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On a sideways parabolic equation

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Abstract. The sideways parabolic equation in the quarter plane

$$u_t = a(x)u_{xx} + b(x)u_x + c(x)u \qquad x \in (0,\infty), t \in (0,\infty)$$

$$u(1,t) = g(t) \qquad t \in (0,\infty)$$

$$u(x,0) = 0 \qquad x \in (0,\infty)$$

is considered. This is a model of a problem where one wants to determine the temperature on both sides of a thick wall, but one side is inaccessible to measurements. This problem is well known to be severely ill-posed: a small perturbation in the data, g, may cause dramatically large errors in the solution. The results available in the literature are mainly devoted to the case of constant coefficients, where one can find an explicit representation for the solution of the problem. In this paper a stability estimate of the Hölder type for the solution of this general problem is established, it is also shown how to apply the mollification method recently proposed by Dinh Nho Háo to solve the problem in a stable way.

1. Introduction

In several engineering contexts there is sometimes a need to determine the temperature on both sides of a thick wall, but one side is inaccessible to measurements (see, e.g. [1, 12]. This problem occasionally leads to the following sideways parabolic equation in the quarter plane

$$u_t = a(x)u_{xx} + b(x)u_x + c(x)u \qquad x \in (0,\infty), t \in (0,\infty) \tag{1.1}$$

$$u(1,t) = g(t) \qquad t \in (0,\infty) \tag{1.2}$$

$$u(x,0) = 0 \qquad x \in (0,\infty). \tag{1.3}$$

Here a, b and c are given functions such that for some $\lambda, \Lambda > 0$

$$\lambda \leqslant a(x) \leqslant \Lambda \qquad x \in (0,\infty) \tag{1.4}$$

and

$$c(x) \leqslant 0. \tag{1.5}$$

For simplicity, we suppose that

$$a(\cdot) \in C^2(0,\infty) \qquad b(\cdot) \in C^1(0,\infty) \qquad c(\cdot) \in C(0,\infty). \tag{1.6}$$

Furthermore, throughout the paper, we suppose

$$g \in L_2(0,\infty) \tag{1.7}$$

which is considered as the measured data in the above-mentioned model.

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The problem (1.1)–(1.3) is well known to be *severely ill-posed*: a small perturbation in the data g may cause dramatically large errors in the solution $u(x, t)$ for $x \in [0, 1)$ (see, e.g. [12] or corollary 2.3 below). There are several methods for solving it in a stable way (see, e.g. our survey in [5], or [7]); however, they are mainly devoted to the case of constant coefficients, as for this one can find an explicit representation for the solution. The aim of this paper is to show how to apply the mollification method suggested in [3, 5, 6] to the general problem. To do this we first solve the *well-posed* problem

$$\begin{aligned} &u_t = a(x)u_{xx} + b(x)u_x + c(x)u \qquad x \in (1, \infty), t \in (0, \infty) \\ &u(1, t) = g(t) \qquad t \in (0, \infty) \\ &u(x, 0) = 0 \qquad x \in (1, \infty) \end{aligned} \tag{1.8}$$

to find $h(t) := u_x(1, t)$. As a result we get the Cauchy problem

$$\begin{aligned} &u_t = a(x)u_{xx} + b(x)u_x + c(x)u \qquad x \in (0, 1), t \in (0, \infty) \\ &u(1, t) = g(t) \qquad t \in (0, \infty) \\ &u_x(1, t) = h(t) \qquad t \in (0, \infty) \\ &u(x, 0) = 0 \qquad x \in [0, 1]. \end{aligned} \tag{1.9}$$

The next step is to solve this Cauchy problem in a stable way. The crucial point is how to do it when the results in [9, 3, 5, 6] allow us to work only with the case when either $h = 0$ or $g = 0$. One immediately has the idea that either one should split this Cauchy problem into two *independent Cauchy problems*

$$\begin{aligned} &u^1_t = a(x)u^1_{xx} + b(x)u^1_x + c(x)u^1 \qquad x \in (0, 1), t \in (0, \infty) \\ &u^1(1, t) = g(t) \qquad t \in (0, \infty) \\ &u^1_x(1, t) = 0 \qquad t \in (0, \infty) \\ &u^1(x, 0) = 0 \qquad x \in [0, 1] \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &u^2_t = a(x)u^2_{xx} + b(x)u^2_x + c(x)u^2 \qquad x \in (0, 1), t \in (0, \infty) \\ &u^2(1, t) = 0 \qquad t \in (0, \infty) \\ &u^2_x(1, t) = h(t) \qquad t \in (0, \infty) \\ &u^2(x, 0) = 0 \qquad x \in [0, 1] \end{aligned} \tag{1.11}$$

and solve them by the technique given in [3, 5, 6] and then take $u = u^1 + u^2$, or one should solve the well-posed problem

$$\begin{aligned} &u^3_t = a(x)u^3_{xx} + b(x)u^3_x + c(x)u^3 \qquad x \in (0, 1), t \in (0, \infty) \\ &u^3(0, t) = 0 \qquad t \in (0, \infty) \\ &u^3_x(1, t) = h(t) \qquad t \in (0, \infty) \\ &u^3(x, 0) = 0 \qquad x \in [0, 1] \end{aligned} \tag{1.12}$$

and then solve the Cauchy problem

$$\begin{aligned} &u^4_t = a(x)u^4_{xx} + b(x)u^4_x + c(x)u^4 \qquad x \in (0, 1), t \in (0, \infty) \\ &u^4(1, t) = g(t) - u^3(1, t) \qquad t \in (0, \infty) \\ &u^4_x(1, t) = 0 \qquad t \in (0, \infty) \\ &u^4(x, 0) = 0 \qquad x \in [0, 1] \end{aligned} \tag{1.13}$$

by the technique given in [3, 5, 6] and take $u = u^3 + u^4$.

