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# STABILITY AND REGULARIZATION OF A DISCRETE APPROXIMATION TO THE CAUCHY PROBLEM FOR LAPLACE'S EQUATION*

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**Abstract.** The standard five-point difference approximation to the Cauchy problem for Laplace's equation satisfies stability estimates—and hence turns out to be a well-posed problem—when a certain boundedness requirement is fulfilled. The estimates are of logarithmic convexity type. Herewith, a regularization method will be proposed and associated error bounds can be derived. Moreover, the error between the given (continuous) Cauchy problem and the difference approximation obtained via a suitable minimization problem can be estimated by a discretization and a regularization term.

**Key words.** Cauchy problem, Laplace's equation, ill-posed, stability, regularization, logarithmic convexity

**AMS subject classifications.** 35R25, 65N99

**PII.** S0036142997316955

**1. Introduction.** In this paper, a numerical method for solving the Cauchy problem for Laplace's equation will be proposed and analyzed. As a model problem, we consider

$$\Delta u = 0 \quad \text{in } [0,1]\times[0,1]; \tag{1}$$

$$u|_{x=0} = u|_{x=1} = 0, \quad y \in [0,1]; \tag{2}$$

$$u|_{y=0} = f_1, \quad x \in [0,1]; \tag{3}$$

$$\left\| \frac{\partial u}{\partial y}\right|_{y=0} - f_2^{\varepsilon}\ |^2_{0,(0,1)} \le \varepsilon_2, \tag{4}$$

with a given function $f_1$ and a perturbation $f_2^\varepsilon$ of $f_2 := \partial u/\partial y|_{y=0}$. For simplicity, let $f_1 = 0$. One knows that this problem is conditionally well posed, which means that the original ill-posed problem becomes well posed if the set of solutions is restricted. Such a restriction can be $\|u(.,1)\|_{0,(0,1)} \le M$ (see [13], [14], [15]) or $J(1;u) \le M$ with any one of the functionals $J_*, J_1, J_2$ given in [9].

An analysis of numerical methods for the above Cauchy problem can rarely be found in the literature though a series of papers contains numerical examples (see, e.g., [1], [3], [2], [4], [5], [6], [8], [9], [10], [11], [12]). Among these, the works of Falk [5], Falk and Monk [6], and Han [8] contain error estimates and convergence results.

Falk and Monk [6] have proposed a choice of an optimal mesh size. The difference in the approaches of Falk [5], Falk and Monk [6], and Han [8] lies in the functional to be miminized. In [5], [6] a defect functional is minimized while in [8] a certain energy norm is minimized. Contrary to [5], [6], no orders of convergence are proved in [8].

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The present work may be considered as a discrete version of [9], where, similar to [8], certain energy functionals are mimimized in order to obtain an optimal regularizing approximation. The crucial idea in [9]—and in discrete form here also—is a certain extension of a three-line theorem for harmonic functions proved by Falk [5]. The numerical example given in [9] demonstrates that the approach may be very well suited for a numerical approximation as well. It turns out, that analogously to [9], the five-point difference approximation for the Cauchy problem of Laplace's equation fulfills stability estimates of logarithmic type and leads to a regularization method including error bounds. Moreover, the error between the solution of the original Cauchy problem and the discrete regularizing solution can be estimated, leading to a suggestion for an optimal mesh size. The numerical computations for the classical Hadamard examples as well as inhomogeneous problems demonstrate the efficiency of our approach.

**Notation.**

$\langle . \, , . \rangle$ = Euclidean scalar product in $\mathbb{R}^{J+1}$;

$|.|_2$ = Euclidean norm in $\mathbb{R}^{J+1}$;

$\Omega$ = $[0,1] \times [0,1]\,,\ \partial\Omega =$ boundary of $\Omega$;

$H^k(\Omega)$ = Sobolev space, $k = 0,1,2$ $(H^0(\Omega) = L^2(\Omega))$;

$H^1_*(\Omega)$ = $\left\{ u \in H^1(\Omega) \,\middle|\, u|_{x=0} = u|_{x=1} = 0 \right\}$;

$[0,1]_h$ = $\{x = ih\,,\ i = 0,\dots,J\}\,, \quad h := 1/J$;

$(0,1)_h$ = $\{x = ih\,,\ i = 1,\dots,J-1\}$,

= $x_i = ih, \quad y_j = jh, \quad i,j = 0,\dots,J$;

$\Omega_h$ = $[0,1]_h \times [0,1]_h\,,\ \partial\Omega_h = \Omega_h \cap \partial\Omega$;

$S_h$ = continuous, piecewise linear functions over a uniform triangulation of $[0,1] \times [0,1]$;

$S_{h,0}$ = $\left\{ v_h \in S_h \,\middle|\, v_h = 0 \text{ on } \partial\Omega \right\}$;

$C[0,1]_h$ = continuous, piecewise linear functions over $[0,1]_h$;

$C_0[0,1]_h$ = $\left\{ \varphi_h \in C[0,1]_h \,\middle|\, \varphi_h(0) = \varphi_h(1) = 0 \right\}$;

grid functions will be denoted by capital letters, $V_i^j = v_h(x_i, y_j)\,,\ v_h \in S_h$;

$\nabla w$ = $\left( \dfrac{\partial w}{\partial x}\,,\ \dfrac{\partial w}{\partial y} \right)^\top$ Gradient;

$D^-_{h,x} V_i^j$ = $\dfrac{1}{h}\left( V_i^j - V_{i-1}^j \right)$ first-order (backwards) difference quotient in $x$-direction;

$D^2_{h,x} V_i^j$ = $\dfrac{1}{h^2}\left( V_{i+1}^j - 2V_i^j + V_{i-1}^j \right)$ central difference quotient of second order in $x$-direction;

$D^+_{h,y} V_i^j$ = $\dfrac{1}{h}\left( V_i^{j+1} - V_i^j \right)$ first-order (forward) difference quotient in $y$-direction;

$$\delta_{h,x} V_i^j = V_i^j - V_{i-1}^j \; (= h D^{h,x} \, V_i^j),$$
first-order difference in $x$-direction;

$$\|w\|_{1,\Omega} = \left( \int_\Omega \left\{ \left( \frac{\partial w}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right\} dx\, dy \right)^{1/2}, \quad w \in H_*^1(\Omega);$$

$$\|w\|_{0,(0,1)} = \left( \int_0^1 w^2 \, dx \right)^{1/2}; \quad w \in L^2(0,1);$$

$$|V^j|_{0,h} = \left( h \textstyle\sum_{i=0}^J |V_i^j|^2 \right)^{1/2} \left( = h^{1/2} |V^j|_2 \right);$$

$$\|V\|_{1,h} = \left\{ h \sum_{j=0}^{J-1} \left( |D_{h,y}^+ V^j|_2^2 + |D_{h,x}^- V^j|_2^2 \right) \right\}^{1/2}.$$

**2. Auxiliary results.** We shall consider the simplest finite difference or finite element approximation to the solution of Laplace's equation. For this, let $S_h \subset H^1(\Omega)$ denote the finite element space of all continuous, piecewise linear function on a uniform grid.

