Approximation of Cauchy problems for elliptic equations using the method of lines

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Abstract: - In this contribution we deal with the development, theoretical examination and numerical examples of a method of lines approximation for the Cauchy problem for elliptic partial differential equations. We restrict ourselves to the Laplace equation. A more general elliptic equation containing a diffusion coefficient will be considered in a forthcoming paper. Our main results are the regularization of the illposed Cauchy problem and the proof of error estimates leading to convergence results for the method of lines. We base them principally on two major foundations. The first one is a conditional stability result for the continuous Cauchy problem. The second one consists of introducing certain finite-dimensional spaces, onto which the possibly perturbed Cauchy data are projected. At the end of this paper we present and discuss results of some of our numerical computations. All proofs are carried out in detail in [1].

Key-Words: - Cauchy problem, elliptic partial differential equation, illposed problem, method of lines

1 The Cauchy problem for Poisson’s equation

We consider the following Cauchy problem for Poisson’s equation on a rectangle

\[ \Delta u = f \text{ in } \Omega = (0, 1) \times (0, r_{\text{max}}) \]  

with given boundary conditions

\[ u = f_i \text{ on } \Sigma_i, \ i = 1, 2, 3, \ \frac{\partial u}{\partial y} = \phi_1 \text{ on } \Sigma_1, \]  

where

\[ \Sigma_1 = \{(x, 0) \in \mathbb{R}^2 | 0 \leq x \leq 1\} \]
\[ \Sigma_2 = \{(0, y) \in \mathbb{R}^2 | 0 \leq y \leq r_{\text{max}}\} \]
\[ \Sigma_3 = \{(1, y) \in \mathbb{R}^2 | 0 \leq y \leq r_{\text{max}}\} \]
\[ \Sigma_4 = \{(x, r_{\text{max}}) \in \mathbb{R}^2 | 0 \leq x \leq 1\}. \]

Here, one tries to identify \( u \) and \( \partial u/\partial y \) on \( \Sigma_4 \). The functions \( f_1, \phi_1 \) are the given Cauchy data.

This is a well-known improperly posed problem. J. Hadamard [6] has given a classical example showing that the solution of the problem is not continuously dependent on the Cauchy data. Without loss of generality, we can set \( f = 0, f_2 = 0, f_3 = 0 \) and \( \phi_1 = 0 \). Otherwise, one has to solve a direct problem for Poisson’s equation beforehand and add its solution to the solution of the Cauchy-problem with the vanishing \( f \)'s. It is impossible to solve this improperly posed problem by the classical theory of partial differential equations. As the last century progressed it became apparent that a growing number of extremely important problems in science and technology are improperly posed. Therefore it has required the attention of many mathematicians. M. M. Lavrent’ev [11] has discussed bounded solutions of the Laplace equation with the Cauchy data in a special two-dimensional domain where the bounded solutions depend continuously on the Cauchy data. Fursikov [5] has extended this approach to domains in \( \mathbb{R}^n \) proving an optimal stability estimate with respect to the \( H^0 \)-norm. The latter is analogous to Hadamard’s classical estimate for analytic functions which forms the content of the three-circles theorem. L. E. Payne [12], [13]
studied solutions of more general second-order elliptic equations which are continuously dependent on the Cauchy data under some restrictions on the domains and on the solutions. In 1975, L. E. Payne outlined this problem in [14]. H. Han has considered the problem (1), (2) in [7] and gave an $H^0(\Omega)$-stability estimate. Namely, suppose that the solution $u(x, y)$ of the problem (1), (2) satisfies the restriction

$$\|u\|_{H^1(\Omega)} \leq M$$

where $M > 0$ is a constant independent of the Cauchy data $(f_1, f_2)$, then $\|u\|_{H^0(\Omega)}$ is continuously dependent on the Cauchy data $(f_1, \phi_1)$. Here $H^k(\Omega)$ denotes the usual Sobolev space on the domain $\Omega$ with an integer $k$.  


In the paper of H. Han and H.-J. Reinhardt [8] a series of stability estimates for the problem (1), (2) in Sobolev spaces are given, from which several regularization methods can be proposed for computing numerical approximations (s. the paper of H.-J. Reinhardt, H. Han and Hao [16]).

2 Method of lines approximation

It is well-known that elliptic equations can be approximated by using the method of lines. One has two choices namely lines parallel to the $x$- or $y$-axis. Here, the problem setting indicates that the lines should be chosen parallel to the $y$-axis (s. Fig. 1).