The second approach is preferential as we have to solve only one ill-posed Cauchy problem. A matter of fact is: *can we do the just-mentioned procedures?* It seems to us that the researchers in the field did not take this question into consideration. First, we note that, in the Cauchy problem (1.9), the Cauchy data g and h cannot be arbitrarily given. For example, Holmgren [8] has proved that a necessary and sufficient condition for the existence of a solution of the problem

$$\begin{aligned} &u_{xx} = u_t \qquad 0 < x < 1, 0 < t < T \\ &u(0,t) = g(t) \qquad 0 < t < T \\ &u_x(0,t) = h(t) \qquad 0 < t < T \end{aligned} \tag{1.14}$$

is that

$$\theta(t) = h(t) + \frac{1}{\sqrt{\pi}} \int_0^t \frac{g'(\tau)\,\mathrm{d}\tau}{\sqrt{t-\tau}}$$

be a function of a Holmgren class 2. A function, $\theta(t)$, defined in (α, β) is said to belong to a Holmgren class 2 if it belongs to $C^\infty(\alpha, \beta)$, and there exist positive numbers $M \geqslant 0$ and s such that $|\theta^{(n)}(t)| \leqslant M(2n)!s^{2n}$, $\forall n \in \{0, 1, 2, \ldots\}$. Secondly, in [4–6] we have also proved that if $h = 0$, then g must be infinitely differentiable and $\|g^{(n)}\|_* \leqslant c(2n)!s^{2n}$, $n = 0, 1, \ldots$, where c and s are some positive constants and $\|\cdot\|_*$ is a defined norm. The same result is also valid for h if $g = 0$. Thus, if g (or h) does not belong to the above-mentioned class, then we cannot split our Cauchy problem (1.9) (or the problem (1.14)) into two independent Cauchy problems (1.10) and (1.11), as they have no solution! Furthermore, in general, if g (or h) belongs to the above-described class it guarantees only *a local solution* of our Cauchy problem (see [4–6]). We note also that here we are working with *non-analytic Cauchy problems* for parabolic equations. Thus, in order to split the Cauchy problem (1.9) into two independent Cauchy problems (1.10) and (1.11) or to deduce it to the Cauchy problem (1.13) we have to prove that g and h are so smooth such that the Cauchy problems (1.10), (1.11) and (1.13) have *global solutions* up to $x = 0$. The main contribution of this paper is to prove this fact as well as to deliver a stability estimate of the Höder type for the solution of the problem (1.1)–(1.3), although we do not claim any originality of our idea.

In this paper we shall use some estimates for the asymptotic behaviour of solutions of a second-order linear ordinary differential equation depending on a parameter in a finite interval given by Tamarkin [13], Coddington and Levinson [2], in the half-axis given by Naimark [10] as well as several estimates of Knabner and Vessella [9].

In the next section we establish some regularity properties for the boundary value problems for parabolic equations in the quarter plane. The last section is devoted to the problem (1.1)–(1.3): a stability estimate, splitting the Cauchy problem (1.9) to the Cauchy problems (1.10), (1.11) and (1.13). Since the mollification method, stable marching difference schemes for the Cauchy problems (1.10), (1.11) and (1.13) as well as numerical experiments for them are given in [5, 6] we do not repeat them in this paper.

Throughout the paper all constants $c_1, c_2, \ldots, c_{27}$, C_1, C_2 are tacitly assumed to be positive.

2. Direct problems

Consider the boundary-value problem for our parabolic equation in the quarter plane

$$u_t = a(x)u_{xx} + b(x)u_x + c(x)u \qquad x \in (0, \infty), t \in (0, \infty) \tag{2.1}$$

$$u(0,t) = f(t) \qquad t \in (0,\infty) \tag{2.2}$$

$$u(x,0) = 0 \qquad x \in (0,\infty). \tag{2.3}$$

We assume

$$f \in L_2(0,\infty). \tag{2.4}$$

As a solution of the problem (2.1)–(2.3) we understand a function $u(x,t)$ satisfying (2.1) and (2.3) in the classical sense; for every fixed $x \in [0,\infty)$, the function $u(x,\cdot)$ and its derivatives $u_x(x,\cdot), u_{xx}(x,\cdot)$ belong to $L_2(0,\infty)$ and $\lim_{x\to 0} \|u(x,\cdot) - f\|_{L_2(0,\infty)} = 0$. Furthermore, for the uniqueness of the solution, we require that $\|u(x,\cdot)\|_{L_2(0,\infty)}$ be bounded.

Since the functions of t in our consideration are defined only in $[0,\infty)$, in order to apply the Fourier transform technique to them we extend their definitions to the whole real t-axis by defining them to be zero for $t < 0$. Further, let

$$\hat{w}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(t) \mathrm{e}^{-\mathrm{i}\xi t}\, \mathrm{d}t$$

denote the Fourier transform of w, and $\|\cdot\|$ denote $\|\cdot\|_{L_2(\mathbb{R})}$.

We associate (2.1)–(2.3) with the following boundary value problem for the ordinary differential equation

$$\begin{aligned} &\mathrm{i}\xi v(x,\xi) = a(x)v_{xx} + b(x)v_x + c(x)v \qquad x \in (0,\infty), \xi \in \mathbb{R} \\ &v(0,\xi) = 1 \qquad \xi \in \mathbb{R} \\ &\lim_{x\to\infty} v(x,\xi) = 0 \qquad \xi \neq 0 \end{aligned} \tag{2.5}$$

for $\xi = 0$ we require $v(x,0)$ be bounded as x tends to ∞.

We note that $|v(x,\xi)| \leqslant 1$ for $\xi \neq 0$. In fact, setting

$$y := A(x) = \int_0^x \frac{\mathrm{d}s}{\sqrt{a(s)}} \tag{2.6}$$

we see that the function $\tilde{v}(y,\xi) := v(x,\xi)$ satisfies the system

$$\begin{aligned} &\mathrm{i}\xi\tilde{v} = \tilde{v}_{yy} + d(y)\tilde{v}_y + \tilde{c}(y)\tilde{v} \qquad \xi \in \mathbb{R}, y \in (0,\infty) \\ &\tilde{v}(0,\xi) = 1 \qquad \xi \in \mathbb{R} \\ &\lim_{y\to\infty} \tilde{v}(y,\xi) = 0 \qquad \xi \neq 0 \end{aligned}$$

where

$$\tilde{c}(y) := c(x) \qquad d(y) := [(b - a_x/2)/\sqrt{a}](x). \tag{2.7}$$

Taking the conditions (1.4)–(1.6) into account and applying an easily modified version of the maximum principle [11, theorem 3, p 6] for the infinite half-axis $[0,\infty)$ to $|\tilde{v}(y,\xi)|^2$, we see that $|\tilde{v}(y,\xi)| \leqslant 1$ for $\xi \neq 0$. Thus, $|v(x,\xi)|$ is uniformly bounded.