A *discrete harmonic function* $w_h \in S_h$ satisfies

$$\int_\Omega \nabla w_h \cdot \nabla \varphi_h \, dx\, dy = 0 \quad \forall \varphi_h \in S_{h,0}. \tag{5}$$

In the present simple geometry and triangulation, $W_i^j = w_h(x_i, y_j)$ satisfies the five-point difference equation at interior mesh points. Hence, one can write (5) as

$$W^{j+1} - 2W^j + W^{j-1} = L_h W^j, \tag{6}$$

with $W^j = (W_0^j, \ldots, W_J^j)^\top$ and $L_h$ the $(J-1) \times (J-1)$ symmetric, tridiagonal matrix with 2 in the diagonal and $-1$ in the off diagonals. With the second-order difference quotient $D_{h,x}^2$, $L_h$ in (6) can be written as $L_h = -h^2 D_{h,x}^2$. We additionally denote the *discrete Laplace operator* by $\Delta_h$,

$$(\Delta_h W)^j = W^{j+1} - 2W^j + W^{j-1} - L_h W^j.$$

In the following, we use the notion "grid function" or "discrete function" for both the vector field $W$ and for $w_h$.

As a discrete analogue to the Cauchy problem (1)–(4), we consider the following discrete boundary value problem:

$$U^{j+1} - 2U^j + U^{j-1} = L_h U^j, \quad j = 1, \ldots, J-1, \tag{7}$$

$$U_0^j = U_J^j = 0, \quad j = 0, \ldots, J, \tag{8}$$

$$U_i^0 = f_{1,h}(x_i), \quad \frac{1}{h}(U_i^1 - U_i^0) = f_{2,h}(x_i), \quad i = 1, \ldots, J-1, \tag{9}$$

with grid functions $f_{1,h}$, $f_{2,h}$. Thus $u_h(x_i, y_j) = U_i^j$ is a discrete harmonic function with zero boundary values at $i = 0, J$ and discrete Cauchy data (9) for $j = 0$.

Let us define

$$D^j := U^{j+1} - U^j, \quad j = 0, \ldots, J-1; \tag{10}$$

then the vectors $D^j = (D_0^j, \ldots, D_J^j)^\top$ also define a discrete harmonic function satisfying the following discrete boundary value problem,

$$D^{j+1} - 2D^j + D^{j-1} = L_h D^j, \quad j = 1, \ldots, J-1, \tag{11}$$

$$D_0^j = D_J^j \ = \ 0, \quad j = 1, \ldots, J, \tag{12}$$

$$D^0 = h f_{2,h}\,, \quad \frac{1}{h}\left(D^1 - D^0\right) \ = \ \frac{1}{h} L_h U^1. \tag{13}$$

In order to define $D^j$ also for $j = J$, we set

$$D^J := 2D^{J-1} - D^{J-2} + L_h D^{J-1}, \tag{14}$$

which means that $U^{J+1}$ is defined by the equation of a discrete harmonic function.

$$U^{J+1} - 2U^J + U^{J-1} = L_h U^J. \tag{15}$$

We now prove a *discrete Lagrange identity* (cf. (16)) and a conclusion thereof for discrete harmonic functions.

Lemma 1. *For any two grid functions $V$ and $W$, with $V_J^j = V_0^j = 0$, $W_J^j = W_0^j = 0$, $j = 0, \ldots, J$, one has*

$$\begin{aligned}
&\sum_{i=1}^{J-1} V_i^j\,(\Delta_h W)_i^j - W_i^j\,(\Delta_h V)_i^j \\
&= \left\langle V^j - V^{j-1}, W^j \right\rangle - \left\langle V^j, W^j - W^{j-1} \right\rangle \\
&\quad - \left(\left\langle V^{j+1} - V^j, W^{j+1}\right\rangle - \left\langle V^{j+1}, W^{j+1} - W^j \right\rangle\right), \quad j = 1, \ldots, J-1.
\end{aligned} \tag{16}$$

*If, additionally, $\Delta_h V = \Delta_h W = 0$, then*

$$\begin{aligned}
&\left\langle V^j, W^j - W^{j-1}\right\rangle - \left\langle V^j - V^{j-1}, W^j \right\rangle \\
&\quad = \ \left\langle V^{j+1}, W^{j+1} - W^j \right\rangle - \left\langle V^{j+1} - V^j, W^{j+1}\right\rangle, \quad j = 1, \ldots, J-1.
\end{aligned} \tag{17}$$

*Proof.*

i. By definition of $\Delta_h$ we obtain

$$\begin{aligned}
&V_i^j(\Delta_h W)_i^j - W_i^j(\Delta_h V)_i^j \\
&= \ - V_i^j(L_h W)_i^j + W_i^j(L_h V)_i^j \\
&\quad + \Big(V_i^j(W_i^{j+1} - W_i^j) - V_i^j(W_i^j - W_i^{j-1}) \\
&\quad - W_i^j(V_i^{j+1} - V_i^j) + W_i^j(V_i^j - V_i^{j-1})\Big)\,.
\end{aligned}$$

ii. By definition of $L_h$,

$$\begin{aligned}
L_h W_i^j &= -W_{i+1}^j + 2W_i^j - W_{i-1}^j \\
&= -\left(\left(W_{i+1}^j - W_i^j\right) - \left(W_i^j - W_{i-1}^j\right)\right).
\end{aligned}$$

Using summation by parts, one obtains

$$\begin{aligned}
\sum_{i=1}^{J-1} V_i^j\left(-L_h W_i^j\right) &= \sum_i V_i^j\left(\left(W_{i+1}^j - W_i^j\right) - \left(W_i^j - W_{i-1}^j\right)\right) \\
&= -\sum_{i=1}^{J}\left(W_i^j - W_{i-1}^j\right)\left(V_i^j - V_{i-1}^j\right) + F_J V_J^j - F_0 V_0^j\,.
\end{aligned}$$

Here, according to our assumption, $V_0^j = V_J^j = 0$, and, in order to define $F_J$, we can arbitrarily extend $W_i^j$ for $i = J+1$. Analogously, because of $W_0^j = W_J^j = 0$,

$$\sum_{i=1}^{J-1} W_i^j \left(-L_h V_i^j\right) = -\sum_{i=1}^{J} \left(V_i^j - V_{i-1}^j\right)\left(W_i^j - W_{i-1}^j\right). \tag{18}$$

We thus have

$$\sum_{i=1}^{J-1} -V_i^j (L_h W)_i^j + W_i^j (L_h V)_i^j = 0, \quad j = 1, \ldots, J-1.$$

iii. Using the notation of the Euclidean scalar product, by summation in part i we obtain

$$\begin{aligned}
&\sum_{i=1}^{J-1} V_i^j \,(\Delta_h W)_i^j - W_i^j \,(\Delta_h V)_i^j \\
&\quad = \left\langle V^j, W^{j+1} - W^j \right\rangle - \left\langle V^j, W^j - W^{j-1} \right\rangle \\
&\qquad - \left\langle W^j, V^{j+1} - V^j \right\rangle + \left\langle W^j, V^j - V^{j-1} \right\rangle, \quad j = 1, \ldots, J-1.
\end{aligned}$$