With mesh points $x_i = i h, i = 0, \ldots, N, h = 1/N$, we approximate the Laplace operator in (1) by the central difference quotient of 2nd order. Therefore, approximations $u_i(y)$ for the solution $u(x_i, y)$ of (1), (2) with $f_2 = 0, f_3 = 0, f = 0$ (s. Fig. 2) can be obtained by the solution of the following system of ode’s, $u_0 = u_N = 0$,

$$u_i'' - \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0$$  \hspace{1cm} (3)

with boundary conditions

$$u_i(0) = f_1(x_i), u_i'(0) = \phi_1(x_i)$$  \hspace{1cm} (4)

for $i = 1, \ldots, N - 1$. This system can be decoupled using the eigenvalues and eigenvectors of the $\mathbb{R}^{N-1,N-1}$ matrix

$$A_h = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

for $j = 1, \ldots, N - 1$. With the orthogonal matrix $W = (\tilde{w}^{(1)}, \ldots, \tilde{w}^{(N-1)})$ consisting of the normalized eigenvektors $\tilde{w}^j = w^j |w^j|_2^2$ as column vectors and the diagonal matrix $D = \text{diag}(\lambda_j)_{j=1,\ldots,N}$, the system (3) is equivalent to

$$V'' + DV = 0$$ \hspace{1cm} (5)

where $V = WU = (u_1, \ldots, u_{N-1})$. Explicit solutions of such a system are well-known and can be written as

$$u_i(y) = \xi_i \exp(\sqrt{-\lambda_i}y) + \eta_i \exp(-\sqrt{-\lambda_i}y), \hspace{1cm} i = 1, \ldots, N - 1.$$
The boundary conditions at $y = 0$ determine the coefficients $\xi_i, \eta_i$,

$$
\Rightarrow \xi_i = \sqrt{\frac{h}{2}} \sum_{j=1}^{N-1} \left( \sin(ijh \pi) f_1(x_j) + \frac{h}{\sin(ih \pi)} \phi_1(x_j) \right)
$$

$$
\eta_i = \sqrt{\frac{h}{2}} \sum_{j=1}^{N-1} \left( \sin(ijh \pi) f_1(x_j) - \frac{h}{2 \sin(ih \pi)} \phi_1(x_j) \right)
$$

This yields the following explicit representation of the solution $u_i$ on the $i$-th line,

$$
u_i(y) = (WV)_i(y) = 2h \sum_{k=1}^{N-1} \left( \sin(ikh \pi) \left( \cosh(\sqrt{-\lambda_k y}) \cdot \sum_{j=1}^{N-1} \sin(kjh \pi) f_1(x_j) + \frac{h}{2 \sin(kh \pi)} \phi_1(x_j) \right) + \sinh(\sqrt{-\lambda_k y}) \sum_{j=1}^{N-1} \sin(kjh \pi) \phi_1(x_j) \right)
$$

In the special case $f_1 = 0$, the solution can be written as

$$
u_i(y) = \sum_{k=1}^{N-1} \frac{h}{\sqrt{-\lambda_k}} w_k^{(i)} \sinh(\sqrt{-\lambda_k y})
$$

where $(\cdot, \cdot)_2$ denotes the Euclidean scalar product $\Phi_1 = (\phi_1(x_1), \ldots, \phi_1(x_{N-1}))^T$ and $w_k^{(i)}$ are the components of $w^{(i)}$. If one chooses $\phi_1(x) = \sin(m \pi x)/m \pi$ for some $m < N$, one ends up with Hadamard’s classical example (s. [6]). In this case, the convergence $u_i(y) \leftarrow u(x_i, y)(h \to 0)$ can be shown with the error estimate (cf. Charton [1], 3.1)

$$
|u(x_i, y) - u_i(y)| \leq \frac{m \pi y}{24} \exp(m \pi y) h^2, \quad i = 1, \ldots, n - 1.
$$

This example is a special case of the situation when the data functions $f_1, \phi_1$ have truncated Fourier series, i.e. $f_1, \phi_1 \in D_M$ for some $M$ where

$$
D_M = \{ \phi \in C^1(0,1) | \phi(0) = \phi(1) = 0, \frac{1}{0} \int \sin(k \pi s) \phi(s) \, ds = 0, k > M \}.
$$

For such $f_1, \phi_1$ the problem is conditionally well-posed and the solution of the original problem (1), (2) is obtained by the formula

$$
u(x, y) = \sum_{k=1}^{M} \left( 2 \sin(k \pi x) (f_1(\cdot), (\sin(k \pi \cdot))_{L^2} \cosh(k \pi y) \right) + \frac{(\phi_1(\cdot), (\sin(k \pi \cdot))_{L^2} \sinh(k \pi y))}{k \pi} \right)
$$

(8)

For technical reasons, $N > M$ should be assumed. We do not deal with this situation further but mention that convergence with order $O(h^2)$ as in (7) can be shown. All details can be found in Charton [1], Chapt 3.