Suppose that the problem (2.1)–(2.3) has a solution u. Then

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi t} v(x,\xi)\hat{f}(\xi)\, \mathrm{d}\xi \qquad \text{for } x \in (0,\infty). \tag{2.8}$$

In fact, for a function h defined on $[0,\infty)$ we denote its Laplace transform by H, and the inverse Laplace transform of H by $\mathcal{L}^{-1}\{H\}$ (see [14, ch 2]). Taking the Laplace transform of both sides of (2.1) and (2.2), having the condition (2.3) in mind, we get

$$\begin{aligned} &sU(x,s) = a(x)U_{xx} + b(x)U_x + c(x)U \qquad x \in (0,\infty), s \in \mathbb{C} \\ &U(0,s) = F(s) \qquad x \in (0,\infty), s \in \mathbb{C}. \end{aligned} \tag{2.9}$$

Furthermore, in order to have a unique solution, according to the maximum principle just mentioned above, we require that $\lim_{x\to\infty} |U(x,s)| = 0$, if Re $s \geqslant 0$. If a solution of the problem (2.1)–(2.3) exists, then

$$u(x,t) = \mathcal{L}^{-1}\{U\}(x,t).$$

According to theorem 7.3 in [14, p 66], we have $\mathcal{L}^{-1}\{U\}(x,t) = 0$ for $t < 0$ and $\mathcal{L}^{-1}\{U\}(x,0) = u(x,+0)/2 = 0$. On the other hand, for a function w which vanishes on the negative t-axis, the Fourier and Laplace transforms are related via

$$W(\mathrm{i}\xi) = \sqrt{2\pi}\,\hat{w}(\xi) \qquad \xi \in \mathbb{R}.$$

Thus, from (2.9), on putting $s = \mathrm{i}\xi$,

$$\hat{u}(x,\xi) = v(x,\xi)\hat{f}(\xi).$$

Hence the equality (2.8).

We see that this u satisfies the condition (2.3). In fact,

$$\begin{aligned} u(x,t) &= \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \mathrm{e}^{\mathrm{i}\xi t}\hat{u}(x,\xi)\,\mathrm{d}\xi \\ &= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \mathrm{e}^{\mathrm{i}\xi t} U(x, i\xi)\,\mathrm{d}\xi \\ &= \lim_{T\to\infty} \frac{1}{2\pi i} \int_{-iT}^{iT} \mathrm{e}^{st} U(x,s)\,\mathrm{d}s \\ &= \mathcal{L}^{-1}\{U\}(x,t) \end{aligned}$$

which has just been proved to satisfy the condition (2.3). For further argument, the reader is referred to [9, p 580].

We note that since $|v(x,\xi)|$ is uniformly bounded, the solution of the problem (2.1)–(2.3) is stable in L_2-norm.

Lemma 2.1. *There exist constants* c_k, $k = 1,2,3,4$, *such that, for* $x \in [0,1]$ *and* $|\xi|$ *large enough, say* $|\xi| \geqslant \xi_0$,

$$c_1\mathrm{e}^{-A(x)\sqrt{|\xi|/2}} \leqslant |v(x,\xi)| \leqslant c_2\mathrm{e}^{-A(x)\sqrt{|\xi|/2}}, \tag{2.10}$$

$$c_3\sqrt{|\xi|}\mathrm{e}^{-A(x)\sqrt{|\xi|/2}} \leqslant |v_x(x,\xi)| \leqslant c_4\sqrt{|\xi|}\mathrm{e}^{-A(x)\sqrt{|\xi|/2}}. \tag{2.11}$$

Proof. Taking $y = A(x)$ as in (2.6), the function d as in (2.7) and performing the transform [9]

$$w(y,\xi) := v(x,\xi)\exp\left(\tfrac{1}{2}\int_0^y \mathrm{d}(s)\,\mathrm{d}s\right) \tag{2.12}$$

in the problem (2.5), we get

$$\begin{aligned} &w_{yy}(y,\xi) = [q(y) + \mathrm{i}\xi]w(y,\xi) \qquad y \in [0,\infty), \xi \in \mathbb{R} \\ &w(0,\xi) = 1 \qquad \xi \in \mathbb{R}. \end{aligned} \tag{2.13}$$

Here,

$$q(y) := \tfrac{1}{16}(-4a'' + 8b' + 3a'^2/a - 8ba'/a + 4b^2/a - 16c)(x). \tag{2.14}$$

Let l be a fixed number greater than $2A(1)$. On the interval $[0,l]$, the function $w(y,\xi)$ satisfies the boundary-value problem

$$w_{yy}(y,\xi) - [q(y) + \mathrm{i}\xi]w(y,\xi) = 0 \qquad y \in (0,l), \xi \in \mathbb{R} \tag{2.15}$$

$$w(0,\xi) = 1 \qquad w(l,\xi) = w(l,\xi) \qquad \xi \in \mathbb{R}. \tag{2.16}$$

We note that, since $|v(x,\xi)| \leqslant 1$ for $\xi \neq 0$,

$$|w(l,\xi)| \leqslant \exp\left(\tfrac{1}{2}\int_0^l |\,\mathrm{d}(s)|\,\mathrm{d}s\right) := e(l) \qquad \text{for } \xi \neq 0.$$