This proves (16), and, for discrete harmonic functions $V$ and $W$, we obtain (17). ∎

LEMMA 2. *For any two discrete harmonic grid functions $V$ and $W$, with $V_0^j = V_J^j = W_0^j = W_J^j = 0$, $j = 0, \ldots, J$, the following relations hold:*

$$\begin{aligned}
&\left\langle V^{j-\nu+1}, W^{j-\nu+1} - W^{j-\nu} \right\rangle - \left\langle V^{j-\nu+1} - V^{j-\nu}, W^{j-\nu+1} \right\rangle \\
&\quad = \left\langle V^{j+1}, W^{j+1} - W^j \right\rangle - \left\langle V^{j+1} - V^j, W^{j+1} \right\rangle \\
&\qquad \forall\, 1 \le j \le J,\ \nu \ge 0 : j - \nu \ge 0.
\end{aligned} \tag{19}$$

*Proof.* Using (16) and summing up from $j-\nu+1$ until $j$ we obtain

$$\begin{aligned}
0 &= \sum_{\mu=j-\nu+1}^{j} \Big(\left\langle V^\mu, W^{\mu+1} - W^\mu \right\rangle - \left\langle V^\mu, W^\mu - W^{\mu-1} \right\rangle \\
&\qquad - \left\langle W^\mu, V^{\mu+1} - V^\mu \right\rangle + \left\langle W^\mu, V^\mu - V^{\mu-1} \right\rangle\Big) \\
&= \sum_{\mu=j-\nu+1}^{j} \Big(\left\langle V^\mu, W^{\mu+1} \right\rangle + \left\langle V^\mu, W^{\mu-1} \right\rangle \\
&\qquad - \left\langle W^\mu, V^{\mu+1} \right\rangle - W^\mu, V^{\mu-1}\Big) \\
&= \left\langle V^{j-\nu+1}, W^{j-\nu} \right\rangle - \left\langle V^{j-\nu}, W^{j-\nu+1} \right\rangle + \left\langle V^j, W^{j+1} \right\rangle - \left\langle V^{j+1}, W^j \right\rangle.
\end{aligned}$$

Obviously,

$$\begin{aligned}
&\left\langle V^{j-\nu+1}, W^{j-\nu} \right\rangle - \left\langle V^{j-\nu}, W^{j-\nu+1} \right\rangle \\
&\quad = -\left\langle V^{j-\nu+1}, W^{j-\nu+1} - W^{j-\nu} \right\rangle + \left\langle V^{j-\nu+1} - V^{j-\nu}, W^{j-\nu+1} \right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle V^j, W^{j+1} \right\rangle - \left\langle V^{j+1}, W^j \right\rangle \\
&\quad = \left\langle V^{j+1}, W^{j+1} - W^j \right\rangle - \left\langle V^{j+1} - V^j, W^{j+1} \right\rangle,
\end{aligned}$$

which proves (19). ∎

Before we prove the last lemma in this section, we remark that the vectors $D^j$—and hence also the vectors $D^+_{h,y}U^j$ of first-order difference quotients—are also defined for $j = J$ because we assume that the auxiliary vector $U^{J+1}$ is defined by (15). Moreover, we assume that any vector under consideration is extended in a constant way in the $x$-direction at $x = 0$, i.e., $V_{-1} = V_0$. Therefore the $\mathbb{R}^J$-vectors $D^-_{h,x}U^j$ can be considered as $\mathbb{R}^{J+1}$-vectors with vanishing zeroth component. We also note that the $U^j_i$ themselves vanish for $i = 0$.

LEMMA 3. *If $U$ satisfies (7)–(9) and the quadratic functional $J(U)$ is defined by*

$$J_h(U)_j := \left|D^+_{h,y}U^j\right|^2_{0,h} + \left|D^-_{h,x}U^j\right|^2_{0,h}, \quad j = 0, \ldots, J, \tag{20}$$

*then*

$$J(U)_j \leq J(U)^{1/2}_{j+\nu} J(U)^{1/2}_{j-\nu} \tag{21}$$

*for every $j = 1, \ldots, J-1$, $\nu > 0$ with $0 \leq j-\nu$, $j+\nu \leq J$.*

*Proof.* With $D^j$ defined in (10) we can take $W^j = D^j$ in Lemma 2, since $D^j$ is a discrete harmonic function with vanishing boundary values for $i = 0$, $i = J$ (see (11) and (12)). Using $V^\ell = U^{2j-\ell+1}$, $\ell = j-1, j, j+1$, and $W^j = \tilde{D}^j := D^{j-1}$ in (19), we have

$$\begin{array}{lll} V^{j+1} = U^j, & V^j = U^{j+1}, & V^{j-1} = U^{j+2}, \\ V^{j-\nu+1} = U^{j+\nu}, & V^{j-\nu} = U^{j+\nu+1} & \end{array}$$

and obtain

$$\begin{aligned} &\left\langle U^{j+\nu}, \tilde{D}^{j-\nu+1} - \tilde{D}^{j-\nu}\right\rangle - \left\langle U^{j+\nu} - U^{j+\nu+1}, \tilde{D}^{j-\nu+1}\right\rangle \\ &\quad = \langle U^j, \tilde{D}^{j+1} - \tilde{D}^j\rangle - \langle U^j - U^{j+1}, \tilde{D}^{j+1}\rangle, \end{aligned}$$

which can be written as ($\tilde{D}^{j+1} = D^j$)

$$\begin{aligned} &\left\langle U^j, D^j - D^{j-1}\right\rangle + \left|D^j\right|^2_2 \\ &\quad = \left\langle U^{j+\nu}, D^{j-\nu} - D^{j-\nu-1}\right\rangle + \left\langle D^{j+\nu}, D^{j-\nu}\right\rangle, \\ &\qquad j = 1, \ldots, J-1,\ \nu > 0\ (0 \leq j-\nu,\ j+\nu \leq J). \end{aligned} \tag{22}$$

Furthermore, according to Lemma 2 (see (18)),

$$\begin{aligned} \left\langle U^j, D^j - D^{j-1}\right\rangle &= \left\langle U^j, U^{j+1} - 2U^j + U^{j-1}\right\rangle \\ &= \left\langle U^j, L_h U^j\right\rangle = \sum_{i=1}^J \left(U^j_i - U^j_{i-1}\right)^2 = \left|\delta_{h,x}U^j\right|^2_2, \\ \left\langle U^{j+\nu}, D^{j-\nu} - D^{j-\nu-1}\right\rangle &= \left\langle U^{j+\nu}, U^{j-\nu+1} - 2U^{j-\nu} + U^{j-\nu-1}\right\rangle \\ &= \left\langle U^{j+\nu}, L_h U^{j-\nu}\right\rangle = \sum_{i=1}^J \left(U^{j+\nu}_i - U^{j+\nu}_{i-1}\right)\left(U^{j-\nu}_i - U^{j-\nu}_{i-1}\right) \\ &= \left\langle \delta_{h,x}U^{j+\nu}, \delta_{h,x}U^{j-\nu}\right\rangle. \end{aligned}$$

