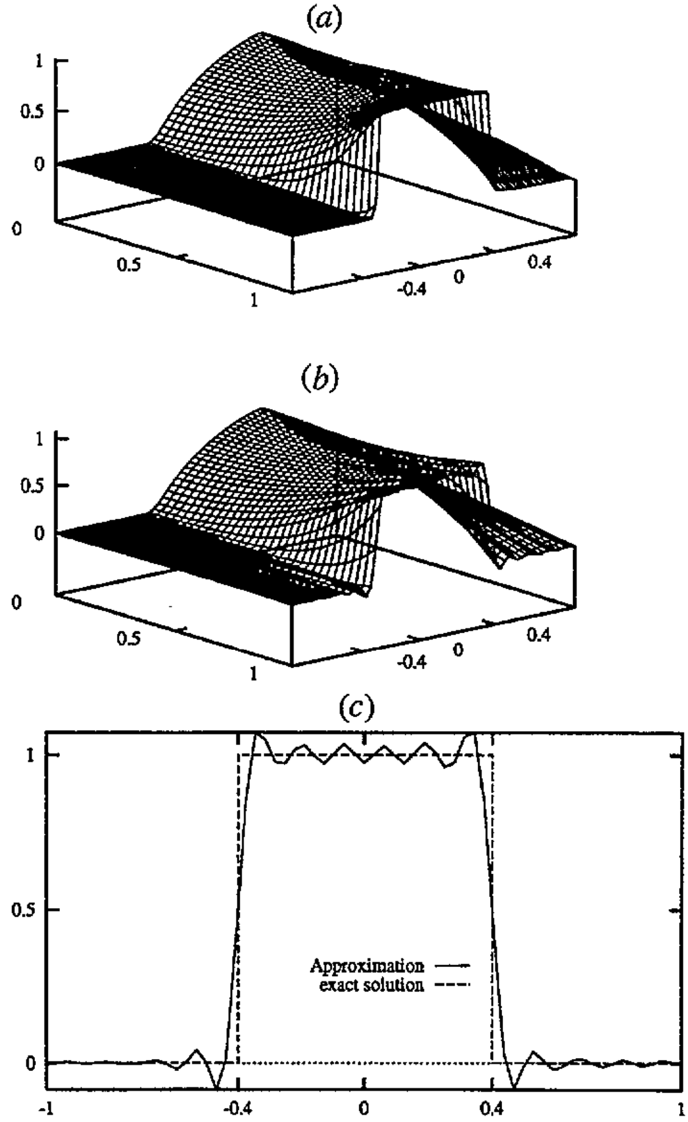

Figure 4. Example 3: $f = \chi_{[-0.4,0.4]}$: (a) approximation with $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant \varepsilon = 10^{-6}$; ($b$) approximation with $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant \varepsilon = 0.001$; and ($c$) exact solution and approximation at $x = 1$ ($\varepsilon = 0.001$).

Example 3. $f = \chi_{[-0.4,0.4]}$, $\varepsilon = 10^{-3}$. This is a hard test example in the inverse heat conduction problems, since $f \in H^s(\mathbb{R})$ only for $s < \frac{1}{2}$. Available methods for this example are not giving good approximations to the exact solution (see, e.g. [13]). It is interesting that although $f \in H^s(\mathbb{R})$ only for $s < \frac{1}{2}$, $u(t, 0)$ is infinitely differentiable, and it satisfies (2.2). The numerical results of the method in section 3 are outlined in figure 4, where the first plot indicates the approximation with noise $\varepsilon = 10^{-6}$, the second one indicates the approximation with noise $\varepsilon = 10^{-3}$, and the last one compares the exact solution at $x = 1$ (that is $u(1, t) = f(t) = \chi_{[-0.4,0.4]}(t)$) with its approximation for $\varepsilon = 10^{-3}$. Numerical results of the stable marching difference scheme of section 4 are outlined in figure 5.

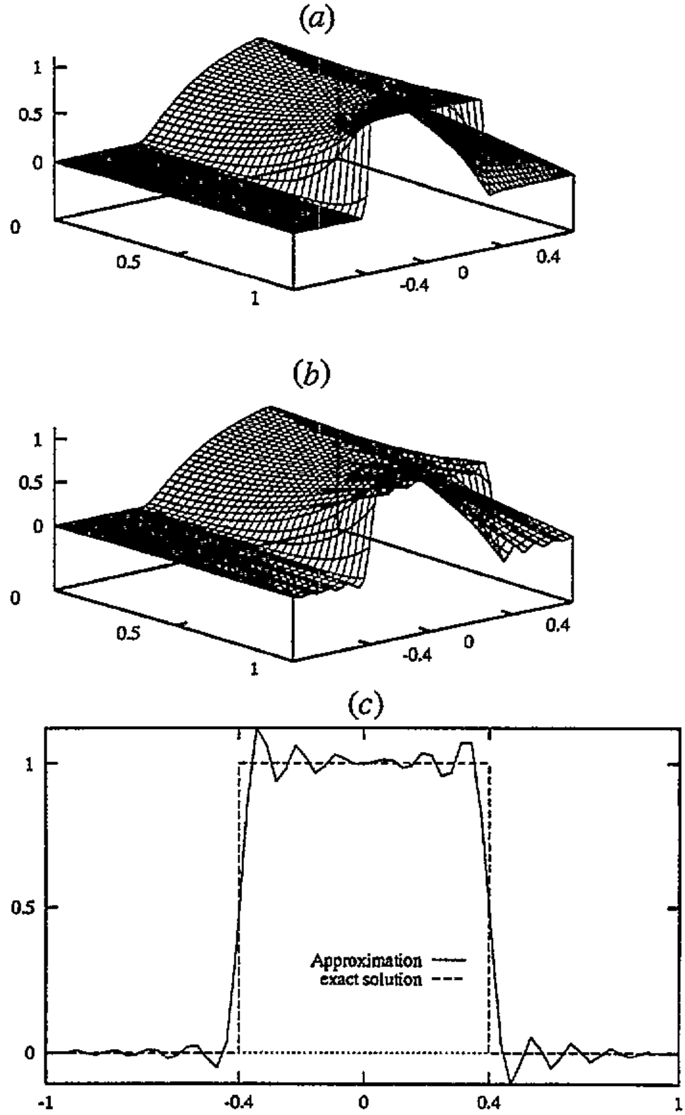

Figure 5. The marching difference scheme (4.2) for example 3: (*a*) approximation with $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant \varepsilon = 10^{-6}$; (*b*) approximation with $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant \varepsilon = 0.001$; and (*c*) exact solution and approximation at $x = 1$ ($\varepsilon = 0.001$).

6. Conclusions

Our regularization method for the severely ill-posed Cauchy problem (1.1) fulfils Hölder-type error estimates. The method can be easily numerically implemented and gives very good numerical results. Our stable marching difference scheme based on the method is convenient for numerical calculations and shows that our method is effective.

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