Following [13] and [2, p 308] the equation (2.15) has two fundamental solutions w^+ and w^- which, for sufficiently large ξ, have the following representations

$$w^{\pm}(y,\xi) = \mathrm{e}^{\pm y\sqrt{\mathrm{i}\xi}}\left\{1 + \frac{[\pm Q(y)]}{\sqrt{\mathrm{i}\xi}}\right\} \tag{2.17}$$

$$w^{\pm}_y(y,\xi) = \pm\sqrt{\mathrm{i}\xi}\,\mathrm{e}^{\pm y\sqrt{\mathrm{i}\xi}}\left\{1 + \frac{[\pm Q(y)]}{\sqrt{\mathrm{i}\xi}}\right\} \tag{2.18}$$

where

$$\sqrt{\mathrm{i}\xi} = \begin{cases} \sqrt{|\xi|/2}(1+\mathrm{i}) & \xi \geqslant 0 \\ \sqrt{|\xi|/2}(1-\mathrm{i}) & \xi < 0 \end{cases}$$

and

$$Q(y) = \tfrac{1}{2}\int_0^y q(\tau)\,\mathrm{d}\tau.$$

By $[\alpha]$ we understand Birkhoff's expression $[\alpha] = \alpha + E/\xi$, where $E = E(y,\xi)$ is a continuous function in both variables and uniformly bounded for large ξ.

Set

$$\Delta(l,\xi) = \begin{vmatrix} w^+(0,\xi) & w^-(0,\xi) \\ w^+(l,\xi) & w^-(l,\xi) \end{vmatrix}$$

$$\Delta_1(l,y,\xi) = \begin{vmatrix} w^+(y,\xi) & w^-(y,\xi) \\ w^+(l,\xi) & w^-(l,\xi) \end{vmatrix}$$

$$\Delta_2(l,y,\xi) = -\begin{vmatrix} w^+(y,\xi) & w^-(y,\xi) \\ w^+(0,\xi) & w^-(0,\xi) \end{vmatrix}.$$

We have

$$w(y,\xi) = \frac{\Delta_1(l,y,\xi)}{\Delta(l,\xi)} + w(l,\xi)\frac{\Delta_2(l,y,\xi)}{\Delta(l,\xi)}.$$

Now we estimate $\Delta(l,\xi)$, $\Delta_1(l,y,\xi)$ and $\Delta_2(l,x,\xi)$. From (2.17),

$$\begin{aligned}\Delta(l,\xi) = w^+(0,\xi)w^-(l,\xi) - w^+(l,\xi)w^-(0,\xi) &= \mathrm{e}^{-l\sqrt{\mathrm{i}\xi}}\left(1 + \frac{[Q(0)]}{\sqrt{\mathrm{i}\xi}}\right)\left(1 + \frac{[-Q(l)]}{\sqrt{\mathrm{i}\xi}}\right) \\ &\quad -\mathrm{e}^{l\sqrt{\mathrm{i}\xi}}\left(1 + \frac{[Q(l)]}{\sqrt{\mathrm{i}\xi}}\right)\left(1 + \frac{[-Q(0)]}{\sqrt{\mathrm{i}\xi}}\right) = \mathrm{e}^{l\sqrt{\mathrm{i}\xi}}\left\{\mathrm{e}^{-2l\sqrt{\mathrm{i}\xi}}\left(1 + \frac{[Q(0)]}{\sqrt{\mathrm{i}\xi}}\right)\right. \\ &\quad \left.\times\left(1 + \frac{[-Q(l)]}{\sqrt{\mathrm{i}\xi}}\right) - \left(1 + \frac{[Q(l)]}{\sqrt{\mathrm{i}\xi}}\right)\left(1 + \frac{[-Q(0)]}{\sqrt{\mathrm{i}\xi}}\right)\right\}.\end{aligned}$$

For ξ large enough, say $|\xi| \geqslant \xi_1$, there exists a constant c_5 such that

$$\begin{aligned}&\left|\left\{\mathrm{e}^{-2l\sqrt{\mathrm{i}\xi}}\left(1 + \frac{[Q(0)]}{\sqrt{\mathrm{i}\xi}}\right)\left(1 + \frac{[-Q(l)]}{\sqrt{\mathrm{i}\xi}}\right) - \left(1 + \frac{[Q(l)]}{\sqrt{\mathrm{i}\xi}}\right)\left(1 + \frac{[-Q(0)]}{\sqrt{\mathrm{i}\xi}}\right)\right\}\right| \\ &\qquad \leqslant \left|\mathrm{e}^{-2l\sqrt{\mathrm{i}\xi}}\left(1 + \frac{[Q(0)]}{\sqrt{\mathrm{i}\xi}}\right)\left(1 + \frac{[-Q(l)]}{\sqrt{\mathrm{i}\xi}}\right)\right| \\ &\qquad\quad + \left|\left(1 + \frac{[Q(l)]}{\sqrt{\mathrm{i}\xi}}\right)\left(1 + \frac{[-Q(0)]}{\sqrt{\mathrm{i}\xi}}\right)\right| \leqslant c_5\end{aligned}$$

since $|e^{-2l\sqrt{i\xi}}| \leqslant e^{-l\sqrt{2\xi_1}}$.

Analogously, there exists a constant c_6 such that for ξ large enough, say $|\xi| \geqslant \xi_2$,

$$\begin{aligned}
&\left|\left\{e^{-2l\sqrt{i\xi}}\left(1+\frac{[Q(0)]}{\sqrt{i\xi}}\right)\left(1+\frac{[-Q(l)]}{\sqrt{i\xi}}\right)-\left(1+\frac{[Q(l)]}{\sqrt{i\xi}}\right)\left(1+\frac{[-Q(0)]}{\sqrt{i\xi}}\right)\right\}\right| \\
&\qquad \geqslant \left|\left|e^{-2l\sqrt{i\xi}}\left(1+\frac{[Q(0)]}{\sqrt{i\xi}}\right)\left(1+\frac{[-Q(l)]}{\sqrt{i\xi}}\right)\right| \right. \\
&\qquad\quad \left. -\left|\left(1+\frac{[Q(l)]}{\sqrt{i\xi}}\right)\left(1+\frac{[-Q(0)]}{\sqrt{i\xi}}\right)\right|\right| \geqslant c_6
\end{aligned}$$

since $|e^{-2l\sqrt{i\xi}}| = e^{-l\sqrt{2|\xi|}}$ is small if ξ is large enough.

