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Criteria for a Philosophy of Mathematics

Taking the Test

One way to test a philosophy of mathematics is to confront it with test questions:

What makes mathematics different?

What is mathematics about?

Why does mathematics achieve near-universal consensus?

How do we acquire knowledge of mathematics, apart from proof?

Why are mathematical results independent of time, place, race, nationality, and gender, in spite of the social nature of mathematics?

Does the infinite exist? How?

The social-historical approach gives better answers than the neo-Fregean, the intuitionist-constructivist, or any other proposed philosophy I know of.

Regarding the infinite, if mathematical objects are like stories we make up, we can make up fantastic weird stories if we want to, just so they fit together with other stories. N, an infinite set, is no harder to accept than $10^{(10^{10})}$. Both are socially validated inventions, real, not as physical objects, but as other socially validated inventions are real. Though they are our inventions, their properties are not arbitrary. They're forced to be what they are, by the purposes for which we invented them.

Guiding Principles

Evaluate a body of thought according to its own goals and presuppositions. Understand it historically, in the sense of history of ideas. Pay attention to its consequences, theoretical and practical. Beneficial consequences don't verify a doctrine. Harmful consequences don't falsify it. But consequences are as important as plausibility, consistency, or explanatory power.

I now list criteria for a philosophy of mathematics. No philosophy, including our own, is satisfactory by all criteria. The list is a vantage point from which to evaluate theories, including our own. We do so in Chapter 13.

1. Breadth

An adequate philosophy of mathematics would be aware at least of *some* active field of mainstream mathematical research (dynamical systems, say, or stochastic processes, or algebraic/differential geometry/topology). It would look at how mathematics is being used somewhere, whether in hydrodynamics, or meteorology, or geophysics. It would notice that theoretical physicists do mathematics differently from either "pure" or "applied" mathematics. It wouldn't ignore computing in mathematics today—real computing with real machines, not just idealized theory of ideal machines. It would be compatible with the history of mathematics and with how people learn mathematics.

Who dares to write philosophy of science without some acquaintance with quantum mechanics and relativity? But you must search long and far to find a philosopher of mathematics who claims a nodding acquaintance with functional analysis or differential topology.

This complaint can be parried by claiming that all mathematics "can be got down" to set theory and arithmetic (quote from Quine in Chapter 9). Such a claim depends on what you mean by "got down." It has a self-serving flavor. The argument, "What I don't know, *ipso facto* doesn't matter" isn't new. Age hasn't made it palatable.

2. Links to Epistemology and Philosophy of Science

Philosophy of mathematics should articulate with epistemology and philosophy of science. But virtually all writers on philosophy of mathematics treat it as an encapsulated entity, isolated, timeless, ahistorical, inhuman, connected to nothing else in the intellectual or material realms. Philosophy of mathematics is routinely done without reference to mind, science, or society. (We'll see exceptions in due course.)

Your view of mathematics should fit your view of physical science. Your view of mathematical knowledge should fit your view of knowledge in general. If you write philosophy of mathematics, you aren't expected simultaneously to write philosophy of science and general epistemology. To write on philosophy of mathematics alone is daunting enough. But to be adequate, it needs a connection with epistemology and with philosophy of science.

3. Valid against Practice

Philosophy of mathematics should be tested against five kinds of mathematical practice: research, application, teaching, history, computing. In all areas of mathematical practice, an essential role is played, one way or another, by something called mathematical intuition. There is an extended discussion of intuition in Chapter 4. Here I want to make clear that an adequate philosophy of mathematics must recognize and deal with mathematical intuition.

The need to check philosophy of mathematics against mathematical *research* doesn't require explication. Many important philosophers of mathematics were mathematical researchers: Pascal, Descartes, Leibniz, d'Alembert, Hilbert, Brouwer, Poincaré, Rényi, and Bishop come to mind.

Applied mathematics isn't illegitimate or marginal. Advances in mathematics for science and technology often are inseparable from advances in pure mathematics. Examples: Newton on universal gravitation and the infinitesimal calculus; Gauss on electromagnetism, astronomy, and geodesy (the last inspired that beautiful pure subject—differential geometry); Poincare on celestial mechanics; and von Neumann on quantum mechanics, fluid dynamics, computer design, numerical analysis, and nuclear explosions.

