mathematics. But they both have the same logic. Both rely on intuition: knowledge coming, not from the senses, study, or learning, but from the nature of Mind. Right and wrong, like time and space, are universal intuitions. Our space intuition leads to geometry, our time intuition leads to arithmetic, our duty intuition leads to Divinity.

In God's mind, the difficulties and puzzles in philosophy of mathematics disappear. How do numbers exist? Why do mathematical facts seem certain and timeless? Why does mathematics work in the "real world"?

In the mind of God, it's no problem.

The trouble with today's Platonism is that it gives up God, but wants to keep mathematics a thought in the mind of God.

Euclid as a Myth. Nobody's Perfect.

The myth of Euclid is the belief that Euclid's *Elements* contain indubitable truths about the universe. Even today, most educated people still believe the Euclid myth. Up to the middle or late nineteenth century, the myth was unquestioned. It has been the major support for metaphysical philosophy—philosophy that sought a priori certainty about the nature of reality.

The roots of our philosophy of mathematics are in classical Greece. For the Greeks, mathematics was geometry. In Plato and Aristotle, philosophy of mathematics is philosophy of geometry.

Rationalism served science by denying the intellectual supremacy of religious authority, while defending the truth of religion. This equivocation gave science room to grow without being strangled as a rebel. It claimed for science the right to independence from the Church. Yet this independence didn't threaten the Church, since science was the study of God's handiwork. "The heavens proclaim the glory of God and the firmament showeth His handiwork."

The existence of mathematical objects as ideas independent of human minds was no problem for Newton or Leibniz; they took for granted the existence of a Divine Mind. In that belief, the problem is rather to account for the existence of nonideal, material objects.

After rationalism displaced medieval scholasticism, it was challenged by materialism and empiricism; by Locke and Hobbes in Britain, by the encyclopedists in France. The advance of science on the basis of the experimental method gave the victory to empiricism. The conventional wisdom became: "The material universe is the fundamental reality. Experiment and observation are the only legitimate means of studying it."

The empiricists held that all knowledge *except mathematical* comes from observation. They usually didn't try to explain how mathematical knowledge originates. In the controversies, first between rationalism and scholasticism, later between rationalism and empiricism, the sanctity of geometry was unchallenged.

Philosophers disputed whether we proceed from Reason (a gift from the Divine) to discover the properties of the world, or whether only our bodily senses can do so. Both sides took it for granted that geometrical knowledge is not problematical, even if all other knowledge is. Hume exempted books of mathematics and of natural science from his outcry, "Commit it to the flames."

For rationalists, mathematics was the main example to confirm their view of the world. For empiricists, it was an embarrassing counter-example, which had to be ignored or explained away. If, as seemed obvious, mathematics contains knowledge independent of sense perception, then empiricism is inadequate as an explanation of all human knowledge. This embarrassment is still with us; it's a reason for the difficulties of philosophy of mathematics.

Mathematics always had a special place in the battle between rationalism and empiricism. The mathematician-in-the-street, with his common-sense belief in mathematics as knowledge, is the last vestige of rationalism.

The modern scientific outlook took ascendancy in the nineteenth century. By the time of Russell and Whitehead, only logic and mathematics could still claim to be nonempirical knowledge, obtained directly by Reason.

From the customary viewpoint among scientists now, the Platonism of most mathematicians is an anomaly. For many years the accepted assumptions in science have been materialism in ontology, empiricism in epistemology. The world is all one stuff, "matter," which physics studies. If matter gets into certain complicated configurations, it falls under a special science with its own methodology—chemistry, geology, and biology. We learn about the world by looking at it and thinking about what we see. Until we look, we have nothing to think about.

Yet in mathematics we have knowledge of things we can never observe. At least, this is the natural point of view when we aren't trying to be philosophical.

Until well into the nineteenth century, the Euclid myth was universal among mathematicians as well as philosophers. Geometry was the firmest, most reliable branch of knowledge. Mathematical analysis—calculus and its extensions and ramifications—derived legitimacy from its link with geometry. We needn't say "Euclidean geometry." The qualifier became necessary only after non-Euclidean geometry had been recognized. Before that, geometry was simply geometry—the study of the properties of space. These were exact, eternal, and knowable with certainty by the human mind.

eight

Mainstream Since the Crisis

How Did We Get Here? Can We Get Out?

Vacillation between two unacceptable philosophies wasn't always the prevalent mode. Where did it come from?

Until the nineteenth century, geometry was regarded by everybody, *including mathematicians*, as the most reliable branch of knowledge. Analysis got its meaning and its legitimacy from its link with geometry.

In the nineteenth century, two disasters befell. One was the recognition that there's more than one thinkable geometry. This was a consequence of the discovery of non-Euclidean geometries.

A second disaster was the overtaking of geometrical intuition by analysis. Space-filling curves** and continuous nowhere-differentiable curves** were shocking surprises. They exposed the fallibility of the geometric intuition on which mathematics rested.

The situation was intolerable. Geometry served from the time of Plato as proof that certainty is possible in human knowledge—including religious certainty. Descartes and Spinoza followed the geometrical style in establishing the existence of God. Loss of certainty in geometry threatened loss of all certainty.