Hence, the left-hand side of (22) is only $hJ_h(U)_j$,

$$\begin{aligned} &\left\langle U^j, D^j - D^{j-1}\right\rangle + \left|D^j\right|^2_2 = \left|\delta_{h,x}U^j\right|^2_2 + |D^j|^2_2 \\ &\quad = h^2\left(\left|D^-_{h,x}U^j\right|^2_2 + \left|D^+_{h,y}U^j\right|^2_2\right) = hJ_h(U)_j. \end{aligned}$$

The right-hand side of (22) can be estimated as follows:

$$\begin{aligned}
&\left\langle U^{j+\nu}, D^{j-\nu} - D^{j-\nu-1}\right\rangle + \left\langle D^{j+\nu}, D^{j-\nu}\right\rangle \\
&\quad = \left\langle \delta_{h,x} U^{j+\nu}, \delta_{h,x} U^{j-\nu}\right\rangle + \left\langle D^{j+\nu}, D^{j-\nu}\right\rangle \\
&\quad \leq \left(\left|\delta_{h,x} U^{j+\nu}\right|_2^2 + \left|D^{j+\nu}\right|_2^2\right)^{1/2} \left(\left|\delta_{h,x} U^{j-\nu}\right|_2^2 + \left|D^{j-\nu}\right|_2^2\right)^{1/2} \\
&\quad = h J_h(U)_{j+\nu}^{1/2} J_h(U)_{j-\nu}^{1/2}.
\end{aligned}$$

Hence (21) is proved. □

**3. Stability.** We are now able to prove a logarithmic convexity-type estimate for the solution of (7)–(9).

THEOREM 1. *With the solution $U$ of* (7)–(9) *and the functional $J_h(U)$ defined in* (20)*, the following estimates hold:*

$$J_h(U)_j \leq J_h(U)_J^{jh} J_h(U)_0^{1-jh}, \quad j = 0, \ldots, J-1. \tag{23}$$

*Proof.*

i. In the case $J_h(U)_0 = 0$, we set $j = \nu$ in (21) and obtain $J_h(U)_j \leq 0 \, \forall j$; thus $J_h(U)_j = 0$, which proves (23) in this case.

ii. In the case $J_h(U)_0 \neq 0$, we set

$$\varphi_j := \ln \left\{J_h(U)_j / J_h(U)_0\right\}, \quad j = 0, \ldots, J,$$

and extend $\{\varphi_j\}_j$ to a continuous, piecewise linear function $F : [0,1] \to \mathbb{R}$,

$$F(\eta) = \frac{\eta - y_{j-1}}{h} \left(\varphi_j - \varphi_{j-1}\right) + \varphi_{j-1}, \quad \eta \in \left[y_{j-1}, y_j\right], \; j = 1, \ldots, J.$$

Obviously, $F(0) = 0$, and we shall prove that

$$F\left(\frac{y + \tilde{y}}{2}\right) \leq \frac{1}{2}\left(F(y) + F(\tilde{y})\right) \quad \forall\, y, \tilde{y} \in [0,1]. \tag{24}$$

The convexity of $F$ and standard arguments (see, e.g., Han and Reinhardt [9, Thm. 2.1]) then ensure that $F(y) \leq y F(1)$. By the definition of $F$ and $\varphi_j$ the desired estimate (23) is hereby proved because at $y = jh$

$$\begin{aligned}
&\ln \left\{J_h(U)_j / J_h(U)_0\right\} \leq jh \ln \left\{J_h(U)_J / J_h(U)_0\right\} = \ln \left\{\left(\frac{J_h(U)_J}{J_h(U)_0}\right)^{jh}\right\} \\
\iff \quad & \frac{J_h(U)_j}{J_h(U)_0} \leq \left(\frac{J_h(U)_J}{J_h(U)_0}\right)^{jh} \\
\iff \quad & J_h(U)_j \;\leq\; J_h(U)_J^{jh} J_h(U)_0^{1-jh}.
\end{aligned}$$

iii. In order to prove (24), we first observe that

$$\varphi_j - \varphi_{j-1} \leq \varphi_{j+1} - \varphi_j, \quad j = 1, \ldots, J-1. \tag{25}$$

This follows from (21) with $\nu = 1$, since

$$\varphi_j \;=\; \ln J_h(U)_j \leq \ln \left\{J_h(U)_{j+1}^{1/2} J_h(U)_{j-1}^{1/2}\right\} \;=\; \frac{1}{2}\left(\varphi_{j+1} + \varphi_{j-1}\right), \quad j = 1, \ldots, J-1.$$

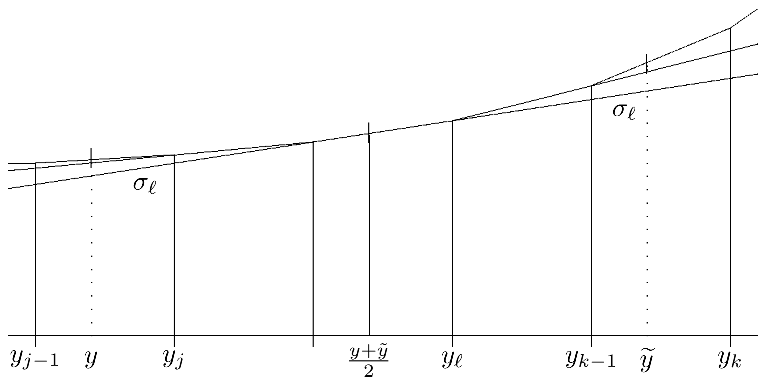


FIG. 1. *The idea of the proof of* (24).

Let us first consider the case of $y, \tilde{y} \in I_j := [y_{j-1}, y_j]$ for one $j \in \{1, \ldots, J\}$. In this case,

$$\begin{aligned}\frac{1}{2}(F(y) + F(\tilde{y})) &= \frac{1}{2}\left(\frac{y - y_{j-1}}{h} + \frac{\tilde{y} - y_{j-1}}{h}\right)(\varphi_j - \varphi_{j-1}) + \varphi_{j-1} \\ &= \frac{1}{2}\left(\frac{y + \tilde{y}}{2} - y_{j-1}\right)(\varphi_j - \varphi_{j-1}) + \varphi_j = F\left(\frac{y+\tilde{y}}{2}\right).\end{aligned}$$

Now, let $y \in I_j$ , $\tilde{y} \in I_k$, and $(y + \tilde{y})/2 \in I_\ell$, where $k > j$ without loss of generality (see Figure 1). Let us denote

$$\sigma_\ell(y) := \varphi_{\ell-1} + \frac{y - y_{\ell-1}}{h}(\varphi_\ell - \varphi_{\ell-1}), \quad y \in [0, 1].$$

By (25), we have

$$\varphi_j - \varphi_{j-1} \le \varphi_{j+1} - \varphi_j \le \cdots \le \varphi_\ell - \varphi_{\ell-1} \le \cdots \le \varphi_k - \varphi_{k-1}$$

and, therefore,

$$\sigma_\ell(y) = \varphi_{\ell-1} + \frac{y - y_{\ell-1}}{h}(\varphi_\ell - \varphi_{\ell-1}) \le F(y).$$