Not only did the same great mathematicians do both pure and applied mathematics, their pure and applied work often fertilized each other. This was explicit in Gauss and Poincaré. Nearer our time is Norbert Wiener. He was generally known for cybernetics, but his life work was mathematical analysis. His study of infinite-dimensional stochastic processes was guided by their physical interpretation as Brownian motion, illuminated by experiments of the French physicist, Jean Perrin (see Wiener, 1948). The standard mathematical model for Brownian motion is the Wiener process. His stochastic processes were useful in controlling anti-aircraft fire. When he renounced military research, he took up prosthetics for the blind—more applied work!

G. H. Hardy "famously" boasted: "I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world." Nevertheless, the Hardy-Weinberg law of genetics is better known than his profound contributions to analytic number theory. What's worse, cryptology is making number theory applicable. Hardy's contribution to that pure field may yet be useful.

Twenty years after the war, mathematical purism was revived, influenced by the famous French group "Bourbaki." That period is over. Today it's difficult to find a mathematician who'll say an unkind word about applied math.

Instead, we see the spectacular merger of elementary-particle physics and high-dimensional differential geometry. One famous practitioner, Ed Witten, is a physicist whose physical insight permits him unexpected discoveries in mathematics.

A philosophy of mathematics that ignores applied mathematics, or treats it as an afterthought, is out of date. The relationship between pure and applied mathematics is a central philosophical question. A philosophy of mathematics blind to this challenge is inadequate.

An interesting fact about mathematics is that it's taught and learned. (Even more interesting—sometimes taught but *not* learned!) A credible philosophy of mathematics must accord with the experience of ***teaching and learning*** mathematics. To a formalist or Platonist who presents an inhuman picture of mathematics, I ask, "If this were so, how could anyone learn it?"

We know a little about how mathematics is learned. It's not done by memorizing the times table or Peano's axioms. Mathematics is learned by computing, by solving problems, and by conversing, more than by reading and listening.

An account of the nature of mathematics incompatible with these facts is wrong. Piaget was one of the few writers on philosophy of mathematics to take teaching seriously. Today Thomas Tymoczko and Paul Ernest are doing so.

There is also ***the historical test.*** Mathematics has seen enormous changes. Its story reaches from Babylonia to Maya, pre-colonial Africa, India, China, Japan. A philosophy about today's mathematics that leaves inexplicable the mathematics of 500 years ago is inadequate.

André Weil, a leading mathematician of the postwar period and now a historian of mathematics, wrote that there could hardly be two disciplines further apart than history of mathematics and philosophy of mathematics. Perhaps in this statement he was identifying philosophy of mathematics with foundations of mathematics.

One philosophical issue in which history is relevant is the reduction of all mathematics to set theory,** as in foundations textbooks, and as mentioned previously in point 1, breadth.

On the basis of this reduction, philosophers of mathematics generally limit their attention to set theory, logic, and arithmetic.

What does this assumption, that all mathematics is fundamentally set theory, do to Euclid, Archimedes, Newton, Leibniz, and Euler? No one dares to say they were thinking in terms of sets, hundreds of years before the set-theoretic reduction was invented. The only way out (implicit, never explicit) is that their own understanding of what they did must be ignored! We know better than they how to explicate their work!

That claim obscures history, and obscures the present, which is rooted in history.

An adequate philosophy of mathematics ***must*** be compatible with the history of mathematics. It ***should*** be capable of shedding light on that history. Why did the Greeks fail to develop mechanics, along the lines that they developed geometry? Why did mathematics lapse in Italy after Galileo, to leap ahead in England, France, and Germany? Why was non-Euclidean geometry not conceived until the nineteenth century, and then independently rediscovered three times? The philosopher of mathematics who is historically conscious can offer such questions to the historian. But if his philosophy makes these questions invisible, then instead of stimulating the history of mathematics, he stultifies it.

Computing is a major part of mathematical practice. The use of computing machines in mathematical proof is controversial. An adequate philosophy of mathematics should shed some light on this controversy.

Formalist and logicist philosophies, each in its own way, picture mathematics as essentially calculation. For the logicist, mathematical theorems are true tautologically, as logical identities. For the formalist, the undefined terms of mathematics are meaningless, so mathematical theorems are meaningless. Both say the essence of mathematics is proof in the sense of formal logic. Proof as a formal calculation in a formal language according to formal rules.