Mathematicians of the nineteenth century rose to the challenge. Led by Dedekind and Weierstrass, they replaced geometry with arithmetic as a foundation for mathematics. This required constructing the continuum—the unbroken line segment—from the natural numbers. Dedekind,** Cantor, and Weierstrass found ways to do this. It turned out that no matter how it was done, building the continuum out of the natural numbers required new mathematical entities—infinite sets.

Foundationism—Our Inheritance

The textbook picture of the philosophy of mathematics is strangely fragmentary. You get the impression that the subject popped up in the late nineteenth century because of difficulties in Cantor's set theory. There was talk of a "crisis in the foundations." To repair the foundations, three schools appeared. They spent thirty or forty years quarreling. But none of the three could fix the foundations. The story ends some sixty years ago. Whitehead and Russell abandoned logicism. Gödel's incompleteness theorem checkmated Hilbert's formalism. Brouwer remained in Amsterdam, preaching constructivism, ignored by most of the mathematical world.

This episode was a critical period in the philosophy of mathematics. By a striking shift in meaning of words, the domination of philosophy of mathematics by foundationism became the *identification* of philosophy of mathematics with foundations. We're left with a peculiar impression: The philosophy of mathematics was awakened by contradictions in set theory. It was active for forty or fifty years. Then it went back to sleep.

Of course there has always been a philosophical background to mathematical thinking. In the foundationist period, leading mathematicians engaged in public controversy about philosophical issues. To make sense of that period, look at what went before and after. Two strands of history have to be followed, philosophy of mathematics and mathematics itself. The "crisis" manifested a longstanding discrepancy between the Euclid myth, and the reality, the actual practice of mathematicians.

In discussions of foundations three dogmas are presented: Platonism, formalism, and intuitionism. Platonism was described in Chapter 1. I remind the reader what it says: "Mathematical objects are real. Their existence is an objective fact, independent of our knowledge of them. Infinite sets, uncountably infinite sets, infinite-dimensional manifolds, space-filling curves—all the denizens of the mathematical zoo—are definite objects, with definite properties. Some of their properties are known, some are unknown. These objects aren't physical or material. They're outside space and time. They're immutable. They're uncreated. A meaningful statement about one of these objects is true or false, whether we know it or not. Mathematicians are empirical scientists, like botanists. We can't invent anything; it's there already. We try to discover."

In recent times Platonism has sometimes been identified with logicism. *If* a Platonist makes an effort to explain the nature of his nonhuman mathematical objects, it's usually in terms of logic and/or set theory.

According to formalism, on the other hand, there are *no* mathematical objects. Mathematics is *axioms, definitions, and theorems*—in brief, formulas. A strong version of formalism says that there are rules to derive one formula from another, but the formulas aren't *about* anything. They're strings of meaningless symbols. Of course the formalist knows that mathematical formulas are being applied to physics. When a formula gets a physical interpretation, *then* it acquires meaning. *Then* it can be true or false. But the truth or falsity refers only to the physical interpretation. As a mathematical formula apart from any interpretation, it has no meaning and can be neither true nor false.

The difference between formalist and Platonist is clear in their attitudes to Cantor's continuum hypothesis. Cantor conjectured that there's no infinite cardinal number greater than $\aleph_0$ (the cardinality of the integers) and smaller than c (the cardinality of the real numbers). Kurt Gödel and Paul J. Cohen showed that on the basis of the Zermelo—Fraenkel axioms of set theory, the continuum hypothesis can neither be disproved (Gödel, 1937) nor proved (Cohen, 1964). To the Platonist, this means our axioms of sets are incomplete. The continuum hypothesis *is* either true or false. We just don't understand the real numbers well enough to tell which is the case.

To the formalist, the Platonist interpretation makes no sense, because there *is* no real number system, except as we "create" it by laying down axioms to describe it. We're free to change these axioms, for convenience, usefulness, or any criterion that appeals to us. But the criterion can't be better correspondence with reality, because there's no reality to correspond with.

Formalists and Platonists take opposite sides on existence and reality. On the principles of mathematical proof, they have no quarrel. Opposed to both of them are the constructivists. Constructivists accept only mathematics that's obtained from the natural numbers by a finite construction. The set of real numbers, and any other infinite set, cannot be so obtained. Consequently, the constructivist accepts neither the Platonist not the formalist view of Cantor's hypothesis. Cantor's hypothesis is meaningless. Any answer is a waste of breath.

Today some mathematicians still call themselves formalists or constructivists, but in philosophical circles one speaks more often of Platonists versus fictionalists. Fictionalists reject Platonism. They can be formalists, constructivists, or something else (see Chapter 10).

Philosophers like to call their arguments and counter-arguments "moves." It's a standard move to finesse a dispute by declaring it meaningless. This was favored by the logical positivists in days of yore. They decreed that the meaning of any statement is no more or less than its truth conditions. It followed, where logical positivists were in charge, that metaphysics, ethics, and much else was thrown out of philosophy. In time, by the same rule, logical positivism was thrown out too.

Two large facts about mathematics are hardly doubted.

Fact one: Mathematics is a human product.

It may seem unfair to expect a Platonist to admit this; it may seem like asking him to give the game away. Nevertheless, contributions to mathematics are made every day, by specific, particular human beings. Many contributions are signed by an author or authors. No one questions the claim of these authors for their results.