Thus, there exist constants c_5 and c_6 such that, for $|\xi| \geqslant \max\{\xi_1, \xi_2\}$,

$$c_6 e^{l\sqrt{|\xi|/2}} \leqslant |\Delta(l,\xi)| \leqslant c_5 e^{l\sqrt{|\xi|/2}}. \tag{2.19}$$

Quite analogously, there exist constants c_7, c_8, c_9, c_{10} such that for ξ large enough

$$c_7 e^{(l-y)\sqrt{|\xi|/2}} \leqslant |\Delta_1(l,y,\xi)| \leqslant c_8 e^{(l-y)\sqrt{|\xi|/2}}, \tag{2.20}$$

$$c_9 e^{y\sqrt{|\xi|/2}} \leqslant |\Delta_2(l,y,\xi)| \leqslant c_{10} e^{y\sqrt{|\xi|/2}}. \tag{2.21}$$

From the last three inequalities we see for ξ large enough there exists a constant c_{11} such that, for $y \in [0, A(1)]$,

$$\begin{aligned}
|w(y,\xi)| &\leqslant \left|\frac{\Delta_1(l,y,\xi)}{\Delta(l,\xi)}\right| + |w(l,\xi)|\left|\frac{\Delta_2(l,y,\xi)}{\Delta(l,\xi)}\right| \\
&\leqslant \frac{c_8}{c_6} e^{-y\sqrt{|\xi|/2}} + \frac{c_{10}}{c_6} e(l) e^{(y-l)\sqrt{|\xi|/2}} \\
&\leqslant c_{11} e^{-y\sqrt{|\xi|/2}}.
\end{aligned}$$

Furthermore, for ξ large enough there also exists a constant c_{12} such that, for $y \in [0, A(1)]$,

$$\begin{aligned}
|w(y,\xi)| &\geqslant \left|\left|\frac{\Delta_1(l,y,\xi)}{\Delta(l,\xi)}\right| - |w(l,\xi)|\left|\frac{\Delta_2(l,y,\xi)}{\Delta(l,\xi)}\right|\right| \\
&\geqslant \left|\frac{c_7}{c_5} e^{-y\sqrt{|\xi|/2}} - \frac{c_{10}}{c_6} e(l) e^{(y-l)\sqrt{|\xi|/2}}\right| \\
&= \left|\frac{c_7}{c_5} - \frac{c_{10}}{c_6} e(l) e^{(2y-l)\sqrt{|\xi|/2}}\right| e^{-y\sqrt{|\xi|/2}} \\
&\geqslant c_{12} e^{-y\sqrt{|\xi|/2}}.
\end{aligned}$$

Thus, there exist constants c_{11} and c_{12} such that, for ξ large enough and $y \in [0, A(1)]$,

$$c_{11} e^{-y\sqrt{|\xi|/2}} \leqslant |w(y,\xi)| \leqslant c_{12} e^{-y\sqrt{|\xi|/2}}.$$

Taking the transform (2.12) into account and noting that $y \in [0, A(1)]$, we thus prove the inequalities (2.10) of the lemma. The inequalities (2.11) are proved quite similarly.

Corollary 2.1. *If the boundary value problem*

$$a(x)v_{xx}(x) + b(x)v_x(x) + c(x)v(x) = 0 \qquad 0 < x < \infty$$

$$v(0) = 1 \qquad v(x) \textit{ is bounded as } x \to \infty.$$

has a unique solution, then there exist constants $\bar{c}_1$ *and* $\bar{c}_2$ *such that*

$$\bar{c}_1 e^{-A(1)\sqrt{|\xi|/2}} \leqslant |v(1,\xi)| \leqslant \bar{c}_2 e^{-A(1)\sqrt{|\xi|/2}} \qquad \forall\xi \in \mathbb{R}. \tag{2.22}$$

Proof. First, we note that $v(1,\xi) \neq 0$ for $\xi \neq 0$, otherwise the Sturm–Liouville problem

$$a(x)v_{xx} + b(x)v_x + c(x)v = \lambda v \qquad x \in (0,1)$$
$$v(0,\lambda) = 1 \qquad v(1,\lambda) = 0$$

has imaginary eigenvalues.

For $\xi = 0$, we have the boundary value problem

$$a(x)v_{xx}(x,0) + b(x)v_x(x,0) + c(x)v(x,0) = 0 \qquad 0 < x < \infty$$
$$v(0,0) = 1 \qquad |v(x,0)| \text{ is bounded as } x \to \infty.$$

We see that $v(1,0) \neq 0$. In fact, if $v(1,0) = 0$, then $v(x,0) = 0$ for $x \geqslant 1$, since this problem is assumed to have a unique solution. Thus, $v(l,0) = 0$, $v_x(l,0) = 0$ for any $l > 1$. The Cauchy problem

$$a(x)v_{xx}(x,0) + b(x)v_x(x,0) + c(x)v(x,0) = 0 \qquad 0 < x < l$$
$$v(l,0) = 0 \qquad v_x(l,0) = 0$$

has a unique solution $v(x,0) \equiv 0$. It is a contradiction, since $v(0,0) = 1$. Thus, $v(1,0) \neq 0$. Consequently, $|v(1,\xi)| > 0$ for $|\xi| \leqslant \xi_0$, where ξ_0 is the constant in lemma 2.1. The function $|v(1,\xi)|$ is continuous in the closed interval $[-\xi_0, \xi_0]$ and therefore attains its minimum at a point $\bar{\xi}$ in this interval. Hence, $|v(1,\xi)| \geqslant |v(1,\bar{\xi})| > 0$. Thus, there exists a constant $\bar{c}_1^1$ such that, for $|\xi| \leqslant \xi_0$,

$$\bar{c}_1^1 \mathrm{e}^{-A(1)\sqrt{|\xi|/2}} \leqslant |v(1,\xi)|.$$

Combining this inequality with the first one in (2.10), we see that there exists a constant $\bar{c}_1$ such that the first inequality in (2.22) is valid. The right-hand side inequality in (2.22) is trivial.