### 3 Conditional well-posedness and convergence under certain boundedness condition

Instead of data with truncated Fournier series, we now allow data such that the solution of the Cauchy problem (1), (2) remains bounded. This is the widely used condition to stabilize the problem (s. section 1 of this paper). Using the technique of logarithmic convexity one can show that, with $f_1 = 0, f_2 = 0, f_3 = 0, f = 0$ under the assumption

$$
\|u\|_{L_2(\Sigma_4)} \leq E
$$

one obtains the following stability estimate for the solution of (1), (2),

$$
\|u(\cdot, y)\|_{L_2} \leq \max(r_{\max}, 1) \|\phi_1\|_{L_2}^{1-y/r_{\max}} E^{y/r_{\max}}.
$$

(10)

In this case the solution has the form (cf. (8))

$$
u(x, y) = \sum_{k=1}^{\infty} \frac{a_k}{k \pi} \sin(k \pi x) \sin(h d \pi y)
$$

(11)
with $a_k = (\phi_1, \sin(k\pi))_{L^2}$. For (9) it is required that $\phi_1$ is such that the above series converges for all $(x, y) \in [0, 1] \times [0, r_{\text{max}}]$. By means of some calculations and results from the theory of Fourier series, a sufficient and, in a certain sense, also necessary condition for the convergence of (10) is the convergence of $\sum_{k=1}^{\infty} a_k^2 \exp(2\pi r_{\text{max}})$. Sketch of the proof of (10).

The convergence for $h \to 0$ of the method of line approximation is assured by the following steps. First, the data function $\phi_1$ – note, that we consider the case $f_1 = 0$ – is projected into the space $D_M$ of functions of truncated Fourier sine series. We even allow perturbed data functions $\phi_1^\varepsilon$ such that $\|\phi_1 - \phi_1^\varepsilon\|_{L^2} \leq \varepsilon$. In this situation it is clear that, in general, $\phi_1^\varepsilon \notin D_M$ even if $\phi_1 \in D_M$. One has to estimate the projection error of the projected data and then the error between the true solution and the method of line approximation with projected data in $D_M$. For the convergence, the magnitude of pertubations should depend on the discretization parameter by $h = 0(\sqrt{\varepsilon})$ and the dimension of $D_M$ has to be chosen in an optimal way. For the orthogonal projection $P_M : D \to D_M$ w.r.t. the $L^2$ scalar product $D = \{ \phi \in C^1(0, 1) | \phi(0)\phi(0) = \phi(1) = 0 \}$ one has the following estimate provided that (10) holds,

$$\|\phi_1 - \phi_1^\varepsilon\|_{L^2(\Sigma_1)} \leq \frac{E}{r_{\text{max}}(1 - \exp(-4\pi r_{\text{max}}))} \cdot \frac{M}{\exp(M\pi r_{\text{max}})}.$$  

Proof: Besides $u$ the solution of (1), (2), with unperturbed data $\phi_1$, let further denote $u^* = \text{solution of (1), (2) with } \phi_1^* = P_M \phi_1$ $u_\varepsilon = \text{solution of (1), (2) with } \phi_1^\varepsilon$ $u^*_\varepsilon = \text{solution of (1), (2) with } (\phi_1^\varepsilon)^* = P_M \phi_1^\varepsilon$ $u^*_{i,\varepsilon} = \text{solution of line method approximation on } i\text{-th line with data } (\phi_1^\varepsilon)_i^* = (\phi_1^\varepsilon)^*(x_i), i = 1, \ldots, N - 1.$ $(\overline{u_\varepsilon})_h = \text{confirmation of } u^*_{i,\varepsilon}(y)_{i=1, \ldots, N-1}$ in $D_M$. Using $u^*_{i,\varepsilon}$, the latter function is given by

$$\overline{(u_\varepsilon^*)_h}(x, y) = \sum_{k=1}^{M} \left( 2h \sum_{j=1}^{N-1} \sin(k\pi j h) u^*_{j,\varepsilon}(y) \right) \sin(k\pi x)$$

The total error essentially consists of three parts which have to be estimated separately,
References