The quibble between discovery and invention or creation was discussed in Chapter 5. But no one doubts that the mathematics we know comes from the work of human beings. In fact, it is sometimes possible to account for features of a mathematical discovery by the interests, tastes, and attitudes of the discoverer and sometimes also by the needs or traditions of his country. This fact is the bulwark of "fictionalism." In its way of coming into being, in the way in which it's thought of by its creators, mathematics is like an art such as fiction or sculpture.

Fact two: We can choose a problem to work on, but we can't choose what the answer should be.

When you resolve a mathematical difficulty, you sense that the answer was already "there," waiting to be found. Even if the answer is, "There's no answer," as in Cantor's continuum problem, *that* is the answer, like it or not.

The number 6,785,123,080,772,901,001 is either prime or composite. I don't know which. But I know that any method I use will give the same answer—prime or composite. 6,785,123,080,772,901,001 is what it is, regardless of what I think or know. In this respect numbers are independent of their creators. Did I just bring 6,785,123,080,772,901 into existence? Or was it waiting and ready, some*how* if not some*where*, along with billions, quadrillions, and quintillions of cousins?

The philosophers more impressed by the *objectivity* of mathematics are Platonists. They say numbers exist apart from human consciousness. Those more impressed with the *human role* in creating mathematics are anti-Platonists. Depending on what they offer to replace Platonism, they may be fictionalists, formalists, constructivists, intuitionists, conventionalists.

What Is logic? What Should It Be?

Is it the rules of correct thinking?

Everyday experience, and ample study by psychologists, show that most of our thinking doesn't follow logic.

This might mean most human thinking is wrong. Or it might mean the scope of logic is too narrow.

Computing machines do almost always obey logic.

That's the answer! Logic is the rules of computing machinery! Logic also applies to people when they try to be computing machines.

Once upon a time logic and mathematics were separate. Then George Boole figured out how to make logic part of mathematics.

Russell claimed the opposite—that mathematics is nothing but logic. But the paradoxes made that idea unpalatable. Far from a solid foundation for mathematics, set theory/logic is now a branch of mathematics, and the least trustworthy branch at that.

Like other branches, logic has expanded greatly in scope and power. It offers problems and challenges, techniques and tools to other parts of mathematics. And it renounces any desire or duty to check up on other branches of mathematics, or to tell people how to think. For today's mathematical logician, logic is just another branch of mathematics like geometry or number theory. He disowns philosophical responsibilities.

"This book does not propose to teach the reader how to think. The word 'logic' is sometimes used to refer to remedial thinking, but not by us" (Enderton).

In U.S. philosophy departments, on the other hand, "analytic philosophy," a kind of left-over from logicism and logical positivism, lingers on. Kitcher gave it the fitting sobriquet "neo-Fregeanism" (see Carnap and Quine in Chapter 9).

Analytic philosophers mustn't be confused with mathematical logicians. A few outstanding logicians do encompass both mathematical and philosophical logic; that is, they are competent by both mathematical and philosophical standards.

Of course logical blunders aren't acceptable in mathematical reasoning. In that sense, mathematicians (and other scientists) are subject to logic. This isn't the business of logicians. It's the business of the mathematician and her referees.

Gottlob Frege (1848–1925)
Grandpa of the Mainstream

Frege is the first *full-time* philosopher of mathematics. According to Baum, "Although Frege is sometimes spoken of as being the first philosopher of mathematics, he was at most the initiator of the recent period of intensive concentration on this area by specialists using the tools of mathematical logic. Frege considered himself to be working entirely within the tradition of Plato, Descartes, Leibniz, etc. with regard to his work on the philosophy of mathematics" (Baum, p. 263). Frege's greatest contribution to learning is the *Begriffsschrift* (*Idea Script*), where he introduced quantifiers—symbols for "there exists" (now written backward E) and for "for all" (now written upside-down A). Quantifiers were independently invented by O. H. Mitchell, a student of Charles Sanders Peirce (Lewis, 1918; Putnam, 1990). Frege's introduction of quantifiers is considered the birth of modern logic. His technical logic is a means to a philosophical end. He wants to establish arithmetic as a part of logic.

It's believed that logic with quantifiers (usually called "predicate calculus") can express any reasoning mathematicians use in strict, formal proof. (We also do heuristic, intuitive, informal reasoning.) In principle, using Frege's notation or others developed later, it seems possible to write any complete mathematical proof in a form that a computer can check.

Kant thought geometry is based on space intuition, and arithmetic on time intuition. That made both geometry and arithmetic "synthetic a priori." About geometry, Frege agreed with Kant that it is a synthetic intuition. About arithmetic, he agreed with Leibniz: It is not *synthetic* but *analytic*. That is, it doesn't depend on an intuition of time. It comes from logic. For Frege and Leibniz, "logic" means the intuitively obvious rules of correct reasoning. These are supposed to be certain and indubitable, independent of anybody's thought or experience. Deriving arithmetic from logic would make arithmetic equally certain and indubitable.

One really cannot speak of Frege's philosophy of mathematics. He had a philosophy of arithmetic, and a different philosophy of geometry. Arithmetic is logic; geometry is space intuition. Fitting them together is as awkward as yoking an ape and an alligator. If arithmetic is part of logic, why not geometry as well (since we construct spaces from numbers by using coordinates)? On the other hand, if geometry is space intuition, why may not arithmetic be time intuition, as Kant had it?