When these conceptions were developed early in the twentieth century, the possibility of realizing them was fantasy. The most famous attempt to formalize statements and proofs of mathematical theorems was the *Principia Mathematica* of Bertrand Russell and Alfred Whitehead. I'm told that finally on page 180 or so they prove 1 is different from 0.

Applied mathematicians have long used computers as a matter of course. Today computers are used more and more in pure mathematics research. Number theorists and algebraists use them to test and make up conjectures. With a computer you sometimes can finish a proof by a calculation impossible by hand. The four-color conjecture** is a famous example (see "Proof" in Chapter 5).

There are differences of opinion about computers. Clifford Truesdell, a leading authority on continuum mechanics, calls them a menace and an abomination. Some say the proof of the four-color theorem violates the traditional idea of proof, since it requires believing that computers work, which is not a mathematical belief. Such objections are ironic. For decades philosophers said valid mathematical proofs should be checkable by machine. Now, when part of a proof *is* done on a machine, some say, "That's not a proof!"

So far our criteria for a philosophy of mathematics have been external—how the philosophy relates to mathematics. There are criteria within the philosophical doctrine itself: consistency, elegance, economy, simplicity, comprehensibility, and precision.

4. *Elegance*

Elegance is more common in mathematical theories than in philosophical ones. Paul Cohen remarked that no one could accuse philosophy of being beautiful. He had in mind foundationist philosophy. Beauty or elegance is desirable, but not prerequisite. Otherwise, Aristotle and Kant would go out the window.

5. *Economy*

Good old Ockham's razor: Use what you need, nothing more. This principle justifies the set-theoretic reduction of mathematics. It's claimed that set membership suffices to define number systems, spaces, geometric figures, and all operations and operands in mathematics past, present, and future. That being so, there should be nothing in mathematics but sets, since there needn't be. But economy can conflict with other criteria such as comprehensibility and applicability. It may be fun to find minimal generators of a theory, even if they're neither unique nor convenient. Economy is like elegance: desirable, but optional.

6. *Comprehensibility*

Comprehensibility is valued by readers, not by all writers.

Philosophy students think that among professional philosophers, incomprehensibility gets "Brownie points," and comprehensibility gets demerits.

Unworthy suspicions aside, it's a question of comprehensibility *to whom.* What's impenetrable to you may be crystalline to the Heidegger expert.

This book aims to be easily comprehensible to anyone. If some allusion is obscure, skip it. It's inessential.

7. *Precision*

Should the philosophy of mathematics be precise? Analytic philosophers sometimes use pseudo-mathematical notation. Call a claim "Claim A" or "Hypothesis B," and obscure a conversation that would have gone better in some natural language, English, for example. If you notice that no other branch of philosophy even hopes for precision, the dispensability of precision in philosophy of mathematics becomes apparent.

Mathematics is precise; philosophy cannot be. Expecting philosophy of mathematics to be a branch of mathematics, with definitions and proofs, is like thinking philosophy of art can be a branch of art, with landscapes and still lives.

Art can be beautiful; philosophy of art cannot be beautiful.

Philosophy of politics cannot win at the ballot. Philosophy of law cannot fill the wallets of lawyers. Philosophical specialties, including the philosophy of mathematics, should be evaluated by philosophical standards, not the standards of the field they critique.

Is philosophy of mathematics, part of mathematics or part of philosophy? It's *about* mathematics, but it's *part* of philosophy. It happens that the creators of foundationist philosophy of mathematics were mathematicians (Hilbert, Brouwer) or mathematically trained (Husserl, Frege, Russell). This training may explain their bias. They sought to turn philosophical problems into mathematical problems, *to make them precise.* This bias was fruitful mathematically. Some of today's mathematical logic descended from the search for mathematical solutions to philosophical problems. But, even though mathematically fruitful, it was philosophically misguided. Today we can turn away from the philosophical failure of the foundationist schools. We can think of philosophy of mathematics, not as a branch of mathematics, but as a philosophical enterprise based on mathematical experience. Give up the illusion of mathematical precision. Aim for insight, enlightenment.

8. *Simplicity*

Both simplicity and precision are desirable, in science and in philosophy of science, especially in philosophy of mathematics.

But in science, philosophy of science, and philosophy of mathematics, there's another desideratum—truthfulness, faithfulness to the facts, simple honesty.