Remark 2.1. From the maximum principle we see immediately that, for $x \in [0,1]$, the right-hand side inequalities in (2.10) and (2.11) are valid for all $\xi \in \mathbb{R}$ with other constants c_2 and c_4. It follows also that for $x \in [0,1]$ there exists a constant c_{13} such that

$$|v_{xx}(x,\xi)| \leqslant c_{13}|\xi|\mathrm{e}^{-A(x)\sqrt{|\xi|/2}} \qquad \forall \xi \in \mathbb{R}.$$

Remark 2.2. If the function $d(\cdot)$ is not positive and the function q is summable in every finite subinterval $(0,l)$, $0 < l < \infty$, then we can very simply prove lemma 2.1. In fact, if $d \leqslant 0$, then $\lim_{y\to\infty} w(y,\xi) = 0$ for $\xi \neq 0$, from theorem 2.5.1 in [10] we have as $\xi \to \infty$

$$w(y,\xi) = \mathrm{e}^{-y\sqrt{\mathrm{i}\xi}}\left[1 + \mathrm{O}\left(\frac{1}{\sqrt{\mathrm{i}\xi}}\right)\right] \tag{2.23}$$

$$w_y(y,\xi) = \sqrt{\mathrm{i}\xi}\,\mathrm{e}^{-y\sqrt{\mathrm{i}\xi}}\left[1 + \mathrm{O}\left(\frac{1}{\sqrt{\mathrm{i}\xi}}\right)\right] \tag{2.24}$$

uniformly with respect to $y \in [0,\infty)$. Hence the lemma.

Corollary 2.2. *For any fixed $x > 0$, the operators*

$$f \in L_2(0,\infty) \to u(x,\cdot) \in L_2(0,\infty)$$

and

$$f \in L_2(0,\infty) \to u_x(x,\cdot) \in L_2(0,\infty)$$

are infinitely smoothing. Furthermore there exist nonnegative constants c_{14} and c_{15} such that

$$\left\| \frac{\partial^n u(x,\cdot)}{\partial t^n} \right\| \leqslant c_{14}(2n)! \left(\frac{\sqrt{2}}{A(x)} \right)^{2n} \tag{2.25}$$

$$\left\| \frac{\partial^{n+1} u(x,\cdot)}{\partial x \partial t^n} \right\| \leqslant c_{15}(2n+1)! \left(\frac{\sqrt{2}}{A(x)} \right)^{2n} \qquad \forall n = 0, 1, 2, \ldots. \tag{2.26}$$

Proof. For any $x > 0$ and $n = 1, 2, \ldots$, we have

$$\begin{aligned}
\left\| \frac{\partial^n u(x,\cdot)}{\partial t^n} \right\| &= \left(\int_{-\infty}^{\infty} |(\mathrm{i}\xi)^n \hat{u}(x,\xi)|^2 \,\mathrm{d}\xi \right)^{1/2} = \left(\int_{-\infty}^{\infty} |(\mathrm{i}\xi)^n v(x,\xi) \hat{f}(\xi)|^2 \,\mathrm{d}\xi \right)^{1/2} \\
&\leqslant c_2 \left(\int_{-\infty}^{\infty} |\xi|^{2n} \mathrm{e}^{-A(x)\sqrt{2|\xi|}} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \right)^{1/2} \\
&\leqslant c_2 \sup_{\xi \in \mathbb{R}} (|\xi|^n \mathrm{e}^{-A(x)\sqrt{|\xi|/2}}) \|f\| \\
&\leqslant c_2 \left(\frac{2n\sqrt{2}}{eA(x)} \right)^{2n} \|f\| \\
&\leqslant c_2 \frac{\mathrm{e}^{2n}(2n)!}{\sqrt{4\pi n}} \left(\frac{\sqrt{2}}{eA(x)} \right)^{2n} \|f\| \\
&\leqslant c_2 \frac{(2n)!}{2\sqrt{\pi}} \left(\frac{\sqrt{2}}{A(x)} \right)^{2n} \|f\|.
\end{aligned}$$

Here we have used the following inequalities:

$$y^p \mathrm{e}^{-cy} \leqslant (p/ec)^p \qquad \text{for any } y > 0, c > 0, p > 0$$

and

$$k^k \leqslant \frac{\mathrm{e}^k}{\sqrt{2\pi k}} k!.$$

Setting

$$c_{14} := \frac{c_2}{2\sqrt{\pi}} \|f\|$$

we get the inequality (2.25).

Thus, for $x > 0$ the function $u(x,t)$ is infinitely differentiable with respect to t and we have the estimates (2.25) for all of its derivatives with respect to t in the L_2 norm. The inequalities (2.26) are proved quite similarly. We note also that $u(x,t) = 0$ for $t \leqslant 0$.

As a direct consequence of this corollary we have the following assertion.

Corollary 2.3. *The problem (1.1)–(1.3) is severely ill-posed.*

To deal with the Cauchy problem (1.13) we need the following result.

Lemma 2.2. *For the problem*

$$\begin{aligned}
&\mathrm{i}\xi \widehat{u^3}(x,\xi) = a(x)\widehat{u^3}_{xx}(x,\xi) - b(x)\widehat{u^3}_x(x,\xi) - c(x)\widehat{u^3}(x,\xi) \qquad x \in (0,1), \xi \in \mathbb{R} \\
&\widehat{u^3}(0,\xi) = 0 \qquad \widehat{u^3}_x(1,\xi) = \hat{h}(\xi) \qquad \xi \in \mathbb{R}.
\end{aligned} \tag{2.27}$$

there exists a constant c_{16} such that

$$|\widehat{u^3}(1,\xi)| \leqslant c_{16} \mathrm{e}^{-A(1)\sqrt{|\xi|/2}} |\hat{f}(\xi)|. \tag{2.28}$$

Proof. Taking $y = A(x)$ as in (2.6), the function d as in (2.7) and performing the transform

$$w^1(y,\xi) := \widehat{u^3}(x,\xi)\exp\left(\tfrac{1}{2}\int_0^y \mathrm{d}(s)\,\mathrm{d}s\right) \tag{2.29}$$

in the problem (2.27), we get

$$\begin{aligned} & w^1_{yy}(y,\xi) = [q(y) + \mathrm{i}\xi]w^1(y,\xi) \qquad y \in [0, A(1)], \xi \in \mathbb{R} \\ & w^1(0,\xi) = 0 \qquad \xi \in \mathbb{R} \\ & w^1_y(A(1),\xi) + Cw^1(A(1),\xi) = c_{17}\hat{h}(\xi) \qquad \xi \in \mathbb{R}. \end{aligned} \tag{2.30}$$

Here $q(y)$ is defined as in (2.14), $C = -\mathrm{d}(A(1))/2$, $c_{17} = \sqrt{a(1)}\exp\left(\tfrac{1}{2}\int_0^{A(1)} \mathrm{d}(s)\,\mathrm{d}s\right)$.