Indeed,

$$\begin{aligned}&\varphi_{\ell-1} + \frac{y - y_{\ell-1}}{h}(\varphi_\ell - \varphi_{\ell-1}) \le \varphi_{\ell-2} + \frac{y - y_{\ell-2}}{h}(\varphi_{\ell-1} - \varphi_{\ell-2}) \\ \iff\ &(\varphi_{\ell-1} - \varphi_{\ell-2}) + \frac{y - y_{\ell-1}}{h}(\varphi_\ell - \varphi_{\ell-1}) \le \frac{y - y_{\ell-2}}{h}(\varphi_{\ell-1} - \varphi_{\ell-2}) \\ \iff\ &\frac{y - y_{\ell-1}}{h}(\varphi_\ell - \varphi_{\ell-1}) \le \left(\frac{y - y_{\ell-2}}{h} - 1\right)(\varphi_{\ell-1} - \varphi_{\ell-2}) \\ &= \frac{y - y_{\ell-1}}{h}(\varphi_{\ell-1} - \varphi_{\ell-2})\end{aligned}$$

and, analogously,

$$\varphi_{\ell-2} + \frac{y - y_{\ell-2}}{h}(\varphi_{\ell-1} - \varphi_{\ell-2}) \le \varphi_{\ell-3} + \frac{y - y_{\ell-3}}{h}(\varphi_{\ell-2} - \varphi_{\ell-3})$$

and so on until

$$\varphi_j + \frac{y - y_j}{h}\,(\varphi_{j+1} - \varphi_j) \le \varphi_{j-1} + \frac{y - y_{j-1}}{h}\,(\varphi_j - \varphi_{j-1}) = F(y).$$

In the same way, one sees that $\sigma_\ell(\tilde{y}) \le F(\tilde{y})$. Combining the first case with these estimates, we finally obtain

$$\begin{aligned} F\left(\frac{y+\tilde{y}}{2}\right) &= \frac{1}{2}\Big(\sigma_\ell(y) + \sigma_\ell(\tilde{y})\Big) \\ &\le \frac{1}{2}\Big(F(y) + F(\tilde{y})\Big), \end{aligned}$$

which completes the proof of (23). ∎

We remark that for the proof of Theorem 1 we need only the basic estimate (21) of logarithmic convexity for $\nu = 1$.

From (23), a stability estimate for the solution of (7)–(9) can be deduced with respect to the seminorm $\|.\|_{1,h}$. We note that because of the vanishing boundary values at $i = 0$ and $i = J$ the discrete Poincaré–Friedrichs inequality ensures that $\|.\|_{1,h}$ is a norm for such grid functions.

THEOREM 2. *If the solution $U$ of* (7)–(9) *satisfies $J_h(U)_J \le M$ with $M > 0$, then*

$$\|U\|^2_{1,h} \le \frac{M - \varepsilon_0}{\ln M - \ln \varepsilon_0}, \tag{26}$$

*where* $\varepsilon_0 := J_h(U)_0$.

*Proof.* Summing up (23), one obtains

$$\begin{aligned} h\sum_{j=0}^{J-1} J_h(U)_j &\le h\sum_j J_h(U)_J^{jh}\, J_h(U)_0^{1-jh} \\ &= J_h(U)_0\, h\sum_j \Big(J_h(U)_J / J_h(U)_0\Big)^{jh} \\ &= J_h(U)_0\, h\sum_j e^{jh\ln\tilde{a}} \qquad \Big(\tilde{a} := J_h(U)_J / J_h(U)_0\Big) \\ &\le J_h(U)_0 \int_0^1 e^{y\ln\tilde{a}}\,dy \\ &= J_h(U)_0 \frac{\tilde{a} - 1}{\ln\tilde{a}} = \frac{J_h(U)_J - J_h(U)_0}{\ln J_h(U)_J - \ln J_h(U)_0}. \end{aligned}$$

If one takes into consideration that

$$h\sum_{j=0}^{J-1} J_h(U)_j = \|U\|^2_{1,h}\,,$$

the stability estimate (26) is proved. ∎

In the trivial case $J_h(U)_0 = 0$, we have $J_h(U)_j = 0\,\forall\, j$, and $\|U\|_{1,h} = 0$. In the case $(M =) J_h(U)_J = J_h(U)_0\,(= \varepsilon_0)$, we can take $\varepsilon_0 = J_h(U)_0$ as the right-hand side in (26) due to l'Hospital's rule. Let us emphasize that up to now there has been no need for restriction on the mesh size $h$.

**4. A regularization method.** Based on the stability estimate (26), we will propose a regularization method for problem (7)–(9). Let $U$ be the unique solution of problem (7)–(9), where, for simplicity, $f_{1,h} = 0$ in (9). In order to check the regularizing properties of our approach we allow perturbations of $f_{2,h}$ (see also (50) for a concrete choice of $f_{2,h}^{\varepsilon}$),

$$\left|f_{2,h} - f_{2,h}^{\varepsilon}\right|_{0,h}^{2} =: \varepsilon_f .$$

Then, instead of (7)–(9) we consider the problem

$$\tilde{U}^{j+1} - 2\tilde{U}^{j} + \tilde{U}^{j-1} = L_h \tilde{U}^{j}, \quad j = 1, \dots, J-1, \tag{27}$$

$$\tilde{U}_0^j = \tilde{U}_J^j = 0, \quad j = 0, \dots, J, \tag{28}$$

$$\tilde{U}_i^0 = 0, \quad i = 1, \dots, J-1, \tag{29}$$

$$\left|\frac{1}{h}\left(\tilde{U}^1 - \tilde{U}^0\right) - f_{2,h}^{\varepsilon}\right|_{0,h}^{2} \le \varepsilon, \tag{30}$$

with an $\varepsilon \ge \varepsilon_f$. Problem (27)–(30) may have many solutions—the solution $U$ of (7)–(9) is one of them. The question arises, which of the solutions of (27)–(30) is an approximation to $U$?

Let $g_h \in C_0[0,1]_h$ and $G$ be the associated grid function, $G_i = g_h(x_i)$, with vanishing boundary values, $G_0 = G_J = 0$. Let $U_G$ be a solution of (27)–(29) with

$$U_{G,i}^{J} = G_i, \quad i = 0, \dots, J; \tag{31}$$

$U_G$ exists and is uniquely determined. Analogously to $U$ given by (7), for $j = J+1$, let $U_G^{J+1}$ be defined by the equation of a discrete harmonic function; i.e., (27) should also hold for $j = J$ (see also (14)).

With $U_G^{J+1}$ defined as in (15), let

$$\begin{aligned} A_0 g_h &:= A_0 G := \tfrac{1}{h}\left(U_G^1 - U_G^0\right), \\ A_J g_h &:= A_J G := \tfrac{1}{h}\left(U_G^{J+1} - U_G^J\right), \end{aligned} \tag{32}$$

which define bounded linear operators from $C_0[0,1]_h$ into $C[0,1]_h$.