Frege's Grundlagen. Logicism's Koran

In his *Grundlagen der Arithmetik* (Foundations of Arithmetic) Frege constructed the natural numbers out of logic. This achievement was ignored for 16 years, until Bertrand Russell took up the same project and made Frege known to the world. "Today, Frege's *Grundlagen* is widely appreciated as a philosophical masterpiece. In retrospect the mathematicians who ignored it appear as men who failed to recognize a pioneering work" (Kitcher, "Frege, Dedekind . . ."). Since all classical mathematics can be built from the natural numbers, Russell claimed that *all mathematics* is logic. This is called logicism.

Before giving his definition of number in the *Grundlagen*, Frege tries to demolish all previous definitions. He carries out a merciless, hilarious campaign against psychologism (numbers are ideas in someone's head—Berkeley, Schloemilch); historicism (numbers evolve); and empiricism (numbers are things in the physical world—Mill). To this day, philosophers of mathematics hardly dare contemplate psychologism or historicism.

First Frege trounces Mill, who based arithmetic on empirical experience (Frege, 1980, pp. 9–10): "The number 3 . . . consists, according to him, in this, that collections of objects exist, which while they impress the senses thus,

0 0 ,
 0

they may be separated into two parts, thus,

00   0.

What a mercy, then, that not everything in the world is nailed down; for if it were, we should not be able to bring off this separation, and 2 + 1 would not be 3!

What a pity that Mill did not also illustrate the physical facts underlying the number 0 and 1! . . . From this we can see that it is really incorrect to speak of three strokes when the clock strikes three, or to call sweet, sour and bitter three sensations of taste, and equally unwarrantable is the expression 'three methods of solving an equation'. For none of these is a parcel which ever impresses the senses thus,

0 0."
 0

(A bit unfair! Mill actually mentions strokes of the clock as an example of counting. See the article on Mill in the next chapter.)

Mill's lack of precision makes him an easy mark for Frege. Nevertheless, Mill is right to say number has something to do with physical reality. Every child learns arithmetic from the pebbles and ginger snaps Frege laughs at. If our ancestors didn't need to keep track of coconuts or fish heads, they wouldn't have invented arithmetic. Much deeper is the discovery, many times repeated, that mathematics is the language of nature. Mill grapples with the relation between numbers and physical reality. Frege brushes it aside. In that respect, Mill did a service to human understanding, Frege a disservice.

Next is Hermann Hankel. Frege writes, "the first question to be faced is whether number is definable." Hankel thought not, and expressed himself in this unfortunate manner: "What we mean by thinking or putting a thing once, twice, three times, and so on, cannot be defined, because of the simplicity in principle of the concept of putting." Replies Frege, "But the point is surely not so much the putting as the once, twice, and three times. If this could be defined, the indefinability of putting would scarcely worry us" (p. 26).

Next Frege turns on George Berkeley (p. 33), whom he quotes: "Number . . . is nothing fixed and settled, really existing in things themselves. It is entirely the creature of the mind. . . . We call a window one, a chimney one, and yet a house in which there are many windows, and many chimneys, hath an equal right to be called one, and many houses go to the making of one city." In fact, as mentioned in an earlier chapter, Berkeley thought numbers don't exist. To him, numerals were meaningless symbols.

Frege's answer: "This line of thought may easily lead us to regard number as something subjective . . . number is no whit more an object of psychology or a product of mental processes than, let us say, the North Sea is. The objectivity of the North Sea is not affected by the fact that it is a matter of our arbitrary choice which part of all the water on the earth's surface we mark off and elect to call the North Sea. This is no reason for deciding to investigate the North Sea by psychological methods. In the same way, number too, is something objective. If we say 'The North Sea is 10,000 square miles in extent' then neither by the 'North Sea' nor by '10,000' do we refer to any state of or process in our minds: on the contrary, we assert something quite objective, which is independent of our ideas and everything of the sort."

His attack on psychologism means to prove that numbers aren't ideas. To do so, he assumes surreptitiously that ideas are property only of individuals, uncorrelated with other people's ideas or with physical reality. An indefensible assumption!

Here is his next assault, against Schloemilch: "I cannot agree with Schloemilch either (p. 36), when he calls number the idea of the position of an item in a series. If number were an idea, then arithmetic would be psychology. But arithmetic is no more psychology than, say astronomy is. Astronomy is concerned, not with ideas of the planets, but with the planets themselves, and by the same token the objects of arithmetic are not ideas either. If the number two were an idea, then it would have straight away to be private to me only. [No! No!] Another man's idea is, *ex vi termini*, another idea. We should then have it might be many millions of twos on our hands. We should have to speak of my two and your two, of one two and all twos. If we accept latent or unconscious ideas, we would have unconscious twos among them, which would then return subsequently to consciousness. As new generations of children grew up new generations of twos would continually be being born and in the course of millennia these might evolve, for all we could tell, to such a pitch that two of them would make five. Yet, in spite of all this, it would still be doubtful whether there existed infinitely many numbers, as we ordinarily suppose. 10^{10}, perhaps, might be only an empty symbol, and there might exist no idea at all, in any being whatever, to answer to the name.