As we have quoted in the proof of lemma 2.1, the equation $w^1_{yy}(y,\xi) = [q(y) + \mathrm{i}\xi]w^1(y,\xi)$ has two fundamental solutions w^+ and w^- which, for sufficiently large ξ, have the representations (2.17) and (2.18).

Set

$$\mathcal{D}(\xi) = \begin{vmatrix} w^+(0,\xi) & w^-(0,\xi) \\ w^+_y(A(1),\xi) + Cw^+(A(1),\xi) & w^-_y(A(1),\xi) + Cw^-(A(1),\xi) \end{vmatrix}.$$

Elementary calculations give

$$w^1(y,\xi) = \frac{w^-(y,\xi)w^+(0,\xi) - w^+(y,\xi)w^-(0,\xi)}{\mathcal{D}(\xi)}c_{17}\hat{h}(\xi).$$

Since

$$\hat{h}(\xi) = v_x(1,\xi)\hat{f}(\xi)$$

and $w^\pm$ have the representations (2.17) and (2.18), by the reasonings in the proof of lemma 2.1, we can prove that for ξ large enough there exists a constant c_{18} such that

$$|\widehat{u^3}(1,\xi)| \leqslant c_{18}\mathrm{e}^{-A(1)\sqrt{|\xi|/2}}|\hat{f}(\xi)|.$$

As in remark 2.1 we see that there is another constant, say c_{16}, such that the last inequality is valid for all ξ. The lemma is proved.

3. Stability results

In this section we shall deliver a stability estimate of the Hölder type for the sideways parabolic equation in the quarter plane (1.1)–(1.3) and prove that we can split it into the two Cauchy problems (1.10) and (1.11) as well as deduce it to the problem (1.13).

Theorem 3.1. *Let u be a solution of the problem (1.1)–(1.3) such that $u(0,\cdot) := f \in L_2(0,\infty)$. Suppose that the problem*

$$a(x)v_{xx} + b(x)v_x + c(x)v = 0 \qquad 0 < x < \infty \tag{3.1}$$

$$v(0) = 0 \qquad v(x) \text{ is bounded as } x \to \infty \tag{3.2}$$

has a unique solution. Then there exist constants C_1, C_2 which are depending only on the coefficients $a(x), b(x), c(x)$ such that, for $x \in [0,1]$,

$$\|u(x,\cdot)\| \leqslant C_1\|g\| + C_2\|f\|^{1-A(x)/A(1)}\|g\|^{A(x)/A(1)}. \tag{3.3}$$

Proof. Since $\hat{u}(x,\xi) = v(x,\xi)\hat{f}(\xi)$ and $|v(1,\xi)| \neq 0$ (corollary 2.1),

$$\hat{f}(\xi) = \frac{1}{v(1,\xi)}\hat{g}(\xi).$$

Thus,

$$\hat{u}(x,\xi) = \frac{v(x,\xi)}{v(1,\xi)}\hat{g}(\xi).$$

Hence

$$\begin{aligned}\|u(x,\cdot)\|^2 &= \int_{-\infty}^{\infty}\left|\frac{v(x,\xi)}{v(1,\xi)}\right|^2 |\hat{g}(\xi)|^2\,\mathrm{d}\xi \\ &= \int_{|\xi|\leqslant\xi_0} + \int_{|\xi|\geqslant\xi_0} := A_1 + A_2.\end{aligned}$$

Here ξ_0 is the constant in lemma 2.1.

In virtue of remark 2.1 and (2.22),

$$A_1 = \int_{|\xi|\leqslant\xi_0}\left|\frac{v(x,\xi)}{v(1,\xi)}\right|^2 |\hat{g}(\xi)|^2\,\mathrm{d}\xi \leqslant \frac{c_2}{\bar{c}_1}\|g\|^2 := C_1^2\|g\|^2.$$

Further, by Hölder's inequality,

$$\begin{aligned}A_2 = \int_{|\xi|\geqslant\xi_0}\left|\frac{v(x,\xi)}{v(1,\xi)}\right|^2 |\hat{g}(\xi)|^2\,\mathrm{d}\xi &= \int_{|\xi|\geqslant\xi_0} |v(x,\xi)|^2|\hat{f}(\xi)|^2\,\mathrm{d}\xi \\ &\leqslant (c_2)^2\int_{|\xi|\geqslant\xi_0} \mathrm{e}^{-A(x)\sqrt{2|\xi|}}|\hat{f}(\xi)|^2\,\mathrm{d}\xi \\ &= (c_2)^2\int_{|\xi|\geqslant\xi_0} \mathrm{e}^{-A(x)\sqrt{2|\xi|}}|\hat{f}(\xi)|^{2A(x)/A(1)}|\hat{f}(\xi)|^{2(1-A(x)/A(1))}\,\mathrm{d}\xi \\ &\leqslant (c_2)^2\left(\int_{|\xi|\geqslant\xi_0} \mathrm{e}^{-A(1)\sqrt{2|\xi|}}|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{A(x)/A(1)}\left(\int_{|\xi|\geqslant\xi_0}|\hat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{1-A(x)/A(1)} \\ &\leqslant (c_2)^2\|g\|^{2A(x)/A(1)}\|f\|^{2(1-A(x)/A(1))}.\end{aligned}$$

Thus,

$$\|u(x,\cdot)\|^2 \leqslant (C_1)^2\|g\|^2 + (c_2)^2\|g\|^{2A(x)/A(1)}\|f\|^{2(1-A(x)/A(1))}.$$

The inequality (3.3) follows immediately. The theorem is proved.