The set

$$K_{\varepsilon,h} := \left\{ g_h \in C_0[0,1]_h \;\middle|\; \left|A_0 g_h - f_{2,h}^{\varepsilon}\right|_{0,h} \le \sqrt{\varepsilon} \right\} \tag{33}$$

defines a closed convex subset of $C_0[0,1]_h$ which is not empty. The latter statement holds because the solution $\hat{U} := U_{\hat{G}}$ of (27)–(29), (31) with $\hat{G}_i = U_i^J$, $i = 1, \dots, J-1$, lies in $K_{\varepsilon,h}$,

$$\begin{aligned} \left|A_0 \hat{G} - f_{2,h}^{\varepsilon}\right|_{0,h}^{2} &= \left|\frac{1}{h}(U^1 - U^0) - f_{2,h}^{\varepsilon}\right|_{0,h}^{2} \\ &= \left|f_{2,h} - f_{2,h}^{\varepsilon}\right|_{0,h}^{2} \le \varepsilon_f \le \varepsilon. \end{aligned}$$

For any $g_h \in K_{\varepsilon,h}$, $U_G$ is obviously a solution of (27)–(30) and, furthermore,

$$I_h(G) := J_h(U_G)_J = \left|A_J g_h\right|_{0,h}^{2} + \left|D_{h,x}^{-} G\right|_{0,h}^{2}. \tag{34}$$

We now consider the following minimization problem:

Find $g_h^\varepsilon \in K_{\varepsilon,h}$, such that

$$I_h(U_{G^\varepsilon}) = \min_{q_h \in K_{\varepsilon,h}} I_h(U_Q). \tag{35}$$

Since

$$a(g_h, q_h) := \langle A_J g_h, A_J q_h \rangle + \left\langle D_{h,x}^- g_h, D_{h,x}^- q_h \right\rangle$$

defines a bounded, coercive bilinear form on $C_0[0,1]_h \times C_0[0,1]_h$, problem (35) has a unique solution, which we denote by $g_h^\varepsilon$, $G_i^\varepsilon = g_h^\varepsilon(x_i)$, $i = 0, \dots, J$. The following theorem shows that $U_{G^\varepsilon}$ is indeed an approximation of the solution $U$ of (7)–(9).

THEOREM 3. *Let $h > 0$ be fixed and $\varepsilon, \varepsilon_f$ be arbitrary constants with $\varepsilon \geq \varepsilon_f \geq 0$. Let $U$ be the solution of* (7)–(9) *with $f_{1,h} = 0$ and $g_h^\varepsilon$ be the solution of the minimization problem* (35). *Then with $G_i^\varepsilon = g_h^\varepsilon(x_i)$, $i = 0, \dots, J$, the solution $U_{G^\varepsilon}$ of* (27)–(29), (31) *is an approximation of $U$ satisfying the error estimate*

$$\|U - U_{G^\varepsilon}\|_{1,h}^2 \leq 4 \frac{M - \varepsilon_0}{\ln M - \ln \varepsilon_0} \tag{36}$$

*provided $J_h(U)_J \leq M$, where $\varepsilon_0 = J_h(U)_0$.*

*Proof.* For $g_h^\varepsilon$ and the associated $U_{G^\varepsilon}$, one has

$$J_h(U_{G^\varepsilon})_J = I_h(g_h^\varepsilon) \leq I_h(\hat{g}_h) = J_h(U)_J \leq M,$$

where $\hat{g}_h(x_i) = U_i$, $i = 0, \dots, J$. Note that $\hat{g}_h \in K_{\varepsilon,h}$ as shown above. Set

$$M_G := J_h(U - U_{G^\varepsilon})_J, \quad \varepsilon_G := J_h(U - U_{G^\varepsilon})_0.$$

Then $M_G \leq 4M$ and $\varepsilon_G \leq 4\varepsilon_0$. Indeed, for any $U$ and $V$,

$$\begin{aligned} hJ_h(U - V)_J &= \left|(U - V)^{J+1} - (U - V)^J\right|_2^2 + \left|\delta_{h,x}(U - V)^J\right|_2^2 \\ &= \left|(U^{J+1} - U^J) - (V^{J+1} - V^J)\right|_2^2 + \left|\delta_{h,x}U^J - \delta_{h,x}V^J\right|_2^2 \\ &\leq 2\left(\left|U^{J+1} - U^J\right|_2^2 + \left|V^{J+1} - V^J\right|_2^2\right) + 2\left(\left|\delta_{h,x}U^J\right|_2^2 + \left|\delta_{h,x}V^J\right|_2^2\right) \\ &= 2h(J_h(U)_J + J_h(V)_J) \end{aligned}$$

and, according to (35), $V = U_{G^\varepsilon}$ satisfies $J_h(U_{G^\varepsilon})_J \leq J_h(U)_J \leq M$; in the same way, the inequality $\varepsilon_G \leq 4\varepsilon_0$ can be proved. Using the representation

$$\frac{M_G - \varepsilon_G}{\ln M_G - \ln \varepsilon_0} = \int_0^1 M_G^s \varepsilon_G^{1-s}\, ds$$

and the inequalities just proved, we finally obtain

$$\|U - U_{G^\varepsilon}\|_{1,h}^2 \leq \frac{M_G - \varepsilon_G}{\ln M_G - \ln \varepsilon_G} \leq 4 \int_0^1 M^s \varepsilon_0^{1-s}\, ds = 4\frac{M - \varepsilon_0}{\ln M - \ln \varepsilon_0}. \qquad \square$$

We close this section by giving sufficient conditions for the stabilizing condition $J_h(U)_J \leq M$. Due to the definition of $U^{J+1}$ (see (15)), this is fulfilled if

$$|D_{h,x}^- U^J|_{0,h} \leq M_0, \quad h|D_{h,x}^2 U^J|_{0,h} \leq M_1, \quad |D_{h,y}^+ U^{J-1}|_{0,h} \leq M_2.$$

**5. Error estimates.** Let us assume that the sufficient smooth function $u^*(x,y)$ is a solution of the Cauchy problem (1)–(4). In this section it is our aim to estimate the error between $u^*(x,y)$ and the numerical approximation $U_{G^\varepsilon}$ obtained via the minimization problem (35). We define $g^* = u^*|_{y=1}$; then $u^*$ satisfies the following properly posed boundary value problem:

$$\Delta u = 0 \quad \text{in } [0,1]\times[0,1], \tag{37}$$

$$u|_{x=0} = u|_{x=1} = 0, \quad y\in[0,1], \tag{38}$$

$$u|_{y=0} = 0, \quad u|_{y=1} = g^*, \quad x\in[0,1]. \tag{39}$$

Consider the discrete approximation of the problem (37)–(39) and denote by $u_h^* \in S_h$ the grid function which solves (27)–(29) together with $u_h^*(x_i,1) = g^*(x_i)$, $i = 0,1,\dots,J$. Using the standard methods, we obtain the following error estimates between $u^*$ and $u_h^*$:

$$\|u^* - u_h^*\|_{1,h} = \mathrm{O}(h), \tag{40}$$

$$|u^*(x_i,y_j) - u_h^*(x_i,y_j)| = \mathrm{O}(h^2), \quad 0\le i,j\le J. \tag{41}$$

Thus

$$\begin{aligned} f_{2,h}^*(x_i) &:= \frac{1}{h}\left(u_h^*(x_i,y_1) - u_h^*(x_i,y_0)\right) \\ &= \frac{1}{h}\left(u^*(x_i,y_1) - u^*(x_i,y_0)\right) + \mathrm{O}(h) \\ &= \frac{\partial u^*(x_i,0)}{\partial y} + \mathrm{O}(h). \end{aligned}$$

We define

$$f_{2,h}(x_i) = f_2(x_i), \quad 0\le i\le J;$$

then

$$\left|f_{2,h} - f_{2,h}^*\right|_{0,h} \le \sqrt{\varepsilon_2} + c_1 h,$$

with $c_1 > 0$.