"Weird and wonderful, as we see, are the results of taking seriously the suggestion that number is an idea. And we are driven to the conclusion that number is neither spatial and physical, like Mill's piles of pebbles and gingersnaps, nor yet subjective, like ideas, but non-sensible and objective. Now objectivity cannot, of course, be based on any sense-impression, which as an affection of our mind is entirely subjective, but only, so far as I can see, on the reason. It would be strange if the most exact of all the sciences had to seek support from psychology, which is still feeling its way none too surely."

It's too late to defend Mill or Hankel or Schloemilch. But we must reject Frege's argument against "psychologism"—the belief that mathematical objects are ideas. He says 2 cannot be an idea, because different people have different ideas, and there is only one 2.

Frege is confounding private and public senses of "idea." It's not unusual to say "We have the same idea." Frege assumes an idea resides only in one person's head (private ideas). But ideas can be shared by several people, even millions of people (public ideas). Cheap and dear, legal and illegal, sacred and profane, patriotic and treasonous—all ideas, but not ideas of a particular person. Public ideas, part of society, history, and culture. (Philosophers say "intersubjective" to avoid "society" and "culture.") The existence of language, society, and all social institutions prove that people sometimes do have the same idea. Not in the sense of subjective inner consciousnesses; in the sense of verbal and practical understanding and agreement. A piece of green paper called a dollar is worth a quart of milk because many people agree that it is. All these people have the same idea—the equal value of a quart of milk and a dollar bill.

Whoever told Frege, "2 is an idea" intended the public meaning of "idea." Frege replaced that with the private meaning of "idea," and then had fun throwing stones at Schloemilch. Could Frege prove that number is *not* an intersubjective, social-cultural object? No. His sarcasm about psychologism has no bearing on my proposal that mathematical objects are ideas on the social level.

After Frege makes mincemeat of empiricism, historicism, psychologism, and (in another book, the *Grundgesetze*) formalism, you are ready for his solution: *NUMBERS ARE ABSTRACT OBJECTS.* Objects which are real, but not physically, not psychologically, real in an *abstract* sense.

What is "abstract"?

Evidently, *not* mental or physical. What qualities are possessed by abstract objects? They're timeless or tenseless. Aren't born, do not die.

What an astonishing kinship to Plato's Ideas! They were neither mental nor physical, but eternal and changeless. Frege is a Platonist as well as a Kantian.

Frege's abstract objects include numbers. Everything else has been proved wrong, so this must be right. But the elimination argument isn't valid, because Frege hasn't considered the alternative we offer: mathematics as part of the social-cultural-historical side of human knowledge. (He did attack the notion of numbers as historical entities.)

Frege's argument against formalism, psychologism, and empiricism comes down to a declaration: "Anyone can see that $7 + 3 = 10$. There's no possible doubt of it. Clearly it's true a priori, now and forever, certainly and indubitably." The same argument was given 1,300 years earlier by Augustine, Bishop of Hippo: "Seven and three are ten, not only now but always; nor was there ever a time when seven and three were not ten, nor will ever be a time when seven and three will not be ten. I say, therefore, that this incorruptible truth of number is common to me and to any reasoning person whatsoever."

Augustine's arithmetical Platonism went with his theology. The certainty of mathematics supported the certainty of religion. By Frege's time, the association between Christian theology and mathematical Platonism had gone underground. The success of secular science made it bad form to bring religion into logic or mathematics. Even so, David Hilbert and Bertrand Russell, unlike Frege, were frank about their religious motives.

The important thing is Frege's analysis of number. *Numbers are classes.* More precisely, *they're equivalence classes of classes*, under the equivalence relation of one-to-one mappings. For example, *two is the class of all pairs.* This class of all pairs exists objectively, timelessly, independently of us. It's an abstract object.

Some readers haven't seen arithmetic built up from logic. It's easy. Frege's logic uses the concept of "class." This is almost what we mean today by set. A set is defined by who are its *members* ("extension"). A class is defined by the *property* that decides whether or not you're a member ("intension").

Frege says "Two is the class of all pairs. Three is the class of all triplets. And so on." The point of this construction is to prove the "a priority" of the numbers, their independence of experience.

One slight problem: These definitions are circular. Knowing what's a "pair" is already knowing what's 2. A better statement is: 2 is by definition the class of all classes equivalent to {A,B}. 2 is called the cardinality of any such class (commonly known as a pair). 3 is the class of all classes equivalent to {A,B,C}. 3 is called the "cardinality" of any such class (commonly known as a triplet).

What's "equivalent"? Classes are equivalent if their members can be matched with nothing left over. Any two pairs are equivalent. Any two triples are equivalent. Any natural number, including 0 and 1, is a class of equivalent classes. The reader is encouraged to think through these two special cases.

A little explanation will help you sympathize with Frege. When he defines 2 as the equivalence class of all pairs, he's assuming that the notion, "equivalence class of all pairs" is free of ambiguity. To *justify* his definition of "2," we have to see if there *is* a "class of all pairs." If it doesn't exist, we needn't bother about it! That such a thing exists may have been crystal clear to Frege. It's not so clear today. Today, with caution learned from Frege's burnt fingers, "the class of all pairs" or "the set of all sets equivalent to {0,1}" would not go down so easily. "Pairs of what?" the students would rightfully demand.