Theorem 3.2. *The Cauchy problems (1.10) and (1.11) have unique solutions up to $x = 0$. Thus, we can split the Cauchy problem (1.9) into the two Cauchy problems (1.10) and (1.11).*

Proof. We associate the Cauchy problem (1.10) with the Cauchy problem

$$a(x)v^1_{xx}(x,\xi) + b(x)v^1_x(x,\xi) + c(x)v^1(x,\xi) = \mathrm{i}\xi v^1(x,\xi) \qquad x \in (0,1), \xi \in \mathbb{R} \tag{3.4}$$

$$v^1(1,\xi) = 1 \qquad v^1_x(1,\xi) = 0. \tag{3.5}$$

From lemma 2.2 of Knabner and Vessella [9], there exist constants c_{19}, c_{20} and c_{21} such that

$$|v^1(x,\xi)| \leqslant c_{19}\mathrm{e}^{\bar{A}(x)\sqrt{|\xi|/2}} \tag{3.6}$$

$$|v^1_x(x,\xi)| \leqslant c_{20}\sqrt{|\xi|}\mathrm{e}^{\bar{A}(x)\sqrt{|\xi|/2}} \tag{3.7}$$

$$|v^1_{xx}(x,\xi)| \leqslant c_{21}|\xi|\mathrm{e}^{\bar{A}(x)\sqrt{|\xi|/2}} \tag{3.8}$$

with

$$\bar{A}(x) := \int_0^{1-x} \frac{\mathrm{d}s}{\sqrt{a(1-s)}}.$$

Taking lemma 2.1, corollary 2.1 and the fact that $A(1) = \bar{A}(0)$, $\bar{A}(x) < \bar{A}(0)$ for $0 < x \leqslant 1$ into account, we have

$$\begin{aligned}\|\widehat{u^1}(x,\cdot)\|^2 &= \int_{-\infty}^{\infty} |v^1(x,\xi)|^2 |\hat{g}(\xi)|^2 \,\mathrm{d}\xi \leqslant \int_{-\infty}^{\infty} |v^1(x,\xi)|^2 |v(1,\xi)|^2 |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \\ &\leqslant (c_{19}c_2)^2 \int_{-\infty}^{\infty} \mathrm{e}^{\bar{A}(x)\sqrt{2|\xi|}} \mathrm{e}^{-A(1)\sqrt{2|\xi|}} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \\ &\leqslant c_{22}\|f\|^2.\end{aligned}$$

Analogously, we can prove that $\widehat{u^1}_x(x,\cdot)$ and $\widehat{u^1}_{xx}(x,\cdot)$ belong to $L_2(\mathbb{R})$ for all $x \in [0,1]$.

For the Cauchy problem (1.11) we associate it with the Cauchy problem

$$av^2_{xx}(x,\xi) + bv^2_x(x,\xi) + cv^2(x,\xi) = \mathrm{i}\xi v^2(x,\xi) \qquad x \in (0,1), \xi \in \mathbb{R} \tag{3.9}$$

$$v^2(1,\xi) = 0 \qquad v^2_x(1,\xi) = 1. \tag{3.10}$$

Following the method of proving lemma 2.2 of Knabner and Vessella [9], we see that there exist constants c_{23}, c_{24} and c_{25} such that, for $|\xi| \geqslant \bar{\xi}$,

$$|v^2(x,\xi)| \leqslant c_{23}\sqrt{|\xi|}^{-1} \mathrm{e}^{\bar{A}(x)\sqrt{|\xi|/2}} \tag{3.11}$$

$$|v^2_x(x,\xi)| \leqslant c_{24}\mathrm{e}^{\bar{A}(x)\sqrt{|\xi|/2}} \tag{3.12}$$

$$|v^2_{xx}(x,\xi)| \leqslant c_{25}\sqrt{|\xi|}\mathrm{e}^{\bar{A}(x)\sqrt{|\xi|/2}}. \tag{3.13}$$

The assertion of the theorem for the Cauchy problem (1.11) is now straightforward, in virtue of lemma 2.1.

Theorem 3.3. *The Cauchy problem (1.13) has a unique solution up to $x = 0$. Thus, we can deduce the Cauchy problem (1.9) into the Cauchy problem (1.13) with the help of the boundary value problem (1.12).*

The proof of this theorem follows immediately from lemma 2.2 and the proof of the first part of theorem 3.2.

In [3, 5, 6] a mollification method has been suggested for solving the Cauchy problems (1.10), (1.11) and (1.13) in a stable way. It is worth noting that if we split the Cauchy problem (1.9) into the two Cauchy problems (1.10) and (1.11), then we have to solve two ill-posed problems. In doing so, we have to take care that

$$\|u^1(0,\cdot)\| = \left(\int_{-\infty}^{\infty} |v^1(0,\xi)|^2 |v(1,\xi)|^2 |\hat{f}|^2 \,\mathrm{d}\xi\right)^{1/2} \leqslant c_{26}\|f\|$$

and

$$\|u^2(0,\cdot)\| = \left(\int_{-\infty}^{\infty} |v^2(0,\xi)|^2 |v_x(1,\xi)|^2 |\hat{f}|^2 \,\mathrm{d}\xi\right)^{1/2} \leqslant c_{27}\|f\|.$$

Thus, we have upper bounds for $\|u^1(0,\cdot)\|$ and $\|u^2(0,\cdot)\|$, if the bound for $\|f\|$ is given. Although these bounds are not explicitly found (as the constants c_{26} and c_{27} are generally not known), the mollification method in [3, 5, 6] is applicable to the problems (1.10) and (1.11).

The approach of solving (1.12) and then (1.13) is preferential, since we have to solve only one ill-posed problem, furthermore for the problem (1.13) we have $\|u^4(0,\cdot)\| = \|f\|$.

The mollification method, stable marching difference schemes for the Cauchy problems (1.10), (1.11) and (1.13) as well as numerical experiments for them have been given in [3, 5, 6].

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