On the other hand,

$$\begin{aligned} \left|f_{2,h}^* - f_{2,h}^\varepsilon\right|_{0,h} &\le \left|f_{2,h}^* - f_{2,h}\right|_{0,h} + \left|f_{2,h} - f_{2,h}^\varepsilon\right|_{0,h} \\ &= c_1 h + \sqrt{\varepsilon_2} + \sqrt{\varepsilon_f}. \end{aligned}$$

We set $\varepsilon_{f,h} = \left(c_1 h + \sqrt{\varepsilon_2} + \sqrt{\varepsilon_f}\right)^2$. For any $\varepsilon \ge \varepsilon_{f,h}$, we know that there exists a unique $g_h^\varepsilon$ as solution of the minimization problem (35) and an associated $u_h^\varepsilon \in S_h$ which solves (27)–(29) and $u_h^\varepsilon(x_i,1) = g_h^\varepsilon(x_i)$, $i = 0,\dots,J$. We now want to estimate the error $u^* - u_h^\varepsilon$ and we have

$$\|u^* - u_h^\varepsilon\|_{1,h} \le \|u^* - u_h^*\|_{1,h} + \|u_h^* - u_h^\varepsilon\|_{1,h}.$$

The first term on the right-hand side can be estimated from (40). We now estimate the second term; $u_h^*$ and $u_h^\varepsilon$ satisfy (27)–(29). Furthermore we have

$$\left|\frac{1}{h}\Big(u_h^*(.,y_1) - u_h^*(.,y_0)\Big) - f_{2,h}^\varepsilon\right|_{0,h}^2 \le \varepsilon;$$

thus

$$J_h(u_h^\varepsilon)_J \leq J_h(u_h^*)_J. \tag{42}$$

On the other hand,

$$\begin{aligned} J_h(u_h^* - u_h^\varepsilon)_0 &= \left| f_{2,h}^* - \frac{1}{h}\Big(u_h^\varepsilon(.,y_1) - u_h^\varepsilon(.,y_0)\Big)\right|_{0,h}^2 \\ &\leq \left\{ \left|f_{2,h}^* - f_{2,h}^\varepsilon\right|_{0,h} + \left| f_{2,h}^\varepsilon - \frac{1}{h}\Big(u_h^\varepsilon(.,y_1) - u_h^\varepsilon(.,y_0)\Big)\right|_{0,h} \right\}^2 \\ &\leq 4\varepsilon. \end{aligned}$$

We utilize the techniques developed in section 4 and obtain

$$\|u_h^* - u_h^\varepsilon\|_{1,h}^2 \leq 4\frac{J_h(u_h^*)_J - \varepsilon}{\ln J_h(u_h^*)_J - \ln\varepsilon}. \tag{43}$$

Together with (40), we have

$$\|u^* - u_h^\varepsilon\|_{1,h} \leq Ch + 4\frac{J_h(u_h^*)_J - \varepsilon}{\ln J_h(u_h^*)_J - \ln\varepsilon}, \tag{44}$$

with a constant $C > 0$. Herein, by using (41),

$$J_h(u_h^*)_J = J_h(u^*)_J + \mathrm{O}(h). \tag{45}$$

It is an open question whether the error bound on the right-hand side of (43) or (45) can be further estimated in powers of $\varepsilon$ and $h$ completely. Presumably, this is not the case since the bound on the right-hand side is quasi-optimal—i.e., optimal up to $\mathrm{O}(h)$—since the stability bound in (26) with $U = u^*$ is quasi-optimal according to the optimality of the related quantity in the continuous case (see Remark 3.1 in [9]). Obviously, the $\mathrm{O}(h)$ always includes bounds for the first or second derivatives of the solution $u^*$ of (37)–(39).

**6. Numerical examples.** In [9] we have applied our regularization method to the classical example of Hadamard [7]. Here, we present two examples for inhomogeneous Cauchy problems of the form

$$\begin{aligned} &\Delta u = f \quad \text{in } [0,1]\times[0,1]; \\ &u|_{x=0} = \gamma_0(y), \quad u|_{x=1} = \gamma_1(y), \quad y \in [0,1]; \\ &u|_{y=0} = f_1(x), \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = f_2(x), \quad x \in [0,1]. \end{aligned} \tag{46}$$

The examples chosen are such that the solutions are known.

Example 1. $u(x,y) = \exp(x+y)$.

Example 2. $u(x,y) = x^{10}y^{10}$.

The solution of (46) is split in two parts, $u = u^{(1)} + u^{(2)}$, namely, the solution $u^{(1)}$ of the direct problem

$$\begin{aligned} &\Delta u^{(1)} = f \quad \text{in } [0,1]\times[0,1], \\ &u^{(1)}|_{x=0} = \gamma_0, \quad u^{(1)}|_{x=1} = \gamma_1, \\ &u^{(1)}|_{y=0} = f_1, \quad u^{(1)}|_{y=1} = \hat{g}, \end{aligned} \tag{47}$$

TABLE 1
*Relative $L^2$-errors at $y = 1$.*

| $h \vert \varepsilon_f$ | $10^0$ | $10^{-2}$ | $10^{-4}$ | $10^0$ | $10^{-2}$ | $10^{-4}$ |
|---|---|---|---|---|---|---|
| $\frac{1}{25}$ | 0.0887367 | 0.0473986 | 0.0472566 | 1.40233 | 1.04704 | 1.04001 |
| $\frac{1}{50}$ | 0.0305681 | 0.0255552 | 0.0240529 | 1.00207 | 0.938912 | 0.933708 |
| $\frac{1}{100}$ | 0.0361468 | 0.0133044 | 0.01309 | 1.06288 | 0.847121 | 0.752594 |
| $\frac{1}{200}$ | 0.0107969 | 0.0095872 | 0.00840506 | 0.956352 | 0.561189 | 0.537604 |
| | Example 1: exp(x+y) | | | Example 2: $x^{10}y^{10}$ | | |

with $\hat{g}(x) = x\gamma_1(1) + (1-x)\gamma_0(1)$, and the solution $u^{(2)}$ of the inverse problem

$$\begin{aligned} \Delta u^{(2)} &= 0 \quad \text{in } [0,1]\times[0,1], \\ u^{(2)}|_{x=0} = u^{(2)}|_{x=1} &= 0, \\ u^{(2)}|_{y=0} = 0, \quad \left.\frac{\partial u^{(2)}}{\partial y}\right|_{y=0} &= \hat{f}_2, \end{aligned} \tag{48}$$

with $\hat{f}_2 = f_2 - \left(\partial u^{(1)}/\partial y\right)|_{y=0}$. We allow perturbations of $f_2$ by adding (pointwise) $\varepsilon_f$ times a random function varying in $[-1,1]$.