Not pairs of numbers; we're trying to *define* numbers. Not shoes or socks; they're too earthy for a transcendental theory. Probably pairs of *abstract* objects. But where are they and what are they? To create the mathematical universe from scratch, I have no ingredients available. If I want to create numbers as collections, I need something to collect! In today's mathematics, one doesn't take for granted that any specification written in English or German is meaningful to define a set. Nowadays we want a set to be *located,* a subset of a given universal set. You may not simply "define" some infinite set. You must show that the definition isn't self-contradictory. Our caution is due partly to the disaster that befell Frege: the Russell paradox.**

This proposal opened a new direction of thinking in foundations. It was the basis of Frege's plan to make arithmetic part of logic/set theory.

Frege's influence is not so much his semi-Kantian philosophy as his statement of the issue—*establish mathematics on a solid, indubitable foundation.*

To an unprepared mind, Frege's definition of number is bizarre. It explains the clear and simple, number, by the complicated and obscure: infinite equivalence classes. The bizarrerie is mitigated if you remember the point—to reduce arithmetic to logic. Mathematicians know how to build analysis and geometry on arithmetic. Frege and Russell believe logic is rock-solid. If they could have built arithmetic on logic, that would have made *all* mathematics as solid as logic itself. It didn't work out that way.

0 is particularly nice. It's the class of sets equivalent to the set of all objects unequal to themselves! *No* object is unequal to itself, so 0 is the class of all empty sets. But all empty sets have the same members—none! So they're not merely *equivalent* to each other—they're all *the same* set. There's only one empty set! (A set is characterized by its membership list. There's no way to tell one empty membership list from another. Therefore all empty sets are the same thing!)

Once I have *the* empty set, I can use a trick of von Neumann as an alternative way to construct the number 1. Consider the class of *all* empty sets. This class has exactly one member: the unique empty set. It's a singleton. "Out of nothing" I have made a *singleton* set—a "canonical representative" for the cardinal number 1. 1 is the class of all singletons—all sets with but a single element. To avoid circularity: "1 is the class of all sets equivalent to the set [{ }]." In words, 1 is the class of all sets equivalent to the set whose only element is the empty set. Continuing, you get pairs, triplets, and so on. Von Neumann recursively constructs the whole set of natural numbers out of sets of sets of sets of nothing.

Set theory was introduced by Georg Cantor as a fundamental new branch of mathematics. The idea of set—any collection of distinct objects—was so simple and fundamental, it looked like a brick out of which all mathematics could be constructed. Even arithmetic could be downgraded (or upgraded) from primary to secondary rank, for the natural numbers could be constructed, as we have just seen, from nothing—i.e., the empty set—by operations of set theory.

At first set theory seemed to be the same as logic. The set-theoretic relation of inclusion, "A is a subset of B," is the same as the logical relation of implication, "If A, then B." "Logic" here means the fundamental laws of reason, of contradiction and implication—the objective, indubitable bedrock of the universe. To show mathematics is part of logic would show it's objective and indubitable. It would justify Platonism, passing to the rest of mathematics the indubitability of logic.

This was the "logicist program" of Russell and Whitehead's *Principia Mathematica*. The logicist school was philosophically (not technically) similar to Hilbert's formalist school. For the logicists it was logic that was indubitable a priori; for the formalists, it was finite combinatorics. The difference between the logicists and Kant is that they give up his claim that mathematics is synthetic a priori. They settle for analytic a priori. According to Bertrand Russell, mathematics is a vast tautology.

The logicists proposed to redeem all mathematics by injecting it with the soundness of logic. First of all, to reduce arithmetic to rock-solid logic. Is the notion of class rock-solid? Even an infinite class? No. If we include infinite sets, logic isn't rock-solid any more. But the class of singletons is already infinite. Without infinite sets, there's no mathematics.

Frege regarded "set" or "class" as equivalent to "property." To any property corresponds the set of things having that property. To any set corresponds the property of membership in it.

Frege's Fifth Basic Law says that to any properly specified property corresponds a set (Furth, 1964). Definition by properties gives a "concept." To Frege, defining numbers as sets is automatically defining them as concepts—not notions in someone's head, but abstract objects.

Frege was about to publish a monumental work in which arithmetic was reconstructed on the foundation of set theory. His hope was shattered in one of the most poignant episodes in the history of philosophy. Russell found a contradiction in the notion of set as he and Frege used it! After struggling for weeks to escape, he sent Frege a letter (van Heijenoort).

Frege added this postscript to his treatise: "A scientist can hardly meet with anything more undesirable than to have the foundations give way just as the work is finished. In this position I was put by a letter from Mr. Bertrand Russell, as the work was nearly through the press."

The axioms from which Russell and Frege attempted to construct mathematics are contradictory!

"My Basic Law concerning courses-of-values (V) . . . the (unrestricted) Axiom of Set Abstraction states that there exists, for any property we describe via an open formula, a set of things which possess the property. From this Axiom we can easily derive Russell's Paradox" (Musgrave, 1964, p. 101). Russell's paradox is catastrophic because it exhibits a legitimate property that is self-contradictory—a property to which no set can correspond.

The Russell paradox and the other "antinomies" showed that intuitive logic is riskier than classical mathematics, for it led to contradictions in a way that never happens in arithmetic or geometry. This was the "crisis in foundations," the central issue in the famous controversies of the first quarter of this century. Three remedies were proposed—logicism, intuitionism, and formalism. As we have already mentioned, all failed.