One wants all three: truthfulness, precision, and simplicity. But one can't usually maximize at once goal A and goal B. If you're not willing to pick one goal and ignore the others (maximum cash flow, for instance—reputation and legality be damned!) then you have to do some balancing or juggling. (Work on cash flow, but don't actually go to jail.)

Precision is easier to achieve in a simple situation than in a complicated one. Some phenomena are inherently imprecise.

Precision in philosophy of mathematics is sought by trying to mathematize it. Axiomatic set theory is a branch of mathematics. If the philosophy of mathematics were no more than a style of doing axiomatic set theory, it would attain mathematical precision. But it's impossible to talk about all interesting aspects of mathematics in the language of axiomatic set theory. And it's not necessary to do so! The notion that philosophy of mathematics is a branch of axiomatic set theory is no divine ordinance or self-enforcing decree. It's just one school, one trend, that hopes to borrow the prestige of mathematics by doing its nonmathematical thing *more mathematico*. In this it's reminiscent of certain "mathematical" specialties in economics, sociology, and psychology.

Simplicity goes with single-mindedness. Where several factors interact to give a complex result, simplicity can be created by ignoring all factors but one. Different scholars may single out different factors. This kind of simplicity leads to fruitless controversy, like between Red Sox fans and White Sox fans. For example, both formalization and construction are central features of mathematics. But the *philosophies* of formalism and constructivism are long-standing rival schools. It would be more productive to see how formalization and construction interact than to choose one and reject the other.

The notions I advocate are less precise and less simple than familiar philosophies. This permits better faithfulness to experience. Some may think no loss of simplicity or precision is acceptable.

Putting simplicity and precision ahead of truthfulness is treating philosophy as a game, *art pour l'art*, like an exotic branch of algebra. Philosophy can be serious—no less than how and why to live. Being serious means putting truthfulness first. First get it right, then go for precision and simplicity.

My first assumption about mathematics is: It's something people do. An account of mathematics is unacceptable unless it's compatible with what people do, especially what mathematicians do.

9. Consistency

This is highly valued by logicians and logic-minded philosophers. Others downplay it. Ralph Waldo Emerson wrote, "A foolish consistency is the hobgoblin of little minds, adored by little statesmen and philosophers and divines."

Walt Whitman wrote:

> Do I contradict myself?
> Very well then I contradict myself.
> (I am large, I contain multitudes.)

Bourbaki wrote:

> Historically speaking, it is of course quite untrue that mathematics is free from contradiction. Non-contradiction appears as a goal to be achieved, not as a God-given quality that has been granted us once for all. There is no sharply drawn line between the contradictions which occur in the daily work of every mathematician, beginner or master of his craft, as the result of more or less easily detected mistakes, and the major paradoxes which provide food for logical thought for decades and sometimes centuries. (N. Bourbaki, 1949)

In a first course in logic, the teacher shows how, from any proposition A together with its negation you can deduce any other proposition B.**

From *any* contradiction, *all* propositions (and their negations) follow! Everything's both true and false! The theory collapses in ruins!

Outside of logic class, it isn't that way. The U.S. Constitution contains contradictions. Any courtroom prosecution or defense probably contains contradictions.

Quantum electrodynamics gives the most precise predictions of any physical theory. Yet physicists have known from its birth that it's self-contradictory. They make *ad hoc* rules for handling the inconsistency. Divergent series of divergent terms are manipulated and massaged. In a *Festschrift* for the famous physicist John Wheeler I found this praise: "He's never stopped by a formal contradiction" (i. e., a mathematical contradiction).

Players in this game know contradiction is seldom fatal. Richard Rorty quotes Aquinas: "Coherence is a matter of avoiding contradictions, and St. Thomas' advice, 'When you meet a contradiction, make a distinction,' makes that pretty easy. As far as I could see, philosophical talent was largely a matter of proliferating as many distinctions as were needed to wriggle out of a dialectical corner."

If A is B and also not B, make a distinction, A_1 and A_2. Example: "Mathematics is precise and imprecise." Whoops, contradiction! What to do? Distinguish between formal and informal mathematics. Formal is precise, informal is imprecise. Contradiction gone!

That contradiction is even useful! It calls attention to an interesting distinction.

Classical logic says all the consequences of a set of axioms exist—are derived—instantly, as soon as the axiom set is laid down. There are infinitely many consequences. This whole infinite theory, created instantly! Consistency holds or it's violated immediately, instantly.