For numerical approximations, we discretize by a uniform mesh size $h$ in the $x$- and $y$-direction and obtain a numerical solution $u_h^{(1)}$ to the direct problem (47). An approximation $u_h^{(2)}$ to the Cauchy problem (48) is then determined by the solution of (27)–(29), where the boundary values $g_h$ at $y = 0$ are obtained via the minimization problem (35). The side condition (see $K_{\varepsilon,h}$ given by (33)) utilizes $u_h^{(1)}$ and is of the form

$$\left| D^+_{h,y} u_h^{(2)}(.,0) - f^\varepsilon_{2,h} \right|_{0,h} \le \varepsilon \tag{49}$$

with $\varepsilon \ge \varepsilon_f, f_2^\varepsilon$ from (4) and

$$f^\varepsilon_{2,h} := f^\varepsilon_2 - D^+_{h,y} u_h^{(1)}(.,0). \tag{50}$$

We know from section 5 that $\varepsilon$ should be also greater than the mesh size $h$ in order to guarantee the error estimate (44). Therefore we have chosen $\varepsilon = \varepsilon_f + ch$, where $c$ is a bound for $u_{yy}|_{y=0}$ ; if one doesn't know such a bound, we suggest choosing $c = 1$.

For computational purposes, $I_h$ in (34), (35) should be written in form of a quadratic functional. Denoting by $\mathcal{A}_h$ the matrix associated with the linear operator $A_J$ (see (32))—e.g., with respect to piecewise constant or piecewise linear basis functions—the functional $I_h$ can be expressed as $I_h(G) = \langle \mathcal{B}_h G, G \rangle$ with

$$\mathcal{B}_h = h\left(-D^2_{h,x} + \mathcal{A}_h^* \mathcal{A}_h\right).$$

From $g_h = G$, the boundary value of $u_h^{(2)}$ at $y = 1$, we obtain the desired boundary values for $u$ via $g_h + \hat{g}$.

We have used the Fortran subroutines QL0001 and QL0002 of Schittkowski based on a computer code of Powell [16] to calculate the solution of the quadratic minimization problem. All calculations were performed in single precision.

Table 1 shows the relative $L^2$-errors at $y = 1$ for various mesh sizes $h$ and different magnitudes of $\varepsilon_f$ in the perturbation of $f$. In Example 1, the decrease of $h$ and $\varepsilon_f$

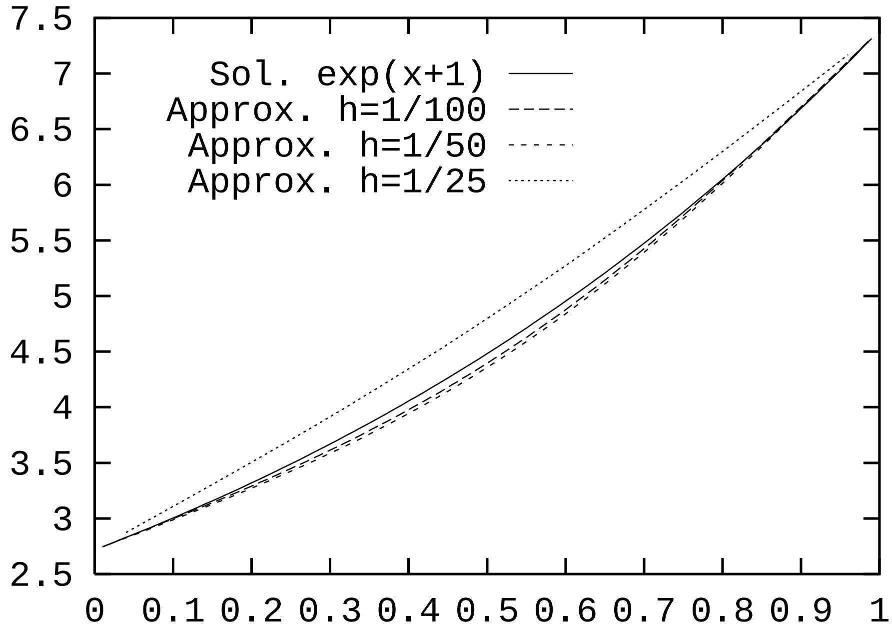


FIG. 2. *Example* 1 *at* $y = 1$.

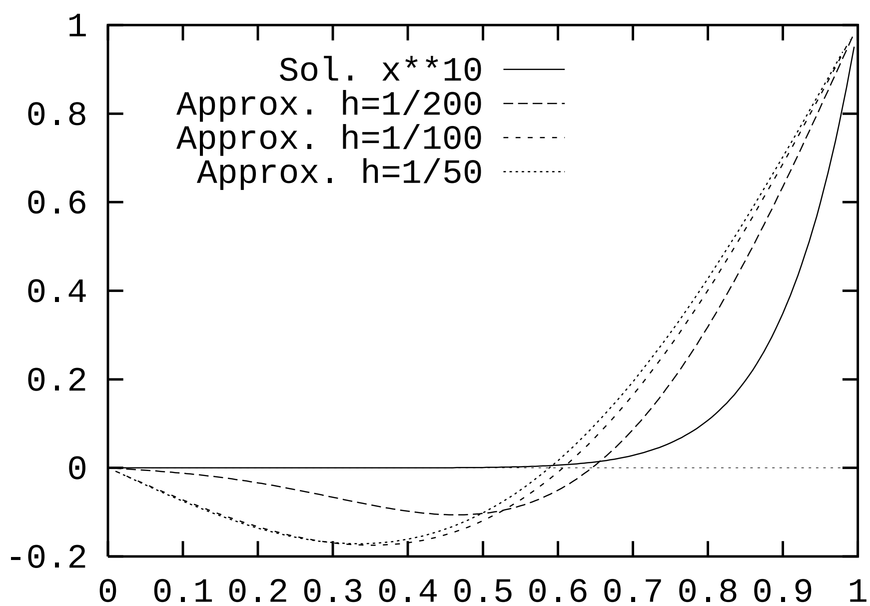


FIG. 3. *Example* 2 *at* $y = 1$.

caused a decrease of the relative errors such that this example behaves nearly well posed. Example 2 behaves differently, which is very likely due to the steep gradients at $x = 1$. The relative errors decreased very slowly and did not become smaller than 53%.

Figures 2 and 3 display the exact solutions at $y = 1$ together with the numerical approximations at grids of size $100 \times 100$, $50 \times 50$, and $25 \times 25$—all with data perturbations of magnitude $\varepsilon_f = 10^{-2}$. As Table 1 has already indicated, the results for

Example 1 are very good and approach the exact solution as $h$ decreases. In Example 2, the errors are relatively large but the shapes of the approximating curves are indeed similar to that of the exact solution and have steep gradients also at $y = 1$. There may be some remedies to improve the results for Example 2 which have to be investigated.

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