The response of "logicism," the school of Frege and Russell, was to reformulate set theory to avoid the Russell paradox, and thereby save the Frege-Russell-Whitehead project of establishing mathematics on logic as a foundation.

Work on this program played a role in the development of logic. But in terms of foundationism, it was a failure. To exclude the paradoxes, set theory had to be patched up into a complicated structure. It acquired new axioms such as the axiom of replacement (a complex recreation of Frege's Axiom 5) and the axiom of infinity (there exists an infinite set).

"There is something profoundly unsatisfactory about the axiom of infinity. It cannot be described as a truth of logic in any reasonable use of that term and so the introduction of it as a primitive proposition amounts in effect to the abandonment of Frege's project of exhibiting arithmetic as a development of logic" (Kneale and Kneale, p. 699).

This patched up set theory could not be identified with logic in the philosophical sense of "rules for correct reasoning." You can build mathematics out of this reformed set theory, but it no longer passes as a foundation, in the sense of justifying the indubitability of mathematics. Mathematics was not shown to be part of logic in the classical sense, as Russell and Whitehead dreamed. It became untenable to claim, as Russell had done, that mathematics is one vast tautology.

"Among all mathematical theories it is just the theory of sets that requires clarification more than any other" (Mostowski).

After Frege's first shock, he continued his foundationalist labors. Russell searched for a way out for a long time. He came up with a modified form of set theory, the theory of types. Zermelo introduced the axiom of foundation, which says any chain of set membership terminates in finitely many steps. This outlaws Russell's paradox by outlawing "Russell sets"—sets that belong to themselves—since the membership relation for a Russell set cycles round ad infinitum. (Recently a British computer scientist, Peter Aczel, published a version of set theory in which Zermelo's axiom of foundation is negated. This theory permits self-membership, and has applications in computer science. It has been proved to be relatively consistent!)

But set theory doctored up with the axiom of infinity, and Zermelo's axiom of foundation was no longer the perspicuous elementary set theory that had aroused foundationist hopes. Russell's paradox was unexpected. Are other paradoxes lurking?

The Russell paradox doomed that hope. Despite this philosophical failure, logico-set theoreticism dominates the philosophy of mathematics today. Philip Kitcher writes that "mathematical philosophy in the last 30 years is a series of footnotes to Frege." This suggests that mathematical philosophy is ready for new ideas and problems. Perhaps ideas and problems rising from today's mathematical practice.

Logicism never recovered from the Russell paradox. Eventually both Frege and Russell gave it up. Set theory had become, not clear and indubitable like elementary logic, but unclear and dubitable. To define a set by a property, I must show the property isn't self-contradictory. Such a demonstration can be harder than the problem it was supposed to clarify. "Reducing" arithmetic to logic was a disappointment. Instead of being anchored to the rock of logic, it was suspended from the balloon of set theory.

Frege's construction of number is defensible. But it's not sufficient to convince doubters that arithmetic is a priori. Its long-range importance wasn't Frege's philosophical goal but the stimulation it gave to logic and foundations. The Frege-Russell definition or the equivalent von Neumann definition let us derive arithmetic from facts about sets. Mainstream mathematicians ignored it for a long time. Frege at Jena was an unknown outsider, but even when the respected Richard Dedekind wrote on the foundation of the natural numbers, he too aroused little interest among mathematicians.

Mathematicians don't regard the natural numbers as a problem. With millennia of experience behind us, and deep, complex problems before us, we're not worrying about elementary arithmetic. Dedekind and Frege may object that we have only vague notions of what's meant by 0 or 1 or 2. Nevertheless, we have no qualms about 0, 1, 2.

Frege and Russell weren't mainly concerned with the opinion of the ordinary, unphilosophical mathematician. They were concerned with establishing mathematics on a solid foundation.

Frege always allowed geometry to rest on space intuition. In his old age he decided arithmetic too was based on geometry and space intuition (1979, pp. 267–81).

Hilbert published his *Grundlagen der Geometrie*, an epoch-making book that led to universal acceptance of the axiomatic method as the right way to present mathematics—in principle. In this book Hilbert (following Pasch and Peano) filled in the gaps in Euclid, making Euclidean geometry for the first time the rigorously logical subject it had always claimed to be. He did more. He showed that the axioms are ***independent*** (can't be deduced from each other) by giving examples in which all the axioms were satisfied except one.

Frege's Kantian views on geometry led him to attack Hilbert. He told Hilbert that Hilbert didn't know the difference between a definition and an axiom. Hilbert answered Frege's first letter or two (1979, pp. 167–73). Thereafter he ignored him. But Frege continued to crow. He even insinuated that Hilbert's failure to keep up the controversy was because Hilbert was afraid his results might be false!

Musgrave: "By 1924 Frege had come to the conclusion that 'the paradoxes of set theory have destroyed set theory.' He continued: 'The more I thought about it the more convinced I became that arithmetic and geometry grew from the same foundation, indeed from the geometrical one; so that the whole of mathematics is actually geometry.'" (These two remarks are quoted by Bynum in his Introduction to Frege [1972], cf. pp 53–54.)