In practice, consequences are derived step by step. At any time only a finite number have been derived. If a contradiction appears, some device is brought in to wall it off, to keep the rest of the theory from infection.

I once wrote that mathematicians hate contradiction. That's not accurate. We love it—like a duck hunter loves ducks. Nothing draws us to the chase like a contradiction in a famous theory.

In evaluating a mathematical theory, consistency is important. But it's less important than fruitfulness (inside and outside of mathematics), imaginative appeal, and linking new mathematical devices to old, respected problems. A contradiction can generally be fixed up one way or another.

As Bourbaki explained, "freedom from contradiction is attained in the process, not guaranteed in advance." They didn't notice that this fact discredits all standard foundationist or logicist theories about mathematics. In practice, we can't always prove in advance the consistency of all possible deductions. Instead, we develop a technique for preserving partial consistency—absence of contradiction up to the latest set of results. In that way we continue to forestall contradiction each time it raises its ugly head.

Frege announced that his building had collapsed. Then after its collapse he tried to patch it up, just as an ordinary, nonfoundationist mathematician would do. Mathematical buildings collapse—lose interest, are forgotten—not because of contradictions, but because their questions are no longer interesting, or because another theory answers them better.

10. Originality, Novelty

This quality is external; it relates to other philosophers and philosophies. It's rare for philosophical writing to be entirely novel. The basic ideas are in Plato and Aristotle. Nothing here is without some antecedent. Presentation, examples, arrangement, and flavor are mine. I hope to be provocative, even convincing to some, but not to all.

11. Certitude and Indubitability

Today the goals of certitude and indubitability are abandoned. The foundationist project has lost its philosophical rationale. "Foundations" (axiomatic set theory and related topics) is just one mathematical specialty of many. It compares axiomatic setups with respect to convenience, effectiveness, or strength. It doesn't pretend to give certainty or indubitability, the founding motive for "foundations."

An adequate philosophy of mathematics must account for the special role of proof in mathematics. Mark Kac, a famous probabilist, asked why mathematicians are obsessed with a need to prove everything. Many people, physicists among them, are willing to believe without proof. Yet mathematicians want proof. They even say, "Without a proof it's nothing." Is this the very nature of mathematics? Or is it one aspect among several of equal importance?

12. *Applicability*

This does not refer to mathematical applications, but to philosophical ones. Your philosophy may increase your feeling of being at home in the universe, or your ability to sleep with a clear conscience. But it should also be helpful in analyzing philosophical problems, perhaps even in solving one or two. If it's useless, who needs it? Chapters 4 and 5 below offer applications of humanist philosophy of mathematics.

13. Acceptability

This criterion is never explicitly demanded. Yet in practice it's the most important. It's why Cartesians prospered while Spinozists were damned (Chapters 6 and 7).

Mathematical theories "ahead of their time" have been ignored for decades, even centuries (Desargues, Grassmann.) In every field of learning, theory can't prosper if it too grossly violates current acceptability. In Europe until the seventeenth century, acceptability was conformity to the Holy Roman Catholic Church. In the Soviet Union, it was Marxism-Leninism-Stalinism, in genetics, linguistics, literature, and music. Here and now, no philosophy penetrates *philosophia academica* without bowing to that establishment's sine qua non.

The acceptability of this book—to mathematicians, philosophers, and the general public—remains to be seen.

Summing Up

Not all of these thirteen criteria are essential.

The first three are essential:

1. Recognize the scope and variety of mathematics.
2. Fit into general epistemology and philosophy of science.
3. Be compatible with mathematical practice—research, application, teaching, history, calculation, and mathematical intuition.

The next five are desirable:

4. Elegance
5. Economy
6. Comprehensibility
7. Precision
8. Simplicity

The next, 9. Consistency, is essential, but not as hard to attain as 1–3.

We reject 10. Novelty, Originality, as inessential and unattainable.

We reject 11. Certainty, Indubitability, as false and misleading.

We recognize 12. Applicability, as essential in practice. If you can't do anything with it, what good is it?

13. Acceptability can't be a goal, yet it can't be evaded.

Notice that in this list moral, ethical, and political considerations are excluded. I don't ask whether a theory is beneficial or harmful, progressive or reactionary, humane or inhumane. The preceding thirteen criteria suffice.

We return to these criteria in Chapter 13, and consider there the linkage between philosophical opinions and political opinions.