Bertrand Russell (1872–1970)

A Loss of Faith

It wouldn't be too wrong to say philosophy of science in the twentieth century is mostly Bertrand Russell. Two other leading thinkers—Frege and Wittgenstein—are both Russell proteges. He didn't create them as philosophers, of course. But his enthusiasm for them is in part their compatibility with his logical atomism. In helping them become influential, he indirectly advances his own point of view.

Russell is frank about his motives, so far as he understands them. In philosophy of science, his leading motive is to establish certainty. In this, he confesses, he's seeking to replace the Christian faith he has rejected. He is also continuing an old tradition: Plato, Descartes, Leibniz, Kant. From "Reflections on My Eightieth Birthday" in *Portraits from Memory*:

"I wanted certainty in the kind of way in which people want religious faith. I thought that certainty is more likely to be found in mathematics than elsewhere. But I discovered that many mathematical demonstrations, which my teachers expected me to accept, were full of fallacies, and that, if certainty were indeed discoverable in mathematics, it would be in a new field of mathematics, with more solid foundations than those that had hitherto been thought secure. But as the work proceeded, I was continually reminded of the fable about the elephant and the tortoise. Having constructed an elephant upon which the mathematical world could rest, I found the elephant tottering, and proceeded to construct a tortoise to keep the elephant from falling. But the tortoise was no more secure then the elephant, and after some twenty years of very arduous toil, I came to the conclusion that there was nothing more that I could do in the way of making mathematical knowledge indubitable."

"Mathematics is, I believe," says Russell, "the chief source of the belief in eternal and exact truth, as well as in a super-sensible intelligible world. Geometry deals with exact circles, but no sensible object is *exactly* circular; however carefully we may use our compasses, there will be some imperfections and irregularities. This suggests the view that all exact reasoning applies to ideal as opposed to sensible objects; it is natural to go further, and to argue that thought is nobler than sense, and the objects of thought more real than those of sense-perception. Mystical doctrines as to the relation of time to eternity are also reinforced by pure mathematics, for mathematical objects, such as number, if real at all, are eternal and not in time. Such eternal objects can be conceived as God's thoughts. Hence Plato's doctrine that God is a geometer, and Sir James Jeans' belief that He is addicted to arithmetic. Rationalistic as opposed to apocalyptic religion has been, ever since Pythagoras, and notably ever since Plato, very completely dominated by mathematics and mathematical method.

"So it compels the soul to contemplate being, it is proper; if to contemplate becoming, it is not proper" (*Republic*, p. 326). For Plato, the "becoming" or the "unreal" is anything visible, ponderable, changeable. The "being," the "real," is invisible, immaterial, unchangeable. That means mathematics.

Russell calls himself a "logical atomist," in opposition to both the classical and evolutionist trends in early twentieth-century philosophy.

"Philosophy is to be rendered scientific" (p. 28).

"The philosophy which is to be genuinely inspired by the scientific spirit . . . brings with it—as a new and powerful method of investigation always does—a sense of power and a hope of progress more reliable and better grounded than any that rests on hasty and fallacious generalization as to the nature of the universe at large. . . . Many hopes which inspired philosophers in the past it cannot claim to fulfil; but other hopes, more purely intellectual, it can satisfy more fully than former ages could have deemed possible for human minds" (p. 20).

He's good at understated sarcasm. "The classical tradition in philosophy is the last surviving child of two very diverse parents: the Greek belief in reason, and the medieval belief in the tidiness of the universe. To the schoolmen, who lived amid wars, massacres, and pestilences, nothing appeared so delightful as safety and order . . . the universe of Thomas Aquinas or Dante is as small and neat as a Dutch interior. . . . To us, to whom safety has become monotony . . . the world of dreams is very different . . . the barbaric substratum of human nature, unsatisfied in action, finds an outlet in imagination (Written before August, 1914)."

Alan Musgrave (1977) quotes Russell, *An Essay on the Foundations of Geometry*, 1897, p. 1: "Geometry, throughout the 17th and 18th centuries, remained, in the war against empiricism, an impregnable fortress of the idealists. Those who held—as was generally held on the Continent—that certain knowledge, independent of experience, was possible about the real world, had only to point to Geometry: none but a madman, they said, would throw doubt on its validity, and none but a fool would deny its objective reference. The English Empiricists, in this matter, had, therefore, a somewhat difficult task; either they had to ignore the problem, or, if, like Hume and Mill, they ventured on the assault, they were driven into the apparently paradoxical assertion that Geometry at bottom, had no certainty of a different *kind* from that of Mechanics."

P. H. Nidditch
Frank Talk on Logicism

The logician and historian P. H. Nidditch gave a fair summing up of the logicist struggle to save the foundations of mathematics. "The effect of these discoveries (Russell & Burali-Forti antinomies) on the development of Mathematical Logic has been very great. The fear that the current systems of mathematics might not have consistency has been chiefly responsible for the change in the direction of Mathematical Logic towards metamathematics, for the purpose of becoming free from the disease of doubting if mathematics is resting on a solid base. A special reason for being troubled is that the theory of classes is used in all parts of mathematics; so if it is wrong in some way, they are possibly in error. Further, quite separately from the theory of classes, might not discoveries of opposite theorems in algebra, geometry or Mathematical Analysis suddenly come into view, as the discoveries of Burali-Forti and Russell had done? It has been seen that common sense is not good enough as a lighthouse for keeping one safe from being broken against the overhanging slope of sharp logic. To become certain